Balanced Vehicle Routing: Polyhedral Analysis and Branch-and-Cut Algorithm☆

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Abstract

This paper studies a variant of the unit-demand Capacitated Vehicle Routing Problem, namely the Balanced Vehicle Routing Problem, where each route is required to visit a maximum and a minimum number of customers. A polyhedral analysis for the problem is presented, including the dimension of the associated polyhedron, description of several families of facet-inducing inequalities and the relationship between these inequalities. The inequalities are used in a branch-and-cut algorithm, which is shown to computationally outperform the best approach known in the literature for the solution of this problem.

Keywords: routing, balanced, polyhedral study, branch-and-cut algorithm.

1. Introduction

The Capacitated Vehicle Routing Problem (CVRP) is concerned with designing routes for a fleet of capacitated vehicles to serve a number of customers. Each customer is in a geographical location and demands a commodity that must be transported by a vehicle from a specific location called the depot. Each vehicle must start from and end at the depot, transporting the demands of a subset of customers without violating the capacity limitations. Travel costs between any pair of locations are known and symmetric, and might be related to the distance between the two locations. The CVRP consists of determining a minimum cost set of routes for the vehicles to serve the demand of each customer with exactly

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one visit. The capacity limitations in the CVRP relate to the maximum load that the vehicle can carry upon leaving the depot; see e.g., [19] for different CVRP formulations. For the variant of the problem where the demand of each customer is equal to one (often called the *unit-demand CVRP*), the capacity limitation can be interpreted as the maximum number of customers that can be visited in each route.

Under the traditional objective of minimizing the travel cost, optimal solutions may contain imbalanced routes, in the sense that some vehicles visit the maximum allowed number of customers while others visit only a few customers. One way to avoid this situation is to impose an additional limitation on the minimum number of customers that should be served on each route. With such a constraint, optimal solutions may result in routes serving similar numbers of customers. We will refer to this extension as the *Balanced Vehicle Routing Problem (BVRP)*.

The BVRP has been studied by [11, 12] under the name of "vehicle routing problem with lower bound on the number of customers per route". The work in [11] describes a branch-and-cut algorithm based on a class of so-called Reverse Multi-Star (RMS) inequalities, which are obtained by projecting out the flow variables from a single commodity flow formulation of the problem. The RMS inequalities are related to the Multi-Star (MS) inequalities considered in [1]. A multi-depot variant of the problem has been studied in [4], describing several formulations and Benders decomposition algorithms.

Alternative approaches described in the literature for finding balanced routes include the use of a multi-objective function, as opposed to enforcing additional constraints in the formulation. Since our paper follows the latter approach to solve the BVRP, we refer the interested reader to [15] for a survey on the former.

The main contribution of our paper is to provide the first investigation of the BVRP polyhedron extending known results on the CVRP polytope. One of the earliest works that conducts a polyhedral study for the cardinality-constrained minimum spanning tree problem and the unit-demand CVRP is [1]. The authors study a variety of MS inequalities, divided into "Large", "Intermediate" and "Small". They present two additional sets of inequalities called Ladybug and Partial MS inequalities, and conclude by analyzing the clique inequalities. The number of routes in a solution is not fixed in their work. Further polyhedral analysis for the unit-demand CVRP where the number of routes is fixed can be found in [7], which presents results on the dimension of the associated polyhedron and facet-inducing properties of the trivial inequalities and the capacity constraints. Most of the proofs presented in [7] are based on what is known as the "indirect method", which consists of concluding the unique representation of the inequality; see e.g. [14]. [9] extend the polyhedral analysis by studying additional inequalities and extensions beyond the unit-demand case. They exploit the fact that the polyhedron of the Graphical Vehicle Routing Problem (GVRP) is a full-dimensional and includes the CVRP polytope as a face. Most of the analysis in that article are about inequalities of the GVRP polyhedron. A generalization of these inequalities for the general-demand CVRP can be found in [17, 18].

Our work aims to contribute to the efforts above and present a polyhedral analysis of the BVRP, which includes the unit-demand CVRP as a special case when the lower limit for the vehicle load is equal to one. We study the dimension of the associated polyhedron and some facet-inducing properties. These results are exploited in a branch-and-cut algorithm to solve the BVRP. Computational experiments show that our implementation is able to solve much larger-scale BVRP instances than previous approaches in the literature.

The rest of the paper is organized as follows. Section 2 provides a formal description of the problem, introduces the notation, and presents a mathematical formulation. Section 3 studies the dimension of the BVRP polyhedron and presents some of its facets. A branch-and-cut algorithm using the new inequalities, together with computational results, are detailed in Section 4. Conclusions are given in Section 5.

2. Problem description

The BVRP is defined on an undirected graph $G = (V, E)$ where $V = \{1, \dots, n\}$ is the set of vertices, each one representing a location, and $E = \{(i, j) | i < j, i \in V, j \in V\}$ is the set of edges. Each edge $(i, j) \in E$ has a travel cost c_{ij}, which is often a function of the distance between the locations. Node 1 represents the depot and the set $V' = \{2, \dots, n\}$ represents the customers, each of which has a unit demand. The minimum and maximum numbers of customers allowed to be served by each vehicle are denoted by $\underline{Q}$ and $\overline{Q}$, respectively. We assume that the number m of vehicle routes is fixed. We also assume that the following conditions hold:

(c.1) $\overline{Q} \leq n - 1 - (m - 1)\underline{Q}$,

(c.2) $\underline{Q} \geq n - 1 - (m - 1)\overline{Q}$, (c.3) $m\underline{Q} \leq n-1 \leq m\overline{Q}$.

Conditions (c.1) and (c.2) can be assumed without loss of generality since, in the event that one of these two conditions is not satisfied, then the bound in that condition can be improved by adjusting the right hand side value. Note that it is not possible to violate (c.1) and (c.2) simultaneously, so only one of the bounds could be improved. Condition (c.3) ensures that the problem is feasible, i.e., the customers can be distributed over the m routes.

For simplicity, we assume that in what follows $m \geq 2$, $\underline{Q} \geq 3$ and $n > 5$, and we will also study separately some cases without these assumptions. When $S \subseteq V'$, the customer set $V' \setminus S$ is denoted by S'. Given a function x on the edges, we write $x(E(S:T))$ instead of $\sum_{i \in S, j \in T: i \neq j} x_{ij}$ for any $S, T \subseteq V$. We also use $x(\delta(S)) = x(E(S : V \setminus S))$ and $x(\gamma(S)) = x(E(S : S))$.

The BVRP can be mathematically formulated as the integer linear program:

$$\text{Minimize} \sum_{(i,j)\in E} c_{ij} x_{ij} \tag{2.1}$$

$$x(\delta(i)) = 2 \qquad i \in V' \tag{2.2}$$

$$x(\delta(1)) = 2m \tag{2.3}$$

$$x(\delta(S)) \geq 2 \left\lceil \frac{|S|}{\overline{Q}} \right\rceil \qquad S \subset V' \tag{2.4}$$

$$x(\delta(S)) \geq 2x(E(\{1\} : S)) \qquad S \subset V' : \ |S| < \underline{Q} \tag{2.5}$$

$$x_{ij} \in \{0, 1\} \qquad (i, j) \in E. \tag{2.6}$$

The objective function (2.1) minimizes the travel cost. Degree equalities (2.2) ensure that each customer is visited exactly once. Equation (2.3) forces to design a route for each vehicle. Constraints (2.4) ensure that no route exceed the maximum capacity $\overline{Q}$. Constraints (2.5) avoid routes that are too small, i.e., those that visit less than $\underline{Q}$ customers. These inequalities can be rewritten as $x(\delta(S \cup \{1\})) \geq 2m$, with the interpretation that if $|S| < \underline{Q}$ then all vehicles are necessary to serve the customers in S'. Finally, constraints (2.6) force the variables to be binary.

3. Polyhedral Analysis

In this section, we first provide results on the dimension of the BVRP polyhedron, defined as

$$P_{BVRP} = \text{Convex.Hull}\{x \in \mathbb{R}^{|E|} \mid x \text{ satisfies (2.2)–(2.6)}\},$$

and then present a number of facet-defining inequalities.

Theorem 3.1. *$dim(P_{BVRP}) = |E| - n$.*

A detailed proof of Theorem 3.1 is presented in Appendix A. It is based on a partition of V' into subsets H^i $(i = 1, \ldots, m)$, each containing a number of customers between $\underline{Q}$ and $\overline{Q}$, and then exploit known results for the polytope associated to the Travelling Salesman Problem (TSP). Note that each route is a TSP solution. The first family of solutions used in the proof is denoted Φ_1, which contains BVRP solutions where all the nodes in H^i are visited by the same route. The remaining families are shown by $\Phi_2, \ldots, \Phi_6$, which are defined in such a way that they use the edges not used by the solutions in Φ_1.

We use the so-called "direct method" which consists in enumerating $|E| - n + 1$ affinely-independent BVRP solutions.

3.1. Facets of the BVRP polytope

This section presents five classes of facet-defining inequalities for the BVRP polyhedron.

3.1.1. Trivial inequalities

We study four sets of trivial inequalities represented by the four theorems below.

