

Dispersion measures and dispersive orderings

Abstract

In this paper, the comparison of random variables according to the functionals of a general class of dispersion measures is characterized in terms of the dilation order. The Gini's mean difference is a particular member of this general class. In addition, a new and weaker order, called the second-order absolute Lorenz ordering, is introduced, and we judge random variables according to certain functionals of this class when the dilation order is not available.

Keywords: Dispersion measures, Dilation ordering, Second-order absolute Lorenz ordering, absolute Lorenz curve.

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1. Introduction

Several approaches have been used in the literature to address the problem of comparing two probability distributions in terms of dispersion. The conventional approach in many empirical works is to compare some associated measures of dispersion, such as the variance or the range. However, such a comparison is based on only two single numbers and therefore, is often not very informative. A second approach is to make ordinal comparisons by means of orderings of distributions that satisfy some suitable conditions. This paper provides a bridge that takes us from the cardinal comparisons (by means of a general class of measures of dispersion) to the safer ordinal ones (by means of dilation and second-order absolute Lorenz orderings).

Let X be a random variable with distribution function F and finite mean μ_X . Let F^{-1} be the left continuous inverse of F , defined by

$$F^{-1}(t) = \inf \{x : F(x) \geq t\}, \quad 0 \leq t \leq 1.$$

An intuitive procedure for measuring the dispersion in X is to average the deviations of $F^{-1}(p)$ from the mean μ_X . If we consider a linear averaging method based on weights that depend on relative ranks, we obtain the class C of functionals I_ω given by

$$I_\omega(X) = \int_0^1 \omega(p) [F^{-1}(p) - \mu_X] dp \quad (1)$$

where $\omega(p)$ is any integrable weight function $\omega : [0, 1] \rightarrow \mathbf{R}$, which is assumed to be independent of F .

The functionals I_ω can be expressed in terms of the function $A_X(p)$ defined as

$$A_X(p) = \int_0^p [F^{-1}(t) - \mu_X] dt, \quad 0 \leq p \leq 1. \quad (2)$$

This function is called the absolute Lorenz curve and is used in economics to compare income distributions (Moyes, 1987). $A_X(p)$ coincides with the horizontal axis when X is a degenerate random variable in μ_X . It is seen that $A_X(p)$ is decreasing for $0 \leq p \leq F(\mu_X)$ and increasing for

$F(\mu_X) < p \leq 1$, it takes the values

$$A_X(0) = A_X(1) = 0 \tag{3}$$

and is a convex function with respect to p (therefore, $A_X(p) \leq 0$ for all $p \in [0, 1]$). Using (2) and the relationship between the Riemann integral and the Riemann-Stieltjes integral, we find that

$$I_\omega(X) = \int_0^1 \omega(t) dA_X(t) \tag{4}$$

and hence, via integration by parts (for Riemann-Stieltjes integrals), we obtain

$$I_\omega(X) = \int_0^1 -A_X(t) d\omega(t). \tag{5}$$

As can be seen from (5), each functional $I_\omega(X)$ is a weighted area between the curve $-A_X(t)$ and the horizontal line.

From (1) and the properties of F^{-1} (see Parzen, 1979) it is easily seen that $I_\omega(aX) = aI_\omega(X)$ for all $a > 0$, $I_\omega(X + b) = I_\omega(X)$ for all b and $I_\omega(c) = 0$ for any degenerate random variable at c . In addition, from (5) it follows that if $\omega(p)$ is non-decreasing, then $I_\omega(X) \geq 0$ for all random variable X . Therefore, the members of the class

$$C_1 = \{I_\omega \in C \text{ such that } \omega \text{ is non-decreasing}\}$$

satisfy the most commonly accepted axioms for all measures of dispersion (see, for example, Bickel and Lehman, 1976).

The class C_1 includes some well known dispersion measures. One of them is the *Gini's mean difference* (Gmd) of X defined as

$$\text{Gmd}(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |y - x| dF(x) dF(y).$$

The Gini's mean difference was discussed in the context of the theory of errors of observations in the late nineteenth century. It was proposed as a measure of dispersion by Gini (1912). It is

given by (1) with $\omega(p) = 4p$. Some theoretical merits of the Gini's mean difference in the context of stochastic orderings can be found in Yitzhaki (1982).

