Stochastic Bounds for Conditional Distributions Under Positive Dependence

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Abstract

We provide stochastic bounds for conditional distributions of individual risks in a portfolio, given that the aggregate risk exceeds its value at risk. Expectations of these conditional distributions can be interpreted as marginal risk contributions to the aggregate risk as measured by the tail conditional expectation. We first provide general lower and upper stochastic bounds and then we obtain further improvements of the bounds in the case of a portfolio consisting of dependent risks. We also derive new characterizations of comonotonic random vectors.

MSC: IM30

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1 Introduction and motivation

For purposes of risk management and insurance pricing, the risk capital of a company has often to be allocated to its different business lines. This process requires decomposing the aggregate risk of the company into individual or marginal risk contributions. Each marginal contribution assigns part of the risk to a particular business line and these contributions are then used to allocate capital. Cummins (2000) provides an overview of the various techniques that have been suggested in the actuarial literature for capital allocating. In this paper, we provide stochastic bounds for conditional distributions whose means can be interpreted in terms of marginal risk contributions.

Consider a portfolio of n individual risks $X_1, ..., X_n$ and let $S = X_1 + ... + X_n$ be the aggregate risk. Assume that the impact of a possible dependence among the individual risks is modeled by a random vector $\mathbf{X} = (X_1, ..., X_n)$ with some dependence structure. Let F_i be the distribution function of X_i and let F_i^{-1} be the corresponding quantile function, defined by $F_i^{-1}(p) = \inf \{x : F_i(x) \geq p\}$, $0 \leq p \leq 1$. The distribution function of S is denoted by F_S and the corresponding quantile function is denoted by F_S^{-1}. In the literature, one finds different methods and formulae to evaluate marginal risk contributions which are associated to several commonly used risk measures (see Dhaene et al., 2011, for a review). One of the most important measures to evaluate the aggregated risk is the tail conditional expectation (or expected shortfall) of S defined by

$$\mathbf{TCE}_S(p) = E\left[S \,\middle|\, S > F_S^{-1}(p)\right], \tag{1}$$

for some $0 < p < 1$. The tail conditional expectation, which coincides with the tail value-at-risk under the assumption of continuous distributions (see Section 2.4 in Denuit et al., 2005) is a coherent risk measure (in the sense of Artzner et al., 1999) that represents the expected risk given that the total risk exceeds its p-quantile. Based on the observation that

$$\mathbf{TCE}_S(p) = \sum_{i=1}^{n} E\left[X_i \,\middle|\, S > F_S^{-1}(p)\right], \tag{2}$$

it is natural to say that the marginal contribution of the risk X_i, $i = 1, ..., n$, to the aggregate risk (as measured by (1)) is given by

$$E\left[X_i \,\middle|\, S > F_S^{-1}(p)\right]. \tag{3}$$

Using this formula, the company allocates capital in a simple way: the capital required for the business line i is its expected contribution to the aggregate risk when the aggregate risk exceeds its value at risk. Contributions of the form (3) are examples of Euler contributions (see Tasche, 1999, and Overbeck, 2000), which satisfy the additivity rule and some other desirable properties from an economic point of view. Further references on this allocation are Venter (2004), Kalkbrener (2005) and, more recently, Dhaene et al. (2011) and Asimit et al. (2011).

Some authors, including Panjer (2002), Landsman and Valdez (2003), Cai and Li (2005), Chiragiev and Landsman (2007), Furman and Landsman (2005, 2008, 2010) and Furman and Zitikis (2008), have obtained explicit expressions for (3) under several different parametric models. Of course, the exact calculation of (3) needs the exact distribution of the vector $\mathbf{X}$. Unfortunately, in many applications only partial information or no information about the dependence structure among the n risks $X_1, ..., X_n$ is available. In practice, when only marginal distributions are available, due to the inequality

$$E\left[X_i \left| S > F_S^{-1}(p)\right.\right] \leq \mathbf{TCE}_{X_i}(p), \text{ for } i = 1, ..., n, \tag{4}$$

(Aubin, 1981), $\mathbf{TCE}_{X_i}(p)$ is taken as a measure of the marginal contribution (3) in the worst case scenario. Inequality (4) motivates the following more general question: can we bound stochastically, from both below and above, conditional random variables of the form

$$\left\{X_i \left| S > F_S^{-1}(p)\right.\right\}, \ i = 1, ..., n \tag{5}$$

if we only know the marginal distribution functions? We address this question in Section 2. Specifically, for $p \in (0, 1)$ and $i = 1, ..., n$, we show that

$$\left\{X_i \left| X_i < F_i^{-1}(1-p)\right.\right\} \leq_{st} \left\{X_i \left| S > F_S^{-1}(p)\right.\right\} \leq_{st} \left\{X_i \left| X_i > F_i^{-1}(p)\right.\right\} \tag{6}$$

where $\leq_{st}$ denotes the usual stochastic order. Inequality (6) presents the following advantages over inequality (4): (i) since stochastic ordering implies ordering of expectations, the second inequality in (6) is clearly stronger and more informative than (4); (ii) in particular, (6) provides bounds for the quantiles (or values at risk) of the random variable (5), for any level of risk; (iii) the fact that (6) provides a lower bound for (5) allows us to control the error when we approximate the marginal risk contribution of X_i by its tail conditional expectation.

