Indian J. Phys. **79**(2), 161-165 (2005)

AN INTERNATIONAL JOURNAL IJP

Thermosolutal convection in compressible, rotating, couple-stress fluid

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Received 2 April 2004, accepted 18 November 2004

Abstract : A layer of compressible, rotating, couple-stress fluid heated and soluted from below is considered. For the case of stationary convection, the compressibility, stable solute gradient and rotation postpone the onset of convection whereas the couple-stress viscosity postpones as well as hastens the onset of convection depending on rotation parameter. The case of overstability is also studied wherein a sufficient condition for the non-existence of overstability is found.

Keywords : Couple-stress fluid, compressibility, rotation, thermosolutal convection.

PACS Nos. : 47.20.Ma, 47.50.+d

1. Introduction

The theoretical and experimental results on thermal convection in a fluid layer, in the absence and presence of rotation, have been given by Chandrasekhar [1]. Veronis [2] has investigated the problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient. Double-diffusive (*e.g.* thermosolutal) convection problems arise in oceanography, limnology and engineering. Brakke [3] explained a double-diffusive instability that occurs when a solution of a slowly diffusing protein is layered over a denser solution of more rapidly diffusing sucrose. Nason *et al* [4] found that this instability, which is deleterious to certain biochemical separations, can be suppressed by rotation in the ultracentrifuge.

The theory of couple-stress fluid has been formulated by Stokes [5]. Walicki and Walicka [6] have modelled synovial fluid as a couple-stress fluid in human joints. One of the applications of couple-stress fluid is its use to the study of the mechanisms of lubrications of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and synovial fluid as the lubricant. When a fluid is generated, squeeze-film action is capable of providing considerable protection to the cartilage surface. The shoulder, ankle, knee and hip joints are the loaded-bearing synovial joints of the human body and these joints have a low friction coefficient and negligible wear. Normal synovial fluid is a viscous, non-Newtonian fluid and is generally clear or yellowish. Lin [7] has studied the couple-stress effect on the squeeze film characteristics of hemispherical bearings with reference to synovial joints. Walicki and Walicka [6] have studied the effects of couple-stresses and inertia effects on the characteristics of squeeze-film behaviour in thrust curvilinear bearings with references to synovial joints. On the basis of Stokes' couple-stress fluid model, Walicki and Walicka [8] have made mathematical modelling of some biological bearings. Sharma *et al* [9] have studied a layer of couple-stress fluid permeated with suspended particles, heated from below. For thermal and thermosolutal convection problems, the Boussinesq approximation has been used, which is well justified in the case of incompressible fluids.

When the fluids are compressible, the equations governing the system become quite complicated. Spiegel and Veronis [10] have simplified the set of equations governing the flow of compressible fluids under the

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assumption that the depth of the fluid layer is much smaller than the scale height as defined by them, if only motions of infinitesimal amplitude are considered. Sharma [11] has studied the thermal instability in compressible fluids in presence of rotation and magnetic field.

Keeping in mind the importance of non-Newtonian fluids, thermosolutal convection and compressibility, the present paper considers a layer of compressible, rotating, couple-stress fluid heated and soluted from below.

2. Perturbation equations

Here, we consider an infinite, horizontal, compressible, couple-stress fluid layer of thickness d bounded by the planes $z = 0$ and $z = d$. This layer is heated and soluted from below so that uniform temperature gradient $\beta\left(=|dT/dz|\right)$ and uniform solute gradient $\beta'\left(=|dC/dz|\right)$ are maintained and the layer is acted on by uniform rotation $\Omega(0,0,\Omega)$ and gravity field $g(0,0,-g)$.

Spiegel and Veronis [10] defined f as any one of the state variables (pressure (p), density (ρ) or temperature (T)) and expressed these in the form

$$f(x,y,z,t) = f_m + f_0(z) + f'(x, y, z, t), \tag{1}$$

where f_m is the constant space average of f, f_0 is the variation in the absence of motion and f' is the fluctuation resulting from motion.