Theorem 3.2. *Inequalities $x_{1i} \geq 0$ for all $i \in V'$ define facets of P_{BVRP} if $m \geq 2$, $\underline{Q} \geq 2$, $\overline{Q} \geq 3$.*

Proof: We use the families of solutions described in the proof of Theorem 3.1 with minor modifications to produce solutions without the edge $(1, i)$. Since $\overline{Q} \geq 3$, we can assume that the partition $\mathcal{P}$ of V' is done such that customer i is in a subset H^i (containing at least 3 customers). To adapt the solutions in Φ_1 we exploit the fact that $x_{1i} \geq 0$ is a facet-inducing inequality in the TSP; hence, we obtain one solution less than in the proof of Theorem 3.8, all of them being BVRP solutions without the edge $(1, i)$. The solutions in Φ_2, Φ_3, Φ_5 and Φ_6

do not need modification by simply setting u_i to be customer i, so no solution includes $(1, i)$. Setting $u = i$, the solutions in Φ_4 do not include the edge $(1, i)$. □

Theorem 3.3. *The inequalities $x_{ij} \geq 0$ for all $i, j \in V'$ define facets of P_{BVRP} if $m \geq 2$ and $\underline{Q} \geq 3$.*

Proof: We select the partition $\mathcal{P}$ such that i and j belong to different subsets, say H^i and H^j, respectively.

All solutions in Φ_1 satisfy $x_{ij} = 0$; thus, all of them can be used in this proof too. To use the solutions in Φ_2, select $u_i \in H^i \setminus \{i\}$ and $v_j \in H^j \setminus \{j\}$; then all the solutions do not use (i, j), except the one with $u = i$ and $v = j$. To use the solutions in Φ_3, select $u = i$ and $v_j = j$. To use the solutions in Φ_4, select $u_i = i$ and $v = j$. To use the solutions in Φ_5, select $u_i = i$ and $v_j = j$. To use the solutions in Φ_6, select $u = i$ and $v = j$. They are all affinely independent and only one of them uses (i, j). □

Theorem 3.4. *The inequalities $x_{1i} \leq 1$ for all $i \in V'$ define facets of P_{BVRP} if $m \geq 2$ and $\underline{Q} \geq 3$.*

Proof: As in the proof of Theorem 3.2, the BVRP solutions in Φ_1 can be constructed so only one does not use $(1, i)$. By selecting appropriately the fixed vertices in H^i and H^j, the solutions of the other families use the edge $(1, i)$. □

The inequalities in the previous theorem are the special case of inequalities (2.5) where $S = \{i\}$.

Theorem 3.5. *The inequalities $x_{ij} \leq 1$ for all $i, j \in V'$ define facets of P_{BVRP} if $m \geq 2$, $\underline{Q} \geq 3$ and $\overline{Q} \geq 5$.*

Proof: Consider the partition $\mathcal{P}$ such that customers i and j are in the same subset, say H^k, with $|H^k| \geq 5$. Then, in the resulting TSP on $H^k \cup \{1\}$ we build one solution less than in the proof of Theorem 3.1 (because $x_{ij} \leq 1$ is facet-defining in the TSP). The solutions in the other families can be easily modified in such a way that edge (i, j) is used in all the solutions. □.

The inequalities in the previous theorem are the special case of (2.4) where $S = \{i, j\}$.

3.1.2. Capacity constraints

Inequalities (2.4) in the BVRP formulation are called *Capacity Constraints*. The authors in [7] show that they are facet-inducing inequalities in the CVRP if there exists an integer α with $1 \leq \alpha \leq K^* - 2$ and $K^* = \lceil (n-1)/\overline{Q} \rceil$ such that $|S| = \alpha\overline{Q} + 1$ and $3 \leq K^* \leq \overline{Q} \leq n - 1 - \alpha(\overline{Q} - 1) - 2$.

The capacity constraints never induce facets for the CVRP polytope when $|S| = \alpha\overline{Q}$ because a solution satisfying (2.4) with equality also satisfies $x(E(S : S')) = 0$, and therefore the constraints associated with S would be dominated by the same constraints associated with the set S without one customer.

In what follows we study the conditions under which the capacity constraints are facet-inducing for P_{BVRP}. First, we look at a special partition of the customers which will be used in some of the ensuing proofs.

Lemma 3.6. *Let $\mathcal{P}_u$ be the partition of the $n-1$ customers into m subsets maximizing the number of subsets with $\overline{Q}$ customers first, and maximizing the number of subsets with $\underline{Q}$ customers second. This partition has a subset with $\overline{Q}$ customers and another subset with $\underline{Q}$ customers. At most one subset in this partition may have a number of customers larger than $\underline{Q}$ and smaller than $\overline{Q}$. Let $\alpha_u, \beta_u, \lambda_u$ non-negative integer numbers such that $n-1 = \alpha_u\overline{Q} + \beta_u\underline{Q} + \lambda_u$, $\underline{Q} \leq \lambda_u < \overline{Q}$ and $\alpha_u + \beta_u = m - 1$.*

In general the capacity constraints are not facet-inducing of the P_{BVRP}. This can be illustrated with an example where $n = 19$, $\overline{Q} = 6$, $\underline{Q} = 3$ and $m = 5$. Then $\alpha_u = 1$, $\beta_u = 3$, $\lambda_u = 3$. Consider a set S with $|S| = 11$, thus $\lceil |S|/\overline{Q} \rceil = 2$. However if only two vehicles are used to visit the 11 customers in S then the remaining three vehicles will need to visit the seven customers in S', which means that at least one vehicle will visit less than $\underline{Q}$ customers. Indeed, the inequality $x(\delta(S)) \geq 6$ is valid in this case, and therefore the capacity constraint (2.4) is dominated.

The capacity inequalities for the BVRP can be improved as follows:

$$x(\delta(S)) \geq R(S) \qquad S \subset V', \tag{3.1}$$

where $R(S)$ is calculated as follows:

$$R(S) = 2\begin{cases} \left\lceil \frac{|S|}{\overline{Q}} \right\rceil & \text{if } |S| \leq \alpha_u\overline{Q} + \lambda_u \\ \alpha_u + 1 + \left\lceil \frac{|S| - \alpha_u\overline{Q} - \lambda_u}{\underline{Q}} \right\rceil & \text{if } |S| > \alpha_u\overline{Q} + \lambda_u, \end{cases} \tag{3.2}$$

and where the coefficient α_u can be calculated as

$$\alpha_u = \left\lfloor \frac{(n-1) - m\underline{Q}}{\overline{Q} - \underline{Q}} \right\rfloor.$$

Theorem 3.7. *Inequalities (3.1) are valid for the BVRP.*

Proof: When $|S| > \alpha_u \overline{Q} + \lambda_u$ then at least $\alpha_u + 1$ vehicles are necessary to visit the customers in S. In addition there are $|S| - (\alpha_u \overline{Q} + \lambda_u)$ customers to be visited by vehicles using the minimum capacity $\underline{Q}$. $\square$

In the case in which $|S| > \alpha_u \overline{Q} + \lambda_u$, the lifted capacity inequalities (3.1) can be rewritten as follows:

$$x(\delta(S)) \geq 2\left(m - \left\lfloor \frac{|S'|}{\underline{Q}} \right\rfloor\right). \tag{3.3}$$

Note that $\lfloor |S'|/\underline{Q} \rfloor$ is a lower bound on the number of vehicles that can be used to serve exclusively customers not in S. Then, the remaining vehicles of the fleet must be used to serve customers in S.

Figure 1 represents the right-hand side value of the inequalities associated with the previous example with $n = 19$, $\overline{Q} = 6$, $\underline{Q} = 3$ and $m = 5$. The red line shows the value of the capacity constraint (2.4) and the green line shows $R(S)$ for the lifted constraints (3.1) in the case where $|S| > \alpha_u \overline{Q} + \lambda_u$. The horizontal axis represents $|S|$ and the vertical axis shows the right-hand side value $x(\delta(S))$. Note that there is an interval (related to $|S|$) in which both values always match; this region corresponds to $\alpha_u \overline{Q} < |S| \leq \alpha_u \overline{Q} + \lambda_u$. To the left of this region, the capacity constraint leads to an equivalent or better inequality than inequality (3.1). To the right of this region inequality (3.1) is stronger than the capacity constraint.

If $|S| = \alpha \overline{Q}$ for an integer number α, then inequality (3.1) never induces a facet of P_{BVRP}. Indeed, let $i \in S$ and consider the inequality (3.1) for $S^* = S \setminus \{i\}$, i.e. $x(\delta(S^*)) \geq R(S^*)$. This inequality is equivalent to

$$x(\delta(S)) - x(\delta(\{i\})) + 2x(E(\{i\} : S^*)) \geq R(S^*),$$

also equivalent to $x(\delta(S)) \geq R(S^*) + 2 - 2x(E(\{i\} : S^*))$. Since $R(S^*) = R(S)$ and $x(E(\{i\} : S^*)) \leq 1$, inequality (3.1) for S^* dominates inequality (3.1) for S.

Theorem 3.8. *Inequalities (3.1) are facet-inducing for P_{BVRP} if $|S| < \alpha_u \overline{Q} + \lambda_u$, $|S| \neq \alpha \overline{Q}$ for any $\alpha \in \mathbb{N}$, and $\overline{Q} \geq 5$ (so we also need $n \geq 9$).*

The proof is presented in Appendix B. If the condition on S does not hold then the inequality would be dominated by the capacity inequality (2.4) given by the subset obtained by either adding or removing a customer in S.