As we show below, the upper bound is reached when the bivariate random vector (X_i, S) is comonotonic, therefore we study in Section 3 the relationship between the comonotonicity of $\mathbf{X}$ and the comonotonicity of the random vectors (X_i, S) for $i = 1, ..., n$. As a consequence, we show that $\mathbf{X}$ is comonotonic if and only if

$$\mathbf{TCE}_S(p) = \sum_{i=1}^{n} \mathbf{TCE}_{X_i}(p) \text{ for all } p \in (0,1).$$

The fact that the lower bound in (6) is not very informative when p is close to 1 (which is often the case in applications) and the intuition that (3) should be greater under positive dependence of $\mathbf{X}$ than under independence, suggest to consider the dependence structure of the vector in order to refine this lower bound. In Section 4, we formalize the idea of "positively dependent" random vectors by considering "conditionally increasing" random vectors (Muller and Scarsini, 2001). Under this assumption, we obtain a lower bound on (5) that is substantially sharper than the lower bound in (6), specifically

$$X_i \leq_{st} \left\{ X_i \,\middle|\, S > F_S^{-1}(p) \right\}, \text{ for } i = 1, ..., n, \text{ for all } p \in (0,1) \tag{7}$$

which formalizes the intuitive idea that X_i needs more capital in the allocation process when X_i is part of a conditionally increasing random vector that when it is considered alone. In Section 5 we illustrate graphically the results and study the closeness of the bounds. Section 6 contains conclusions.

Throughout this paper, expected values are assumed to exist whenever they are mentioned. Note that, given a random vector $\mathbf{X} = (X_1, ..., X_n)$ with continuous marginal distribution function, the distribution function of the sum $S = X_1 + ... + X_n$ is either continuous or degenerated (this is the case, for instance, of the random vector $(X, 1 - X)$, where X is uniform on $(0, 1)$). However, if S is degenerated, the event $\left\{S > F_S^{-1}(p)\right\}$ has null probability for all $p \in (0, 1)$ and (5) is not defined. Consequently, unless otherwise stated, we assume that the sum S is continuous. We use $\equiv_{st}$ to denote equality in distribution.

2 Stochastic bounds and characterizations

Let X and Y be two continuous risks with distribution functions F and G, respectively. Our first result provides stochastic upper and lower bounds on conditional distributions of the form $\{X \,|Y > G^{-1}(p)\}$ for any probability level $p \in (0,1)$. Recall that, given two random variables X and Y with respective survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, we say that X is smaller than Y in the stochastic order, denoted by $X \leq_{st} Y$, if $\overline{F}(x) \leq \overline{G}(x)$ for all x.

Theorem 1 *Let (X,Y) be an absolutely continuous random vector with respective marginal distribution functions F and G. Given $p \in [0,1]$ we have*

$$\left\{X \,\middle|X < F^{-1}(1-p)\right\} \leq_{st} \left\{X \,\middle|Y > G^{-1}(p)\right\} \leq_{st} \left\{X \,\middle|X > F^{-1}(p)\right\}. \quad (8)$$

Proof. Given $p \in (0,1)$, we denote by

$$\overline{F}_{\{X|Y>G^{-1}(p)\}}(x), \quad x \in \mathbf{R},$$

the survival function of the conditional random variable $\{X \,|Y > G^{-1}(p)\}$, given by

$$\begin{aligned}\overline{F}_{\{X|Y>G^{-1}(p)\}}(x) &= P\left[X > x \,\middle|Y > G^{-1}(p)\right] \\ &= \frac{P\left[X > x, Y > G^{-1}(p)\right]}{1-p} \\ &= \frac{\overline{F}(x) - p + P\left[X \leq x, Y \leq G^{-1}(p)\right]}{1-p}. \end{aligned} \quad (9)$$

Using the well-known Fréchet-Hoeffding bounds inequality (Fréchet, 1951) the joint distribution function in the numerator of (9) satisfies

$$\max\{F(x) + p - 1, 0\} \leq P\left[X \leq x, Y \leq G^{-1}(p)\right] \leq \min\{F(x), p\}. \quad (10)$$

Therefore,

$$\begin{aligned}\frac{\overline{F}(x) - p + \max\{F(x) + p - 1, 0\}}{1-p} &\leq \overline{F}_{\{X|Y>G^{-1}(p)\}}(x) \\ &\leq \frac{\overline{F}(x) - p + \min\{F(x), p\}}{1-p},\end{aligned}$$

or, equivalently,

$$\max\left\{\frac{\overline{F}(x) - p}{1-p}, 0\right\} \leq \overline{F}_{\{X|Y>G^{-1}(p)\}}(x) \leq \min\left\{\frac{\overline{F}(x)}{1-p}, 1\right\}, \text{ for all } x. \quad (11)$$

It is easy to see that the lower bound in (11) is the survival function of the random variable

$$\left\{ X \left| X < F^{-1}(1-p) \right. \right\}$$

and the upper bound is the survival function of the random variable

$$\left\{ X \left| X > F^{-1}(p) \right. \right\},$$

therefore (11) is the same as (8). ■

Now consider a portfolio of n individual risks $X_1, ..., X_n$ with respective distribution functions $F_1, ..., F_n$ and let $S = X_1 + ... + X_n$ be the aggregate risk with distribution function F_S. In this context, we are interested in obtaining stochastic bounds on conditional distributions of the form (5). The following result is a direct application of Theorem 1.

Corollary 2 *Let* $\mathbf{X} = (X_1, ..., X_n)$ *be an absolutely continuous random vector with marginal distribution functions* $F_1, ..., F_n$. *Let* $S = X_1 + ... + X_n$ *be the aggregate risk with distribution function* F_S. *Then,*

$$\left\{ X_i \left| X_i < F_i^{-1}(1-p) \right. \right\} \leq_{st} \left\{ X_i \left| S > F_S^{-1}(p) \right. \right\} \leq_{st} \left\{ X_i \left| X_i > F_i^{-1}(p) \right. \right\} \tag{12}$$

for $p \in (0, 1)$ *and* $i = 1, ..., n$.

Since the bounds in (12) only depend on marginal distributions, they are much more tractable than the distribution of the bounded random variable, which depends on the joint distribution of the vector. Now, taking into account that

$$E\left[X_i \left| X_i < F_i^{-1}(1-p) \right. \right] = -\mathbf{TCE}_{-X_i}(p), \text{ for all } p \in (0, 1), \tag{13}$$

the following corollary easily follows from the previous one. This result provides lower and upper bounds for the marginal risk contributions (3) in terms of the tail conditional expectations of the marginals.

