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Control and Inference of Structured Markov Models

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Abstract

This Ph.D. thesis studies structured Markov models (SMMs) in the context of applied probability, stochastic modelling and applied statistics. SMMs are Markov chains on countable state spaces, whose transition rate matrices have particular structures, such as bidiagonal or tridiagonal block matrices. From a modelling point of view, applicability, robustness, simplicity, and tractability of SMMs make them a very important tool for studying complex systems. This is mainly due to the fact that for evaluation of performance measures of SMMs, there are efficient numerical analysis methods, commonly called matrix analytic methods (MAM). Some popular classes of SMMs are phase-type (PH) distributions, Markovian arrival processes (MAPs) and quasi-birth-and-death (QBD) processes. This thesis describes the outcomes of three research projects dealing with SMMs and their applications in stochastic modelling and applied statistics.

In the first project, we introduce the notion of burstiness for a MAP. We call a stationary MAP bursty if both the squared coefficient of variation of inter-arrival times and the asymptotic index of dispersion of counts (IDC) are greater than unity. The simplest bursty MAP is a Hyperexponential renewal process with further classes, as we establish, being the Markov modulated Poisson process (MMPP), the Markov transition counting process (MTCP) and the Markov switched Poisson process (MSPP). Of these, MMPP has been used most often in applications for modelling bursty phenomena. Much of the popularity of MMPP stems from the intuition that it serves as a good model of bursty traffic. However, when MMPPs are viewed through the lens of the inter-arrival process, there is no proof to show that MMPPs are bursty. We provide analytical proofs to show that all of the MAPs mentioned above are bursty.

Further, we investigate relations between these bursty MAPs. One of our main results is establishing a duality in terms of first and second moments of counts between MTCPs and a rich class of MMPPs which we refer to as slow-MMPPs (modulation is slower than the arrivals). Such a duality further confirms the applicability of MTCP as an alternative to MMPP. We augment our analytic results with numerical illustrations.

In the second project, we consider a simple discrete-time controlled queueing system, where the controller has a choice of which server to use at each time slot, and server performance varies according to a Markov modulated random environment. We explore the role of information on the system stability region. At the extreme cases of information availability, that is when there is either full information or no information, stability regions and maximally stabi-

lizing policies are trivial. But in the more realistic cases where only the environment state of the selected server is observed, only the service successes are observed or only the queue length is observed, finding throughput maximizing control laws is a challenge. To handle these situations, we devise a partially observable Markov decision process (POMDP) formulation of the problem and illustrate properties of its solution. We further model the system under given decision rules, using a QBD structure to find a matrix analytic expression for the stability bound. We use this formulation to illustrate how the stability region grows as the number of controller belief states increases.

The third project focuses on the statistical methodology of semi-Markov processes, as motivated by the study of the trajectory of patients in intensive care units (ICUs) in hospitals. The main result of this project is the comparison of two approaches for defining and estimating semi-Markov models that are applied in ICUs. One approach is based on sojourn times, and the other approach is based on transition rates of the Markov jump process. We show that the second model has fewer parameters and its likelihood can be considered as the product of likelihoods of simpler two-state models. The comparison of these approaches helps to build models for predicting risks and chances of expected trajectories of patients through ICUs. Moreover, we extend the model to the case of having a multi-absorption PH distribution as the sojourn time distribution or intensity distribution of the semi-Markov model.

The result of the projects mentioned above have been published or submitted to prestigious journals and peer-reviewed conferences, see [8–11, 146]. Further papers are currently in final stages of preparation.

Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

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Contributor	Statement of contribution
Azam Asanjarani (Candidate)	Problem identification and Concept design 70% Methodology design (70%) Experimental evaluation (100%) Paper writing and editing (70%)
Yoni Nazarathy	Problem identification and Concept design (30%) Methodology design (30%) Paper writing and editing (30%)

Contributions by others to the thesis

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Keywords

Structured Markov models, Control, Inference, Matrix analytic methods, Markovian arrival processes, Quasi-birth-and-death (QBD) processes, Semi-Markov models, Partially observable Markov decision processes (POMDPs), Stability, Queueing systems.

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Table of Contents

List of Figures

List of Tables

List of Abbreviations

BD	birth-and-death, page 24
CDF	cumulative distribution function, page 15
CTMC	continuous time Markov chain , page 13
EM	expectation-maximization, page 43
FCFS	first come, first serve, page 21
H_p	hyperexponential distribution of order p, page 27
IDC	index of dispersion for counts, page 36
IDI	index of dispersion for intervals, page 35
iid	independent and identically distributed, page 17
IPP	interrupted Poisson process, page 73
KPC	Kronecker product composition, page 85
MAM	matrix analytic methods, page 25
MAP	Markovian arrival process, page 29
MDP	Markov decision process, page 50
MLE	maximum likelihood estimation, page 43
MMPP	Markov modulated Poisson process, page 31
MSPP	Markov switched Poisson process, page 74
PDF	probability density function, page 15

PMF probability mass function, page 18

POMDP partially observable Markov decision process, page 53

RORMAB reward observing restless multi armed bandits, page 96

SCV squared coefficient of variation, page 15

SMM structured Markov model, page 25

SMP semi-Markov process, page 20

Chapter 1

Introduction

This Ph.D. thesis lies in the intersection of applied probability, stochastic modelling and applied statistics and consists of three main projects associated with structured Markov models (SMMs). The models considered are continuous or discrete time Markov chains, typically on countable state spaces, whose transition rate matrices are configured in special structures, such as bidiagonal or tridiagonal block matrices. Markov models of this type have the virtue of being amenable to performance analysis using algorithmic methods that utilize the structured nature of their associated matrices.

SMMs have been heavily analysed in the field of applied probability, often under the title matrix analytic methods (MAM). Since SMMs preserve the Markovian structure, by using MAM, the results are algorithmically tractable. This makes them a valuable tool for studying complex dynamics such as congestion phenomena in queueing systems. Other types of applications include storage, reliability, epidemics, and population processes. There are three popular classes of SMMs which we consider in this thesis: Phase-type (PH) distributions, Markovian arrival processes (MAPs), and quasi-birth-and-death (QBD) process. A PH distribution is a generalisation of the exponential distribution, and a MAP is a generalisation of the Poisson process. The QBD process is a generalisation of a birth-and-death process and a MAP.

In this thesis, we investigate the role of applying SMMs for stochastic modelling, in the framework of three research projects. Here, an overview of these projects is outlined, providing the reader with motivation for the need of continued research in this area. More background

details are provided in Chapter 2.

1.1 Bursty MAPs

Point processes on the line generated by transitions of CTMCs have been studied intensely by the applied probability community over the past few decades under the umbrella of matrix analytic methods (MAM), see for instance [119]. These have been implemented to Teletraffic [3], business networks [88], social operations research [199], and biological systems [159]. The typical model referred to as the Markovian arrival process (MAP) is comprised of a finite state irreducible CTMC which generates arrivals at selected instances of state change according to Poisson processes modulated by the CTMC. So, a MAP can be considered as a generalisation of the Poisson process where the inter-arrival times of a MAP are not necessarily independent of each other, nor exponentially distributed. Introducing correlation between times of events for point processes often makes them more complex models and usually computationally intractable. However, MAPs are Markovian and hence analytically more tractable than general point processes. MAPs have been shown to be dense in the class of point processes on the line so that they can essentially approximate any point process, see [18].

A standard way to characterise a MAP is through the matrices C and D whose sum is the Q-matrix of the background CTMC, see Section 2.3.2. The matrix D is non-negative and contains intensities of transitions that count as arrivals or events. The matrix C has strictly negative diagonal elements and non-negative off-diagonal elements and represents transitions of intensities that do not count as events. Some notable descriptions of MAPs are in Chapter XI of [15], Chapter 2 of [83], Chapter 3 of [119], and [130, 149].

Let $N(A)$ denote the number of events of the MAP occurring on the set $A \subset \mathbb{R}$. As with general point processes, a MAP is time-stationary if the distribution of $N(A)$ and $N(A+t)$ is the same, where $A + t := \{u : u - t \in A\}$. Further use $\{T_n\}$ to denote the sequence of inter-event times. As with general simple point processes on the line, a MAP is event-stationary if the joint distribution of $T_{k_1}, \ldots, T_{k_m}$ is the same as that of $T_{k_1+\ell}, \ldots, T_{k_m+\ell}$ for any integer sequence of indices $k_1 \ldots, k_m$ and any integer shift ℓ. For MAPs, time-stationarity and

event-stationarity are easily characterized by the initial distribution of the background CTMC. Starting the CTMC at its stationary distribution yields time-stationarity and starting at the stationary distribution of the embedded Markov chain (jump chain) yields event-stationarity.

In describing and measuring simple stationary point processes on the line, first and second order quantities are often very useful. The first order measure is the rate of the point process, λ^*, which specifies the mean number of events occurring during a unit time. For the time stationary version, $\mathbb{E}[N(t)] = \lambda^* t$ and for the event-stationary version, $\mathbb{E}[T_n]^{-1} = \lambda^*$. Then, typical second order measures of interest are the limiting index of dispersion of counts and the squared coefficient of variation, given respectively by:

$$d^2 = \lim_{t\to\infty} \frac{\mathrm{Var}\big(N(t)\big)}{\mathbb{E}\big[N(t)\big]}, \quad \text{and} \quad c^2 = \frac{\mathrm{Var}(T_1)}{\mathbb{E}^2\,[T_1]},$$

where T_1 is taken from the event-stationary version.

In point process modelling, the Poisson process, often used as a benchmark, exhibits both $d^2 = 1$ and $c^2 = 1$. Values greater than unity indicate high variability (burstiness) and values less than unity are nearing more deterministic arrival patterns. Although there is not a single precise definition for describing burstiness, see [98] and [121], yet in applications $c^2 \geq 1$ is often considered as a sign of burstiness by practitioners. Here, we call a MAP bursty if it satisfies $c^2 \geq 1$ and $d^2 \geq 1$.

Our main contribution is in identifying the following classes of MAPs as being bursty and further in finding relationships between them. Here, we take the view of presenting a MAP based on its parameters, even though it is known that MAPs are not identifiable, see [170].

1. The (well-known) Markov modulated Poisson process (MMPP) – A MAP with a diagonal matrix D, so when there is an event, there is no transition between states of the background CTMC.
2. The Markov transition counting process (MTCP) – This is a MAP with diagonal C where the diagonal elements of D are all 0. We call such a MAP, MTCP since this process exactly counts all ordinary transitions of a CTMC. Note that MTCPs have been analysed in classic works such as [175] as well as [147] under different names.
3. Hyperexponential (H) renewal process – A MAP with a diagonal matrix C and rank-one

matrix D. In fact, $D = -C\mathbf{1}\eta$, where $\mathbf{1}$ is a column vector of 1's and η is the initial distribution of the hyperexponential distribution.

4. The Markov switched Poisson process (MSPP) – This is a MAP with a diagonal C, so all transitions are counted (both ordinary and self-transitions). Note that, both MTCP and hyperexponential renewal process are sub-classes of the MSPP.

Between the above-mentioned MAPs, MSPPs (the class of MAPs with diagonal matrix C) have received much less attention in the literature. On the other hand, the MMPP (the class of MAPs with diagonal matrix D), as a doubly stochastic Poisson process, is a highly popular model from both a theoretical and applicative point of view. For a detailed outline of a variety of classic MMPP results, see [70] and references therein. Regarding applications, MMPPs are useful for modelling phenomena where bursty point processes are present such as in telecommunications, health-care, earth-quakes modelling and finance. To date, MMPPs have been used in thousands of research papers with hundreds of new papers appearing yearly. In addition, theoretical properties of MMPP generalizations are of recent research interest, as in [165].

