ON A DENSITY CONDITION FOR ${K_0}^+$ OF VON NEUMANN REGULAR RINGS.

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Abstract

P.Ara and K.R.Goodearl, in [1], introduced and studied the concept of a regular ring R satisfying the following condition, which they called *condition (D)*: $\Phi(K_0(R)^+)$ is dense in $\mathrm{Aff}(S(K_0(R),[R]))^+$, where Φ denotes the natural map from $K_0(R)$ to $\mathrm{Aff}(S(K_0(R),[R]))$. They proved that every nonartinian, stably finite, strictly unperforated, simple regular ring satisfies condition (D). In this note we prove that *a regular ring R satisfies condition (D) if and only if R has no nonzero artinian homomorphic image.* We then obtain as a consequence that every nonartinian, simple regular ring satisfies condition (D).

Introduction

One of the most famous and difficult problems related to the structure of von Neumann regular rings is the third question in Goodearl's book, which asks if every directly finite, simple regular ring must be unit-regular. In the past few years, some advances have been made in the direction of an affirmative answer.

The methods involved in those advances are mainly derived from the fact that the unit-regular rings are those for which finitely generated projective modules cancel from direct sums. An easy consequence of this last statement is that the Grothendieck group, K_0, has the interpolation property. Thus, a standard technique is to study decompositions of finitely generated projective modules, or of elements of K_0, trying to obtain analogous properties to the cancellation or the interpolation property.

P.Ara and K.R.Goodearl, in [1], show that, if R is a directly finite, simple regular ring, then $(\Phi(K_0(R)), \ll)$ is a simple dimension group (and therefore an interpolation group) provided that the image of $K_0(R)^+$ under the natural map

$$\Phi : K_0(R) \to \text{Aff}(S(K_0(R), [R]))$$

is dense in $\text{Aff}(S(K_0(R), [R]))^+$, and they give the name *condition (D)* to this property. Under this hypothesis, they obtain decomposition properties of modules quite similar to the unit-regular case.

Two interesting questions then are:

a) When does a directly finite, simple regular ring satisfy condition (D), and more generally, what kind of regular rings satisfy condition (D)?

b) What conditions on $K_0(R)$ imply that $K_0(R)$ is an interpolation group?.

In this paper we give a partial answer to *b)* and a complete answer to *a)* by showing that a regular ring R satisfies condition (D) if and only if R has no nonzero artinian homomorphic image.

1 Preliminaries

Throughout this work, we follow the notation of [4] and [5], except that we shall use P_R to denote the class of finitely generated projective right modules over a regular ring R.

We will also use s_N to denote the state associated to $N \in \mathbb{P}(R)$ by the affine homeomorphism $\Theta : S(K_0(R), [R]) \to \mathbb{P}(R)$ described in [4, Proposition 17.12]. In particular, $s_N([xR]) = N(x)$ for all $x \in R$.

We recall some definitions that we will use later, as:

Definition 1.1 *Let R be any ring. Then we say that R is* **directly finite** *provided that for any $x, y \in R$, if $xy = 1$, then $yx = 1$; and we say that R is* **stably finite** *if $M_n(R)$ is directly finite for all $n \in \mathbb{Z}^+$.*

Definition 1.2 *Let R be a regular ring, and let $\mathbb{P}(R)$ be the Choquet simplex of pseudo-rank functions on R. Then, we define $\operatorname{Ker}\mathbb{P}(R) = \bigcap_{N\in\mathbb{P}(R)}\operatorname{Ker}N$ if $\mathbb{P}(R) \neq \emptyset$, where $\operatorname{Ker}N = \{x \in R \mid N(x) = 0\}$ for $N \in \mathbb{P}(R)$, and $\operatorname{Ker}\mathbb{P}(R) = R$ if $\mathbb{P}(R) = \emptyset$. By [4, 16.7], every $\operatorname{Ker}N$ is a two-sided ideal in R, and hence so is $\operatorname{Ker}\mathbb{P}(R)$.*

Definition 1.3 *Let R be a regular ring. Then we say that R is* **N^*-torsion free** *if $\operatorname{Ker}\mathbb{P}(R) = 0$.*

Notice that, if $\mathbb{P}(R) \neq \emptyset$, we can define

$$N^*(x) = \sup\{N(x) \mid N \in \mathbb{P}(R)\}.$$

Hence, R is N^*-torsion free if $N^*(x) > 0$ for all nonzero $x \in R$, and, in particular, it is not difficult to see that R is stably finite.