Other members of C_1 are summarized in Table 7.8 of Nygard and Sandström (1981).

Comparisons of functionals of two random variables sometimes produce stochastic orders (see Shaked and Shanthikumar (1994) for a detailed treatment of this topic). One of the most used stochastic order for comparing two random variables in terms of dispersion is the dilation order. Following Hickey (1986), we say that the random variable Y is more dispersed than X in the dilation sense (denoted by $X \leq_{dil} Y$) if

$$E[\Phi(X - \mu_X)] \leq E[\Phi(Y - \mu_Y)]$$

for all convex functions Φ , provided that these expectations exist. This notion generalizes the use of the variance for comparing distributions in terms of dispersion. Note that dilation involves dispersion from the mean of a distribution, as in (1). This leads us to consider whether the members of C_1 are preserved under this ordering. However, it should be mentioned here that members of C_1 cannot in general be written in the form $E[\Phi(X - \mu_X)]$, with Φ convex (this is the case, for example, of the Gini's mean difference, as shown by the example of Newbery, 1970).

In Section 2, we characterize the comparison of random variables according to the measures I_ω of C_1 in terms of dilation order. From this result we deduce that if X and Y are ordered in the dilation sense, then we can judge between them according to any measures I_ω without needing to agree on the form of $\omega(p)$ (except that it be non-decreasing). In addition, this characterization, together with the definition of the dilation order, suggests that this order preserves most of the measures involving dispersion about the mean of a random variable, independently of its functional form. This confirms the leading role that the dilation order plays in measures of dispersion.

Let C_2 be the class of measures I_ω of C such that ω is non-decreasing and convex. Obviously, $C_2 \subset C_1$. In Section 3, we introduce a new criterion for evaluating the dispersion of random variables that is consistent and is implied by the unanimous order generated by the class C_2 .

The corresponding dispersive ordering, called the second-order absolute Lorenz order, is based on comparisons of absolute Lorenz areas. We deduce that if X and Y are ordered in the second-order absolute Lorenz order, then we can judge between them according to any measure I_ω without needing to agree on the form of $\omega(p)$ (except that it be non-decreasing and convex).

In order to prove our first result, we require the notion of the decreasing rearrangement of a function (on this topic, see Hardy et al. (1929) and Chong (1974)). Denote by $L^1(\Omega, \mu)$ the set of all extended real-valued integrable functions on a measure space (Ω, Λ, μ) . The decreasing rearrangement of $f \in L^1(\Omega, \mu)$ is defined by

$$f^*(t) = \inf \{s \in \mathbf{R} : D_f(s) \leq t\}, \quad t \in [0, \mu(\Omega)]$$

where

$$D_f(s) = \mu(\{x : f(x) > s\}),$$

for each $s \in [-\infty, \infty]$. Denote by m the Lebesgue measure on \mathbf{R} . We have the following result from Chong (1974).

Theorem 1.1. *Let $f \in L^1(\Omega, \mu)$, $g \in L^1(\Omega', \mu')$, where $\mu(\Omega) = \mu'(\Omega') = a < \infty$. Then,*

$$\int_0^t f^* dm \leq \int_0^t g^* dm \quad \text{for all } t \in [0, a)$$

and

$$\int_0^a f^* dm = \int_0^a g^* dm$$

if and only if

$$\int_\Omega \Phi(f) d\mu \leq \int_{\Omega'} \Phi(g) d\mu'$$

for all convex functions $\Phi : \mathbf{R} \rightarrow \mathbf{R}$.

Before ending this introduction, we note that integrals occurring in this paper are interpreted in the sense of Riemann-Stieltjes. The Riemann-Stieltjes notation allows the simultaneous treatment of the purely discrete and absolutely continuous cases (as well as combinations thereof).