Corollary 3 *Let* $\mathbf{X} = (X_1, ..., X_n)$ *be an absolutely continuous random vector with marginal distribution functions* $F_1, ..., F_n$. *Let* $S = X_1 + ... + X_n$ *be the aggregate risk with distribution function* F_S. *Then,*

$$-\mathbf{TCE}_{-X_i}(p) \leq E\left[X_i \left| S > F_S^{-1}(p) \right. \right] \leq \mathbf{TCE}_{X_i}(p) \tag{14}$$

for $p \in (0, 1)$ *and* $i = 1, ..., n$.

The second inequality in (14) was first obtained by Aubin (1981) in the framework of game theory. Combining (2) and (14) we easily obtain lower and upper bounds for the tail conditional expectation of the aggregate risk

Corollary 4 *Let* $\mathbf{X} = (X_1, ..., X_n)$ *be an absolutely continuous random vector and let* $S = X_1 + ... + X_n$ *be the aggregate risk. Given* $p \in (0, 1)$ *we have*

$$-\sum_{i=1}^{n} \mathbf{TCE}_{-X_i}(p) \leq \mathbf{TCE}_S(p) \leq \sum_{i=1}^{n} \mathbf{TCE}_{X_i}(p). \tag{15}$$

The second inequality in (15) is a well-known consequence of the subadditivity of the tail conditional expectation. Since X_i is continuous, we can write

$$E\left[X_i \middle| X_i < F_i^{-1}(1-p)\right] = \frac{\int_0^{1-p} F^{-1}(t)dt}{1-p},\ 0 < p < 1,$$

$$E\left[X_i \middle| X_i > F_i^{-1}(p)\right] = \frac{\int_p^{1} F^{-1}(t)dt}{1-p},\ 0 < p < 1,$$

and (15) can be expressed in the following terms:

$$\sum_{i=1}^{n} \int_0^{1-p} F_i^{-1}(t)dt \leq \int_p^1 F_S^{-1}(t)dt \leq \sum_{i=1}^{n} \int_p^1 F_i^{-1}(t)dt.$$

3 The case of comonotonic random vectors

The concept of comonotonicity plays an important role in actuarial theory (see Dhaene et al., 2002ab). A random vector $(X_1, ..., X_n)$ is said to be comonotonic if there exists a random variable Z and non-decreasing functions $f_1, ..., f_n$ on $\mathbf{R}$ such that

$$(X_1, ..., X_n) \equiv_{st} (f_1(Z), ..., f_n(Z)).$$

Therefore, comonotonicity is used for modelling situations where individual risks are subject to the same external mechanism. In the bivariate case, there exists a kind of opposite of comonotonicity, namely countermonotonicity. A random vector (X_1, X_2) is said to be countermonotonic if it is distributed as $(g_1(Z), g_2(Z))$ for some random variable Z, an increasing function g_1 and a decreasing function g_2 (this concept does not extend to higher dimensions). We have the following result as a consequence of Theorem 1.

Corollary 5 *Let X and Y be two continuous risks with distribution functions F and G, respectively.*
(a) If (X,Y) is comonotonic, then

$$\left\{X \left| Y > G^{-1}(p) \right.\right\} \equiv_{st} \left\{X \left| X > F^{-1}(p) \right.\right\}, \quad \textit{for all} \;\; p \in (0,1).$$

(b) If (X,Y) is countermonotonic, then

$$\left\{X \left| Y > G^{-1}(p) \right.\right\} \equiv_{st} \left\{X \left| X < F^{-1}(1-p) \right.\right\}, \quad \textit{for all} \;\; p \in (0,1).$$

Proof. It is well-known (see, for example, section 1.9.2 in Dhaene et al., 2005) that a random vector is comonotonic if and only if its joint distribution function is the Fréchet-Hoeffing upper bound distribution function. Therefore, if (X,Y) is comonotonic, the second inequality in (10) becomes equality and (a) follows. Part (b) follows similarly by taking into account that the random vector (X,Y) is countermonotonic if and only if its joint distribution is the Fréchet-Hoeffing lower bound distribution. ■

The following corollary is an immediate consequence of the previous one.

Corollary 6 *Let $\mathbf{X} = (X_1, ..., X_n)$ be an absolutely continuous random vector with marginal distribution functions $F_1, ..., F_n$. Let $S = X_1 + ... + X_n$ be the aggregate risk with distribution function F_S.*
(a) If (X_i, S) is comonotonic, then

$$\left\{X_i \left| S > F_S^{-1}(p) \right.\right\} \equiv_{st} \left\{X_i \left| X_i > F_i^{-1}(p) \right.\right\}, \quad p \in (0,1), \; i = 1, ..., n.$$

(b) If (X_i, S) is countermonotonic, then

$$\left\{X_i \left| S > F_S^{-1}(p) \right.\right\} \equiv_{st} \left\{X_i \left| X_i < F_i^{-1}(1-p) \right.\right\}, \quad p \in (0,1), \; i = 1, ..., n.$$

Corollary 6 suggests to study the relationship between the comonotonicity of the initial vector $\mathbf{X}$ and the comonotonicity of the pairs (X_i, S) for $i = 1, ..., n$. It is easy to show that if $\mathbf{X}$ is comonotonic then the bivariate vectors (X_i, S) are comonotonic for $i = 1, ..., n$. However, in general, comonotonicity of the pairs (X_i, S), for $i = 1, ..., n$, will not necessary imply comonotonicity of $\mathbf{X}$. Consider, for example, the random vector $(X, 1-X)$, where X is uniformly distributed on $(0,1)$. Then $S = 1$ and $(X, 1)$ and $(1-X, 1)$ are both comonotonic, but $(X, 1-X)$ is not comonotonic. However, if F_S is non-degenerate and the vectors (X_i, S) are comontonic for $i = 1, ..., n$, then $\mathbf{X}$ is also comonotonic.