Definition 1.4 *Let R be any ring, and let A, B be right R-modules. Then we write $A \lesssim B$ if A is isomorphic to a submodule of B, and we write $A \prec B$ if A is isomorphic to a proper submodule of B.*

Recall that when R is regular and $A, B \in P_R$, if $A \lesssim B$ then A is isomorphic to a direct summand of B. Similarly, if $A \prec B$ then A is isomorphic to a proper direct summand of B.

Definition 1.5 *Let R be a regular ring. Then we say that R satisfies the* **unperforation property** *if for all $A, B \in P_R$ and for all $n \in \mathbb{N}$, $nA \lesssim nB$ if and only if $A \lesssim B$. If this property holds with $\lesssim$ replaced by $\prec$, then R is said to satisfy the* **strict unperforation property**.

Definition 1.6 *Let G be a preordered abelian group with order-unit, and let $S(G, u)$ be the space of states on G. Then, we say that a state s on G is* **discrete** *if $s(G)$ is a cyclic subgroup of $\mathbb{R}$, that is, if there exists an $m \in \mathbb{N}$ such that $s(G) = \frac{1}{m}\mathbb{Z}$, and we say that s is* **indiscrete** *if it is not discrete, that is, if $s(G)$ is a dense subgroup of $\mathbb{R}$.*

Definition 1.7 *Let G be a partially ordered abelian group. Then G is said to satisfy the* **(Riesz) interpolation property** *provided that, for all $x_1, x_2, y_1, y_2 \in G$ such that $x_i \leq y_j$ for $i, j = 1, 2$, there exists $z \in G$ such that $x_i \leq z \leq y_j$ for $i, j = 1, 2$. Equivalent conditions are given in [5, 2.1]. If this property holds for strict inequalities, then G is said to satisfy the* **strict interpolation property**. *Further, G is said to be a* **Riesz group** *provided that G is a directed group satisfying the Riesz interpolation propety.*

Definition 1.8 *Let G be a partially ordered abelian group. Then G is said to be* **unperforated** *provided that for all $x \in G$ and for all $n \in \mathbb{N}$, $nx \in G^+$ implies $x \in G^+$, and G is said to be* **strictly unperforated** *provided that for all $x \in G$ and for all $n \in \mathbb{N}$, $nx > 0$ implies $x > 0$.*

Definition 1.9 *Let G be a partially ordered abelian group. Then G is said to be a* **dimension group** *if G is an unperforated, Riesz group.*

We will recall some results from [2], [7] and [10] that will be useful to prove the main theorem.

Remark 1.10 *If R is an N^*-torsion free regular ring, then its Hausdorff completion, S, with respect to the N^*-metric, satisfies:*

a) *S is an unperforated unit-regular ring; [10].*

b) *The natural map $\phi : R \to S$ is an injective ring homomorphism; [7].*

c) *For $N \in \mathbb{P}(R)$ and $s \in S$, define $\overline{N}(s) = \lim_{n\to\infty} N(s_n)$, where $(s_n)_{n\geq 1} \subseteq R$ and $\phi(s_n) \to s$ in S. Then:*

   1) *$\overline{N} \in \mathbb{P}(S)$; [2, Lemma 1.3].*

   2) *The map $\psi : \mathbb{P}(R) \to \mathbb{P}(S)$ defined by the rule $\psi(N) = \overline{N}$ is an affine homeomorphism; [10, Theorem 2.4]. The inverse map is given by the rule $\psi^{-1}(P) = P\phi$*

   3) *S is complete with respect to the N^*-metric; [10].*

   And so,

d) *$K_0(S)$ is an archimedean norm-complete dimension group; [7, Theorem 2.11].*

e) *$K_0(S)$ is the norm-completion of $K_0(R)$ with respect to the order-unit norm; [2, Cororollary 1.15].*

We will also use:

Proposition 1.11 ([5, Corollary 13.6]) *Let (G, u) be a nonzero dimension group with order-unit, and let $\Phi : G \to \text{Aff}(S(G, u))$ be the natural map. Set $A = \{p \in \text{Aff}(S(G, u)) \mid p(s) \in s(G)$ for all discrete $s \in \partial_e S(G, u)\}$. Then, $\Phi(G^+)$ is dense in A^+.*

Corollary 1.12 ([5, Corollary 13.7]) *Let* (G,u) *be a nonzero dimension group with order-unit, and let* $\Phi : G \to \mathrm{Aff}(S(G,u))$ *be the natural map. If* $\partial_e S(G,u)$ *contains no discrete states, then* $\Phi(G^+)$ *is dense in* $\mathrm{Aff}(S(G,u))^+$.

Finally, as we will frequently need to know whether states are discrete or indiscrete, we establish a relationship between this property and some chain conditions on the ring.

Lemma 1.13 (P.Ara, unpublished) *Let* R *be a regular ring, and let* $N \in \mathbb{P}(R)$. *Then* s_N *is a discrete state if and only if* $R/\mathrm{Ker}N$ *is artinian and* N *is a rational convex combination of extremal pseudo-rank functions.*

Proof: Suppose that s_N is discrete. Then, clearly $N(R) \subseteq s_N(K_0(R)) \bigcap [0,1]$ is a finite set. By [4, 16.7], if $\pi : R \to \overline{R} := R/\mathrm{Ker}N$ is the natural projection map, there exists a unique $N' \in \mathbb{P}(\overline{R})$ such that $N'\pi = N$, and moreover, N' is a rank function.

Since $N'(\overline{R})$ is a finite subset of $[0,1]$, we have that $\overline{R}$ contains no infinite sequences of nonzero orthogonal idempotents, and so, by [4, Corollary 2.16], $\overline{R}$ is semisimple artinian. By the assumption that s_N is discrete it is clear that N', and so N, is a rational convex combination of extremal pseudo-rank functions.

The converse is clear. *Q.E.D.*

Corollary 1.14 *If* R *is a regular ring, then* R *has no nontrivial artinian homomorphic image if and only if* $S(K_0(R),[R])$ *contains no discrete states.*

2 The Main Result

Lemma 2.1 *Let* R *be an* N^**-torsion free regular ring. Let* S *be the* N^**-metric completion of* R*, and let* $\phi : R \to S$ *be the natural inclusion. Then:*

a) The induced homomorphism

$$\mathrm{Aff}(\phi_*) : \mathrm{Aff}(S(K_0(R),[R])) \to \mathrm{Aff}(S(K_0(S),[S]))$$

is an isomorphism of partially ordered abelian groups with order-unit.

b) $\mathrm{Aff}(\phi_*)$ *is an isometry with respect to the supremum norms.*

c) $K_0(\phi)(K_0(R)^+)$ *is norm-dense in* $K_0(S)^+$.

Proof: a) Notice that, by 1.10,c2, $\psi : \mathbb{P}(R) \to \mathbb{P}(S)$ is an affine homeomorphism, which is the inverse of $\mathbb{P}(\phi) : \mathbb{P}(S) \to \mathbb{P}(R)$. As there is a natural affine homeomorphism $\Theta : S(K_0(R),[R]) \to \mathbb{P}(R)$, and similarly for S, it is clear that the map induced by $K_0(\phi)$, $\phi_* : S(K_0(S),[S]) \to S(K_0(R),[R])$, is an affine homeomorphism. Then, as Aff is a functor, we conclude that $\mathrm{Aff}(\phi_*)$ is an isomorphism of partially ordered abelian groups with order unit, as desired.
b) Since $\mathrm{Aff}(\phi_*)(1) = 1$, the isomorphism $\mathrm{Aff}(\phi_*)$ must be an isometry with respect to the order-unit norms in $\mathrm{Aff}(S(K_0(R),[R]))$ and $\mathrm{Aff}(S(K_0(S),[S]))$. However, in these groups the order-unit norms (with respect to the order-unit 1) coincide with the supremum norms $\|\cdot\|_\infty$.
c) We have the following commutative diagram:

$$(*)\qquad \begin{array}{ccc} \mathrm{Aff}(S(K_0(R),[R])) & \stackrel{\mathrm{Aff}(\phi_*)}{\rightarrow} & \mathrm{Aff}(S(K_0(S),[S])) \\ \Phi_R \uparrow & & \uparrow \Phi_S \\ K_0(R) & \stackrel{K_0(\phi)}{\rightarrow} & K_0(S). \end{array}$$

where $\mathrm{Aff}(\phi_*)$ is an isomorphism by (a) and Φ_R, Φ_S are the natural maps. Moreover Φ_S is a monomorphism, since $K_0(S)$ is archimedean. We claim that $K_0(\phi)$ is an isometry.To see this, observe that for any $x \in K_0(R)$,

$$\|x\|_{[R]} = \|\Phi_R(x)\|_\infty = \|\mathrm{Aff}(\phi_*) \circ \Phi_R(x)\|_\infty =$$

$$= \|\Phi_S \circ K_0(\phi)(x)\|_\infty = \|K_0(\phi)(x)\|_{[S]}.$$

Therefore, $K_0(\phi)$ is an isometry as claimed. The rest of the proof is essentially contained in the proof of [2, Corollary 1.15]. *Q.E.D.*

Lemma 2.2 *Let R be an N^*-torsion free regular ring. Let S be the N^*-metric completion of R, and let $N \in \mathbb{P}(R)$. Then, the state associated to N, s_N, is indiscrete if and only if the state associated to $\overline{N}$, $s_{\overline{N}}$, is indiscrete.*

Proof: Take any $N \in \mathbb{P}(R)$, $\overline{N} = \psi(N) \in \mathbb{P}(S)$, and let $s_N, s_{\overline{N}}$ be the associated states. It is clear that $s_N(K_0(R)) = s_{\overline{N}}(K_0(\phi)(K_0(R))) \subseteq s_{\overline{N}}(K_0(S))$. Since $K_0(\phi)(K_0(R)))$ is dense in $K_0(S)$ and $s_{\overline{N}}$ is continuous in the induced metric, we have that $s_N(K_0(R))$ is dense in $\mathbb{R}$ if and only if $s_{\overline{N}}(K_0(S))$ is dense in $\mathbb{R}$. Thus s_N is indiscrete if and only if $s_{\overline{N}}$ is indiscrete. *Q.E.D.*

Lemma 2.3 *Let R be a regular ring such that $\mathbb{P}(R) \neq \emptyset$ and let $\pi : R \to \overline{R} := R/\mathrm{Ker}\,\mathbb{P}(R)$ be the natural projection. Then the following hold:*

a) $K_0(\pi) : K_0(R) \to K_0(\overline{R})$ *is an epimorphism of partially ordered abelian groups with order-unit, and* $\mathrm{Ker}\,(K_0(\pi))$ *is the ideal generated by*

$$H = \{[xR] \mid x \in \mathrm{Ker}\,\mathbb{P}(R)\}.$$

b) $\mathbb{P}(\pi) : \mathbb{P}(\overline{R}) \to \mathbb{P}(R)$ *is an affine homeomorphism.*

c) $\mathrm{Aff}(\pi_*) : \mathrm{Aff}(S(K_0(R), [R])) \to \mathrm{Aff}(S(K_0(\overline{R}), [\overline{R}]))$ *is an isomorphism of partially ordered abelian groups with order-unit.*

d) If $N \in \mathbb{P}(R)$ and $\overline{N} = \mathbb{P}(\pi)^{-1}(N) \in \mathbb{P}(\overline{R})$, then s_N is indiscrete if and only if $s_{\overline{N}}$ is indiscrete.

Proof: a) By [4, Proposition 15.15].
b) By [4, Prop.16.19], $\mathbb{P}(\pi) : \mathbb{P}(\overline{R}) \to \mathbb{P}(R)$ is an affine homeomorphism from $\mathbb{P}(\overline{R})$ onto a closed face F of $\mathbb{P}(R)$. Let $N \in \mathbb{P}(R)$ be a pseudo-rank function.