The initial state is therefore a state in which the density, pressure, temperature, solute concentration and velocity at any point in the fluid are given by

$$\rho = \rho(z),\ p = p(z),\ T = T(z),\ C = C(z),\ v = 0 \tag{2}$$

respectively, where

$$T(z) = T_0 - \beta z,$$

$$C = C_0 - \beta' z,$$

$$p(z) = p_m - g\int_0^d (\rho_m + \rho_0)\,dz,$$

$$\rho(z) = \rho_m\,[1 - \alpha_m(T - T_m) + \alpha'(C - C_m) + K_m(p - p_m)], \tag{3}$$

$$\alpha_m = -\left(\frac{1}{\rho}\frac{\partial\rho}{\partial T}\right)_m (= \alpha\text{, say}),\ \alpha'_m = -\left(\frac{1}{\rho}\frac{\partial\rho}{\partial C}\right)_m$$

$$(= \alpha'\text{, say}),\ K_m = \left(\frac{1}{\rho}\frac{\partial\rho}{\partial p}\right)_m$$

Then the linearized perturbation equations, relevant to the problem (Stokes [7], Veronis [2], Sharma [11]), are

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_m}\nabla\delta p - g(\alpha\theta - \alpha\gamma)$$

$$+\left(\nu - \frac{\mu'}{\rho_m}\nabla^2\right)\nabla^2 q + 2(v\times\Omega), \tag{4}$$

$$\nabla . v = 0, \tag{5}$$

$$\frac{\partial\theta}{\partial t} = \left(\beta - \frac{g}{c_p}\right)w + \kappa\nabla^2\theta, \tag{6}$$

$$\frac{\partial\gamma}{\partial t} = \beta' w + \kappa'\nabla^2\gamma, \tag{7}$$

where θ, γ, $v(u, v, w)$, δp and $\delta\rho$ denote respectively the perturbations in temperature T, solute concentration C, fluid velocity (0,0,0), pressure p and density ρ. υ, μ', c_p, κ and κ' stand for kinematic viscosity, couple-stress viscosity, specific heat at constant pressure, thermal diffusivity and solute diffusivity respectively. The equation of state

$$\rho = \rho_m[1 - \alpha(T - T_m) + \alpha'(C - C_m)], \tag{8}$$

contains the thermal coefficient of expansion α and an analogous solute coefficient α', as the density primarily depends on temperature and solute concentration. The change in density $\delta\rho$, caused by the perturbations θ and γ, is given by

$$\delta\rho = -\rho_m\,(\alpha\theta - \alpha'\gamma), \tag{9}$$

and has been used in writing eq. (4).

Writing the scalar components of eq. (4) and eliminating u, v, δp between them, by using eq. (5), we obtain

$$\frac{\partial}{\partial t}\nabla^2 w - g\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\alpha\theta - \alpha'\gamma)$$

$$-\left(\nu - \frac{\mu'}{\rho_m}\nabla^2\right)\nabla^4 w + 2\Omega\frac{\partial\zeta}{\partial z} = 0 \tag{10}$$

eqs. (4)–(7) also yield

$$\frac{\partial\zeta}{\partial t} - \left(\nu - \frac{\mu'}{\rho_m}\nabla^2\right)\nabla^2\zeta = 2\Omega\frac{\partial w}{\partial z}, \tag{11}$$

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\theta = \left(\beta - \right. \tag{12}$$

$$\left(\frac{\partial}{\partial t} - \kappa'\nabla^2\right)\gamma = \beta' w. \qquad (13)$$

Here, we consider the case in which both the boundaries are free and the temperatures, solute concentrations at the boundaries are kept constant. Then the boundary conditions appropriate to the problem are

$$w = \frac{\partial^2 w}{\partial z^2} = 0,\ \frac{\partial \zeta}{\partial z} = 0,\ \theta = 0,\ \gamma = 0 \text{ at } z = 0 \text{ and } z = d. \qquad (14)$$

3. The dispersion relation

Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, \gamma, \zeta] = [W(z), \Theta(z), \Gamma(z), Z(z)] \exp(ik_x x + ik_y y + nt), \qquad (15)$$

where k_x, k_y are wave numbers along x- and y-directions respectively, $k\left(=\sqrt{k_x^2 + k_y^2}\right)$ is the resultant wave number and n is, in general, a complex constant.