However, are all the elementary properties of the basic MMPP known? In Chapter 3, we establish an elementary result for MMPPs that has often been taken for granted in modelling folklore. Modellers often use MMPPs to represent bursty temporal patterns as in for example [49], [90] and [127], among hundreds of other important research works. In fact, showing that $d^2 \geq 1$ is straightforward (for example Chapter 6 of [108]). However, to the best of our knowledge, a proof that for MMPPs, $c^2 \geq 1$ has been lacking to date and is not simple. We provide such a proof, together with stochastic order relations between T_1^a and T_1^b, corresponding to the first inter-event time in the time-stationary and event-stationary cases, respectively. Namely, we show that $T_1^a \geq_{\mathsf{st}} T_1^b$, that is $\mathbb{P}(T_1^a > t) \geq \mathbb{P}(T_1^b > t)$ for all t.

Moreover, we demonstrate that although from a path-wise perspective, MMPPs and MTCPs are in a sense the exact opposites (in MTCPs only ordinary events are counted while in MMPPs only self-transitions are counted), they can be used as alternative models for modelling bursty phenomena. This is the result of applying MAM to show that their related counting processes have the same first and second moments. When it comes to parameter estimation, in the literature, there are a number of methods for parameter estimation

of MMPPs. However, most of these methods are applied to a special case of a two-state MMPP, see for instance [177, 183]. As we will show in Chapter 3, for general MAPs, typical quantities that one needs to compute are C^{-k} and e^{Cx} for some integer $k > 0$ and real $x > 0$. Here, by using MAPs with diagonal matrix C, we get better computational performance. Specifically, considering MTCP as an alternative model for MMPP, may reduce computational time and makes certain algebraic quantities easier to compute. In addition, we consider these alternative models as the arrival stream of a MAP/PH/1 queueing system. We show that the MTCP$_4$/PH/1 queue has almost the same basic steady state characteristics (mean and variance of the queue length) as the MMPP$_2$/PH/1 queue for different choices of PH distributions. This provides more insights about these kinds of queuing systems and gives modellers more freedom in model choice.

1.2 Control of Queuing Systems

Performance evaluation and control of queueing systems with servers operating in random environments have been widely studied during the past few decades, see [54, 180] and references therein. This is mainly with respect to applications in various areas such as telecommunications, supply chain logistics, healthcare, manufacturing, and transportation. In all of these applications, a controller needs to decide how to apply resources optimally when system conditions, such as service rates, change randomly. A natural choice for modelling these situations is considering Markovian random environments. This is motivated by applicability and tractability of Markov models, see Section A.1 in [144] for a general discussion on the ubiquity of Markov models.

In Chapter 4, we consider a two-server single-queue system where the servers' environment states vary in a Markovian fashion and are not explicitly observed. The controller sequentially allocates servers (for example communication channels, transmitters, manufacturing machines) to units requiring processing (for example file transfers, widgets). The goal is to design an optimal policy to have a stable system; that is an efficient queueing system that reduces the number of units requiring processing and keeps that finite under an online controller.

These kinds of systems have been partially addressed in the literature. For instance, see [78, 106, 188] where the stability region of a fully observed system was characterised. Even in this situation, performing explicit system analysis and finding an optimal policy is not a trivial task, see for instance [23, 107, 188], or the more recent [78]. However, in practice, the environment states are often not directly observed. The stability condition for a partially observable system is considered in [123] for a single-queue multi-server system and more recently, in [142, 146]. In [146], the authors considered the simplest possible model with a single-queue, a (fixed-state) safe server, and a two-state bandit server. In [142], authors analysed the structure of optimal policies at each time slot, using the Whittle index. Also see [113], where much previous literature is surveyed.

In reality, controller decisions affect both the immediate reward (service success) and the observation made, and this makes the system more complex. Here, we consider such a situation. The main contribution in this area is [110], where from first principals, the structure of optimal policies is found, and the more recent [126], generalizes the setting and utilizes the celebrated Whittle index, [198] for this kind of partially observable system. Related recent results dealing with more general systems (multi-armed bandits) are in [116, 117]. Of further interest is the latest rigorous account on the asymptotic optimality of the Whittle index, [194], as well as in the context of partially observable two state Markov chains [163].

Our focus in this area of research is on the role of information in designing the optimal controller policy. We consider a discrete time controlled queueing system, where server environment varies according to a two-state Markov chain. The controller has a partial observation of the system and chooses servers only based on this partial information. Moreover, we assume that a job in the queue is served by either Server 1 or Server 2 which are independent of each other.

For exploring the role of information in controlling such a system, we consider different observation schemes. At one extreme, the controller has full information of the servers' environment states (full observation). At the other extreme, the controller has no information of the servers' environment states (no observation). Obviously, the stability region of the system in the latter situation is a subset of the former. Our contribution is in considering additional more realistic observation schemes:

1. State observation: the controller is only aware of the state of the currently chosen server. This type of situation has been widely studied in some of the references mentioned above and surveyed in [113]. However, most of the literature dealing with this situation does not consider a queue.

2. Output observation: the controller is only aware of the success or failure of service at every time slot. Such a situation was recently introduced in [146] in the context of stability and analysed in [142] with respect to the Whittle index. In [123], the stability of a related multi-server system was analysed.

3. Queue observation: the controller is only aware of the queue length. Such an observation scheme in a (non-degenerate) continuous time system, is identical to the former scheme. However, in the case of our discrete time model, since both an arrival and a departure may coincide, going unnoticed by the controller, queue observation reveals less information.

For analysing the effect of information, based on the above three partial observation schemes, we formulate the system as a partially observable Markov decision processes (POMDP). For POMDPs, since the controller is unaware of the actual current state, the policy depends on the history of observations. So, we need to determine the controller's actions in terms of the probability distribution of states, called belief states. For each of the observation schemes, we develop recursive equations for belief state updates. These are then embedded in Bellman equations describing optimal solutions of associated POMDP. Numerical solution of the POMDPs then yields insight on structural properties and achievable stability regions for different observation schemes. By construction, two-state Markov server environments are more predictable when the mixing times of the Markov chains increase. We quantify this use of channel-memory, through numerical and analytic results.

Furthermore, we formulate the above controlled queuing system as a QBD process, where the only observation is the previous output of the system. We present a detailed QBD model of the system. Applying QBDs, as a special class of infinite state Markov chains, guarantees a large degree of expressiveness in modelling and at the same time benefits from the efficient MAM. Here, using MAM, we find an upper bound for stability region and quantify the effect of a finite state controller on the achievable stability region. Our numerical approximations to the POMDP Bellman equations and the numerical solutions of the QBDs

hint at a variety of structural results. Furthermore, using the QBD formulation illustrates how the stability region grows as the number of controller belief states increases. For more details on applying QBD for Markov decision processes (or POMDPs) associated with queueing models, see for example [143].

1.3 Semi-Markov Processes

Major advances in techniques of analysis of semi-Markov processes (SMPs) have occurred in the context of applied probability models with a large range of applications in queueing theory, reliability, control theory, and health care, see for instance [99].

Multi-state Markov models (finite-state Markov models) are the most appropriate models for developing maintenance strategies, predicting the reliability of systems, and modelling the survival time in manufacturing and healthcare industries. This is mainly due to the state-dependent behaviour of Markov chains and the well-developed algorithms for analysing them. However, in reality, some aspects of systems' behaviour can not be captured well by Markov chains with a finite or countable number of states. For instance, the risk of chronic disease such as AIDS essentially depends on the time since infection, see [105], or mechanical or electronic components failure time are usually do not follow an exponential distribution and often have a heavy-tailed distribution such as Weibull or lognormal distribution, see for instance Chapter 2 of [25]. Therefore, semi-Markov processes (SMPs), as an extension of ordinary Markov processes, where the transition probability between two states depends only on the sojourn times, and the clock is reset to zero after each transition into a new state, seems a suitable choice. In fact, in semi-Markov processes states of the process have Markov property but sojourn times are not necessarily memoryless.

For biomedical applications, especially those concerned with characterizing an individual's progression through various stages of a disease, the three-state semi-Markov models are of more interest, see for instance [36, 105] . Here, we consider a special three-state semi-Markov model known as illness-death model which is applied for modelling the trajectory of patients in intensive care units (ICUs) of hospitals. Our main focus is on the statistical methodology of semi-Markov processes for this model. We compare and contrast two

approaches to defining the likelihood function and estimating the parameters of the illness-death model. The comparison of these approaches provides a contribution of building a prediction model that predicts risks and chances of expected trajectories of patients through ICUs.

Another problem with modelling and analysis of survival, is that observations are often very coarse, see for instance [1, 95]. Therefore, it may often be natural to consider a PH distribution to explain that for instance, behind the stages of a disease there is a background (Markov) process going through a set of stages which are only partially observed, see [135]. Moreover, PH distributions are a versatile class of distributions that are dense in the class of all distributions defined on the non-negative real line. So, applying them for approximating an unknown survival (or failure) time distribution seems quite suitable. In addition to the above reasons, applying MAM for analysing PH distributions, makes them an interesting tool in medical statistics, see for instance [24, 64, 129]. We find the related formulas to extend the illness-death model to the case of having a multi-absorption phase-type (MAPH) distribution as the sojourn time distribution or intensity distribution of the semi-Markov model.

1.4 Thesis Structure

The structure of the remainder of this thesis is as follows:

- Chapter 2 provides a relevant background on mathematical objects and computational methods in stochastic modelling. Moreover, in that chapter, we present the literature review of the most closely related works.
- Chapter 3 overviews and summarizes MAP results and establishes the class of busty MAPs including the MMPP, the MSPP, the MTCP, and the H-renewal process. Further, the relations between these bursty MAPs is investigated and their moment results are presented. Here, we show that for many instances of MMPPs, one can find an associated MTCP with the same first and the second moments of the counting process. Numerical results for approximating a given MMPP_2/PH/1 with an MTCP_4/PH/1 are presented. Further, we compare the above approximation with the result of a nonlinear

optimisation procedure for matching inter-arrival process of an $MTCP_4/M/1$ queue with the inter-arrival process of a given $MMPP_2/M/1$ queue.

- Chapter 4 presents the problem of finding an optimal controller for a discrete-time controlled queueing system. We introduce the system model and different observation schemes, put forward corresponding recursions for belief state updates, and present Bellman equations for different observation schemes. Furthermore, we construct a QBD representation of the system to find the upper bound for the stability criterion. The numerical results for both methods are presented and indicate that the throughputs Bellman equations and those obtained by a QBD solution are matched for different set of parameters.

- Chapter 5 focuses on statistical methodology of semi-Markov processes (SMPs), as motivated by study the trajectory of patients in intensive care units (ICUs) in hospitals. After a brief review of SMPs, we present two different approaches for a SMP depends on known parameters. The first approach is based on sojourn times and second approach is based on transition rates. Relations between some quantities of interest and parameters of these two approaches are obtained. Inference for finding parameters, based on MLE method and by applying the related R packages, are presented. Moreover, the multi-absorption PH distribution and its related formulas are presented.

- Chapter 6 concludes the discussions made throughout the thesis with an outlook towards future work.

Chapter 2

Mathematical Objects and Methods in Stochastic Modelling

2.1 Introduction

The purpose of this chapter is to introduce the mathematical objects and computational methods in stochastic modelling which play a key role in this thesis.

In Section 2.2, we introduce stochastic processes which are used in this thesis. These processes include the Markov processes, semi-Markov processes, general point processes, and the queueing systems. The renewal process and the Poisson process as two important examples of point processes and the M/M/1 queue as the basic example of queueing systems are mentioned. Then, a birth-and-death process as a natural generalisation of the M/M/1 queue is introduced. Further, we add a discussion about the stationarity and the measures of variability for point processes.

Structured Markov models (SMMs) are introduced in Section 2.3. SMMs are continuous or discrete time Markov chains on countable state spaces, where the transition rate matrix features special structures such as being block diagonal or sparse. SMMs are suitable for modelling queueing, storage, reliability, epidemics, and population processes. Three classes of SMMs are defined in this chapter: phase-type (PH) distributions, Markovian arrival processes (MAPs) and quasi-birth-and-death (QBD) processes.