Theorem 7 *Let* $\mathbf{X} = (X_1, ..., X_n)$ *be an absolutely continuous random vector with marginal distribution functions* $F_1, ..., F_n$ *and let* $S = X_1 + ... + X_n$ *be the aggregate risk with distribution function* F_S*. Then*
(a) If $\mathbf{X}$ *is comonotonic then the vectors* (X_i, S) *are comonotonic for* $i = 1, ..., n$*.*
(b) If F_S *is non-degenerate and the vectors* (X_i, S) *are comonotonic for* $i = 1, ..., n$*, then* $\mathbf{X}$ *is comonotonic.*

Proof. Suppose that $\mathbf{X}$ is comonotonic or, equivalently (see Proposition 2.1.a in Cuesta-Albertos, Rüschendorf and Tuero-Díaz, 1993), that there exists a random variable $U \equiv_d U(0,1)$ such that for some non-decreasing functions $f_1, ..., f_n$,

$$X_1 = f_1(U), ..., X_n = f_n(U), \tag{16}$$

almost surely. Then $S = f_1(U) + .. + f_n(U) = g(U)$ almost surely, where g is non-decreasing and, therefore, (X_i, S) are also comonotonic for $i = 1, ..., n$. Now suppose that F_S is non-degenerate and the bivariate vectors (X_i, S) are comonotonic for $i = 1, ..., n$. Then, from Proposition 2.1.d in Cuesta-Albertos, Rüschendorf and Tuero-Díaz (1993)

$$X_i = F_i^{-1} \circ F_S(S) \text{ almost surely for } i = 1, ..., n.$$

Thus (16) holds almost surely, where $f_i = F_i^{-1}$, for $i = 1, ..., n$ and $U = F_S(S)$, which implies that $\mathbf{X}$ is comonotonic. ■

Now we provide two characterizations of comonotonic random vectors. The first one shows that a random vector is comononotonic if and only if the upper bounds in (12) are attained. This result extends, in particular, a result of Dhaene et al. (2008, Theorem 3.1) who showed that for a comonotonic random vector with continuous marginals,

$$E\left[X_i \left| S > F_S^{-1}(p)\right.\right] = E\left[X_i \left| X_i > F_i^{-1}(p)\right.\right],$$

for all $p \in (0,1)$ and $i = 1, ..., n$.

Corollary 8 *Let* $\mathbf{X} = (X_1, ..., X_n)$ *an absolutely continuous random vector with marginal distribution functions* $F_1, ..., F_n$ *and let* $S = X_1 + ... + X_n$ *be the aggregate risk with distribution function* F_S*. Then* $\mathbf{X}$ *is comonotonic if and only if*

$$\left\{X_i \left| S > F_S^{-1}(p)\right.\right\} \equiv_{st} \left\{X_i \left| X_i > F_i^{-1}(p)\right.\right\}, \text{ for all } p \in (0,1), \; i = 1, ..., n. \tag{17}$$

Proof. Suppose that $\mathbf{X}$ is comonotonic. Then, from Theorem 7(a), (X_i, S) are comonotonic for $i = 1, ..., n$ and (17) follows from Corollary 6(a). Conversely, assume that (17) holds. This assumption implies: (i) that F_S is non-degenerate (otherwise $\left\{X_i \left| S > F_S^{-1}(p)\right.\right\}$ is not defined); (ii) that the bivariate vectors (X_i, S) are comonotonic for $i = 1, ..., n$. The result follows from Theorem 7(b). ■

Corollary 9 *Let $\mathbf{X} = (X_1, ..., X_n)$ be an absolutely continuous random vector and assume that the aggregate risk $S = X_1 + ... + X_n$ has a non-degenerate distribution function. Then, $\mathbf{X}$ is comonotonic if and only if*

$$\mathbf{TCE}_S(p) = \sum_{i=1}^{n} \mathbf{TCE}_{X_i}(p) \text{ for all } p \in (0,1)\,. \tag{18}$$

Proof. If $\mathbf{X}$ is comonotonic, (18) follows from Corollary 8. Now suppose that (18) holds. Combining (18) and (14), it follows that

$$E\left[X_i \left| S > F_S^{-1}(p)\right.\right] = \mathbf{TCE}_{X_i}(p), \text{ for } i = 1, ..., n. \tag{19}$$

Taking into account that two stochastically ordered random variables with the same mean have the same distribution, it follows from (12) and (19) that

$$\left\{X_i \left| S > F_S^{-1}(p)\right.\right\} \equiv_{st} \left\{X_i \left| X_i > F_i^{-1}(p)\right.\right\}, \text{ for } i = 1, ..., n,$$

and the result follows from Corollary 8. ■

4 Improvement of the bounds under positive dependence

Since any random variable X is stochastically smaller than $\{X \left| X > t\right.\}$ for all t, it follows from Corollary 6(a) that if (X_i, S) is comonotonic, then

$$X_i \leq_{st} \left\{X_i \left| S > F_S^{-1}(p)\right.\right\} \quad \text{for all } p \in (0,1)\,,\ i = 1, ..., n. \tag{20}$$

In particular, (20) implies that

$$E[X_i] \leq E\left[X_i \left| S > F_S^{-1}(p)\right.\right] \text{ for all } p \in (0,1)\,,\ i = 1, ..., n, \tag{21}$$

which means that X_i needs a larger amount of capital in the allocation process when X_i is part of a comonotonic random vector that when it is considered alone. In general, not every marginal risk satisfies (21). For example, if the random vector (X_i, S) is countermonotonic, it follows from Corollary 6(b) that the marginal risk contribution of X_i is less than the mean. Intuitively, however, we may expect that a positive dependence structure of the random vector $\mathbf{X}$ (another weaker than comonotonicity) will also imply (20). We show that this conjecture is true if we formalize the idea of "positively dependent structure" by considering "conditionally increasing random vectors". In order to introduce this structure we need a previous notion.