Putting $x = x^* d$, $y = y^* d$, $z = z^* d$, $D = \frac{d}{dz^*}$, $a = kd$, $\sigma = \frac{nd^2}{\nu}$ and using expression (15), eqs. (10)–(13) in non-dimensional form become

$$\sigma(D^2 - a^2)W + \frac{gd^2}{\nu} a^2(\alpha\Theta - \alpha'\Gamma) + T_A^{1/2} d\, DZ = \left[1 - F\left(D^2 - a^2\right)\right]\left(D^2 - a^2\right)^2 W, \qquad (16)$$

$$(D^2 - a^2 - p_1\sigma)\Theta = -\left(\frac{G-1}{G}\right)\frac{\beta d^2}{\kappa} W, \qquad (17)$$

$$(D^2 - a^2 - q\sigma)\Gamma = -\frac{\beta' d^2}{\kappa'} W, \qquad (18)$$

$$\left[\{1 - F(D^2 - a^2)\}(D^2 - a^2) - \sigma\right] Z = -\frac{T_A^{1/2}}{d} DW, \qquad (19)$$

where $p_1 = \frac{\nu}{\kappa}$ is the Prandtl number, $q = \frac{\nu}{\kappa'}$ is the Schmidt number, $T_A = \frac{4\Omega^2 d^4}{\nu^2}$ is the Taylor number, $G = \frac{c_p\beta}{g}$ is the dimensionless compressibility parameter and $F = \frac{\mu^*}{\rho_m d^2 \nu}$ is the dimensionless couple-stress viscosity. We shall suppress the stars (*) in distances for convenience hereafter.

Eliminating Θ, Γ, Z between eqs. (16)–(19), we obtain

$$\begin{aligned}
&\sigma(D^2 - a^2)(D^2 - a^2 - p_1\sigma)(D^2 - a^2 - q\sigma) \\
&\quad \times [\{1 - F(D^2 - a^2)\}(D^2 - a^2) - \sigma)]W - Ra^2\left(\frac{G-1}{G}\right) \\
&\quad \times (D^2 - a^2 - q\sigma)\left[\{1 - F(D^2 - a^2)\}(D^2 - a^2) - \sigma)\right] \\
&\quad \times W + Sa^2(D^2 - a^2 - p_1\sigma) \\
&= \left[\{1 - F(D^2 - a^2)\}(D^2 - a^2) - \sigma)\right] \\
&\quad - T_A(D^2 - a^2 - p_1\sigma)(D^2 - a^2 - q\sigma)D^2 W \\
&= (D^2 - a^2 - p_1\sigma)(D^2 - a^2 - q\sigma) \\
&\quad \times \left[\{1 - F(D^2 - a^2)\}(D^2 - a^2) - \sigma)\right] \\
&\quad \times \{1 - F(D^2 - a^2)\}(D^2 - a^2)^2 W,
\end{aligned} \qquad (20)$$

where $R = \frac{g\alpha\beta d^4}{\nu\kappa}$ is the Rayleigh number and $S = \frac{g\alpha'\beta' d^4}{\nu\kappa'}$ is the solute Rayleigh number.