2. Characterizations in terms of the dilation order

We need to state the following result before obtaining the main theorem of this section.

Theorem 2.1. *Let X and Y be random variables with respective finite means μ_X and μ_Y and let the corresponding distribution functions be F and G , respectively. Then, $X \leq_{dil} Y$ if and only if $A_X(p) \geq A_Y(p)$, $\forall p \in [0, 1]$.*

Proof. Let $(\Omega_X, \mathcal{B}_X, P_X)$ and $(\Omega_Y, \mathcal{B}_Y, P_Y)$ be the probability spaces on which X and Y , respectively, are defined. Define $f(\omega) = X(\omega) - \mu_X$ for all $\omega \in \Omega$ and $g(\omega) = Y(\omega) - \mu_Y$ for all $\omega \in \Omega_Y$. The decreasing rearrangements of f and g are given, respectively, by $f^*(x) = F^{-1}(1 - x) - \mu_X$ and $g^*(x) = G^{-1}(1 - x) - \mu_Y$, for all $x \in [0, 1]$. The result is now obtained as a direct application of Theorem 1.1.

The following result characterizes the comparison of random variables according to the measures I_ω of C_1 in terms of the dilation order.

Theorem 2.2. *Let X and Y be random variables with finite means. Then*

$$I_\omega(X) \leq I_\omega(Y) \text{ for all } I_\omega \in C_1 \iff X \leq_{dil} Y.$$

Proof. (\implies) Suppose that

$$\int_0^1 \omega(t) dA_X(t) \leq \int_0^1 \omega(t) dA_Y(t) \tag{6}$$

holds for all non-decreasing functions ω (where we have used the Riemann-Stieltjes notation obtained in (4) for $I_\omega(X)$). The function $\omega_p(t)$ defined by

$$\omega_p(t) = \begin{cases} 0 & \text{if } t < p \\ 1 & \text{if } t \geq p \end{cases}$$

is a non-decreasing function of t for each $p \in [0, 1]$, so we have

$$\int_0^1 \omega_p(t) dA_X(t) \leq \int_0^1 \omega_p(t) dA_Y(t), \quad \forall p \in [0, 1]. \tag{7}$$

Using (3) it is seen that an equivalent form for (7) is

$$A_X(p) \geq A_Y(p), \quad \forall p \in [0, 1] \tag{8}$$

and the relation $X \leq_{dil} Y$ follows from Theorem 2.1.

(\Leftarrow) Suppose now that $X \leq_{dil} Y$ or, equivalently, that (8) holds and take an arbitrary $I_\omega \in C_1$. Since $A_X(p) \leq 0$ for all $p \in [0, 1]$ and $A_Y(p) \leq 0$ for all $p \in [0, 1]$, it follows from (8) that

$$\int_0^1 -A_X(t)d\omega(t) \leq \int_0^1 -A_Y(t)d\omega(t),$$

because the monotonic nature of ω ensures that the increments $d\omega$ are non-negative. Using (5) it is seen that $I_\omega(X) \leq I_\omega(Y)$ holds.

Remark 2.1. *Many examples of dilation and stronger orderings within parametric families of distributions can be found in Saunders and Moran (1978), Lewis and Thompson (1981), Shaked (1982) and Hickey (1986). Often, these orderings are related to the value of a real parameter. It follows from Theorem 2.2 that the corresponding orderings with respect to the measures $I_\omega \in C_1$ also hold.*

Remark 2.2. *Fagiuoli et al. (1999) proved, for random variables with continuous distribution functions, a result that corresponds to our Theorem 2.1. However, since we do not impose constraints on the class of distribution functions to be compared, our result is more general. Moreover, our proof follows different lines from the ones followed by these authors.*

3. Characterization in terms of the second-order absolute Lorenz order

As can be seen from Theorem 2.1, only random variables that have associated absolute Lorenz curves that do not intersect are ordered in the dilation sense. It follows from Theorem 2.2 that only if the absolute Lorenz curves of two random variables do not intersect, can we judge between them according to any measures I_ω without needing to agree on the form of ω (except that it be non-decreasing). We must now ask: Under what conditions can we judge between two random variables when the corresponding absolute Lorenz curves intersect? Can we find a simple criterion

that is necessary and sufficient for judging between them according to any I_ω without specifying the particular weight function ω ? The answer is “yes” if we restrict our attention to the class of functions $\omega(p)$ that are non-decreasing and convex. The criterion is a new partial ordering based on the stochastic comparison of absolute Lorenz areas.