Definition 10 *A random vector $(X_1, ..., X_n)$ is conditionally increasing in sequence (CIS) if*

$$\{X_i | X_1 = x_1, ..., X_{i-1} = x_{i-1}\} \leq_{st} \left\{X_i | X_1 = x'_1, ..., X_{i-1} = x'_{i-1}\right\} \qquad (22)$$

$$\textit{whenever } x_j \leq x'_j, \; j = 1, 2, ..., i-1.$$

When (22) holds for $n = 2$, we say that X_2 is stochastically increasing (SI) in X_1.

The CIS notion is a concept of positive dependence that was studied, among others, by Lehmann (1966) and Barlow and Proschan (1975).

Definition 11 *A random vector $(X_1, ..., X_n)$ is conditionally increasing (CI) if, and only if, the random vector*

$$\mathbf{X}_\pi = (X_{\pi(1)}, ..., X_{\pi(n)})$$

is CIS for all permutations $\pi \in \Pi_n$.

The CI notion was studied by Müller and Scarsini (2001). This notion is related to the notion of multivariate totally positive of order 2 (MTP_2, see Karlin and Rinott, 1980). Müller and Scarsini (2001) prove that MTP_2 is a sufficient condition for CI. Another useful concept of bivariate dependence is positive quadrant dependency (Lehmann, 1966).

Definition 12 *We say that the random vector (X_1, X_2) is positively quadrant dependent (PQD) if*

$$X_1 \leq_{st} \{X_1 \,|X_2 > t\} \textit{ for all } t \;\; \textit{such that } P\,[X_2 > t] > 0.$$

Note that the vector (X_i, S) is PQD if and only if (20) holds. Thus, in the rest of this section we will show that if $(X_1, ..., X_n)$ is CI, then (X_i, S) is PQD. In order to prove it, we introduce the following standard construction. Given $\mathbf{u} = (u_1, ..., u_n)$ in $[0,1]^n$, the standard construction for an absolutely continuous random vector $\mathbf{X}$, denoted by

$$\hat{x}(\mathbf{u}) = (\hat{x}_1(u_1), \hat{x}_2(u_1, u_2), ..., \hat{x}_n(u_1, ..., u_n)),$$

is defined as follows

$$\begin{aligned} \hat{x}_1(u_1) &= F_{X_1}^{-1}(u_1), \\ \hat{x}_i(u_1, ..., u_i) &= F^{-1}_{\{X_i | \bigcap_{j=1}^{i-1} X_j = \hat{x}_j(u_j)\}}(u_i), \text{ for } i = 2, ..., n, \end{aligned}$$

where F_X^{-1} denotes the quantile function of X. This construction is widely used in simulation theory and plays the role of the quantile function in the multivariate case. It is well-known that

$$\hat{x}(\mathbf{U}) \equiv_{st} \mathbf{X}, \tag{23}$$

where $\mathbf{U}$ is a random vector with n independent uniform distributed components on $[0,1]$ (see Li, Scarsini and Shaked, 1996).

Before obtaining the main result, we will show that under a CI dependence structure of the vector $\mathbf{X}$, the aggregate risk S is stochastically increasing in X_i. More generally, the following result shows that, under a CI structure, $\psi(\mathbf{X})$ is stochastically increasing in X_i for any increasing ψ from $\mathbf{R}^n$ to $\mathbf{R}$.

Theorem 13 *Let $\mathbf{X} = (X_1, ..., X_n)$ be an absolutely continuous CIS random vector and let ψ be an increasing real function from $\mathbf{R}^n$ to $\mathbf{R}$. Then, $\psi(\mathbf{X})$ is stochastically increasing in X_1.*

Proof. For all $0 \leq u_1 \leq v_1 \leq 1$, we just need to prove that

$$\{\psi(\mathbf{X}) | X_1 = \hat{x}_1(u_1)\} \leq_{st} \{\psi(\mathbf{X}) | X_1 = \hat{x}_1(v_1)\}. \tag{24}$$

First, note that the vector

$$\hat{z}(u_i, ..., u_n) = (\hat{x}_i(u_1, ..., u_i), ..., \hat{x}_n(u_1, ..., u_n))$$

represents the standard construction evaluated at $(u_i, ..., u_n)$ for the conditional random vector

$$\left\{ (X_i, ..., X_n) \left| \bigcap_{j=1}^{i-1} X_j = \hat{x}_j(u_1, ..., u_j) \right. \right\},$$

for $i = 2, ..., n$, where he have omitted $(u_1, ..., u_{i-1})$ in the notation of $\hat{z}$ for simplicity. Hence, using (23), it easily holds that

$$\hat{z}(U_i, ..., U_n) \equiv_{st} \left\{ (X_i, ..., X_n) \left| \bigcap_{j=1}^{i-1} X_j = \hat{x}_j(u_1, ..., u_j) \right. \right\}, \qquad (25)$$

where $(U_i, ..., U_n)$ is a random vector with $n - i + 1$ independent uniform distributed components on $[0, 1]$. Using (25) for $i = 2$ we note that

$$\begin{aligned} \{\psi(\mathbf{X})|X_1 = \hat{x}_1(u_1)\} &\equiv_{st} \psi(\hat{x}_1(u_1), \{(X_2, \ldots X_n)|X_1 = \hat{x}_1(u_1)\}) \\ &\equiv_{st} \psi(\hat{x}_1(u_1), \hat{x}_2(u_1, U_2), \ldots, \hat{x}_n(u_1, U_2, \ldots, U_n))] \\ &\equiv_{st} \psi(\hat{x}(u_1, U_2, \ldots, U_n)) \qquad (26) \end{aligned}$$