Using expression (15), the boundary conditions (14) in non-dimensional form, transform to

$$W = D^2W = 0,\ \Theta = 0,\ \Gamma = 0,\ DZ = 0 \text{ at } z = 0 \text{ and } z = 1. \qquad (21)$$

Using the boundary conditions (21), it can be shown with the help of eqs. (16)–(19) that all the even order derivatives of W must vanish at $z = 0$ and 1. Hence, the proper solution of W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \qquad (22)$$

where W_0 is a constant. Substituting the proper solution (22) in eq. (20), we obtain the dispersion relation

$$\begin{aligned}
R_1 = &\left(\frac{G}{G-1}\right) \times i\sigma_1(1 + x)(1 + x + ip_1\sigma_1)(1 + x + iq\sigma_1) \\
&\times [\{1 + F_1(1 + x)\}(1 + x) + i\sigma_1] \\
&+ S_1 x(1 + x + ip_1\sigma_1) \\
&\times [\{1 + F_1(1 + x)\}(1 + x) + i\sigma_1] \\
&+ T_1(1 + x + ip_1\sigma_1)(1 + x + iq\sigma_1) \\
&+ (1 + x + ip_1\sigma_1)(1 + x + iq\sigma_1) \\
&\times [\{1 + F_1(1 + x)\}(1 + x) + i\sigma_1]
\end{aligned}$$

$$\times\{1+F_1(1+x)\}(1+x)^2/x(1+x+iq\sigma_1)$$
$$\times\,[\{1+F_1(1+x)\}(1+x)+i\sigma_1] \quad (23)$$

where $R_1=\frac{R}{\pi^4}$, $S_1=\frac{S}{\pi^4}$ $F_1=\pi^2F$, $T_1=\frac{T_A}{\pi^2}$ and $i\sigma_1=\frac{\sigma}{\pi^2}$.

4. The stationary convection

For the stationary convection, $\sigma=0$ and eq. (23) reduces to

$$R_1=\left(\frac{G}{G-1}\right)(1+x)^3\{1+F_1(1+x)\}^2$$
$$+S_1x\{1+F_1(1+x)\}+T_1/x[\{1+F_1(1+x)\}]. \quad (24)$$

Eq. (24) expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters G, F_1, S_1 and T_1. For fixed F_1, S_1 and T_1, let G (accounting for the compressibility effects) also be kept fixed. Then we find that

$$\bar{R}_c=\left(\frac{G}{G-1}\right)R_c \quad (25)$$

where $\bar{R}_c$ and R_c denote respectively the critical Rayleigh numbers in the presence and absence of compressibility. $G>1$ is relevant here. The cases $G<1$ and $G=1$ correspond to negative and infinite values of the critical Rayleigh numbers in the presence of compressibility, which are not relevant in the present study. The effect of compressibility is, therefore, to postpone the onset of thermosolutal convection and so has a stabilizing effect.

Eq. (24) yields

$$\frac{dR_1}{dS_1}=\frac{G}{G-1}, \quad (26)$$

$$\frac{dR_1}{dT_1}\quad\frac{G}{G-1}\quad /x[\{1+F_1(1+x)\}], \quad (27)$$

which imply that stable solute gradient and rotation have stabilizing effects on the thermosolutal convection. Eq. (24) also yields

$$\frac{dR_1}{dF_1}=\left(\frac{G}{G-1}\right)(1+x)\left[(1+x)^3\{1+F_1(1+x)\}^2-T_1\right]\Big/$$
$$x\{1+F_1(1+x)\}^2. \quad (28)$$

It is evident from eq. (28) that the couple-stress viscosity has a stabilizing effect if .

$$T_1<(1+x)^3\ \{1+F_1\ (1+x)\}^2, \quad (29)$$

and has a destabilizing effect also if

$$T_1>(1+x)^3\ \{1+F_1\ (1+x)\}^2. \quad (30)$$

The couple-stress viscosity thus, has a dual role on the thermosolutal convection in compressible, rotating, couple-stress fluid. It has a stabilizing effect if $T_1<(1+x)^3\{1+F_1(1+x)\}^2$, and also in the absence of rotation ($T_1\to0$) whereas the couple-stress viscosity has a destabilizing effect if $T_1>(1+x)^3\{1+F_1(1+x)\}^2$.