As $\operatorname{Ker}\mathbb{P}(R) \subseteq \operatorname{Ker} N$, [4, Prop. 16.7] implies that there exists a unique pseudo-rank function N' on $\overline{R}$ such that $N'\pi = N$. Thus, $F = \mathbb{P}(R)$ and the result holds.

c) Replacing ϕ_* with π_* and S with $\overline{R}$ in the proof of 2.1, we obtain (c).

d) Take $e \in R$, and observe that

$$s_N([eR]) = N(e) = \overline{N}\pi(e) = s_{\overline{N}}([\pi(e)(\overline{R})]).$$

As $K_0(R)^+ = \langle [eR] \mid e \in R\rangle$ and $K_0(R)$ is directed as ordered group, we conclude that $s_N(K_0(R)) = s_{\overline{N}}(K_0(\overline{R}))$. Thus, the result holds. *Q.E.D.*

Notice that if $\mathbb{P}(R) = \emptyset$, then $\operatorname{Ker}\mathbb{P}(R) = R$ and so we have that $\overline{R} := R/\operatorname{Ker}\mathbb{P}(R) = (0)$, $\mathbb{P}(\overline{R}) = \emptyset$ and $\operatorname{Aff}(S(K_0(R),[R])) = \operatorname{Aff}(S(K_0(\overline{R}),[\overline{R}])) = 0$, whence (a),(b),(c) and (d) obviously hold.

We are now ready to show the main result of the paper.

Theorem 2.4 *Let R be a regular ring, let $K_0(R)$ be its Grothendieck group, let $S = S(K_0(R),[R])$ be the space of states on $K_0(R)$, and let $\Phi : K_0(R) \to \operatorname{Aff}(S)$ be the natural map. For each $s \in S$, set $B_s = s(K_0(R))$ if s is discrete, $B_s = \mathbb{R}$ if s is indiscrete. Set $A = \{p \in \operatorname{Aff}(S) \mid p(s) \in B_s \,\forall s \in \partial_e S\}$. Then, $\Phi(K_0(R)^+)$ is dense in A^+.*

Proof: The general case is easily reduced to the case where R is N^*-torsion free by using 2.3. Therefore, we will assume that R is an N^*-torsion free regular ring. Also, it is not difficult to see that $\{p \in \operatorname{Aff}(S) \mid p(s) \in s(K_0(R))$ for all discrete $s \in \partial_e S\} = \{p \in \operatorname{Aff}(S) \mid p(s) \in B_s \,\forall s \in \partial_e S\}$.

Let S be the N^*-metric completion of R. Then, $K_0(S)$ is a nonzero dimension group with order-unit, whence the result holds for S by 1.11. Denote by A_R the set A in $\operatorname{Aff}(S(K_0(R),[R]))$, and A_S the set A in $\operatorname{Aff}(S(K_0(S),[S]))$. By 2.1 and 2.2, it is clear that $\operatorname{Aff}(\phi_*)(A_R^+) = A_S^+$.

Let $p \in A_R^+$, $\epsilon > 0$. Then $\mathrm{Aff}(\phi_*)(p) \in A_S^+$, and so there exists $B \in P_S$ such that

$$\|\Phi_S([B]) - \mathrm{Aff}(\phi_*)(p)\|_\infty < \frac{\epsilon}{2}.$$

By 2.1, there exists $A \in P_R$ such that

$$\|K_0(\phi)([A]) - [B]\|_{[S]} < \frac{\epsilon}{2},$$

and so, by the triangle inequality

$$\|\mathrm{Aff}(\phi_*)(p) - \Phi_S \circ K_0(\phi)([A])\|_\infty < \epsilon.$$

Then, as $\mathrm{Aff}(\phi_*)$ is an isometry, the commutative diagram (*) (proof of 2.1) gives us

$$\|p - \Phi_R([A])\|_\infty < \epsilon.$$

So, $\Phi(K_0(R)^+)$ is dense in A_R^+. *Q.E.D.*

Corollary 2.5 *A regular ring R satisfies condition (D) if and only if R has no nonzero artinian homomorphic image.*

Proof: Assume that R has no nonzero artinian homomorphic image. Then, by 1.14, $S(K_0(R), [R])$ contains no discrete states, whence $A_R = \mathrm{Aff}(S)$, and thus the result holds directly from 2.4.