where $(U_2, ..., U_n)$ is a random vector with $n - 1$ independent uniform distributed components on $[0, 1]$ and analogously for $\{\psi(\mathbf{X})|X_1 = \hat{x}_1(v_1)\}$. On the other hand, using both that ψ is increasing and the fact that the CIS property implies that the standard construction $\hat{x}(\mathbf{u})$ is increasing in $\mathbf{u} \in (0, 1)^n$, (see Rubinstein, Samorodnitsky and Shaked, 1985), we obtain that

$$\psi(\hat{x}(u_1, u_2, ..., u_n)) \leq \psi(\hat{x}(v_1, u_2, ..., u_n)) \qquad (27)$$

for all $0 \leq u_1 \leq v_1 \leq 1$. Using both (26) and (27), then (24) follows directly from Theorem 1.A.1 in Shaked and Shanthikumar (2007). ■

Corollary 14 *Let $\mathbf{X} = (X_1, ..., X_n)$ be an absolutely continuous CI random vector and let ψ be an increasing real function from $\mathbf{R}^n$ to $\mathbf{R}$. Then, $\psi(\mathbf{X})$ is stochastically increasing in X_i for $i = 1, ..., n$.*

Proof. For each permutation $\pi \in \Pi_n$, we consider the orthogonal matrix A_π in $M_{n \times n}$, defined by $a_{\pi(j)j} = 1$ for all $j = 1, ..., n$ and zero for the rest of components, such that $\mathbf{X}_\pi = (X_{\pi(1)}, ..., X_{\pi(n)}) = \mathbf{X}A_\pi$. By hypothesis assumption, the random vector $\mathbf{X}_\pi$ is CIS for all permutations, π. Then, using Theorem 13, $\phi(\mathbf{X}_\pi)$ is stochastically increasing in $X_{\pi(1)}$ for all increasing real functions $\phi : \mathbf{R}^n \longrightarrow \mathbf{R}$ and for all permutations, π. Let us consider now ψ an increasing real function on $\mathbf{R}^n$. The proof follows directly just considering any permutation such that $\pi(1) = i$ and taking into account that $\psi(\mathbf{x})$ can be rewritten as $\phi(\mathbf{x}_\pi) = \psi(\mathbf{x}_\pi A_\pi^t)$, which, due to the fact that A_π^t just permutes the components, is also trivially increasing. ∎

We are now in conditions to state the main result in this section.

Corollary 15 *Let $\mathbf{X} = (X_1, ..., X_n)$ be an absolutely continuous CI random vector and let $S = X_1 + ... + X_n$ be the aggregate risk. For $i = 1, ..., n$ and $p \in (0,1)$ we have:*
(a) S is stochastically increasing in X_i,
(b) the vector (X_i, S) is PQD, that is,

$$X_i \leq_{st} \left\{ X_i \left| S > F_S^{-1}(p) \right. \right\}. \tag{28}$$

Proof. By taking in Corollary 14 the increasing function $\psi(X_1, ..., X_n) = X_1 + ... + X_n$, we see that S is stochastically increasing in X_i for $i = 1, ..., n$ and this implies that (X_i, S) is PQD (in general, SI implies PQD, see Section 5.2 in Nelsen, 1999). ∎

Corollary 15 formalizes the following intuition: when X_i is part of a conditionally increasing random vector, it needs more capital in the allocation process that when it is considered alone. As a consequence, we state the following result, which improves the bounds given in Corollary 15 for the conditional distribution $\left\{ X_i \left| S > F_S^{-1}(p) \right. \right\}$ under a CI structure.

Corollary 16 *Let $\mathbf{X} = (X_1, ..., X_n)$ be an absolutely continuous CI random vector with marginal distribution functions $F_1, ..., F_n$. Let $S = X_1 + ... + X_n$ be the aggregate risk with distribution function F_S. Then,*

$$X_i \leq_{st} \left\{ X_i \left| S > F_S^{-1}(p) \right. \right\} \leq_{st} \left\{ X_i \left| X_i > F_i^{-1}(p) \right. \right\}$$

for $p \in (0,1)$ and $i = 1, ..., n$.

5 An Ilustration

In order to illustrate graphically the results, we consider different marginal random variables under a common CI dependence structure or copula. By using copulas, we can separate the marginal distributions from the dependence structure of the vector. A copula C is a cumulative distribution function with uniform marginals on $[0,1]$. It is well-known that if H is a n-dimensional distribution function with marginal distribution functions $F_1, ..., F_n$, then there exists a n-copula C such that, for all $(x_1, ..., x_n) \in \mathbb{R}^n$, we have $H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n))$. Moreover, if $F_1, ..., F_n$ are continuous, then C is unique (see Nelsen, 1999). As noted by Müller and Scarsini (2001), most of the multivariate dependence structure properties (included MTP_2 and CI) of a distribution are in the copula. Therefore, given that MTP_2 is a sufficient condition for CI, any distribution with a MTP_2 density has a CI copula. Many examples of MTP_2 distributions (and, therefore, of CI copulas) can be found in Karlin and Rinott (1980), Sarkar and Chang (1997) and Shaked and Spizzichino (1998). Remarkable examples include the case of independent risks, the multivariate gaussian copula with nonnegative correlations and certain archimedean copulas (see Müller and Scarsini, 2005, for details). For some recent applications of CI copulas to insurance, see Balakrishnan et al. (2012), Belzunce, Suárez-Llorens and Sordo (2012), Cai and Wei (2012a,b) and Lu et al. (2012).