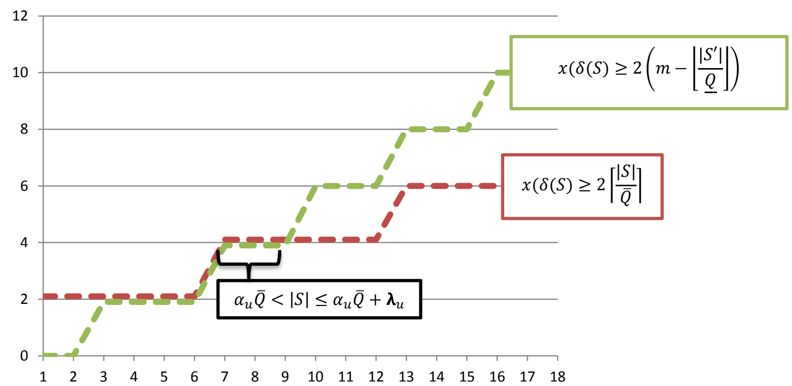

Figure 1: Comparison between the capacity constraint and the lifted capacity inequality in an example with $n = 19$, $\overline{Q} = 6$, $\underline{Q} = 3$ and $m = 5$.

Theorem 3.9. *Inequalities (3.3) are facet-inducing for P_{BVRP} if $|S| \neq c\underline{Q} + \alpha_u\overline{Q} + \lambda_u$, where $c \in \mathbb{N}$ and $\overline{Q} \geq 5$ (so $n \geq 9$).*

Proof: It is analogous to Theorem 3.8, using the same partition and selecting the vertices in S as before. By definition of the right-hand side value, it can be obtained by solutions such that all the subsets $H^i \subset S$ define one route, and the route associated with a subset H^k ($H^k \cap S \neq \emptyset$ and $H^k \setminus S \neq \emptyset$) has to satisfy $x(\delta(H^k \cap S)) = 2$. To this end, the same families of solutions can be used. □

3.1.3. Lower Capacity constraints

In case that $|S| \geq \underline{Q}$, inequalities (2.5) can be generalized as follows:

$$x(\delta(S)) \geq 2\left(x(E(\{1\} : S)) - \left\lfloor \frac{|S|}{\underline{Q}} \right\rfloor\right). \tag{3.4}$$

These inequalities are the undirected version of the Enhanced Rounded Multi-Star (ERMS) inequalities presented in [11] for the asymmetric variant of the problem.

Inequalities (3.4) can be lifted as follows:

$$R^l(S) = 2\begin{cases} \left\lfloor \frac{|S|}{\underline{Q}} \right\rfloor & \text{if } |S| < \beta_u\underline{Q} + \lambda_u, \\ m - \left\lceil \frac{n-1-|S|}{\overline{Q}} \right\rceil & \text{if } |S| > \beta_u\underline{Q} + \lambda_u. \end{cases} \tag{3.5}$$

In what follows we show the relationship between these inequalities and the lifted capacity constraints (3.1) in Table 1 by exploiting the fact that the degree of the depot is fixed. The first row of this table indicates that, under the condition that $|S| < \alpha_u \overline{Q} + \lambda_u$, the capacity constraints correspond to the lower capacity constraints written for $S' = V' \setminus S$. This is easy to check by observing that

$$x(\delta(S')) = x(\delta(S)) - x(E(\{1\} : S)) + x(E(\{1\} : S')),$$

and

$$x(E(\{1\} : S)) + x(E(\{1\} : S')) = 2m.$$

Similarly, it can be proved that when $|S| > \alpha_u \overline{Q} + \lambda_u$, the lifted capacity inequalities correspond to the lower capacity inequalities written for S'. Consequently, the lower capacity inequalities induce the same facets as the lifted capacity constraints (3.1).

Note that when $|S| = \alpha \underline{Q}$ for any $\alpha \in \mathbb{N}$ and $|S| \leq \beta_u \underline{Q} + \lambda_u$, then inequalities (3.1) with (3.5) instead of $R(S)$ would be dominated. Let S be a subset of customers such that $|S| = \alpha \underline{Q}$, and let $S^* = S \setminus \{i\}$ for some customer $i \in S$. Since $R^l(S) = R^l(S^*) - 2$, and using

$$x(\delta(S)) = x(\delta(S^*)) + x(\delta(\{i\})) - 2x(E(\{i\} : S^*)),$$

the inequality with (3.5) for S^* gives

$$x(\delta(S)) \geq R^l(S) + 2 - 2x(E(\{i\} : S^*)),$$

which clearly dominates $x(\delta(S)) \geq R(S)$. Similarly, it can be checked that the inequalities with (3.5) never induce facets of P_{BVRP} when $|S| = \alpha \overline{Q}$, $\alpha \in \mathbb{N}$ and $|S| \geq \beta_u \underline{Q} + \lambda_u$.

3.1.4. Multi-Star (MS) inequalities

The Multi-Star (MS) inequalities for the CVRP are defined as follows. Let $S \subset V'$ be a subset of customers and let $N \subseteq S'$ where $S' = V' \setminus S$. Then, the MS inequality has the structure $x(\delta(S)) \geq \rho + \sigma x(E(S : N))$ where ρ and σ are constants depending on N and S. In this paper we only address the case in which $N = S'$.

An analysis of the cases in which the MS inequalities are facet-inducing for the minimum spanning tree and minimum spanning forest polytopes can be found in [1]. The authors define three different types of MS inequalities by

Table 1: Relationships between the Capacity and Lower Capacity constraints

Condition in $\lvert S\rvert$	(Upper) Capacity	Lower Capacity	Conditions in $\lvert S'\rvert$
$\lvert S\rvert < \alpha_u\overline{Q} + \lambda_u$	$x(\delta(S)) \geq 2\left\lceil \frac{\lvert S\rvert}{\overline{Q}} \right\rceil$	$x(\delta(S')) \geq 2x(E(\{1\}:S')) - 2\left(m - \left\lceil \frac{n-1-\lvert S'\rvert}{\overline{Q}} \right\rceil\right)$	$\lvert S'\rvert > \beta_u\underline{Q} + \lambda_u$
$\lvert S\rvert > \alpha_u\overline{Q} + \lambda_u$	$x(\delta(S)) \geq 2\left(\alpha_u + 1 + \left\lceil \frac{\lvert S\rvert - \alpha_u\overline{Q} - \lambda_u}{\underline{Q}} \right\rceil\right)$	$x(\delta(S')) \geq 2x(E(\{1\}:S')) - 2\left\lfloor \frac{\lvert S'\rvert}{\underline{Q}} \right\rfloor$	$\lvert S'\rvert < \beta_u\underline{Q} + \lambda_u$

Table 2: Relationships between the lifted MS and the lifted RMS inequalities

Condition in $\lvert S\rvert$	MS	RMS	Conditions in $\lvert S'\rvert$
$\lvert S\rvert < \alpha_u\overline{Q} + \lambda_u$	$x(\delta(S)) \geq 2\left(\frac{\lvert N\rvert + x(E(N:N'))}{\overline{Q}}\right)$	$x(\delta(S')) \geq 2\left(m - \beta_u - 1 - \frac{\lvert S'\rvert - \beta_u\underline{Q} - \lambda_u - x(E(S:S'))}{\overline{Q}}\right)$	$\lvert S'\rvert > \beta_u\underline{Q} + \lambda_u$
$\lvert S\rvert \geq \alpha_u\overline{Q} + \lambda_u$	$x(\delta(S)) \geq 2\left(m - \frac{\lvert N'\rvert - x(E(N:N'))}{\underline{Q}}\right)$	$x(\delta(S')) \geq 2\left(m - \frac{\lvert S'\rvert - x(E(S:S'))}{\underline{Q}}\right)$	$\lvert S'\rvert \leq \beta_u\underline{Q} + \lambda_u$

taking into account different conditions on the subset N. A wide variety of MS constraints is presented in [1].

Using $N' = V' \setminus N$, the MS inequalities can be written as:

$$x(\delta(N)) \geq 2\left(\frac{|N| + x(E(N:N'))}{\overline{Q}}\right) \qquad N \subset V'. \tag{3.6}$$

Since the right hand side is divided by $\overline{Q}$ and the number of routes with $\overline{Q}$ customers is limited by α_u, inequalities (3.6) can be lifted as follows:

$$x(\delta(N)) \geq R^{MS}(N) \qquad N \subset V', \tag{3.7}$$

where

$$R^{MS}(N) = 2\begin{cases} \frac{|N|+x(E(N:N'))}{\overline{Q}} & \text{if } |N| \leq \alpha_u\overline{Q} + \lambda_u \\ m - \frac{|N'|-x(E(N:N'))}{\underline{Q}} & \text{if } |N| > \alpha_u\overline{Q} + \lambda_u. \end{cases}$$

3.1.5. Enhanced Reverse Multistar (ERMS) inequalities

The Enhanced Reverse Multi-Star (ERMS) inequalities were introduced in [11]. For the asymmetric BVRP, these inequalities take the following form:

$$\underline{Q}x(E(\{1\}:S)) + x(E(S':S)) \leq (\underline{Q}-1)x(E(S:S')) + |S| \qquad S \subset V'.$$

Using (2.3), these inequalities are equivalent to

$$x(\delta(S)) \geq 2\left(m - \frac{|S'| - x(E(S:S'))}{\underline{Q}}\right) \qquad S \subset V'. \tag{3.8}$$

These inequalities correspond to (3.7) when $|S| > \alpha_u\overline{Q} + \lambda_u$. Therefore, the ERMS inequalities dominate the MS inequalities when $|N| > \alpha_u\overline{Q} + \lambda_u$.