From a historical point of view, Phase-type (PH) distributions and a particular kind of MAPs, were introduced in the 1970s [7]. A PH distribution is a generalisation of the exponential distribution, and a MAP is a generalisation of the Poisson process.

Applying PH distributions and MAPs in modelling stochastic processes led to the development of efficient numerical analysis methods commonly called matrix-analytic methods (MAM)[1]. The idea of MAM is that for models with a structured transition rate matrix (for example block diagonal), one can replace scalar quantities by matrices. We show this idea in the following of this chapter by presenting an example of applying MAM for finding the stationary distribution of a QBD.

A brief literature review on inference methods for SMMs, with an emphasis on parameter estimation methods, is presented in Section 2.4. Methods for parameter estimation of PH distributions have been a research topic for more than 30 years. However, parameter estimation for MAPs is more challenging and most available methods have been developed during the past decade. The related section starts with common methods of parameter estimation (fitting) of SMMs based on data. Then, the relevant challenges to applying these methods for SMMs are considered.

The last section of this chapter, Section 2.5, is about Markov decision processes (MDPs). A Markov decision process is an automated system for choosing the best service at each time spot and consists of environment states, actions, rewards, and transition probabilities. At each time spot, an action is chosen and depend on the action the state of the environment changes and a reward is generated. The current state and the granted reward determine the next action through a conditional transition probability function. Solving an MDP is equivalent to find the optimal policy for choosing actions. MDPs are applied in many operations research problems in ecology, manufacturing, economics, and communications engineering. Here, we briefly introduce MDPs and the way of finding their optimal policy. For the special case of partially observable Markov decision process (POMDP), the average reward optimality is considered.

[1] Sometimes, MAM is called matrix geometric methods, see [15].

2.2 General Stochastic Processes

In this section, we present a short introduction to stochastic processes with a focus on the processes using in this thesis. For more details see [15, 61, 62, 156] and [83].

2.2.1 Markov Processes

Stochastic processes are the collection of random variables applying for modelling of random manner phenomena or systems. A stochastic process $X(t)$ is a collection of random variables that take values in a state space $\mathcal{S}$ and is indexed by a variable $t \in \mathcal{T}$. The set $\mathcal{S}$ is called the state space of the process, and the set $\mathcal{T}$ is called the index set. Both sets $\mathcal{S}$ and $\mathcal{T}$ can be discrete or continuous. Whenever t is discrete, $X(t)$ is referred to as a discrete-time process, and whenever t is continuous, $X(t)$ is referred to as a continuous-time process. In this thesis, we concern a discrete and countable state space $\mathcal{S}$ and the index set $\mathcal{T}$ represents the time where $\mathcal{T} \subseteq [0, \infty)$ or $\mathcal{T} \subseteq \mathbb{N}$ for the continuous or discrete case, respectively.

A stochastic process is said to be *memoryless* or have the *Markov property* if the conditional probability distribution of future states of the process depends only upon the current state. More precisely, a stochastic process $X(t)$ has Markov property if

$$\mathbb{P}\Big(X(t_{n+1}) = x_{n+1} \;\Big|\; X(t_0) = x_0,\, X(t_1) = x_1,\, \cdots,\, X(t_n) = x_n\Big) = \mathbb{P}\Big(X(t_{n+1}) = x_{n+1} \;\Big|\; X(t_n) = x_n\Big),$$

where $t_i \in \mathcal{T}$ and $x_i \in \mathcal{S}$, for $i = 0, 1, \cdots, n$ and $n \geq 0$. Here, for any $n \geq 0$, we have $t_n < t_{n+1}$. Such a process is said to be a Markov process.

Definition 2.2.1. *A continuous-time stochastic process $X(t)$ that satisfies the Markov property is called a **continuous-time Markov chain** (CTMC) .*

Now consider the case that for all $i, j \geq 0$, the transition rate from J_i to J_j is independent of i, j and only depends on their difference. This property is called the time-homogeneity and gives Markov processes a high degree of analytical tractability.

Definition 2.2.2. *A Markov process $X(t)$ with state space $\mathcal{S}$ is said to be **time-homogeneous***

if for all time slots $s, t > 0$ *and* $i, j \in \mathcal{S}$*,*

$$p_{ij}(t) := \mathbb{P}\Big(X(t+s) = j \mid X(s) = i\Big) = \mathbb{P}\Big(X(t) = j \mid X(0) = i\Big).$$

Here, $p_{ij}(t)$ is called the transition probability from state i to state j at time t. The corresponding matrix is called the *transition probability matrix* and denoted by P, that is $P = \big(p_{ij}(t)\big)$.

The matrix representing transition rates of a Markov chain is called the *transition rate matrix* and denoted by $Q = (q_{ij})$, where for any $i, j \in \mathcal{S}$, the element q_{ij} is the transition rate from state i to state j. For the sake of simplicity, we may refer to the transition rate matrix of a process as the Q-matrix of the process.

For a CTMC, the Q-matrix elements have the following properties:

- $-\infty < q_{ii} \leq 0$ for $i \in \mathcal{S}$,
- $0 \leq q_{ij} < +\infty$ for $i, j \in \mathcal{S}$ and $i \neq j$,
- $\sum_{j \in S} q_{ij} \leq 0$ for $i \in \mathcal{S}$.

Here, we assume that all row sums are equal to 0 and $q_i < +\infty$, where $q_i = \sum_{j \neq i} q_{ij}$.

The relationship between the transition probability matrix and the transition rate matrix of a CTMC is given by *Kolmogorov forward and backward equations*:

$$\frac{P(t)}{dt} = P(t)Q, \quad P(0) = I, \qquad \text{(forward equation)}$$

and

$$\frac{P(t)}{dt} = QP(t), \quad P(0) = I, \qquad \text{(backward equation)}$$

where I is the identity matrix. If the state space $\mathcal{S}$ is a finite set, then both forward and backward Kolmogorov equations have a unique solution, see [6]:

$$P(t) = \exp\big(Qt\big) = \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n.$$

For a time-homogeneous CTMC, the realisation of the process is as follows. The process

starts from a state i and remains there for a while and then jumps to state j. The process goes from state i to state j with the probability $p_{ij} = \frac{q_{ij}}{q_i}$, $\big(q_i = \sum_{j \neq i} q_{ij}\big)$. The time that the process spends in state i before jump to another state is called the *sojourn time* of state i. For a time-homogeneous CTMC, sojourn times are *exponential random variables*. In general, a non-negative random variable X is said to be an *exponential random variable* with parameter λ if its (cumulative) distribution function (CDF) is given by $F(t) = \mathbb{P}\big(X \leq t\big) = 1 - e^{-\lambda t}$ for $t \geq 0$, where λ is a positive real number. For an exponential random variable X with parameter λ, the probability density function is given by $f(t) = \frac{dF(t)}{dt} = \lambda e^{-\lambda t}$. Further, its mean, variance and squared coefficient of variation (SCV) are given as follows. $\mathbb{E}[X] = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = \frac{1}{\lambda^2}$, $SCV(X) = \frac{\mathrm{Var}(X)}{\big(\mathbb{E}[X]\big)^2} = 1$. Using the conditional probability, we have the memoryless property for exponential random variables:

$$\mathbb{P}\Big(X > t + s \;\Big|\; X > s\Big) = \mathbb{P}\big(X > t\big), \quad t, s \geq 0\,.$$

It can be shown that the exponential random variables are the only non-negative, non-zero, and finite continuous random variables that possess the memoryless property, see Chapter 1 of [83].

Classification of States

Depending on the states' connections, a CTMC exhibits different long term behaviours. Therefore, it is beneficial to consider the classification of a CTMC states in terms of the elements of its Q-matrix. A state j is said to be *accessible* from state i if $q_{ij} > 0$. If state i is accessible from state j and state j is accessible from state i, then, state i and state j *communicate* each other. A *communicating class* is the set of all states where pairs communicate each other. A communicating class $\mathcal{A} \subseteq \mathcal{S}$ is said to be closed, if $i \in \mathcal{A}$ and i communicates j implies that $j \in \mathcal{A}$. Therefore, there is no escape from a closed set and a closed set is called an *absorbing set*. If there is only one state in an absorbing set, that state is called an *absorbing state*. The states of a communicating class are either transient or recurrent. For a given state i, assume that the first return time to state i is denoted by the random variable $T_i = \inf\{t > 0 : X(t) = i \;\big|\; X(0) = i\}$. If $q_i > 0$ or $\mathbb{P}\big(T_i < \infty\big) = 1$, then the state i is said to be a *transient state* and if $q_i = 0$ or $\mathbb{P}\big(T_i < \infty\big) < 1$, then state i is said to be a *recurrent state*. A recurrent state i is said to be a *positive recurrent* state if $\mathbb{E}\big(T_i\big) < \infty$. Otherwise the state i is called a *null recurrent* state. If all the states of a CTMC belong to a single class, then the CTMC is named after that class.

A CTMC is said to be *irreducible* if its state space consists of a single communicating class. The long term behaviour of an irreducible CTMC is determined by the *stationary distribution* of the CTMC which is denoted by π and is the unique solution of the following equation:

$$\pi Q = \mathbf{0}',$$

where $\mathbf{0}'$ is a p-dimensional row vector of zeros, Q is the Q-matrix of the CTMC and π is a probability row vector, that is $0 \leq \pi_i \leq 1$ for $i = 1, \cdots, p$ such that $\sum_{i=1}^{p} \pi_i = 1$. Here, π_i is the i-th element of the vector π and p denotes the number of CTMC states. Throughout this thesis, we consider irreducible CTMCs. The exceptions are mentioned in the text.

Notation: Henceforth, for the sake of consistency in notation, all matrices are shown with capital Latin letters and bold notation is used for column vectors. Moreover, vectors of probabilities are row vectors and shown with Greek letters. Also, note that the vectors $\mathbf{0}$ and $\mathbf{1}$ have different dimensions in different contexts (sometimes may consider as infinite dimension vectors).

2.2.2 Point Processes

Point processes are stochastic processes for modelling the random distribution of points in a space and are applied to solve a variety of modelling problems. For instance, modelling the arrival or departure time pattern in a queueing system, the location of trees in a forest, and the repair time of machines in a factory, see [60]. Here, we consider the point processes on the line with non-decreasing non-negative integer values. In general, we can specify a point process in three ways:

1. By the joint distribution of event epochs $T_0, T_1, \cdots$.
2. By the joint distribution of inter-event times $\tau_1, \tau_2, \cdots$.
3. By the joint distribution of the counting random variables $N(t)$ for $t > 0$, where $N(t) = \max\{n : T_n \leq t\}$, represents the number of events during the interval $(0, t]$.

As the relation between (1) and (2), we have $T_n = \tau_1 + \tau_2 + \cdots + \tau_n$.

Definition 2.2.3. *A stochastic process $N(t)$ for $t \geq 0$ is said to be a **counting process** if $N(t)$ represents the total number of events that have occurred up to time t. Therefore,*

- $N(t) \geq 0$,
- $N(t)$ *is an integer,*
- *If $s \leq t$, then $N(s) \leq N(t)$,*
- *For $s < t$, $N(t) - N(s)$ is the number of events that have occurred during the interval $(s, t]$.*

A counting process $N(t)$ has *independent increments* if $N(s,t)$ and $N(u,v)$ are independent for all disjoint intervals (s,t) and (u,v), where $N(s,t) = N(t) - N(s)$ for $s < t$. Further, the counting process $N(t)$ has *stationary increments* if the distribution of $N(s, s+t)$ is independent of s. The Poisson process and the renewal process are examples of counting processes.