5. The case of overstability

Here, we discuss the possibility of whether instability may occur as an overstability. We have put $i\sigma_1=\sigma/\pi^2$ in eq. (23), it being remembered that σ may be complex. Since for overstability, we wish to determine the critical Rayleigh number for the onset of instability, a state of pure oscillations, it suffices to find conditions for which eq. (23) will admit of solutions with σ_1 real. Equating real and imaginary parts of eq. (23) and eliminating R_1 between them, we obtain

$$Ac_1^2+Bc_1+C=0, \quad (31)$$

where we have put $c_1=\sigma_1^2$ $\alpha=1+x$ and

$$A=q^2\alpha^2(1+p_1+p_1F_1\alpha),$$
$$B=\alpha^4(1+p_1+p_1F_1\alpha)+q^2\alpha^4(1+F_1\alpha)(p_1+1)$$
$$+q^2\alpha^5F_1(1+F_1\alpha)\{2+p_1(1+F\alpha)\}$$
$$+S_1\alpha(\alpha-1)(p_1-q)+T_1q^2\alpha\{(p_1-1)+p_1F_1\alpha\}, \quad (32)$$

$$C=\alpha^6(1+F_1\alpha)^2(1+p_1+p_1F_1\alpha)+S_1\alpha^3(\alpha-1)$$
$$\times(1+F_1\alpha)^2(p_1-q)+T_1\alpha^3\{(p_1-1)+p_1F_1\alpha\}.$$

As σ_1 is real for overstability, the two values of $c_1=(\sigma_1^2)$ are positive. Eq. (31) is quadratic in c_1 and does not involve any of its roots to be positive if

$$p_1>1 \text{ and } p_1>q,$$
$$\text{if } \kappa<\nu \text{ and } \kappa<\kappa', \quad (33)$$

for then the coefficients of c_1^2, c_1 and the constant term are all positive and there is no change of sign in eq. (31). $\kappa<\min\{\nu,\kappa'\}$ is, therefore, a sufficient condition for the non-existence of overstability, the violation of which does not necessarily imply occurrence of overstability.

6. Conclusion

Stommel and Fedorov [12] and Linden [13] have remarked that the length scales characteristic of double diffusive convecting layers in the ocean may be sufficiently large that the ocean may be sufficiently large that the Earth's rotation might be important in their formation. Moreover, the rotation of the Earth distorts the boundaries of a hexagonal convection cell in a fluid through a porous medium and distortion plays an important role in the extraction of energy in the geothermal regions.

Couple-stress fluid is an important and useful non-Newtonian fluid. Due to importance of non-Newtonian fluids, compressibility and thermosolutal convection, a layer of compressible, couple-stress fluid heated and soluted from below is studied. During the study it is found that presence of compressibility postpones the onset of convection. The presence of rotation and the stable solute gradient also postpones the onset of convection. In the absence of rotation as well as stable solute gradient the Rayleigh number increases with the increase in couple stress parameters thus postpones the onset of convection.

The couple stress viscosity postpones as well as hastens the onset of convection depending on rotation parameter. If $T_1 < (1 + x)^3\{1 + F_1(1 + x)\}^2$, it has a stabilizing effect, whereas it has a destabilizing effect if $T_1 > (1 + x)^3\{1 + F_1(1 + x)\}^2$. In the absence of rotation, couple stress viscosity always postpone the onset of thermosolutal convection in the presence of compressibility. Overstable comes into play and sufficient condition for the non-existence of overstability is found. For $\kappa < \nu$ and $\kappa < \kappa'$, overstability cannot occur and the principle of exchange of stabilities is valid. In the absence of couple stress viscosity ($F_1 = 0$) the sufficient condition for non-existence of overstability are same as in its presence.

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