Inequalities (3.8) can be improved by exploiting the fact that the number of routes that one can build with exactly $\underline{Q}$ customers is limited, and is related to m and $\overline{Q}$. Consider an example with $n = 22$, $m = 5$, $\overline{Q} = 5$ and $\underline{Q} = 3$. If we consider a subset S with $|S| = 4$ it is easy to check that in all the solutions satisfying $x(\delta(S)) = 2$, the solution which maximizes $x(E(S:S'))$ uses only one edge of $E(S:S')$. Therefore, since $|S'| = 17$, the resulting left hand side would be negative, so it could be improved considerably.

The partition $\mathcal{P}_u$ allows computing the number of routes that can visit $\underline{Q}$ customers. If $|S'| > \beta_u\underline{Q} + \lambda_u$ (or equivalently $|S| < \alpha_u\overline{Q}$) these inequalities can be improved as follows:

$$x(\delta(S)) \geq 2\left(m - \beta_u - 1 - \frac{|S'| - \beta_u \underline{Q} - \lambda_u - x(E(S:S'))}{\overline{Q}}\right) \qquad S \subset V'.$$

These inequalities coincide with inequalities (3.7) when $|S| \leq \alpha_u \overline{Q} + \lambda_u$. Therefore, the MS inequalities dominate the ERMS inequalities when $|S| \leq \alpha_u \overline{Q} + \lambda_u$. Table 2 shows the relationship between the lifted MS inequalities and the lifted ERMS inequalities.

3.2. Comb inequalities

The comb inequalities were first introduced by [8] for the TSP and further generalized in [14]. Extensions of the comb inequalities were later described for the CVRP [see 16, 2, 20], location-routing problems [see, e.g., 6] and the two level single truck and trailer problem [see, e.g., 5]. We first present comb inequalities for the sake of completeness, and then present conditions under which they define facets for P_{BVRP}.

The comb inequalities are defined by a set $H \subset V'$ of customers, called the handle, and $t \geq 3$ sets of customers, called teeth, such that:

1. t is odd,
2. $H \cap T_j \neq \emptyset$ and $T_j \setminus H \neq \emptyset$ for each $j = 1, \ldots, t$,
3. $T_i \cap T_j = \emptyset$ for each $i, j \in \{1, \ldots, t\}$.

The comb inequalities are as follows:

$$x(\delta(H)) + \sum_{i=1}^{t} x(\delta(T_i)) \geq 3t + 1. \tag{3.9}$$

Theorem 3.10. *Inequalities (3.9) induce facets of P_{BVRP} if $|H| + \sum_{i=1}^{t} |T_i| \leq \overline{Q}$.*

Proof: The comb inequalities are facet-inducing inequalities for the TSP [14]. The proof of this theorem is similar to that of Theorem 3.1. All the customers in $H, T_1, \ldots, T_t$ can be visited by one route, and therefore the TSP on the union of these subsets can be used to build the solutions in Φ_1. The solutions built on the rest of families of solutions remain the same. □

The comb inequalities can be extended by taking into account the number of customers in the subsets. In the case that $|H| + \sum_{i=1}^{t} |T_i| > \overline{Q}$ then inequalities

(3.9) can be improved as follows ([20]). If $\sum_{i=1}^{t} R(T_i \cap H) + R(T_i \setminus H) + R(T_j)$ is odd, where $R(S)$ is defined in (3.2), then:

$$x(\delta(H)) + \sum_{i=1}^{t} x(\delta(T_i)) \geq \sum_{i=1}^{t} R(T_i \cap H) + R(T_i \setminus H) + R(T_j) + 1. \qquad (3.10)$$

In [20] the authors relaxed conditions 1 and 2 of the definition of *combs*. Condition 1 is modified to ensure that $\sum_{i=1}^{t} R(T_i \cap H) + R(T_i \setminus H) + R(T_j)$ should be odd, and condition 2 allows some intersection between each pair of teeth in either the handle or out of the handle, but not in both. If the value $R(T_i \cap H) + R(T_i \setminus H) + R(T_j)$ is even then the right hand side cannot be increased by one, and the inequalities are a linear combination of capacity constraints.

4. Branch-and-Cut Algorithm and Computational Analysis

This section first describes a branch-and-cut algorithm that uses the inequalities studied in the previous sections, and then presents computational results on solving the BVRP using benchmark instances from the literature.

4.1. Branch-and-Cut algorithm

The branch-and-cut algorithm was implemented in C++ using Visual Studio 2012 and CPLEX 12.5 on a personal computer with core i7 2600 processor and 4 GB memory. We use the *strong branching* strategy in CPLEX to explore the branch-and-bound tree.

The branch-and-cut algorithm operates on the basis of solving (2.1), (2.2), (2.3) and $0 \leq x_{ij} \leq 1$ for all $1 \leq i < j \leq n$. This linear program is denoted by L_r. At each node of the branch-and-bound tree, separation algorithms are invoked to find any violated inequalities from the following set of families, which are then added dynamically in a cutting-plane fashion:

- Capacity constraints (3.1) described in Section 3.1.2. The improved version of these inequalities are the lower capacity constraints. In the light of the equivalence between the capacity and the lower capacity constraints (see Table 1 in Section 3.1.3), by separating the improved capacity constraints, one also separates the lower capacity constraints.

- MS Inequalities (3.7). The improved version of these inequalities are the RMS inequalities, which have been obtained by analyzing the reverse multistar inequalities. Note that the RHS of inequalities (3.7) depend on $|N|$, and there is one case that coincides with the RMS inequalities as shown in Table 2. It therefore suffices to separate (3.7) alone.

- Comb inequalities (3.9) and (3.10).

The separation of these inequalities is done using the following procedure. Let x^* be an optimal solution of the linear program L_r at a given node of the branch-and-bound tree, possibly augmented with inequalities added in previous nodes, where the solution may be fractional or integer. We now describe exact and heuristic algorithms to find violated inequalities, collectively named as a *separation procedure*. These routines are applied sequentially, with each being invoked only if the preceding routine did not return a violated inequality. In each iteration of the cutting-plane phase, the number of cuts of each type to be included in the LP is limited to 100. The limit was not achieved in our computational experiments, but we find it useful when solving larger instances so the size of L_r does not increase too much.

In what follows we present the routines in the order applied within the cutting-plane phase.

1. First, let G^* be the weighted graph G where the capacity of edge (i, j) is x^*_{ij}, and shrink all edges (i, j) with $x^*_{ij} = 1$. The aim is to find subsets of vertices as candidates for generating violated lower capacity or upper capacity constraints.
2. We use a tabu search algorithm based on [3] for separating both the rounded capacity inequalities (3.1) and the MS inequalities. In the initialization phase, we generate an initial subset $S \subset V'$, starting from a single customer, where a new customer $v^* = \text{argmax}_{v \in V' \setminus S}\{x^*(E(S : \{v\}))\}$ is added to S at each iteration. In an interchange phase, we either add to S or remove from S a customer depending on whether both the resulting lifted capacity constraint and the MS inequality are closer to be violated for the modified subset.
3. We use the min-cut algorithm in [10] to separate the subtour elimination constraints. They are the capacity constraints (3.1) when $|S| \leq \overline{Q}$.
4. We use the exact procedure in [12] to separate the multistar inequalities, with an explicit focus on identifying RMS inequalities. Initially, a graph G' is constructed by adding a new dummy vertex (say 0) to graph G, which is connected to all the customers $V \setminus \{1\}$ using dummy edges. The

capacity of any edge $(1, j)$ for $j \in V \setminus \{0, 1\}$ is set to $(\underline{Q} - 1)x^*_{1j}$. The capacity of any edge (i, j) for $i, j \in V \setminus \{0, 1\}$ is set to $(\underline{Q} - 2)x^*_{1j}$. Finally, the capacity of any edge $(i, 0)$ for $i \in V \setminus \{0, 1\}$ is set to $1 - x^*_{i1}$. Then, if the capacity of the optimal cut separating vertex 1 and 0 in G' is smaller than $(\underline{Q} - 1)x^*(\{1\} : V \setminus \{1\})$ then the optimal cut yields a set S that violates a MS inequality. The optimal cut is calculated by solving a max-flow problem in the undirected network G'.

5. We use the heuristic procedures in [20][1] to separate the comb inequalities (3.9) and (3.10)
6. Finally, we use the procedure in [13][2] to separate the comb inequalities (3.9).