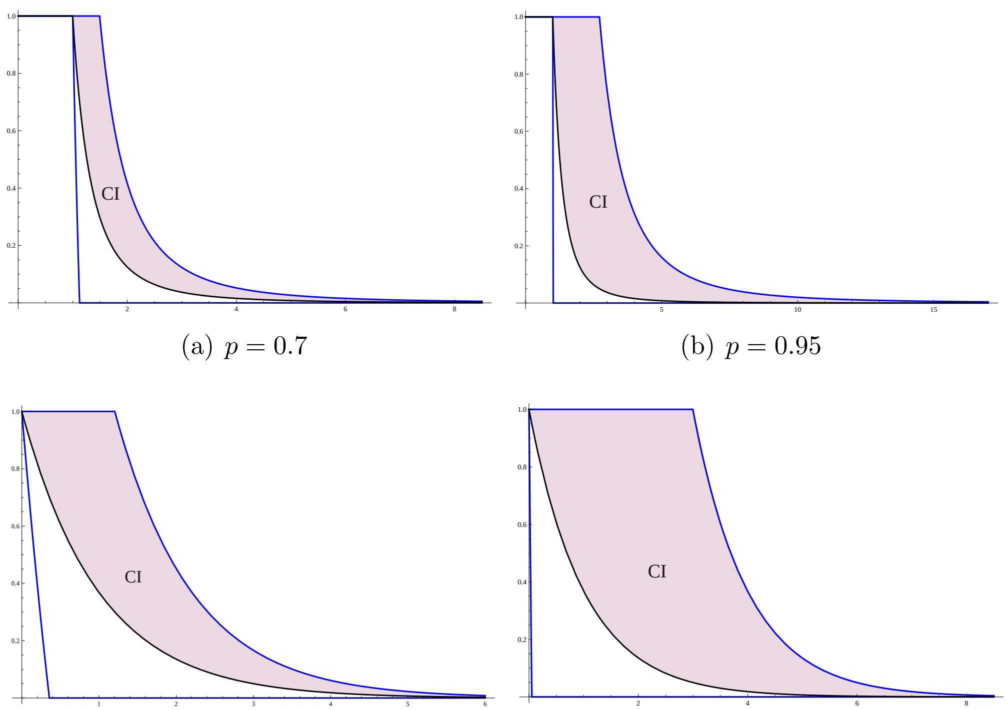
Let $\mathbf{X} = (X_1, ..., X_n)$ be a random vector with a copula C and let $S = X_1 + ... + X_n$ be the aggregate risk. As a first example we suppose that the marginal X_i follows a Pareto distribution $P(\varepsilon, \alpha)$ with survival function

$$\overline{F}(x) = \left(\frac{x}{\varepsilon}\right)^{-\alpha}, \quad x \geq \varepsilon > 0, \ \alpha > 0.$$

For the second example, we assume that the marginal X_i follows an exponential distribution $Exp\,(\lambda)$ with mean $\lambda = 1$. Figure 1 shows the bounds on the survival function of $\left\{X_i \,\middle|\, S > F_S^{-1}(p)\right\}$ when $X_i \sim P(1,3)$ and $X_i \sim Exp\,(1)$, respectively, for $p = 0.7$ and $p = 0.95$, respectively. The graphs illustrate how the lower bound in Corollary 2 becomes less significant as p increases and how this lower bound is substantially refined under a CI copula.

When the random vector $\mathbf{X}$ is CI, the filled area represents a "distribution band" for the conditional random variable $\left\{X_i \,\middle|\, S > F_S^{-1}(p)\right\}$, where X_i and $\left\{X_i \,\middle|\, X_i > F_i^{-1}(p)\right\}$ are, respectively, the lower bound and the upper bound of the band. This means that, given $p \in (0,1)$, the survival function of the random variable $\left\{X_i \,\middle|\, S > F_S^{-1}(p)\right\}$ lies on the set of survival functions $\{\overline{F} : \overline{F}_L \leq \overline{F} \leq \overline{F}_U\}$, where $\overline{F}_L$ is the survival function of $X_L = X_i$ given by

$$\overline{F}_L(x) = \overline{F}_i(x) \text{ for all } x \tag{29}$$



(a) $p = 0.7$ (b) $p = 0.95$

(c) $p = 0.7$ (d) $p = 0.95$

Figure 1: Graphs (a) and (b) show the lower and upper bounds on the survival function of $\left\{X_i \middle| S > F_S^{-1}(p)\right\}$ when $X_i \sim P(1, 3)$ for $p = 0.7$ and $p = 0.95$, respectively. The filled area corresponds to CI random vectors. Graphs (c) and (d) show the bounds when $X_i \sim Exp\,(1)$.

and $\overline{F}_U$ is the survival function of $X_U = \left\{X_i \middle| X_i > F_i^{-1}(p)\right\}$ given by

$$\overline{F}_U(x) = \begin{cases} 1, & x < F_i^{-1}(p), \\ \dfrac{\overline{F}_i(x)}{1-p}, & x > F_i^{-1}(p). \end{cases} \tag{30}$$

A natural way to evaluate the uncertainty of the band is to use a probability metric to measure the closeness between the bounds. There are several metrics commonly applied to measure distances between random variables (see Gibbs and Su, 2002, for a summary and Chapter 9 in Denuit et al., 2005, for applications in actuarial sciences). One possibility is to consider the Kolmogorov (or uniform) metric, given by

$$K(X_L, X_U) = \sup_{x \in \mathbf{R}} |\overline{F}_L(x) - \overline{F}_U(x)|,$$

which represents the largest absolute difference between $\overline{F}_L$ and $\overline{F}_U$. A straightforward computation yields

$$|\overline{F}_L(x) - \overline{F}_U(x)| = \begin{cases} 1 - \overline{F}_i(x) & \text{if} \quad x \leq F_i^{-1}(p), \\ \\ \overline{F}_i(x)\frac{p}{1-p} & \text{if} \quad x \geq F_i^{-1}(p). \end{cases} \tag{31}$$

It is easy to see that the supremum of (31) over $\mathbf{R}$ is achieved at $F_i^{-1}(p)$. Therefore, $K(X_L, X_U) = p$, which reflects that the uncertainty of the band increases as p increases.