Definition 3.1. *Let X and Y be two random variables with absolute Lorenz curves $A_X(t)$ and $A_Y(t)$, respectively. We say that X is smaller than Y in the second-order absolute Lorenz order if $\int_p^1 A_X(t)dt \geq \int_p^1 A_Y(t)dt$ for all $p \in [0, 1]$.*

Note that the dilation order implies the second-order absolute Lorenz order.

Let C_2 be the class of measures I_ω of C such that ω is non-decreasing and convex. The following result characterizes the comparison of random variables according to the measures I_ω of C_2 in terms of the second-order absolute Lorenz order.

Theorem 3.1. *Let X and Y be two random variables with absolute Lorenz curves $A_X(t)$ and $A_Y(t)$, respectively. Then*

$$I_\omega(X) \leq I_\omega(Y) \text{ for all } I_\omega \in C_2$$

if and only if

$$\int_p^1 A_X(t)dt \geq \int_p^1 A_Y(t)dt, \text{ for all } p \in [0, 1]. \quad (9)$$

Proof. (\implies) Note that for a fixed $p \in [0, 1]$, the function $\omega(t) = (t - p)^+ = \max\{t - p, 0\}$ is non-decreasing and convex. In addition, from the integration by parts formula, we have that

$$\int_0^1 (t - p)^+ dA_X(t) = \int_p^1 -A_X(t)dt. \quad (10)$$

Thus, the result easily follows .

(\impliedby) Let $\omega : [0, 1] \longrightarrow \mathbf{R}$ be a non-decreasing and convex function. Then, there exists a non-decreasing, non-negative, integrable function φ such that

$$\omega(t) - \omega(0) = \int_0^t \varphi(p)dp, \quad \forall t \in [0, 1]$$

(see Zygmund, 1959). Using integration by parts, we have, for $t \in [0, 1]$,

$$\omega(t) - \omega(0) = \int_0^1 (t-p)^+ d\varphi(p) + t\varphi(0). \quad (11)$$

On the other hand, from (3) it follows that an alternative expression for (4) is

$$I_\omega(X) = \int_0^1 [\omega(t) - \omega(0)] dA_X(t). \quad (12)$$

Therefore, combining (11) and (12), shows that

$$\begin{aligned} I_\omega(X) &= \int_0^1 \left[\int_0^1 (t-p)^+ d\varphi(p) + t\varphi(0) \right] dA_X(t) = \\ &= \int_0^1 \left[\int_0^1 (t-p)^+ dA_X(t) \right] d\varphi(p) + \varphi(0) \int_0^1 t dA_X(t) \end{aligned} \quad (13)$$

where the last equality follows from the additivity properties of integrals and from Fubini's theorem. Using integration by parts again, we have that

$$\int_0^1 t dA_X(t) = \int_0^1 -A_X(t) dt. \quad (14)$$

Therefore, combining (13), (10) and (14) it is seen that

$$I_\omega(X) = \int_0^1 \left[\int_p^1 -A_X(t) dt \right] d\varphi(p) + \varphi(0) \int_0^1 -A_X(t) dt. \quad (15)$$

Taking into account that

$$\int_p^1 -A_X(t) dt \geq 0 \quad \forall p \in [0, 1],$$

$$d\varphi(p) \geq 0,$$

and

$$\varphi(0) \geq 0,$$

from (9) and (15) the result holds.