4.2. Computational results

Two sets of computational experiments are conducted. The first set uses the symmetric instances for the BVRP described and tested in [11], generated from the instance `eilA101`[3], each with 100 customers but assuming different values for $\underline{Q}$, $\overline{Q}$ and m. As these instances have the number of vehicles (m) fixed, we use the degree equation (2.3) in the formulation. We set a solution time limit of two hours when solving each instance. Table 3 shows the results of this experiment, in comparison with the results obtained in [11] for the same instances. The results presented in [11] were run on a slightly slower computer, Intel(R) Core(TM)2 CPU 6700 at 2.66 GHz desktop computer with 2 GB RAM, using CPLEX 12.1. The first three columns of this table show $\underline{Q}$, $\overline{Q}$ and m. The remaining columns are the best lower bound (LB) and upper bound (UB) on the objective function value obtained at the end of the time limit, the total time spent (T_t), the total time required by the separation procedures (T_s), the number of explored nodes in the tree (Nodes), and the lower bound obtained at the root node (LB0), the total number of cuts of each type generated by the separation procedures, corresponding to columns #CC (capacity constraints ((3.1))), #MS (multistar (3.7)) and #COMBS (combs (3.10)), respectively. The table also reports the number of these inequalities found at the root node, denoted by #CC0, #MS0 and #COMBS0, respectively. The notation '2h' indicates that the corresponding instance was not solved to optimality within two hours.

[1] Available at `http://www.hha.dk/~lys/CVRPSEP.htm`
[2] Available at `http://www.math.uwaterloo.ca/tsp/concorde/downloads/downloads.htm`
[3] Available at `http://neo.lcc.uma.es/vrp/vrp-instances/capacitated-vrp-instances/`

Table 3: Comparison results on instances generated from `eilA101`

			[11]					Branch-and-Cut Algorithm											
$\overline{Q}$	$\underline{Q}$	m	LB	UB	T_t	nodes	LB0	LB	UB	T_t	T_s	Nodes	LB0	#CC0	#MS0	#COMBS0	#CC	#MS	#COMBS
55	45	2	640	640	11.4	169	636.94	640	640	0.47	0.02	0	640	50	8	12	50	8	12
54	46	2	640	640	13.4	206	637.17	640	640	0.51	0.08	0	640	50	10	12	50	10	12
53	47	2	640	640	10.5	150	637.17	640	640	0.44	0.05	0	640	46	5	15	46	5	15
52	48	2	640	640	12.2	192	636.97	640	640	0.70	0.05	0	640	47	5	13	47	5	13
51	49	2	643	643	120.2	3873	637.17	643	643	3.49	1.08	8	639.75	74	21	113	389	37	697
38	24	3	655	655	34	818	652.02	655	655	6.29	0.86	0	655	140	7	190	140	7	190
37	26	3	657	657	39.7	1385	650.80	657	657	22.06	10.90	129	654.53	336	24	305	591	31	539
36	28	3	657	657	174.9	4394	652.01	657	657	7.82	1.98	12	654.63	339	23	307	1491	42	2631
35	30	3	661	661	595.1	17386	652.79	661	661	27.25	10.81	186	655.96	356	25	322	855	64	1272
34	32	3	662.5	665	2h	197585	653.50	665	665	588.01	133.12	1745	657.01	393	36	335	3583	75	9689
28	16	4	674	674	15	432	672.08	674	674	1.40	0.06	0	674	146	33	55	146	33	55
27	19	4	676	676	33.5	895	671.38	676	676	2.57	0.27	0	676	244	73	69	244	73	69
26	22	4	682	682	688.6	17968	675.00	682	682	85.83	22.70	164	678.77	537	32	444	1731	140	1725
22	11	5	704	704	730.4	9741	699.48	704	704	123.43	35.99	156	702.38	714	72	522	1657	267	1592
21	16	5	709.07	712	2h	120963	698.91	712	712	111.65	39.84	380	705.41	829	71	546	2535	266	2473
18	10	6	734.87	741	2h	98801	725.28	739	739	399.50	141.32	1653	733.03	951	42	655	4838	356	5743
17	15	6	735.23	805	2h	49901	727.73	744	744	567.69	199.93	1317	736.56	1055	55	730	4954	340	5923

The results reported in Table 3 show that all instances with 100 customers are solved to optimality by our branch-and-cut algorithm. They include the four instances that were not solved to optimality in [11] within two hours of computation time, but which were solved by our algorithm in less than 10 minutes. The difference in performance of the two algorithms can be attributed to the lifted lower and upper bound capacity constraints used within the cutting plane phase. The general trend on this set of instances is that those with tighter bounds (i.e., $\overline{Q}$ close to $\underline{Q}$) are more difficult to solve. It is worth highlighting that both the lower-bound and upper-bound capacity constraints (3.1) are separated first and usually have a more significant impact on the LB in comparison to the comb inequalities (3.10). The number of MS inequalities (3.7) used is quite low and generally do not have as big an impact on LB as they are separated after the lower bound and upper bound capacity constraints (3.1).

The second set of experiments are aimed at looking at the effect of introducing various inequalities into the branch-and-cut algorithm, namely capacity inequalities (3.1) (denoted by CC), multistar inequalities (3.7) (denoted by MS) and comb inequalities (3.10) (denoted by COMBS). We use the combinations CC+MS, CC+COMBS and CC+MS+COMBS to obtain three variants of the branch-and-cut algorithm. We use the same instances shown in Table 3 as well as additional larger BVRP instances generated from the CVRP instance `M-n151-k12` with 150 customers. The results for the 100-customer and 150-customer instances are presented in Table 4 and Table 5, respectively. For convenience, we indicate the lowest computational time needed to solve each instance in bold.

Table 4 shows that all the instances with 100 customers were solved to optimality by the three variants of the algorithm. In this case, the variant CC+COMBS proved to be the most effective, requiring the lowest average computational time to solve the instances to optimality and the smallest number of nodes evaluated within the exploration of the branch-and-cut tree. The variant CC+MS+COMBS exhibits a similar behavior, but is slightly worse than the variant CC+COMBS with respect to the same instances. The variant CC+MS showed the worst behavior out of the three, particularly on the number of nodes evaluated, implying roughly a four-fold increase in the number of nodes over the other two variants. As far as the root lower bounds are concerned, CC+COMBS and CC+MS+COMBS show a similar performance, which is better than that of CC+MS.

The three variants of the algorithm exhibit a slightly different behaviour for the instances with 150 customers, shown in Table 5. In this case, all instances were solved to optimality by the three variants of the algorithm, with the single exception that the variant CC+MS did not converge to an optimal solution for instance with $n = 151$, $\overline{Q} = 42$, $\underline{Q} = 36$ and $m = 4$ within the two-hour time limit. As in Table 4, the variant of the algorithm with capacity inequalities and combs inequalities enabled (i.e., CC+COMBS) shows the most effective compared to the other two variants in relation to the average solution time, whereas CC+MS+COMBS performed the best in terms of the number of nodes explored in the branch-and-bound tree. Interestingly, the average solution time required by CC+MS+COMBS is not significantly different than that of CC+COMBS, and in some of the instances, such as ($\overline{Q} = 76$, $\underline{Q} = 74$, $m = 2$), ($\overline{Q} = 42$, $\underline{Q} = 32$, $m = 4$) and ($\overline{Q} = 30$, $\underline{Q} = 22$, $m = 6$), the effect of the MS inequalities seems to be quite significant in reducing the computational time needed. In fact, the longest time needed to solve any of the 150-customer instances in the variant CC+MS+COMBS is lower than that of CC+COMBS, suggesting that the former algorithm is more robust. This result shows the relevance and effectiveness of separating the improved version of both the lower and upper bound capacity constraints first.

The results obtained on the set of instances with seven vehicles suggest that the problem is easier to solve if the interval ($\overline{Q}$, $\underline{Q}$) is either fairly loose or very tight, whereas in all other cases the complexity of the problem increases significantly.

The two sets of results shown in Tables 4 and 5 suggest that all the inequalities considered in the paper are shown to be useful when dealing with larger-scale instances, whereas it might suffice to only use CC+COMBS for smaller-size instances.

5. Conclusions

This paper presented polyhedral results and an exact algorithm for a unit-demand vehicle routing problem with lower and upper bounds on the number of customers that can be visited in each route. The computational results showed that the proposed algorithm outperformed a previously described algorithm on instances with up to 100 customers, and was able to solve instances with up to 150 customers for the first time in the literature. The computational tests suggested that all the inequalities described in the paper contribute positively to the performance of the branch-and-cut algorithm.