The Poisson Process

Definition 2.2.4. *The counting process $N(t)$ for $t \geq 0$ is said to be a **Poisson process** if the inter-event times, τ_n ($n = 0, 1, 2, \cdots$), are independent and identically distributed (iid) exponential random variables with a common parameter λ.*

The random variables τ_n in the above definition denote the time between two consecutive events and are called the inter-event (inter-arrival) times of the counting process $N(t)$. The above definition implies the following properties of the Poisson process, see [83]:

- $N(0) = 0$,
- the process has independent increments,
- the process has stationary increments,

- the number of events in any interval of length t is a Poisson distributed random variable with mean λt, that is for any $s, t \geq 0$:

$$\mathbb{P}\Big(N(s+t) - N(s) = n\Big) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \cdots .$$

In general, a discrete random variable X is said to be a **Poisson distributed random variable** with parameter λ if its probability mass function (PMF) is given by:

$$f(k) = \mathbb{P}\big(X = k\big) = e^{-\lambda}\frac{(\lambda)^k}{k!},$$

where $\lambda > 0$ and $k = 0, 1, 2, \cdots$. For a Poisson distributed random variable X, we have $\mathbb{E}[X] = \mathrm{Var}(X) = \lambda$.

Renewal Processes

Definition 2.2.5. *The counting process $N(t)$ is said to be a **renewal process** if the inter-event intervals are positive iid random variables.*

Note that for a renewal process $N(t)$, given $T_n = t_n$ be the time that the n-th event occurs, $N(t_n + s) - N(t_n)$ for $s \geq 0$, is again a counting process with iid inter-event intervals of the same distribution as the original renewal process. This motivates the title "renewal processes" for these kinds of processes.

Markov Renewal Processes

Markov chains, Poisson processes, and renewal processes are special cases of a Markov renewal process. Consider a stochastic process (J_n, τ_n), where $J_n = X(T_n)$ is the sequence of states of a CTMC $X(t)$ and $\tau_n = T_n - T_{n-1}$ (here, $T_0 = 0 < T_1 < T_2 < T_3 < \cdots$, where T_n denotes the sequence of jump times for $n = 0, 1, 2, \cdots$). Then, the stochastic process (J_n, τ_n) is said to be a *Markov renewal process* if the process for any $t \geq 0$, $n \geq 1$ and

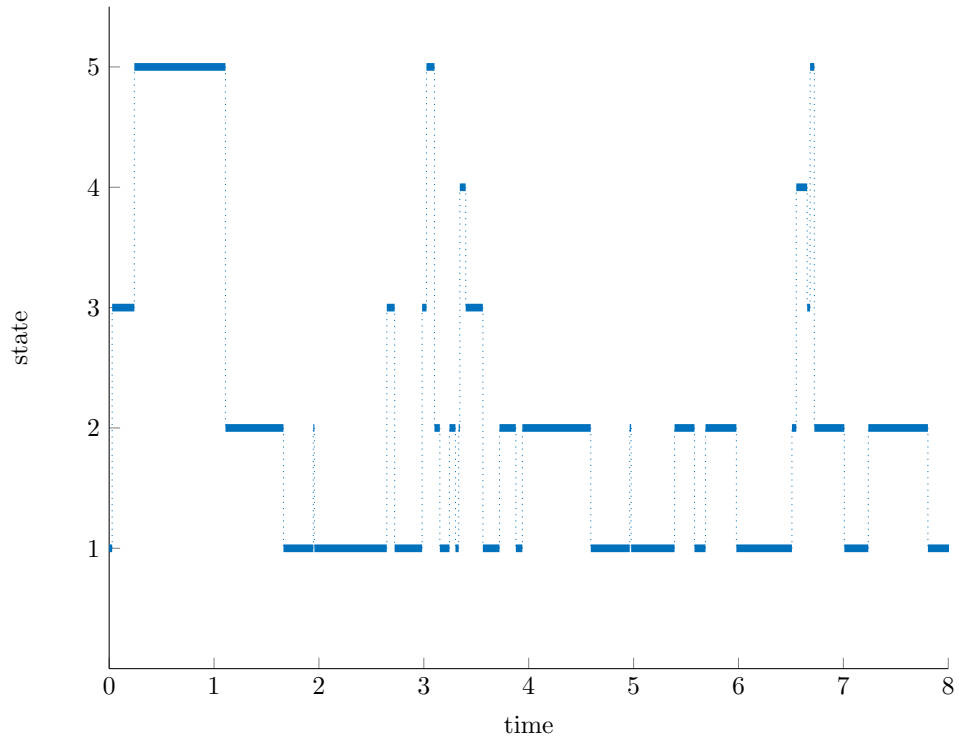


Figure 2.1: Demonstration of the jump process of a CTMC with 5 states during the time interval $[0, 8]$.

$i, j \in \mathcal{S}$ satisfies:

$$\mathbb{P}\big(\tau_{n+1} \leq t, J_{n+1} = j \mid (J_0, T_0), (J_1, T_1), \cdots, (J_n = i, T_n)\big) = \mathbb{P}(\tau_{n+1} \leq t, J_{n+1} = j \mid J_n = i),$$

Note that if the states ignored and just the sequence of iid inter-events τ_n considers, then we have a renewal process $N(t) = \sum_n \mathbb{1}_{\{T_n \leq t\}}$. Moreover, the process $J_n = X(T_n)$ is said to be the *embedded Markov chain* or the *jump process* related to the CTMC $X(t)$.

2.2.3 Semi-Markov Processes

Consider a Markov renewal process (J_n, τ_n). The process $J(t) := J_{N(t)}$, where for any $t \geq 0$, $N(t) = \max\{n : T_n \leq t\}$ is a Markov renewal counting process that counts the number of renewals in an interval $[0, t]$ and is said to be a *semi-Markov process* (SMP). The process $J(t)$ is not necessarily Markovian (memoryless) and has the Markov property only when t is confined to jump times, T_n. More precisely, we have:

Definition 2.2.6. *(Chapter 4 of [174]) A stochastic process $J(t)$ with state space $\mathcal{S}$ is said to be a **semi-Markov process (SMP)** if whenever the process enters state i, the next state is j with probability p_{ij} and given that the next state to be entered is j, the time until the*

transition from i to j occurs has distribution $F_{ij}(t)$, where $P = (p_{ij})$ and $F(t) = \big(F_{ij}(t)\big)$ are transition probability matrix and (cumulative) distribution function of the Markov chain on the state space $\mathcal{S}$.

In fact, for semi-Markov processes we consider a relaxation of the Markov property for sojourn times and only the chain of states has Markov property. It means that the next state of the process depends only on the current state. But, the sojourn times may depend on other characteristics of the trajectory, and therefore a semi-Markov process is not necessarily a Markov process. SMPs are applied for modelling a variety of processes in different areas such as economics, reliability, and health care, see [99].

2.2.4 Queueing Systems

Queueing phenomena occur whenever there are some servers which can not immediately provide the requested service to their users. The examples can be visible in our daily lives, such as in check-out counters, in airport check-in system, traffic intersections, and manufacturing systems, or be hidden in modern telecommunication systems and computer networks.

Queueing systems first introduced by A. K. Erlang in 1917 [68] for mathematical modelling of telephone conversations. From then on, queueing systems have been received considerable interest from both researchers and practitioners. This leads to a rich collection of papers and books in this area. From application point of view, *queue length* (number of customers in the system), waiting times (waiting time for a customer is the time between arriving into the system and start of service for that customer), and *busy periods* (a measure of how busy a system is defined as the ratio of mean service time to mean inter-arrival time) are of primary interest. The most important performance measures are *traffic intensity*, mean and distribution of queue length, mean and distribution of *waiting times*, mean sojourn time (the time between beginning of service for a customer and departing the system after being served), and mean busy period. See [11, 28], and their references.

A queueing system can be demonstrated by Figure 2.2. Some arrivals/customers come into the system, may wait for service and after being served to depart the system. Therefore, the two main processes in a queueing system are the arrival process and the departure process.

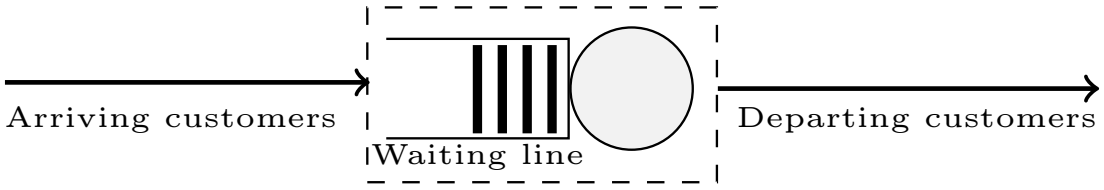


Figure 2.2: A basic queueing system.

Moreover, the number of servers, queue discipline, the manner of choosing customers for service such as first come, first serve (FCFS), and system capacity are important factors. The *Kendall's notation* is a standard way of representing a queueing system. In Kendall's notation, a queue is represented by a notation A/B/k/c/C, where A indicates the inter-arrival time distribution, B indicates the service time distribution, k is the number of servers, c is the system capacity, and C is the queue discipline.

In this thesis, we consider just the first three factors in Kendall's notation (A/B/k) unless mentioned otherwise. Moreover, by default, we assume that we have a single queue with a single class of customers and customers are served on an FCFS basis unless otherwise stated. **Notation**: Some of the most commonly used symbols for distributions in Kendall's notation are: M for Poisson process or exponential distribution (here, M refers to the fact that they follow Markov property or they are memoryless), D for deterministic distribution, G for a general/unspecific distribution, and GI for a general independent distribution.

Traffic intensity In a G/G/1 queueing system, assume that the arrival rate is denoted by λ and the mean service time is denoted by $\mathbb{E}[S]$. Since the server can handle just 1 unit job per unit time, for avoiding system explosion (where the number of customers goes eventually to infinity), the mean number of arrivals should be less than the mean service time or $1 < \lambda\,\mathbb{E}[S]$. Moreover, where $\lambda\,\mathbb{E}[S] = 1$, the mean queue length also explodes (the only exception is the D/D/1 queue, where there is no randomness). Therefore, it would be quite useful to consider the arrival rate multiplied by the mean service time as a measure of traffic in queueing systems. This value is called the *offered load* or *traffic intensity* and is given by:

$$\rho = \lambda\,\mathbb{E}[S].$$

Little's law In a queueing system, there are two induced processes of the queue. Namely, $Q(t)$, the number of items in the system at time t, and the sequence $W_1, W_2, \cdots$, where W_n for $n \geq 1$, indicates the sojourn time of the $n-$th customer. Often their expectations converge to a stationary random variable:

$$L = \lim_{t\to\infty} \mathbb{E}[Q(t)], \qquad W = \lim_{n\to\infty} \mathbb{E}[W_n]$$

Little's law implies a very important relationship between the long-term average number of customers in the queueing system, L, the mean sojourn time, W, and the average arrival rate, λ:

$$L = \lambda W.$$

Here, as an example of queueing systems, we consider the M/M/1 queue.

The M/M/1 queue

Consider a queueing system consists of a single server where customers arrive according to a Poisson process and service times follow an exponential distribution. This queueing system is said to be an M/M/1 queue.

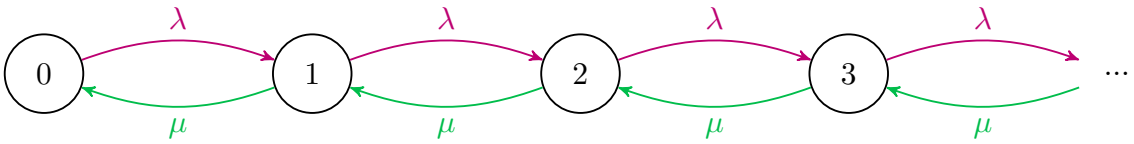


Figure 2.3: Transition diagram of an M/M/1 queue. Circles show queue lengths. The red arrows show transitions to one level up, and the green arrows show transitions to one level down. λ is the Poisson rate of inter-arrival times and μ is the exponential rate of service times.