Conversely, if R satisfies condition (D), then all states on $K_0(R)$ are indiscrete, and so by 1.14 R has no nonzero artinian homomorphic image. *Q.E.D.*

In particular, if R satisfies condition (D), then $\mathrm{soc}(R_R) \subseteq \mathrm{Ker}\,\mathbb{P}(R)$. The converse is false, as is easy to see by considering any free regular Λ-algebra with Λ a commutative ring [6], or any regular ring of the form $(\prod_{i=1}^\infty F_i)/(\bigoplus_{i=1}^\infty F_i)$, where $F_1, F_2, \ldots$ are fields.

As a corollary, we obtain the following:

Corollary 2.6 *If R is a nonartinian, simple regular ring, then R satisfies condition (D).*

Proof: As R is a simple ring, R has no nonzero artinian homomorphic image. Hence, by 2.5, the result holds. *Q.E.D.*

It is well known that every strictly unperforated, directly finite, simple regular ring is unit-regular [9, Corollary 4], and particularly its Grothendieck group, $K_0(R)$, is a simple strictly unperforated Riesz group , a class of groups which has recently been studied by Elliot in [3]. Our next result shows that, in order to prove that $K_0(R)$ is a simple strictly unperforated Riesz group, we only need that $K_0(R)$ is strictly unperforated.

Proposition 2.7 *Let (G, u) be a simple, strictly unperforated, partially ordered abelian group with order-unit, and let $\Phi : G \to \mathrm{Aff}(S(G, u))$ be the natural homomorphism, and suppose that $\Phi(G^+)$ is dense in $\mathrm{Aff}(S(G, u))^+$. Assume also that $S(G, u)$ is a Choquet simplex. Then G is an interpolation group.*

Proof: Clearly, it suffices to prove that G has the strict interpolation property. Let $x_1, x_2, y_1, y_2 \in G$ such that $x_i < y_j\ \forall i, j$. Then $\Phi(x_i) \ll \Phi(y_j)$. Choose $\epsilon > 0$ such that $\Phi(x_i) + \epsilon \ll \Phi(y_j) - \epsilon\ \forall i, j$. As $\mathrm{Aff}(S(G, u))$ is an interpolation group [5, Thm 11.4], there exists $f \in \mathrm{Aff}(S(G, u))$ such that $\Phi(x_i) + \epsilon \leq f \leq \Phi(y_j) - \epsilon\ \forall i, j$. Since $\Phi(G^+)$ is dense in $\mathrm{Aff}(S(G, u))^+$, there exists $z \in G$ such that $\|\Phi(z) - f\| < \epsilon$, and so $\Phi(x_i) \leq f - \epsilon \ll \Phi(z) \ll f + \epsilon \leq \Phi(y_j)\ \forall i, j$. As G is strictly unperforated, then by [5, 7.8], $x_i < z < y_j\ \forall i, j$, whence G has the strict interpolation property. *Q.E.D.*

Corollary 2.8 *Let R be a nonartinian stably finite regular ring, and suppose that for all $N \in \mathbb{P}(R)$, $\mathrm{Ker}N = 0$, and that $K_0(R)$ is strictly unperforated. Then, $K_0(R)$ is a simple strictly unperforated Riesz group.*

Proof: As all pseudo-rank functions on R are actually rank functions, R has no nontrivial stably finite homomorphic image, that is, $K_0(R)$ has no nontrivial ideals. So, $(K_0(R), [R])$ is a directed, strictly unperforated, simple, partially ordered abelian group with order-unit. Moreover, R satisfies condition (D), by 2.5, and $S(K_0(R), [R])$ is a Choquet simplex by [4, 17.5]. Now by 2.7 the result holds. *Q.E.D.*

The last result can fail if the ring has pseudo-rank functions that are not rank functions, because the condition (D) can fail. An example of this situation is given by the ring T defined in [8] with constant $n = 2$: $K_0(T)$ is a strictly unperforated group with torsion, but it is not an interpolation group, and it is easy to see that $\mathbb{P}(T) = \{N\}$, where $\mathrm{Ker}N \neq 0$ and $T/\mathrm{Ker}N$ is a commutative field, whence T has a nontrivial artinian homomorphic image and condition (D) fails.

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