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Table 4: The effect of different inequalities on the performance of the algorithm on 100-customer instances

			CC+MS+COMBS				CC+MS				CC+COMBS			
$\overline{Q}$	$\underline{Q}$	m	T_t	T_s	nnodes	LB0	T_t	T_s	nnodes	LB0	T_t	T_s	nnodes	LB0
55	45	2	0.47	0.02	0	640.00	**0.42**	0.03	0	640.00	0.66	0.02	0	640.00
54	46	2	0.51	0.08	0	640.00	**0.47**	0.03	0	640.00	0.66	0.02	0	640.00
53	47	2	**0.44**	0.05	0	640.00	0.45	0.06	0	640.00	0.66	0.02	0	640.00
52	48	2	**0.70**	0.05	0	640.00	0.90	0.02	0	640.00	0.83	0.12	0	640.00
51	49	2	**3.49**	1.08	8	639.75	3.96	1.26	94	638.75	11.06	3.71	25	640.25
38	24	3	6.29	0.86	0	655.00	9.63	3.84	340	652.63	**4.23**	0.48	0	655.00
37	26	3	22.06	10.90	129	654.53	**8.03**	2.03	143	652.88	9.44	1.79	19	654.50
36	28	3	**7.82**	1.98	12	654.63	15.54	5.04	269	652.88	11.56	3.62	33	654.61
35	30	3	**27.25**	10.81	186	655.96	44.66	12.07	1013	653.92	136.05	52.63	1037	655.82
34	32	3	588.01	133.12	1745	657.01	1336.94	147.80	10400	655.00	**327.38**	96.58	1156	657.08
28	16	4	1.40	0.06	0	674.00	1.61	0.11	0	674.00	**0.73**	0.08	0	674.00
27	19	4	2.57	0.27	0	676.00	2.31	0.16	0	676.00	**1.81**	0.11	0	676.00
26	22	4	85.83	22.70	164	678.77	**12.98**	2.23	86	676.75	104.01	36.61	278	678.77
22	11	5	123.43	35.99	156	702.38	**7.83**	1.72	45	700.10	43.03	11.70	35	702.38
21	16	5	111.65	39.84	380	705.41	**106.78**	31.78	1482	702.66	218.35	65.61	628	705.35
18	10	6	**399.50**	141.32	1653	733.03	601.81	152.09	3679	730.85	443.74	115.52	1066	732.90
17	15	6	567.69	199.93	1317	736.56	462.06	101.45	2208	732.92	**444.87**	139.26	1118	736.58
Average			114.65	35.24	338.24	669.59	153.90	27.16	1162.29	668.20	**103.47**	31.05	317.35	669.60

Table 5: The effect of different inequalities on the performance of the algorithm on 150-customer instances

			CC+MS+COMBS				CC+MS				CC+COMBS			
$\overline{Q}$	$\underline{Q}$	m	T_t	T_s	nnodes	LB0	T_t	T_s	nnodes	LB0	T_t	T_s	nnodes	LB0
80	70	2	19.25	1.37	9	708.00	10.81	2.04	153	706.00	**4.13**	0.56	9	708.36
78	72	2	16.61	1.48	9	708.41	4.85	0.39	18	706.00	**3.29**	0.58	5	708.33
77	73	2	6.93	0.44	10	708.43	**4.18**	0.27	8	705.85	5.88	0.23	5	708.25
76	74	2	**15.54**	1.26	10	708.74	58.30	8.60	482	706.08	193.89	21.11	266	708.70
55	45	3	**0.70**	0.02	0	719.00	0.83	0.05	0	719.00	0.73	0.06	0	719.00
53	47	3	1.39	0.09	0	720.00	1.67	0.05	0	720.00	**1.06**	0.03	0	720.00
52	48	3	1.19	0.09	0	721.00	**1.17**	0.11	0	721.00	1.62	0.11	0	721.00
51	49	3	1020.17	107.05	1252	722.00	**47.30**	5.21	124	719.67	1007.49	120.45	1507	722.33
44	34	4	1093.74	337.70	1733	740.25	1397.63	279.01	8872	736.58	**554.41**	142.69	1262	740.24
42	36	4	**1443.17**	403.29	3125	740.56	2h	1053.64	11264	737.24	6253.16	690.03	5705	740.72
35	25	5	1756.32	426.85	2008	761.57	1387.29	354.34	8848	757.09	**1334.78**	430.44	2418	761.20
33	27	5	4524.36	887.94	2977	761.02	**2241.72**	441.12	10363	757.57	4178.69	837.13	2834	761.81
30	22	6	**22.15**	7.27	35	781.76	390.94	85.32	1219	777.53	199.71	76.88	244	782.34
28	23	6	147.06	36.75	60	785.16	**102.52**	39.52	418	782.47	205.02	25.21	23	785.22
27	17	7	856.32	260.80	1826	809.95	2169.18	420.84	2745	806.89	**344.00**	135.38	2066	807.94
26	17	7	1334.18	473.31	1842	805.75	2105.37	533.62	2652	806.50	**280.55**	126.56	1653	804.73
25	17	7	1354.46	375.59	1482	808.02	1050.84	400.30	1942	808.28	**533.01**	181.10	2269	806.53
24	17	7	407.40	124.86	417	810.20	393.42	122.45	691	810.38	**178.00**	62.48	760	809.21
23	17	7	224.81	45.52	156	812.49	424.73	92.51	486	812.50	**107.31**	20.51	179	812.23
25	18	7	1233.67	412.97	2161	807.40	4218.02	777.84	5393	806.94	**701.82**	290.90	3693	806.04
25	19	7	5757.33	1567.05	10273	809.13	5328.55	951.33	12587	808.47	**3306.10**	976.52	13283	807.84
25	20	7	**2506.65**	599.15	3653	807.92	6831.24	1126.89	7459	807.61	2704.65	887.93	11216	806.84
23	19	7	786.65	222.85	760	812.92	**195.55**	42.85	514	810.80	631.29	92.06	438	813.00
22	20	7	1063.91	178.65	276	818.60	6197.23	904.54	6921	813.99	**886.48**	146.03	199	818.07
Average			1066.42	269.68	1419.8	766.18	1502.75	318.45	3465	764.35	**984.04**	219.37	2084.8	765.83

Appendix A. Proof of Theorem 3.1

Proof: Consider a partition $\mathcal{P}$ of the customers V' into m disjoint subsets H^i (with $i = 1, \ldots, m$) such that $\underline{Q} \leq |H^i| \leq \overline{Q}$. Consider a sequence of the customers in each subset H^i, and a route visiting these customers in that sequence. Let $n_i = |H^i|$ for each $i = 1, \ldots, m$, x' be the feasible BVRP solution defined by the m routes, and $r = |E| - n = n(n-3)/2$.

Observe that $\dim(P_{BVRP}) \leq r$ since each vertex has an associated degree equality (2.2)–(2.3), which are linearly independent. We now apply the "direct method" (see [14]) which consists in enumerating $r + 1$ affinely independent solutions. The solution x' is a first one, and we now need r others. To this end we describe here six families of BVRP solutions, denoted $\Phi_1, \ldots, \Phi_6$.

We start creating a first family Φ_1 with BVRP solutions based on x' where the routes used in H^i are modified iteratively to guarantee that the generated solutions are affinely independent. Consider any fixed $i = 1, \ldots, m$, and let us consider the subgraph of G induced by $H_i \cup \{1\}$. The dimension of the TSP polytope associated to this subgraph is known to be the number of edges minus the number of vertices of this subgraph, which is $(n_i + 1)n_i/2 - (n_i + 1) = (n_i - 1)n_i/2 - 1$. Thus, using the routes in x' to visit the customers in $V' \setminus H^i$, there are $(n_i - 1)n_i/2$ affinely independent BVRP solutions, one of them being x'. Figure A.2 shows the subset H^i where the edges shown in dashed lines may vary to produce the solutions. Including x' and enumerating over all i, we have $1 + \sum_{i=1}^{m}((n_i - 1)n_i/2 - 1)$ solutions in Φ_1.

The BVRP solutions in family Φ_2 are obtained by using edges whose endpoints belong to different subsets of the partition $\mathcal{P}$. Consider any fixed pair (i, j) with $i, j = 1, \ldots, m$ and $i < j$. Let $u_i \in H^i$ and $v_j \in H^j$. We now build $(n_i - 1)(n_j - 1)$ BVRP solutions using each edge (u, v) in the set $E(H^i \setminus \{u_i\} : H^j \setminus \{u_j\})$. We exclude u_i and v_j to have the edge (u_i, v_j) in all the solutions in Φ_2. Figure A.3 shows a generic solution in the family. All the customers in $V' \setminus (H^i \cup H^j)$ are visited exactly as in x'. The customers in $H^i \cup H^j$ can be visited with the following two routes. The first route starts from the depot, then visits v_j, then u_i, and then visits all the customers in $H^i \setminus \{u, u_i\}$, closing the route at the depot. The second route starts from the depot, then visits u, then v, and then visits all the customers in $H^j \setminus \{v, v_j\}$ before returning to the depot. The numbers of customers in these routes are $|H^i|$ and $|H^j|$, respectively; thus, they are valid BVRP solutions. In addition, they are affinely independent when considering also x' and the solutions in Φ_1. Finally, enumer-

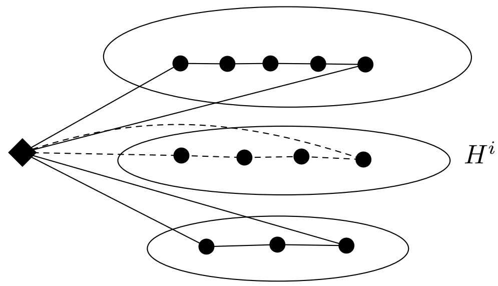


Figure A.2: Solution in Φ_1 obtained by considering the TSP in $H^i \cup \{1\}$.

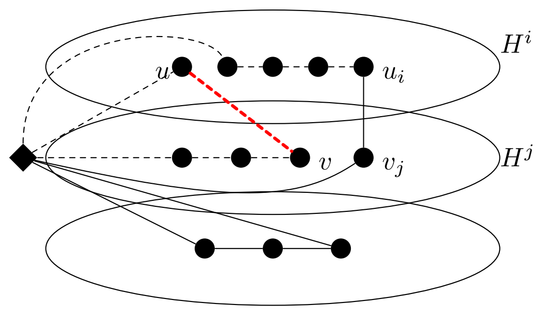


Figure A.3: Solution in Φ_2 using one edge between in $E(H^i \setminus \{u_i\} : H^j \setminus \{v_j\})$.

ating over all pairs (i, j) with $i < j$, we have $\sum_{1 \leq i < j \leq m} (n_i - 1)(n_j - 1)$ BVRP solutions in Φ_2.