For an M/M/1 queue with arrival rate λ and service rate μ, the traffic intensity is

$$\rho = \frac{\lambda}{\mu}. \tag{2.2.1}$$

As we mentioned before, the distribution of inter-arrival times in a Poisson process is exponential. Therefore, since the service time also has exponential distribution, we can consider

an $M/M/1$ queueing system as a Markov process with the following Q-matrix:

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -(\mu+\lambda) & \lambda & 0 & \dots \\ 0 & \mu & -(\mu+\lambda) & \lambda & \dots \\ 0 & 0 & \mu & -(\mu+\lambda) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This approach let us apply matrix analytic methods to find the performance measures of this queueing system. For obtaining the long term behaviour of the M/M/1 queue, we need to solve the equation $\pi Q = 0$. Consequently, we have the following system of equations:

$$\begin{aligned} &\pi_0 \lambda = \pi_1 \mu, \\ &\pi_n(\lambda+\mu) = \pi_{n-1}\lambda + \pi_{n+1}\mu, \qquad \text{for } n \geq 1, \\ &\sum_{n=0}^{\infty} \pi_n = 1. \end{aligned}$$

Using (2.2.1), the first equation can be rewritten as $\pi_1 = \rho\pi_0$. Then, from the second equation, the mathematical induction results in:

$$\pi_n = \pi_{n-1}\rho \qquad n \geq 1, \qquad \text{or} \qquad \pi_n = \pi_0 \rho^n \qquad n \geq 1. \tag{2.2.2}$$

If we consider that $\rho < 1$, the geometric series $\sum_{n=0}^{\infty} \rho^n$ is convergent. On the other hand, $\sum_{n=0}^{\infty} \pi_n = 1$. Therefore, the above equation results in $\pi_0 = 1-\rho$ and so,

$$\pi_n = \pi_0 \rho^n = (1-\rho)\rho^n.$$

Consequently, the mean and variance of the number of customers in the system, is given by $\frac{\rho}{1-\rho}$ and $\frac{\rho}{(1-\rho)^2}$, respectively. So, by applying the Little's law, the mean sojourn time is $W = \frac{L}{\lambda} = \frac{\mu^{-1}}{1-\rho}$. Note that when $\rho \to 1$, both quantities of L and W grow to infinity. The dramatic behaviour is the result of the variation in the arrival and service process. This type of behaviour with respect to ρ is characteristic of almost every queueing system.

Now consider an M/M/1 queue where the inter-arrival and departure rates vary for different levels. The resulted queueing system is called a *birth-and-death process*.

Birth-and-Death processes

A continuous-time *birth-and-death (BD) process* is an irreducible CTMC on the countably infinite state space $\mathbb{Z}_+$ for which the transition rate matrix has the following tridiagonal structure (2.2.3). Figure 2.4 illustrates the transition diagram of a BD process, where for all values of $n = 0, 1, \cdots$, both λ_n and μ_n are positive real numbers called the *birth rate* and the *death rate*, respectively. In BD processes, the only possible transitions from a given state n are to the state $n-1$ with rate μ_n, or to the state $n+1$ with rate λ_n. When the process is in the state 0, it remains there for a period of time which is distributed exponentially with rate λ_0, then the process moves to state 1 with probability one. The process stays in the state n ($n \geq 1$) for an exponentially distributed period of time with parameter $(\mu_n + \lambda_n)$. At the end of this period, the process moves either to the state $n-1$ with probability $\frac{\mu_n}{\mu_n+\lambda_n}$ (this corresponds to a death event), or to the state $n+1$ with probability $\frac{\lambda_n}{\mu_n+\lambda_n}$ (this corresponds to a birth event). Therefore, the Q-matrix for this process is given by:

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & & \\ & \mu_2 & -(\mu_2+\lambda_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.2.3}$$

where the missing elements are all zeros. From now on, if an element or a block of elements of a matrix is missing, then it is zero or a block of zeros. The exceptions are distinguished in the text.

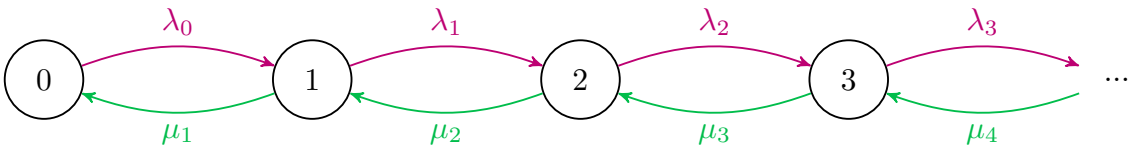


Figure 2.4: Transition diagram of a BD process. Circles show states, the red arrows show transitions accompanied with a birth/arrival, and the green arrows show transitions accompanied with a death/departure. λ_is are birth rates and μ_is death rates.

When the process is positive-recurrent, the stationary distribution vector $\pi = (\pi_0, \pi_1, \pi_2, \cdots)$, where $\pi_n = \lim_{t\to\infty} \mathbb{P}(X(t) = n)$ for $n \geq 0$, satisfies the system of equations: $\pi Q = \mathbf{0}'$ and $\pi \mathbf{1} = 1$. Therefore, for any $n \geq 1$, π_n can be expressed in terms of π_0 by (see Chapter 4

of [119] for more details):

$$\pi_n = \pi_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}. \tag{2.2.4}$$

Note that where $\mu_n = \mu$ and $\lambda_n = \lambda$, the above equation reduces to $\pi_n = \pi_0 \rho^n$ and the BD process is the M/M/1 queue.

2.3 Structured Markov Models

Structured Markov models (SMMs) are Markov chains on countable state spaces, with a special structure for the Q-matrix. For instance, their Q-matrix is block-diagonal or sparse. SMMs are suitable for modelling queueing, storage, reliability, epidemics, and population processes.

The advantage of modelling processes with a SMM is that for evaluation of performance measures of SMMs, there are efficient numerical analysis methods, commonly called *matrix analytic methods* (MAM). Further, SMMs preserve the Markovian structure and by using MAM the results are algorithmically tractable. This makes them an important tool for studying complex systems.

From a historical point of view, Phase-type (PH) distributions and Markovian Arrival Processes MAP was introduced in the 1970s [7]. A PH distribution is a generalisation of the exponential distribution and a MAP is the generalisation of the Poisson process. In this section, we construct the above SMMs from the probabilistic point of view. First, we show that how starting from a CTMC, we can construct a PH distribution. Then, we build up a *Markovian arrival process* (MAP) by starting from a Poisson process. Furthermore, we introduce another type of SMMs, the quasi-birth-and-death process as a generalisation of both a BD process and a MAP.

2.3.1 Phase-Type (PH) Distributions

The method of phases (stages) was introduced by A. K. Erlang at the beginning of the 20th century [67], developed by Jensen into the modern language [101], and was extensively generalised by M.F. Neuts [150, 152, 153] and others more than half a century later. The class of PH distributions is very versatile and dense in the class of distributions defined on the non-negative real numbers. So, any probability distribution on the non-negative half-line can be approximated by a PH distribution, see for instance [15]. This fact provides one of the main reasons for widely applying PH distributions in stochastic modelling.

Moreover, PH distributions preserve the Markov structure of stochastic models and therefore, are algorithmically tractable. Another useful advantage of PH distributions is their closure properties. The class of PH distributions is closed under finite convolution, finite mixture, and finite maxima and minima of the random variables [14]. These are motivations behind a wide range of applications of PH distributions in telecommunications [35, 71, 92, 155, 191], teletraffic modelling [57, 100], queueing theory [12, 19, 79, 118], reliability theory [21, 154], and biostatistics [1, 69, 128]. Here, we present a probabilistic definition of a PH distribution presented by Neuts in 1975, see [148].

Definition 2.3.1. *Let $X(t)$ be a CTMC with finite state space $\mathcal{S} = \{0, 1, 2, \cdots, p\}$, where states $1, \cdots, p$ are transient, and the state 0 is an absorbing state. The time until absorption into state 0 is said to have a **phase-type (PH) distribution** of order p represented by $\tau \sim PH_p(\eta, T)$, where:*

$$\tau = \inf\{t \geq 0 \mid X(t) = 0\},$$

and $\eta = (\eta_1, \cdots, \eta_p)$ is the initial distribution of CTMC $X(t)$ (that is $\eta_i = \mathbb{P}\big(X(0) = i\big)$, for $i = 1, \cdots, p$). Here, matrix $T = (t_{ij})$ for $i, j = 1, \cdots, p$ represents the transition rates between transient states.

In the above definition, we assume that $\eta_0 = 0$ which means that there is no possibility that the process starts at its absorbing state. The Q-matrix for Markov chain of a PH distribution of order p with parameters (η, T) has the following form:

$$Q = \begin{pmatrix} 0 & \mathbf{0}' \\ \mathbf{t} & T \end{pmatrix}, \tag{2.3.1}$$

where $\mathbf{0}'$ is a p-dimensional row vector of zeros, and the *exit vector* $\mathbf{t} = (t_1, \cdots, t_p)'$ satisfies $\mathbf{t} = -T\mathbf{1}$ and denotes the intensities (rates) of going to the absorption state from each of the transient states. Here, $\mathbf{1}$ is a p-dimensional column vector of 1's.

For a PH random variable X (see for example [19] or [30]),

- the probability density function is given by $f(x) = \frac{dF(x)}{dx} = \eta \exp(Tx)\,\mathbf{t}$,
- the Laplace transform has the form $\mathbb{E}[e^{-Xs}] = \eta\ (sI - T)^{-1}\mathbf{t}$, and
- the n-th moment has the form $\mathbb{E}[X^n] = (-1)^n\ n!\ \eta\ T^{-n}\mathbf{1}$.

Extending many results on exponential distributions, to more complex models without losing computational tractability is another rationale behind using PH distributions. The next example shows that many distributions derived from the exponential distribution can be formulated as a PH distribution. For more on PH distributions, see Chapter 1 of [83].

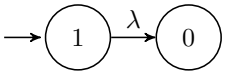
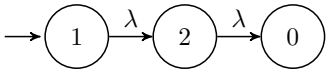
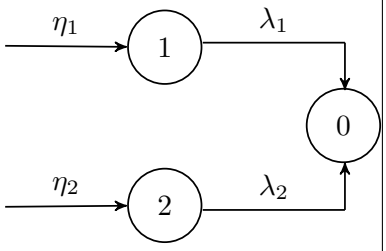
Example 2.3.2. *Here we present three basic examples of PH distributions:*

1. *the exponential distribution is a PH distribution of order 1.*
2. *the Erlang-p distribution (E_p) that is the distribution of a sum of p ($p > 1$) iid exponential random variables is a PH distribution of order p. Since the SCV of an Erlang random variable is less than 1, an alternative name for Erlang distribution is hypoexponential distribution.*
3. *the hyperexponential distribution (H_p) that is the distribution of mixture of p independent exponential random variables is a PH distribution of order p. The SCV of a hyperexponential random variable is always greater than (or equal to) 1.*

Considering the SCV of the above three basic examples of PH distributions implies that the SCV of a PH random variable could be any positive real number. Table 2.1 demonstrates the main characteristics of the basic examples of PH distributions given in Example 2.3.2.

Note that in the above example the Erlang distribution is a special version of a gamma distribution where the shape parameter is a positive integer. In general, a positive random

Table 2.1: Three basic examples of PH distributions.

Distribution	**Initial distribution**	**Transition rate matrix Q**	**SCV**	**Transition diagram of CTMC**
Exponential	$\eta = (1)$	$\left(\begin{array}{c\|c} 0 & 0 \\ \hline \lambda & -\lambda \end{array}\right)$	1	
Erlang-2 (E_2)	$\eta = (1, 0)$	$\left(\begin{array}{c\|cc} 0 & 0 & 0 \\ \hline 0 & -\lambda & \lambda \\ \lambda & 0 & -\lambda \end{array}\right)$	<1	
Hyperexponential (H_2)	$\eta = (\eta_1, \eta_2)$	$\left(\begin{array}{c\|cc} 0 & 0 & 0 \\ \hline \lambda_1 & -\lambda_1 & 0 \\ \lambda_2 & 0 & -\lambda_2 \end{array}\right)$	>1	

variable X is said to have a *gamma distribution* with parameters $\alpha > 0$ (shape parameter) and $\lambda > 0$ (rate), if its density function (PDF) is given by:

$$f(x) = \frac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \lambda e^{-\lambda x}.$$

(For the Erlang-p distribution, $f(x) = \frac{(\lambda x)^{p-1}}{(p-1)!} \lambda e^{-\lambda x}$, where $\lambda > 0$ and $p \in \mathbb{Z}^+$). For a gamma distributed random variable X, we have: $\mathbb{E}[X] = \frac{\alpha}{\lambda}$, $\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$, $SCV(X) = \frac{\mathrm{Var}(X)}{\big(\mathbb{E}[X]\big)^2} = \frac{1}{\alpha}$.