Example 3.1. Let X be a uniform random variable with distribution function $F(x) = \frac{x}{3}$, $0 < x < 3$ and let Y be a power random variable with distribution function $G(x) = \left(\frac{x}{3}\right)^{1/2}$, $0 < x < 3$. The absolute Lorenz curves are, respectively, $A_X(p) = \frac{3}{2}(p^2 - p)$, $0 \leq p \leq 1$,

and $A_Y(p) = p^3 - p$, $0 \leq p \leq 1$. It is easy to verify that $A_X(p) < A_Y(p)$ for $0 < p < 1/2$ and $A_X(p) > A_Y(p)$ for $1/2 < p < 1$. Consequently, (8) fails and Theorem 2.2 cannot be used. Nevertheless, relation (9) is easily verified and from Theorem 3.1 it follows that $I_\omega(X) \leq I_\omega(Y)$ for all $I_\omega \in C_2$.

4. Concluding remarks and related topics

In this paper we have studied the consistency of a family of functionals of the form (1), defined on the class of random variables, with two stochastic orderings. The first of these orderings is the well known dilation ordering and the other one is new and weaker.

As a first step, we have restricted our attention to the class C_1 of functionals with non-decreasing weight function ω . We have connected this class with the dilation order, that has been proved to be consistent. This result gives us the possibility of ranking parametric families of distributions according to any $I_\omega \in C_1$, by using well known results about the dilation ordering in these families.

The weakest ordering, that has been called the second-order absolute Lorenz, it enables us to judge between two random variables according to any I_ω without needing to agree on the form of ω (except that it be non-decreasing and convex) when dilation order is not possible.

Some properties of the class C of functionals given by (1) have been stated in this paper. Other properties of these functionals have been discussed by Nygard and Sandström (1981, Sec.7.4) in the context of income distributions, when the underlying random variables are non-negative. They considered each $I_\omega \in C$ as an absolute-invariant measure of income inequality (i.e., a measure that is invariant under a constant addition to incomes). Some particular members of C used in this context can be found in Table 7.8 of their book. In particular, they discussed functionals belonging to the class C_2 , with power weight functions of the form $\omega(p) = ap^n$, with $a > 0$ and $n > 1$. The Piesch' absolute measure, that is defined with $\omega(p) = \frac{3}{2}p^2$, is an example of such functionals. The Nygard and Sandström's approach and the results of this paper suggest that the dilation order and

the second-order absolute Lorenz are powerful tools for comparing absolute income inequalities.

If each $I_\omega \in C$ is normalized by mean income, we obtain the “linear measures of inequality” discussed by Mehran (1977) in the same context of income distribution (the term “linear” in the literature on income distribution means linear after arranging incomes in an increasing order).

The basic estimators of the measures I_ω we form in practice are linear functions of order statistics. This topic has an extensive literature (see Chernoff et al. (1967), Moore (1968), Shorack (1972) and Stigler (1974)). Observing that $\mu_X = \int_0^1 F^{-1}(t)dt$, it is easily seen that (1) can be rewritten as

$$\int_0^1 u(t)F^{-1}(t)dt \tag{16}$$

where $u(t) = \omega(p) - \int_0^1 \omega(p)dp$. If $X_{i:n}$ denotes the i th order statistic of a random sample of size n from X , a natural estimator of (16) is

$$\hat{I}_\omega = \frac{1}{n} \sum_{i=1}^n u\left(\frac{i}{n}\right) X_{i:n}.$$

It follows from Theorem 1 of Shorack (1972) and Proposition 2 of Sandler (1979) that \hat{I}_ω is asymptotically normal under quite general conditions. Examples of such estimators for particular members of C_1 and C_2 together with expressions for their asymptotic variances can be found in Table 10.1 of Nygard and Sandström (1981).

Finally, it should be noted that the class of functionals discussed in this paper can be generalized to include measures with weight function ω that may depend on the distribution function F . As examples of measures of this generalized class, we have the mean deviation $E[|X - \mu_X|]$, by taking

$$\omega(p) = \begin{cases} -1 & \text{if } p \leq F(\mu_X) \\ 1 & \text{if } p > F(\mu_X) \end{cases}.$$

Choosing in (1) the weight function $\omega(p) = F^{-1}(p)$, we obtain the variance of X .

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