To build Φ_3, let us fix again a pair (i, j) with $i, j = 1, \ldots, m$ and $i < j$. Fix any $u_i \in H^i$, $v_j \in H^j$, $u \in H^i \setminus \{u_i\}$ and consider each $v \in H^j \setminus \{v_j\}$. We now built a BVRP solution with four edges between H^i and H^j as in Figure A.4. In that solution, again, all the customers in $V' \setminus (H^i \cup H^j)$ are visited exactly as in x'. The customers in $H^i \cup H^j$ are instead visited with the following two routes. The first route starts from the depot, then visits v_j, then u_i, then v, and finally $n_i - 3$ additional customers in H^i starting from u and ending at the depot. The second route visits the remaining n_j customers. The edges (u_i, v) ensures that these solutions are affinely independent respect to the previous families of solutions. By enumerating over $v \in H^j \setminus \{v_j\}$ we have $n_j - 1$ BVRP solutions, and by also enumerating over the pairs (i, j) we have $\sum_{1 \leq i < j \leq m} (n_j - 1)$ in Φ_3, all affinely independent.

Similarly we construct the family Φ_4 with $(n_i - 1)$ BVRP solutions for each

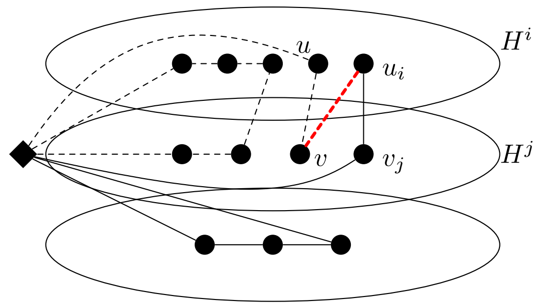


Figure A.4: Solution in Φ_3 using one edge in $E(\{u_i\} : H^j \setminus \{v_j\})$.

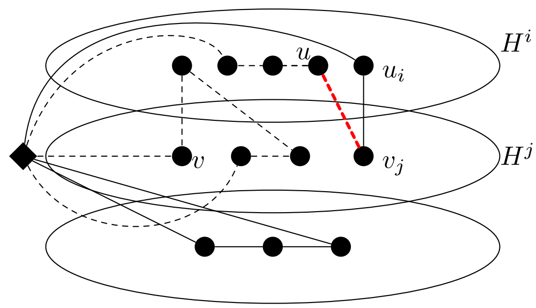


Figure A.5: Solution in Φ_4 using one edge in $E(H^i \setminus \{u_i\} : \{v_j\})$.

pair (i, j) with $i, j = 1, \dots, m$ and $i < j$. Fix any $u_i \in H^i$, $v_j \in H^j$, $v \in H^j \setminus \{v_j\}$ and consider each $u \in H^i \setminus \{u_i\}$. We build one solution containing four edges between H^i and H^i as in Figure A.5. In that solution, again, all the customers in $V' \setminus (H^i \cup H^j)$ are visited exactly as in x'. The customers in $H^i \cup H^j$ are instead visited with the following two routes. The first route starts from the depot, then visits u_i, then v_j, then u, and finally $n_i - 3$ additional customers in H^i, ending at the depot. The second route visits the remaining n_j customers starting from v, then the unvisited vertex in H^i, and then the unvisited vertices in H^j, ending at the depot. The edges (u, v_j) ensures that these solutions are affinely independent respect to the previous families of solutions. By enumerating over $u \in H^i \setminus \{u_i\}$ we have $n_i - 1$ solutions, and by also enumerating over (i, j) we have $\sum_{1 \le i < j \le m} (n_i - 1)$ solutions in Φ_4.

Family Φ_5 contains one solution for each pair (i, j) with $i, j = 1, \dots, m$ and $i < j$. Note that any solution in families Φ_1, Φ_2, Φ_3 and Φ_4 satisfies the following

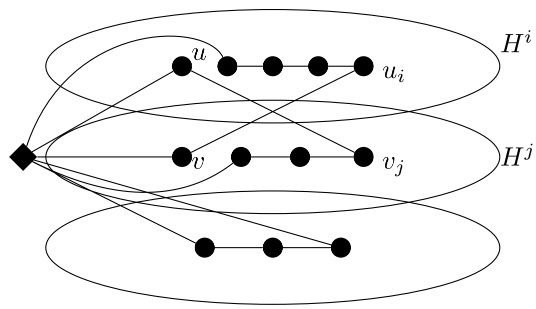


Figure A.6: Solution in Φ_5.

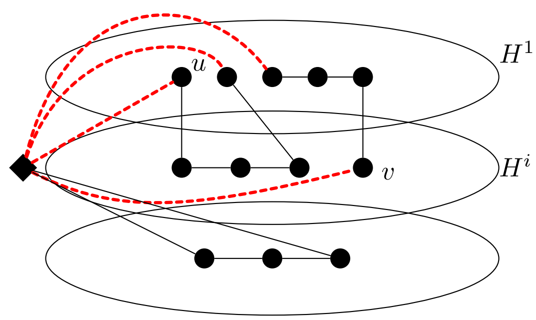


Figure A.7: Solution in Φ_6 using three edges in $E(\{1\} : H^1)$.

equality:

$$x(E(H^i \backslash \{u_i\} : H^j \backslash \{v_j\})) = x(E(H^i \backslash \{u_i\} : \{v_j\})) + x(E(\{u_i\} : H^j \backslash \{v_j\})) + x_{u_i v_j}. \quad \text{(A.1)}$$

To construct a solution violating this equation, we simply swap two customers $u \in H^i \setminus \{u_i\}$ and $v \in H^j \setminus \{v_j\}$ between the routes in x' as shown in Figure A.6.

Finally, we now construct $m - 1$ solutions in family Φ_6 as follows. Observe that the previous solutions satisfy $x(E(H^i : \{1\})) = 2$ for each $i = 1, \ldots, m$. Now, for each $i = 2, \ldots, m$ let us create a BVRP solution by merging the subsets H^1 and H^i, and building two routes visiting these customers with three customers in H^1 directly connected to the depot, thus $x(E(H^i : \{1\})) < 2$ for the generated BVRP solution; see Figure A.7.

Summarizing, we have created the following number of BVRP inequalities:

$$1+\sum_{i=1}^{m}\left(\frac{(n_i-1)n_i}{2}-1\right)+\sum_{1\le i<j\le m}\left((n_i-1)(n_j-1)+(n_i-1)+(n_j-1)+1\right)+m-1=$$

$$=1-m+\frac{1}{2}\left(\sum_{i=1}^{m}n_i^2-\sum_{i=1}^{m}n_i+2\sum_{1\le i<j\le m}n_in_j\right)+m-1=$$

$$=\frac{1}{2}\left(\big(\sum_{i=1}^{m}n_i\big)^2-\sum_{i=1}^{m}n_i\right)=\frac{(n-1)^2-(n-1)}{2}=r+1,$$

and all of them are affinely independent. □

Appendix B. Proof of Theorem 3.8

We use the same notation as in Appendix A. The aim is to build r BVRP solutions satisfying (3.1) with equality. These solutions are constructed with slight modifications of the solutions built in Appendix A.

Let us consider the partition $\mathcal{P}_u$ in Lemma 3.6 such that the customers in S are assigned to subsets H^i containing $\overline{Q}$ customers and, if $|S|>\alpha_u\overline{Q}$ then the other customers in S (which are $|S|-\alpha_u\overline{Q}$) are assigned to the subset with λ_u customers. No customer in S belongs to any subset considered in β_u (by condition c.1). Therefore we have

The first family of solutions are obtained similarly to Φ_1 in Theorem 3.1. We build one route for each subset in $\mathcal{P}_u$ where the route built from subset H^k satisfies $x(\delta(S\cap H^*))=2$. Then, we use the same procedure as in Appendix A for all the subsets H^i with $i\in\{1,\dots,m\}\setminus\{k\}$. In subset H^k we use the fact that $x(\delta(S\cap H^k))\ge 2$ is a subtour-elimination inequality in the TSP, which is facet inducing if there are more than six vertices. Since the depot is considered when building the routes, and using condition c.3, we have $(n_k+1)(n_k-2)/2$ solutions affinely independents satisfying $x(\delta(S\cap H^k))=2$. (one solution less than all the other subsets). Therefore, our first family has $\sum_{i=1}^{m}((n_i+1)(n_i-2))/2$ solutions, which is one solution less than in Appendix A.

We now construct $\sum_{1\le i<j\le m}n_in_j$ affinely independent solutions using edges whose endpoints belongs to different subsets, that is, edges in $E(H^i:H^j)$ with $i<j$. Let us fix arbitrary vertices $u_i\in H^i$, $v_j\in H^j$, $u\in H^i\setminus\{u_i\}$ and $v\in H^j\setminus\{v_j\}$. We construct families of solutions Φ_2, Φ_3, Φ_4 and Φ_5 similar to

the ones in Appendix A, but with slight modifications according to the following cases:

Case (a): $H^i \subset S$ and $H^j \subset S$. The $n_i n_j$ solutions can be obtained using exactly the same families Φ_2, Φ_3, Φ_4 and Φ_5 as defined in Appendix A.