The pdf of mixture of n probability density functions $f_1(x), \cdots, f_n(x)$ with corresponding weights $\omega_1, \cdots, \omega_n$ ($\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$) is given by $f(x) = \sum_{i=1}^{n} \omega_i f_i(x)$. The mean and variance of a mixture of n distributions each with mean μ_i and variance σ_i^2, with corresponding weights ω_i, are:

$$\mu = \sum_{i=1}^{n} \omega_i \mu_i \,, \qquad \sigma^2 = \sum_{i=1}^{n} \omega_i \Big((\mu_i - \mu)^2 + \sigma_i^2 \Big) \,. \tag{2.3.2}$$

2.3.2 Markovian Arrival Processes (MAPs)

The MAP[2] is a generalisation of the Poisson process in the sense that the inter-arrival times of a MAP are not necessarily independent of each other, nor exponentially distributed. Introducing correlation into the input process often makes queueing models more complex and usually computationally intractable. The attempts to analyse queueing networks with correlated input traffic dates back to the last two decades of the 20th century [166]. In 1979, MAPs were introduced by Neuts [149] and later on in 1991 by Lucantoni [130] in a modern form.

Definition 2.3.3. *A **Markovian arrival process (MAP)** of order p (MAP$_p$) with parameters (η, C, D), is a two-dimensional Markov process $\big(N(t), X(t)\big)$. The counting process $N(t)$ counts the number of "arrivals" in $[0,t]$ and $\mathbb{P}\big(N(0) = 0\big) = 1$. The phase process $X(t)$ is an irreducible CTMC with the state space $\mathcal{S} = \{1, \cdots, p\}$, the initial distribution η, and the Q-matrix Q. The matrices C and D are transition rate matrices such that $C + D = Q$. The matrix C has negative diagonal elements and non-negative off-diagonal elements and records the phase transitions with no arrival (event). The event intensity matrix D has non-negative elements and describes changes of the phase process accompanied with an arrival (event).*

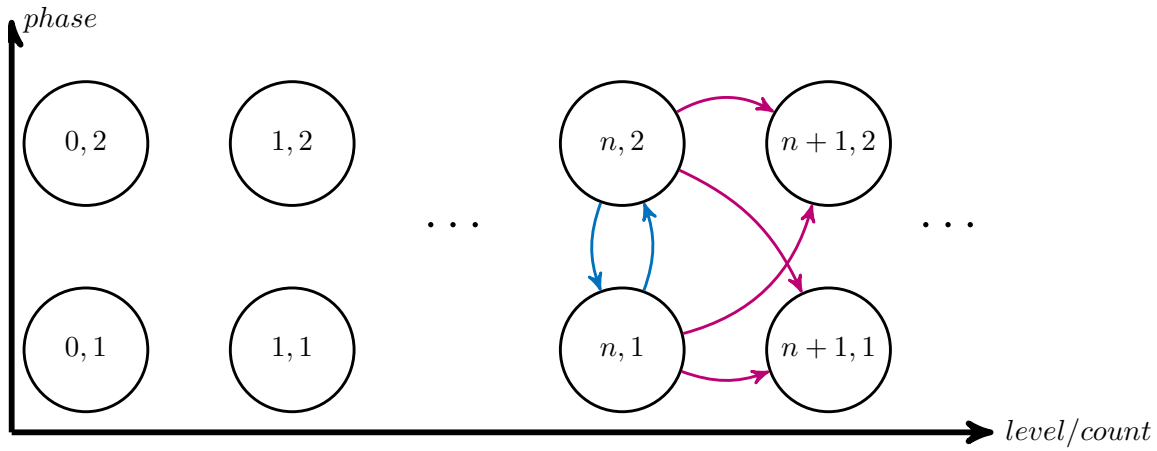


Figure 2.5: Demonstration of the MAP$_2$. Circles show states of the MAP$_2$ and numbers in each state is a pair showing the level and the phase of that state, respectively. Red arrows show transitions accompanied with an arrival and blue arrows show transitions between phases with no arrival.

Since it is often the case that the counting/level process is observed, but the phase process is not, we sometimes refer to the phase process as the *background Markov chain*. As illustrated

[2] Note that in other texts, the acronym "MAP" used to refer to "Markov Additive Process" [15] and in Bayesian statistics MAP stands for "Maximum A Posteriori Probability".

in Figure 2.5, from a given state (n, i), where n shows the level (number of arrivals) and i shows the phase, the only possible one-step transitions are to the states

- $\{(n, j) : 1 \leq j \leq p,\ j \neq i\}$
- $\{(n+1, j) : 1 \leq j \leq p\}$

So, listing the elements of the state space of $\big(N(t), X(t)\big)$ in lexicographic order (similarly to the example shown later in (2.3.6)), the transition rate matrix of a MAP$_p$ has a block bidiagonal form:

$$A = \begin{pmatrix} C & D & & \\ & C & D & \\ & & \ddots & \ddots \end{pmatrix}, \tag{2.3.3}$$

where C and D are $p \times p$ matrices. In each phase transition $i \to j$, the probability of having an arrival is $\frac{D_{ij}}{D_{ij}+C_{ij}}$ and the probability of no arrival is $\frac{C_{ij}}{D_{ij}+C_{ij}}$. The arrivals during the sojourn time ("self-transitions") when the process is in state i (equivalently $X(t) = i$) come from a Poisson process with rate D_{ii}.

Therefore, if we consider the probability transition matrix $P = (p_{ij})$ of the embedded Markov chain $J_n = X(T_n)$ (see Subsection 2.2.2), then the probability of a jump from i to j accompanied with no arrival during the time interval $(t, t+dt]$ (we call these transitions "ordinary" transitions) is $q_{ij}dt\frac{C_{ij}}{D_{ij}+C_{ij}} = q_{ij}dt\frac{C_{ij}}{q_{ij}} = C_{ij}dt$. This implies that given $J_0 = i$, the distribution of the embedded Markov chain at time t is $\mathbf{e}_i' \exp(Ct)$, where $\mathbf{e}_i$ is the i-th unit vector of the standard basis for $\mathbb{R}^p$. So, for each state $j \in \mathcal{S}$, the probability that the phase process is in state j at time t is $\mathbf{e}_i' \exp(Ct)\mathbf{e}_j$. Similarly, the probability that the process goes from state j to state k during the time dt and at the same time, an arrival occurs is $D_{jk}dt$. Hence

$$\mathbb{P}\Big(T_{n+1} - T_n \in dt, J_{n+1} = j \;\Big|\; J_n = i\Big) = \mathbf{e}_i' \exp(Ct) D \mathbf{e}_j dt.$$

The above probability implies that $F_{ij}(t) = \int_0^t \mathbf{e}_i' \exp(Ct) D \mathbf{e}_j dt$, and so:

$$F(t) = \int_0^t \exp(Ct)Ddt \qquad \text{and} \qquad P = \lim_{t\to+\infty} F(t) = (-C)^{-1}D. \tag{2.3.4}$$

Note that $D \neq 0$ and the sub-intensity matrix C is invertible. It is useful to recall the definition and properties of an M-matrix to establish non-singularity of the matrix $-C$ as well as for

results appearing in Chapter 3. Recall that a real-valued square matrix $A = (a_{ij})$ is an M-matrix if the diagonal elements of A are all positive and its off-diagonal elements are all non-positive ($a_{ii} > 0$ and $a_{ij} \leq 0, i \neq j$) and there exists a vector $\nu > 0$ such that $A\nu > 0$. Note that for a vector x, $x > 0$ implies that $x_i \geq 0$ for all i and $x_j > 0$ for at least one j. For MAPs, the matrix $-C$ is an M-matrix since $-C\mathbf{1} = D\mathbf{1} > 0$. The following is well known (see for example [168]):

Proposition 2.3.4. *Suppose A is an M-matrix. Then the following statements are equivalent:*

(i) $A = sI - B$ *for some non-negative matrix* B *and some* $s > \rho(B)$ *, where* $\rho(B)$ *is the spectral radius of B.*

(ii) A^{-1} *exists and is a non-negative matrix.*

Since Q is assumed irreducible and finite, it has a unique stationary distribution π satisfying $\pi Q = \mathbf{0}'$, $\pi\mathbf{1} = 1$. Of further interest is the stationary distribution α of the embedded discrete-time Markov chain $J_n = X(T_n)$ satisfying $\alpha P = \alpha$ and $\alpha\mathbf{1} = 1$. More details on MAPs are in [15] (Chapter XI), [83] (Chapter 2) and [32] (Chapter 10).

MAPs are widely used in application areas such as queueing, reliability, manufacturing, communication systems, and insurance problems, see [7]. This popularity is not only because they provide a natural generalisation of the Poisson process and capture correlations between arrivals (which arises naturally in many applications where the arrival flow is bursty). The more important fact is that using them for modelling often leads to a matrix structured formalism, to which powerful MAM can be applied. Moreover, the class of MAPs is very versatile and even dense in the set of point processes on the real line, see [18]. The class of MAPs contains many of the commonly used arrival processes such as:

- Poisson process: the matrix C is the element $-\lambda$ and the matrix D is reduced to λ, where λ is the rate of the Poisson process.
- PH renewal process: the renewal process where the sojourn times (inter-arrival times) are PH distributed random variables. Consider that parameters of the PH distribution are η and T, and $\boldsymbol{t}$ is the exit vector. Then, the PH renewal process is a MAP with matrices $C = T$ and $D = \boldsymbol{t}\,\eta$.

- Markov modulated Poisson process (MMPP): an arrival process which consists of a finite number of different Poisson processes, modulated by a Markov process. In other words, the MMPP is a particular case of doubly stochastic Poisson processes whose arrival rate is directed by transitions of a finite-state CTMC. For an MMPP$_p$, in terms of Eq. (3.4.3), $D = \text{diag}(\lambda)$, where diag$(\cdot)$ is an operation taking a vector and resulting in a diagonal matrix with the vector in the diagonal, and $\lambda = (\lambda_1, \cdots, \lambda_p)'$. Here, when the phase process, $X(t)$, is in state i, the Poisson rate of arrivals is λ_i. The matrix C is specified by the equation $C = Q - D$.

One of our contributions is in analysing other types of MAPs (MTCP), see Chapter 3.

2.3.3 Quasi-Birth-and-Death (QBD) Processes

A QBD process can be considered as a generalisation of a MAP with a BD structure. For the sake of simplicity, sometimes QBD refers to "QBD process". For a BD process, the distribution of time between level transitions is exponential. But, for a QBD, the time between level transitions has a more complicated distribution. From another point of view, we can consider a QBD as a MAP where transitions to one level down are also possible.

Definition 2.3.5. *A two-dimensional Markov process*

$$\big(N(t), X(t)\big) = \{(n, i) : n \geq 0,\, i = 1, \cdots, p\},$$

including a so-called level process $N(t)$ and a finite-state phase process $X(t)$, is said to be a continuous-time, ***homogeneous QBD***$_p$ *if it satisfies the following properties.*

(I) Transitions from a state are restricted to states in the same level or in the two adjacent levels. In other words, a transition from (n, i) to (n', j) is possible only when $|n' - n| < 2$.

(II) (homogeneity) For $n \geq 1$, the transition rate from (n, i) to (n', j) may depend on i, j and $|n' - n|$, not on the specific values of n and n'.