The Kolmogorov metric suffers from the shortcoming that it is completely insensitive to the losses in the tail of the distributions (this is because the difference $|\overline{F}_L(x) - \overline{F}_U(x)|$ converges to zero as x increases or decreases). Another possibility to evaluate the closeness between the bounds is to use the Kantorovich metric, defined by

$$d(X_L, X_U) = \int_{-\infty}^{\infty} |\overline{F}_L(x) - \overline{F}_U(x)|\, dx, \tag{32}$$

which provides aggregate information about the deviations between the bounds. Observe, from (29) and (30), that $\overline{F}_U(x) = h(\overline{F}_L(x))$, where h is a concave distortion function[1] given by $h(t) = \min\left(\frac{t}{1-p}, 1\right)$ and, consequently, (32) can be expressed as

$$d(X_L, X_U) = \int_{-\infty}^{\infty} |\overline{F}_L(x) - h(\overline{F}_L(x))|\, dx.$$

López-Díaz, Sordo and Suárez-Llorens (2012) interpret the Kantorovich metric between the survival function of X_i and its distortion as a characteristic of the variability[2] of X_i. In fact, using expression (21) in that paper, we obtain

$$d(X_L, X_U) = E[X_U] - E[X_L] = \mathbf{TCE}_{X_i}(p) - E[X_i].$$

It is interesting to note that this distance is consistent with the dilation order, which is defined as follows. Given two random variables X and Y with finite

[1] A distortion function is a non-decreasing mapping $h : [0, 1] \longmapsto [0, 1]$ such that $h(0) = 0$ and $h(1) = 1$.

[2] Other variability measures based on distorted distributions can be found in Sordo and Suárez-Llorens (2011).

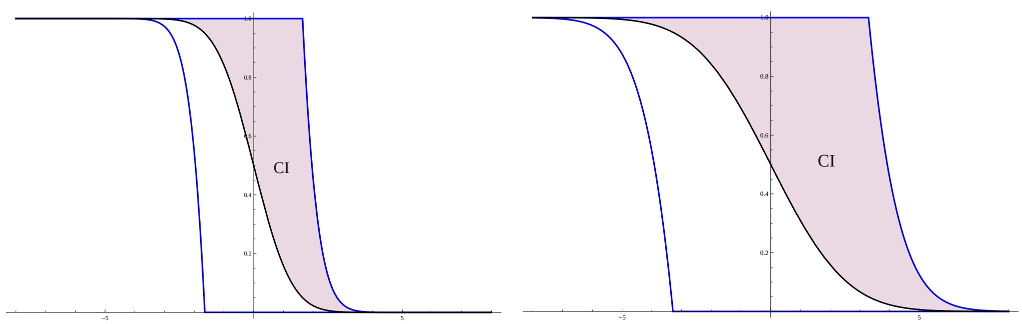
expectations, X is said to be smaller than Y in the dilation order, denoted by $X \leq_{dil} Y$, if

$$E\left[\Phi\left(X - E\left[X\right]\right)\right] \leq E\left[\Phi\left(Y - E\left[Y\right]\right)\right] \text{ for all convex functions } \Phi,$$

provided that these expectations exist (see Ramos and Sordo, 2003, for some applications of this order). The following result is a direct application of Proposition 4.4 in López-Díaz, Sordo and Suárez-Llorens (2012).

Theorem 17 *Let X_i and X_j be two components of a CI random vector. Let (X_L, X_U) and (X'_L, X'_U) be, respectively, the distributional bands for the random variables $\left\{X_i \left| S > F_S^{-1}(p)\right.\right\}$ and $\left\{X_j \left| S > F_S^{-1}(p)\right.\right\}$, respectively, for some $p \in (0,1)$. If $X_i \leq_{dil} X_j$ then $d(X_L, X_U) \leq d(X'_L, X'_U)$, where d is the Kantorovich metric defined by (32).*

Theorem 17 reflects the idea that higher marginal variability increases the gap between the bounds of the distributional band. This is illustrated with an example in Figure 2: given two random variables $X_1 \sim N(0,1)$ and $X_2 \sim N(0,2)$, it is well-known that $X_1 \leq_{dil} X_2$. The graphs show that, under a CI copula, the uncertainty of the ditributional band for $\left\{X_2 \left| S > F_S^{-1}(p)\right.\right\}$ is higher than for $\left\{X_1 \left| S > F_S^{-1}(p)\right.\right\}$.



(a) $N(0,1)$ (b) $N(0,2)$

Figure 2: Graphs (a) and (b) show, respectively, the lower and upper bounds on the survival function of $\left\{X_i \left| S > F_S^{-1}(p)\right.\right\}$ when $X_i \sim N(0,1)$ and $X_i \sim N(0,2)$, respectively, for $p = 0.95$. The filled area corresponds to CI random vectors. It is clear that a higher marginal variance increases the gap between the bounds.

6 Conclusions

The tail conditional expectation is one of the most commonly used risk measures. The calculation of this measure is often followed by a process of allocating the aggregate risk of the portfolio to individual risks based on their marginal contributions to the total. In this process, conditional distributions of the form $\left\{ X_i \left| S > F_S^{-1}(p) \right. \right\}$, with $p \in (0,1)$, where $S = X_1 + ... + X_n$, play a fundamental role. In this paper, we have obtained general lower and upper stochastic bounds for these conditional distributions and we have shown that the lower bound can be improved under a conditionally increasing structure of the vector. The improved lower bound is interpreted as meaning that the individual risks require, under the CI assumption, more capital than the mean in the allocation process. We have also shown that the largest marginal risk distribution, in the stochastic order, of an individual risk with a given distribution function F_i will be obtained in a random vector such that (X_i, S) is comonotonic.

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