Case (b): H^k and $H^i \subset S$. In that case, in order to get a solution satisfying (3.1) with equality we need exactly two paths to visit all the customers in $S^* = H^i \cup (S \cap H^k)$ such that $x(\delta(S^*)) = 4$ (which means $x(\delta(S)) = 2\lceil |S|/\overline{Q} \rceil$). We assume that $u_i, v_k \in S$. To build the solution obtained in Φ_2 we distinguish two cases illustrated in Figure B.8:

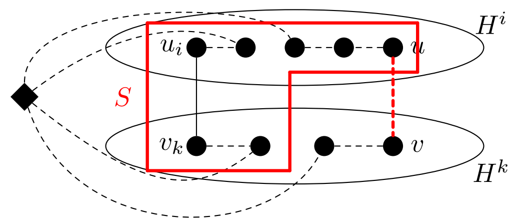
If $v \in H^k \setminus S$ we build the routes as in Figure 8(a). The first path with n_i customers, and which visits the customers in $H^k \setminus S$, then uses the edge (u, v) and then visits $n_i - |H^k \setminus S|$ customers in $H^i \setminus \{u_i\}$. The second path uses edge (u_i, v_j) and then, starting from u_i (resp. v_j), we build a path which visits all the remaining customers in H^i (resp. H^k), connecting the last customer of the path with the depot. Note that this construction satisfies:

- The two routes are feasible.
- The solution obtained by visiting the rest of customers in the different subsets by routes using as in solution x' satisfies $x(\delta(S)) = 2\lceil |S|/\overline{Q} \rceil$.
- The solution is affinely independent to the previous solutions since a new edge is used.
- The solution satisfies equation (A.1).

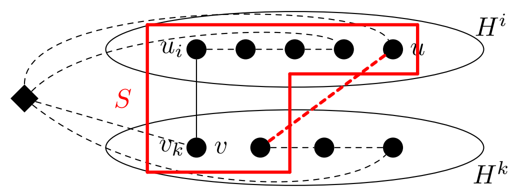
If $v \in H^k \cup S$ we build the routes as is shown in Figure 8(b). The first route visits $\{v_k\} \cup H^i \setminus \{u\}$. The second route visits $\{u\} \cup H^k \setminus \{v_k\}$. It is easy to check that the previous properties hold.

Solutions in Φ_3 are obtained as shown in Figure B.9. The aim of these solutions is that all of these solutions uses edge (u_i, v). Again we differentiate two cases ($v \in H^k \cap S$ and $v \in H^k \setminus S$) and the solutions are drawn in Figures 9(a) and 9(b), respectively. Note that all these solutions can be built satisfying the previous properties. Solutions in Φ_4 and Φ_5 are depicted in Figures B.10 and B.11, respectively.

Case (c): H^k and $H^i \cap S = \emptyset$. The solutions are built similarly, but now the solutions must satisfy $x(\delta(H^k \cap S)) = 2$. In that case we assume that $u_k \in H^k \setminus \{S\}$.

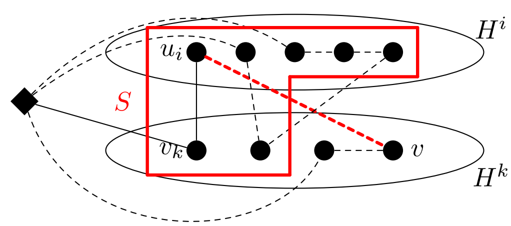


(a) Case $v \in H^k \setminus S$

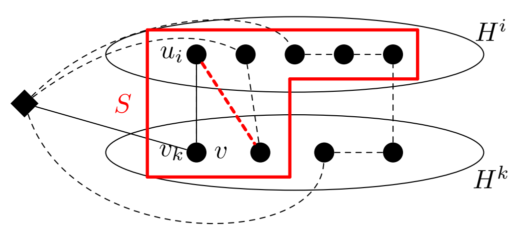


(b) Case $v \in H^k \cap S$

Figure B.8: Solutions in Φ_2.



(a) Case $v \in H^k \setminus S$



(b) Case $v \in H^k \cap S$

Figure B.9: Solutions in Φ_3.

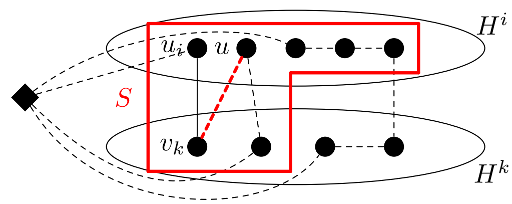


Figure B.10: Solutions in Φ_4.

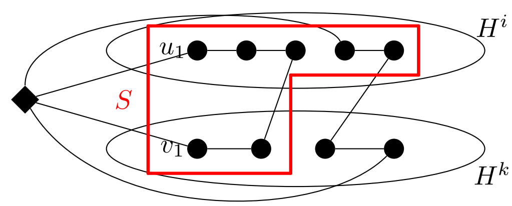


Figure B.11: Solution in Φ_5.

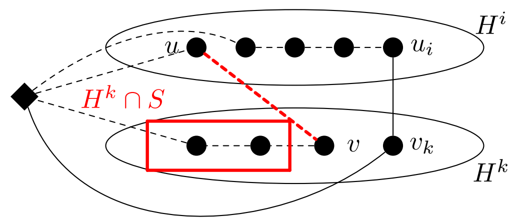
Figure B.12 shows the solution in Φ_2. Again, we distinguish two cases: $v \in H^k \cap S$ and $v \in H^k \setminus S$. The solutions are depicted in Figures 12(a) and 12(b), respectively. In both cases we use two fixed edges $(1, v_k)$ and (v_k, u_i). The difference is that in Figure 12(b) there is an edge connecting customer u to a customer in S.

Solutions corresponding to family Φ_3 are depicted in Figure B.13. The two cases, $v \in H^k \cap S$ and $v \in H^k \setminus S$, are shown in Figures 13(a) and 13(b) respectively. Solutions in Φ_4 and Φ_5 are shown in Figures B.14 and Figure B.15, respectively.

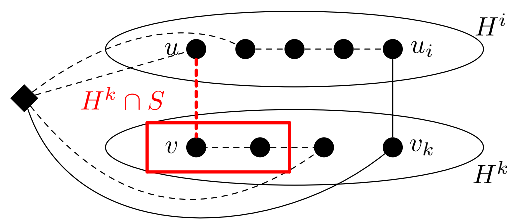
Case (d): $H^i \cap S = \emptyset$ and $H^j \cap S = \emptyset$. The $n_i n_j$ solutions can be obtained using exactly the same families Φ_2, Φ_3, Φ_4 and Φ_5 in Appendix A.

Case (e): $H^i \subset S = \emptyset$ and $H^j \cap S = \emptyset$. This case implies to change the partition for each edge (u, v) with $u \in H^i$ and $v \in H^j$. We swap customer u with one customer on $H^k \cap S$ and then we build a solution using edge (u, v) as depicted in Figure B.11. Then, we can build one solution affinely independent from the previous solutions for each edge (u, v).

Finally, it is easy to adapt the $m - 1$ solutions in Φ_6 in Appendix A. □

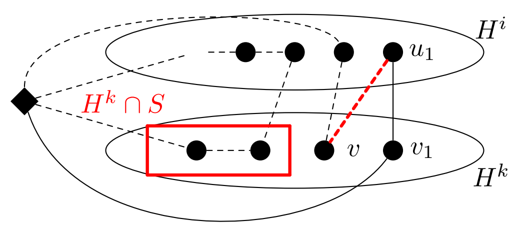


(a) Case $v \in H^k \setminus S$

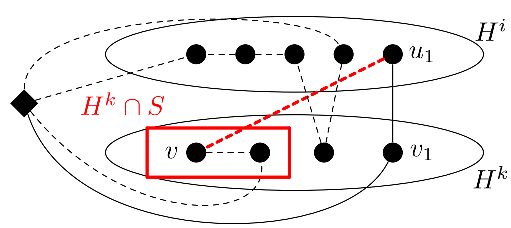


(b) Case $v \in H^k \cap S$

Figure B.12: Solutions in Φ_2.



(a) Case $v \in H^k \setminus S$



(b) Case $v \in H^k \cap S$

Figure B.13: Solutions in Φ_3.

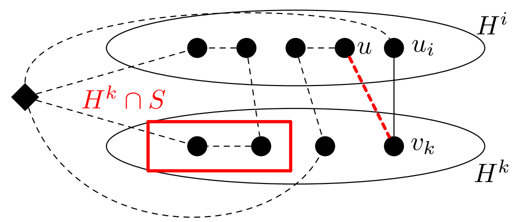


Figure B.14: Solution in Φ_4.

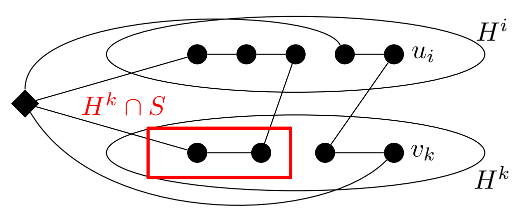


Figure B.15: Solution in Φ_5.