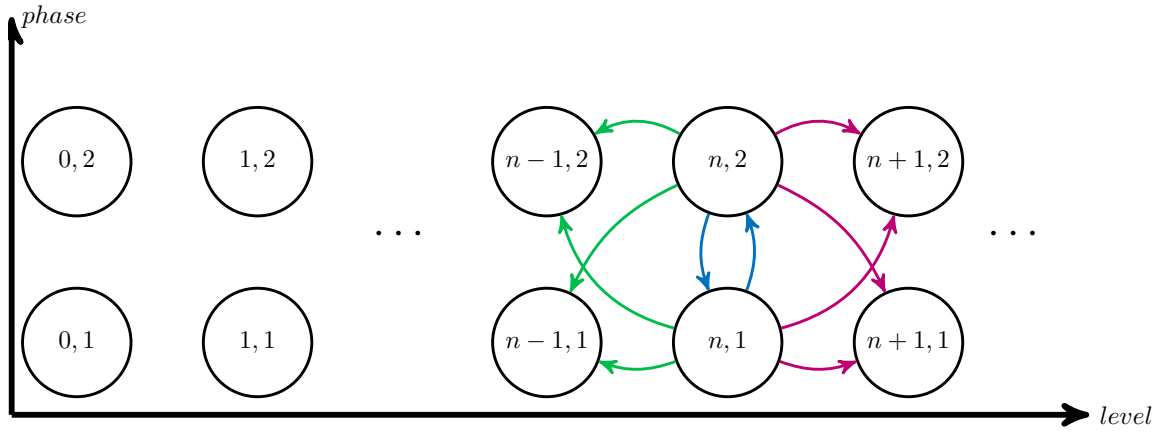


Figure 2.6: Demonstration of the QBD_2. Circles show the state of the QBD_2 and numbers in each state is a pair showing the level and the phase of that state, respectively. Red arrows indicate transitions accompanied with an arrival, blue arrows show transitions between phases with no arrival, and green arrows show transitions with a departure.

The transition rate matrix of a QBD_p has the tridiagonal form

$$A = \begin{pmatrix} B_0 & B_1 & & & & 0 \\ B_{-1} & A_0 & A_1 & & & \\ & A_{-1} & A_0 & A_1 & & \\ & & A_{-1} & A_0 & A_1 & \\ 0 & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.3.5}$$

Here, A_{-1}, A_0, A_1, B_{-1}, B_0, and B_1 are square matrices of order p. The elements of matrix A_{-1} record the transition rates accompanied with a departure, the elements of matrix A_1 record the transition rates accompanied with an arrival, and the matrix A_0 consists of transition rate between phases with no arrival or departure. The matrices B_{-1}, B_1, and B_0 are corresponding matrices for level zero [3]. Figure 2.6 illustrates the diagram of a QBD_2.

Example 2.3.6. ***(the E_p/M/1 queue)*** *Consider an $E_p/M/1$ queue where the service times are exponentially distributed with rate μ and the inter-arrival times have the Erlang distribution with p phases and rate $p\lambda$. Here, we set up the inter-arrival rate parameter such that its mean is not affected by p. We can illustrate this example by Figure 2.7, where there are some transitions between arrival phases and the process remains in each phase for an exponentially distributed interval of time with rate $p\lambda$. Whenever the process goes to absorption state, a new customer joins the system.*

[3] Note also that in a slightly more general formulation the phase at level zero is allowed to have a different number of states. In this case, the matrices B_{-1}, B_1, and B_0 may have a different dimension.

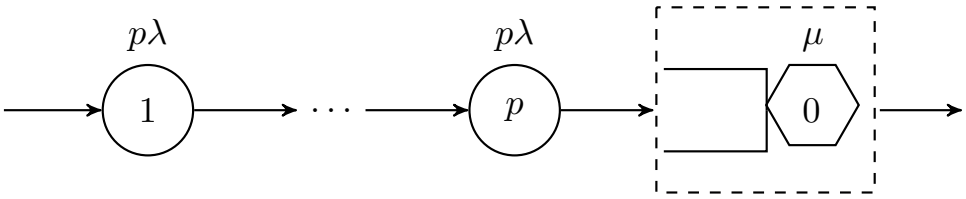


Figure 2.7: Diagrammatic representation of the E_p/M/1 queue. Circles show the arrival phases and the hexagone illustrates the absorption state or the server. The time spent in each phase is exponentially distributed with rate $p\lambda$. The service time is exponentially distributed with rate μ.

The transition rate matrix for the special case of $p = 2$ *is given by:*

$$A = \begin{array}{c} \\ (0,1) \\ (0,2) \\ \hline (1,1) \\ (1,2) \\ \hline (2,1) \\ (2,2) \\ \hline \vdots \end{array} \begin{array}{c} \begin{array}{cc|cc|cc|c} (0,1) & (0,2) & (1,1) & (1,2) & (2,1) & (2,2) & \cdots \end{array} \\ \left(\begin{array}{cc|cc|cc|c} -2\lambda & 2\lambda & 0 & 0 & 0 & 0 & \cdots \\ 0 & -2\lambda & 2\lambda & 0 & 0 & 0 & \cdots \\ \hline \mu & 0 & -(2\lambda+\mu) & 2\lambda & 0 & 0 & \cdots \\ 0 & \mu & 0 & -(2\lambda+\mu) & 2\lambda & 0 & \cdots \\ \hline 0 & 0 & \mu & 0 & -(2\lambda+\mu) & 2\lambda & \\ 0 & 0 & 0 & \mu & 0 & -(2\lambda+\mu) & \ddots \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & & \ddots \end{array}\right) \end{array}. \tag{2.3.6}$$

Here we have:

$$B_0 = \begin{pmatrix} -2\lambda & 2\lambda \\ 0 & -2\lambda \end{pmatrix}, \qquad B_{-1} = A_{-1} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix},$$
$$A_0 = \begin{pmatrix} -(2\lambda+\mu) & 2\lambda \\ 0 & -(2\lambda+\mu) \end{pmatrix}, \quad B_1 = A_1 = \begin{pmatrix} 0 & 0 \\ 2\lambda & 0 \end{pmatrix}.$$

Note that the matrix B_0 *is the transition rate matrix of the PH distribution* E_2 *with rate* 2λ *(see Table 2.1).*

2.4 Computational and Estimation Methods

We now explore some inference and computational aspects of SMMs. We begin by considering second order measures of variability.

2.4.1 Measures of Variability

For carrying statistical inference, we need some assumptions of stochastic regularity or stationarity.

Consider that Y_t is a time series, and $Y_{\mathcal{T}}$ is a finite subset of it. The series is said to be *strictly stationary* if the distribution of $Y_{\mathcal{T}}$ and $Y_{\mathcal{T}+s}$ are the same for any s. In other words, strict stationarity implies that the joint moment generating functions of $Y_{\mathcal{T}+s}$ are independent of s if they exist. Although strict stationarity yields powerful theoretical results, checking it from data is usually impossible. Therefore, in practice, second order stationary is considered. The time series Y_t is said to be *second order stationary* or *weakly stationary*, if its *autocovariance* is time independent. Equivalently, $\mathrm{Cov}(Y_t, Y_{t+s})$ does not depend on t.

The *covariance* is the first measure of variability of two random variables and is a measure of the amount of linear dependence between them. The covariance of two random variables X and Y is given by:

$$\mathrm{Cov}(X, Y) = \mathbb{E}\big[(X - \mu_X)(Y - \mu_Y)\big],$$

where μ_X and μ_Y are their expected values.

The normalized version of the covariance is given by the *correlation coefficient* that is a number between -1 and 1 (a result of the Cauchy-Schwarz inequality) and shows the strength of the linear relationship between two random variables X and Y:

$$\rho_{X,Y} = \mathrm{corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\mathbb{E}[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y},$$

where, μ_X and μ_Y are their expected values and $\sigma_X = \sqrt{\mathrm{Var}(X)}$ and $\sigma_Y = \sqrt{\mathrm{Var}(Y)}$ are their standard deviations.

From definition of the correlation coefficient, it is obvious that for independent random variables, the correlation coefficient is 0, but the converse statement is not necessarily true. Two random variables X and Y are said to be *positively correlated* if $\rho_{X,Y} > 0$, *negatively correlated* if $\rho_{X,Y} < 0$ and *uncorrelated* if $\rho_{X,Y} = 0$.

The *autocorrelation* of a stochastic process Y_t is defined as the correlation between values of the process at different time spots.

$$\operatorname{corr}(Y_t, Y_s) = \frac{\mathbb{E}[(Y_t - \mu_t)(Y_s - \mu_s)]}{\sigma_t \sigma_s},$$

where μ_t and σ_t are mean and standard deviation of the process at time t. For a stationary process, we have:

$$\operatorname{corr}(Y_0, Y_s) = \operatorname{corr}(Y_t, Y_{t+s}).$$

Therefore, it would be more convenient to express the autocorrelation of a stationary process as a function of time-lag:

$$\rho_s = \frac{\operatorname{Cov}(Y_0, Y_s)}{\operatorname{Var}(Y)} = \frac{\mathbb{E}[(Y_t - \mu)(Y_{t+s} - \mu)]}{\sigma^2},$$

where μ and σ^2 are the common mean and variance of the process. The above expression is usually referred as the *lag-s autocorrelation coefficient*.

In general, treating point processes as *stationary* often yields a useful mathematical perspective which matches scenarios when there is no known dependence on time. Roughly, a point process is *time-stationary* if the distribution of the number of events within a given interval does not depend on the location of the interval; that is if $N(t_1 + s) - N(t_1)$ is distributed as $N(t_2+s)-N(t_2)$ for any t_1, t_2 and s. A point process is *event-stationary* if the joint distribution of $T_{k_1}, \cdots, T_{k_n}$ is the same as that of $T_{k_1+\ell}, \cdots, T_{k_n+\ell}$ for any integer sequence of indices $k_1 \cdots, k_n$ and any integer shift ℓ. For stationary point processes, in addition to variance, standard deviation, and SCV, there are two other measures of variability/dispersion: *the index of dispersion for intervals* and *the index of dispersion for counts*.

The *index of dispersion for intervals* (IDI) represents the variability of an event-stationary

inter-arrival process and is given by:

$$\mathcal{J}_n = \frac{\mathrm{Var}\big(T_{i+1}, \cdots, T_{i+n}\big)}{n\,\mathbb{E}^2[T]} = \frac{n\mathrm{Var}(T) + 2\sum_{j=1}^{n-1}\sum_{k=1}^{j}\mathrm{Cov}(T_j, T_{j+k})}{n\,\mathbb{E}^2[T]}, \qquad n = 1, 2, \cdots,$$

where $\mathbb{E}[T]$ and $\mathrm{Var}(T)$ are the common mean and variance of inter-arrival times, respectively.

Since the process is stationary, the above formula can be written regarding autocorrelation coefficients ρ_j and the SCV of intervals, c_J^2, (see [77]) as:

$$\mathcal{J}_n = c_J^2\,[1 + 2\sum_{j=1}^{n-1}(1 - \frac{j}{n})\rho_j],$$

which shows that stationary point processes with positive correlation coefficients have monotonically increasing IDI curves. Moreover, when the number of inter-arrivals goes to infinity, we have

$$\lim_{n\to\infty}\mathcal{J}_n = c_J^2\,[1 + 2\sum_{j=1}^{\infty}\rho_j], \tag{2.4.1}$$

which shows that the IDI is proportional to the sum of all correlation coefficients (plus 1), see [77]. If we consider a time-stationary counting process $N(t)$, then the variability of number of arrivals in the interval $[0, t]$ is given by the *index of dispersion for counts* (IDC):

$$\mathcal{I}_t = \frac{\mathrm{Var}\big(N(t)\big)}{\mathbb{E}[N(t)]},$$

where $\mathbb{E}[N(t)]$ and $\mathrm{Var}\big(N(t)\big)$ are the mean and variance of the number of arrivals till time t. It can be shown that in the limit, IDI and IDC are equal, see [77]:

$$\lim_{n\to\infty}\mathcal{J}_n = \lim_{t\to\infty}\mathcal{I}_t\,.$$

Further, in the case of renewal processes, since inter-arrival times are iid, we have (see Chapter V of [15] or [77]):

$$\lim_{n\to\infty}\mathcal{J}_n = \lim_{t\to\infty}\mathcal{I}_t = c_J^2\,.$$

In practice, usually, the limit of indices of dispersion is applied. For instance, for modelling bursty traffic, the values of the limiting indices of dispersion can be estimated from empirical observations and then a suitable model (for example an MMPP) which is consistent with the

observed index of dispersion (see [77]) be fitted. Although there are other powerful burstiness descriptors such as *peakedness* [98] and the *index of variability* [121], it seems that in practice, the indices of dispersion are very effective to characterize workload burstiness, see [48].

2.4.2 Matrix Analytic Methods

Matrix analytic methods (MAM) or matrix geometric methods (see [15]) first were applied in the study of queueing systems by Neuts in the 1970s. Since then, matrix analytic methods have become a significant tool in the study of stochastic processes and have had a variety of applications in construct and analysis of queueing systems [17, 124, 185], telecommunications networks [122, 138–140, 172, 173], supply chain systems, risk and insurance models [16, 20], reliability models and manufacturing systems [74, 75].

This popularity and power of MAM comes from its ability to analyse a wide class of stochastic models in a unified way, its capacity for analytic exploration, and its algorithmically tractable manner. The idea of MAM is that for models with a structured transition rate matrix (for example block-diagonal), one can replace the scalar quantities by matrices. However, this matrix formalism for complex models may come at the expense of computational problems resulting from high dimensionality. For more references about the theory and applications of MAM, the reader is referred to [4, 56, 83, 119, 161] and references therein. An earlier review about the development of MAM can be found in [151].

In the following, we present an example of applying MAM to find the stationary distribution of a QBD$_p$.

Assume that π is the stationary distribution of a QBD$_p$. We can rewrite the equation $\pi A = \mathbf{0}'$ in terms of the block matrices of A as in Eq. (2.3.5) to obtain the following matrix equations

(here for the sake of simplicity we assume that $B_1 = A_1$ and $B_{-1} = A_{-1}$):

$$\begin{aligned}
\mathbf{0}' &= \pi_0 B_0 + \pi_1 A_{-1}, \\
\mathbf{0}' &= \pi_0 A_1 + \pi_1 A_0 + \pi_2 A_{-1}, \\
\mathbf{0}' &= \pi_1 A_1 + \pi_2 A_0 + \pi_3 A_{-1}, \\
\mathbf{0}' &= \pi_2 A_1 + \pi_3 A_0 + \pi_4 A_{-1}, \\
&\vdots
\end{aligned} \tag{2.4.2}$$

where π_i for $i = 0, 1, \cdots$ is a row vector of dimension p. The general idea behind using MAM for QBDs is to express π_n in terms of π_{n-1} by using a matrix R instead of a scalar ρ in Eq. (2.2.2) for the M/M/1 queue. Here, we assume that there is a matrix R, such that $\pi_n = \pi_{n-1} R$, which yields:

$$\pi_n = \pi_0 R^n, \quad \forall n > 0, \tag{2.4.3}$$

Substituting Eq. (2.4.3) into Eq. (2.4.2) results in:

$$\begin{aligned}
\mathbf{0}' &= \pi_0 B_0 + \pi_0 R A_{-1} && \Rightarrow \quad \mathbf{0}' &&= \pi_0 (B_0 + R A_{-1}), \\
\mathbf{0}' &= \pi_0 A_1 + \pi_0 R A_0 + \pi_0 R^2 A_{-1} && \Rightarrow \quad \mathbf{0}' &&= \pi_0 (A_1 + R A_0 + R^2 A_{-1}), \\
\mathbf{0}' &= \pi_1 A_1 + \pi_1 R A_0 + \pi_1 R^2 A_{-1} && \Rightarrow \quad \mathbf{0}' &&= \pi_1 (A_1 + R A_0 + R^2 A_{-1}), \\
\mathbf{0}' &= \pi_2 A_1 + \pi_2 R A_0 + \pi_2 R^2 A_{-1} && \Rightarrow \quad \mathbf{0}' &&= \pi_2 (A_1 + R A_0 + R^2 A_{-1}), \\
&\vdots
\end{aligned}$$

Therefore, the common term is:

$$A_1 + R A_0 + R^2 A_{-1} = \mathbf{0}. \tag{2.4.4}$$

In general, there is no known closed-form solution for the above matrix quadratic equation. However, where the QBD$_p$ is positive-recurrent (stable), it can be shown that Eq. (2.4.4) has a unique non-negative solution R. In that case, by assuming that the matrix A_0 is a non-singular matrix, from Eq. (2.4.4) we have:

$$R = -(R^2 A_{-1} + A_1) A_0^{-1}.$$

This equation can be solved for R to an arbitrary accuracy ε (for example 10^{-7}) by the following iteration algorithm (here R_n denotes the n-th iteration of R):

Algorithm 1 Iteration algorithm for finding R.

1: Let $R_0 = 0$ (or a better guess, if available),
2: While $\|R_{n+1} - R_n\| > \varepsilon$, set $R_{n+1} = -(R_n^2 A_{-1} + A_1)A_0^{-1}$.

There are several possible definitions of the metric $\|R_{n+1} - R_n\|$, for instance, the maximum absolute value of all elements in the matrix $R_{n+1} - R_n$. For more details, see Chapter 21 of [79].

Once $\{R_n\}$ converges to R, by finding π_0 and putting it in (2.4.3), we can find the other vectors π_n for $n > 0$. For finding π_0, note that there are two equations involving π_0: the first matrix equation in (2.4.2) and the normalizing equation $\pi\mathbf{1} = 1$, where $\pi = (\pi_0, \pi_1, \pi_2, \cdots)$. Putting Eq. (2.4.3) in the normalizing equation, gives:

$$\sum_{n=0}^{\infty} \pi_0 R^n \mathbf{1} = 1, \quad \text{or} \quad \pi_0 \left(\sum_{n=0}^{\infty} R^n \right) \mathbf{1} = 1.$$

If R has a spectral radius[4] less than one, then $\sum_{n=0}^{\infty} R^n = (I - R)^{-1}$ (note that the positive recurrence of the QBD is a necessary and sufficient condition for matrix R to have a spectral radius less than one). Therefore, the above equation can be written as:

$$\pi_0 (I - R)^{-1} \mathbf{1} = 1.$$

This equation and the first matrix equation in (2.4.2), provide a system of equations which has a unique solution π_0:

$$\begin{aligned} \pi_0 \Psi &= 1, \\ \pi_0 \Phi &= \mathbf{0}', \end{aligned} \tag{2.4.5}$$

where for notational simplicity, we put $\Psi = (I - R)^{-1} \mathbf{1}$ and $\Phi = B + RA_{-1}$.

Consider that we apply this method for the $E_2/M/1$ queue. Then, expanding out the second equation in (2.4.5) results in:

$$[\pi_0^1 \quad \pi_0^2] \begin{bmatrix} \Phi_{00} & \Phi_{01} \\ \Phi_{10} & \Phi_{11} \end{bmatrix} = [0 \quad 0].$$

[4] The *spectral radius* of a square matrix R is $\max\{|\gamma| : \gamma \in \sigma(R)\}$, where $\sigma(R)$ is the set of all eigenvalues of R.

After replacing one column with the normalizing equation, we get:

$$[\pi_0^1 \quad \pi_0^2] \begin{bmatrix} \Psi_0 & \Phi_{01} \\ \Psi_1 & \Phi_{11} \end{bmatrix} = [1 \quad 0].$$

This system of equations has a unique solution. These quantities describe the stationary distribution of the E_2/M/1 according to Eq. (2.4.3).

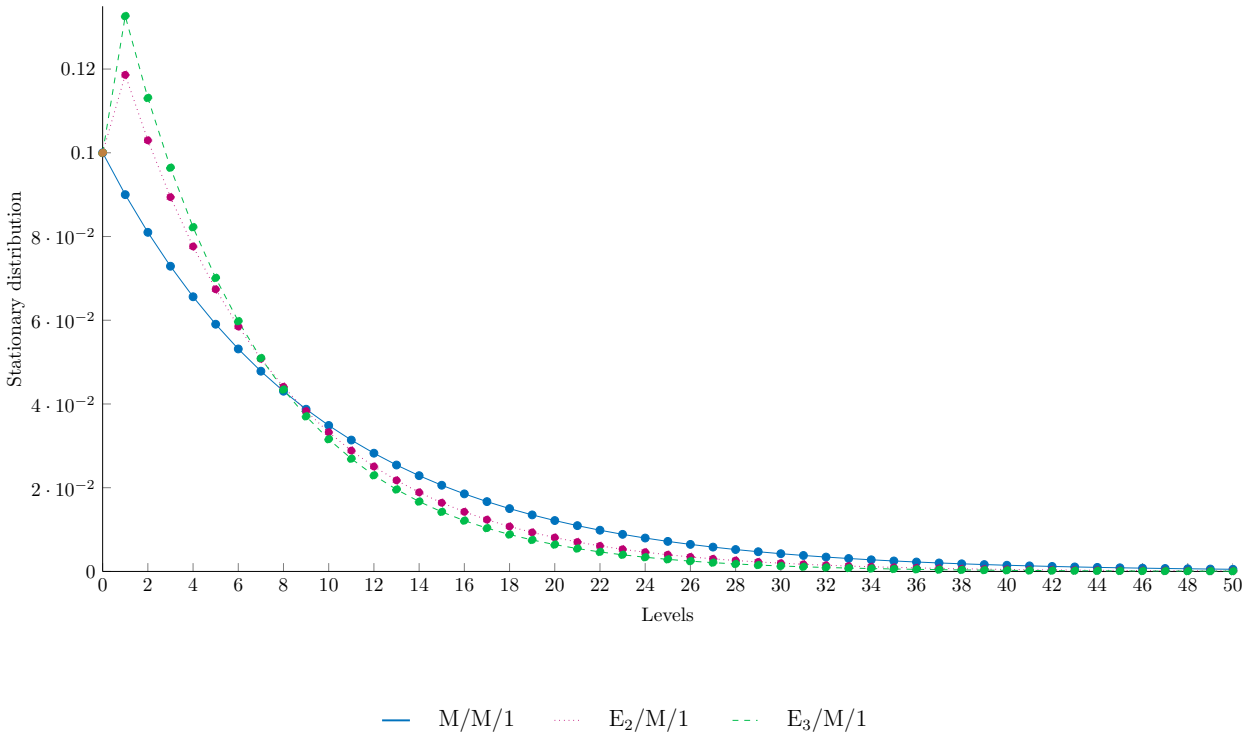


Figure 2.8: The stationary distribution of levels for E_p/M/1 queueing system. The stationary distribution is obtained for $\rho = 0.9$ and for a different number of phases where $p = 1, 2, 3$.

Using the obtained stationary distribution, we can find the probability distribution of the number of customers in a queuing system or in general, the probability distribution of levels in a QBD system. For instance, if $\rho = \frac{\lambda}{\mu} = 0.9$ (see Example 2.3.6),

- for the E_3/M/1 queueing system, we obtain:

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.9482 & 0.8992 & 0.8526 \end{bmatrix}, \quad \pi_0 = (0.0173 \;\; 0.0336 \;\; 0.0491).$$

- for the E_2/M/1 queueing system, we obtain:

$$R = \begin{bmatrix} 0 & 0 \\ 0.9318 & 0.8682 \end{bmatrix}, \quad \pi_0 = (0.0341 \;\; 0.0659).$$

- for the M/M/1 (E_1/M/1) queueing system, we obtain:

$$R = 0.9 \quad \text{(the same as } \rho\text{)} \quad \pi_0 = 0.1.$$

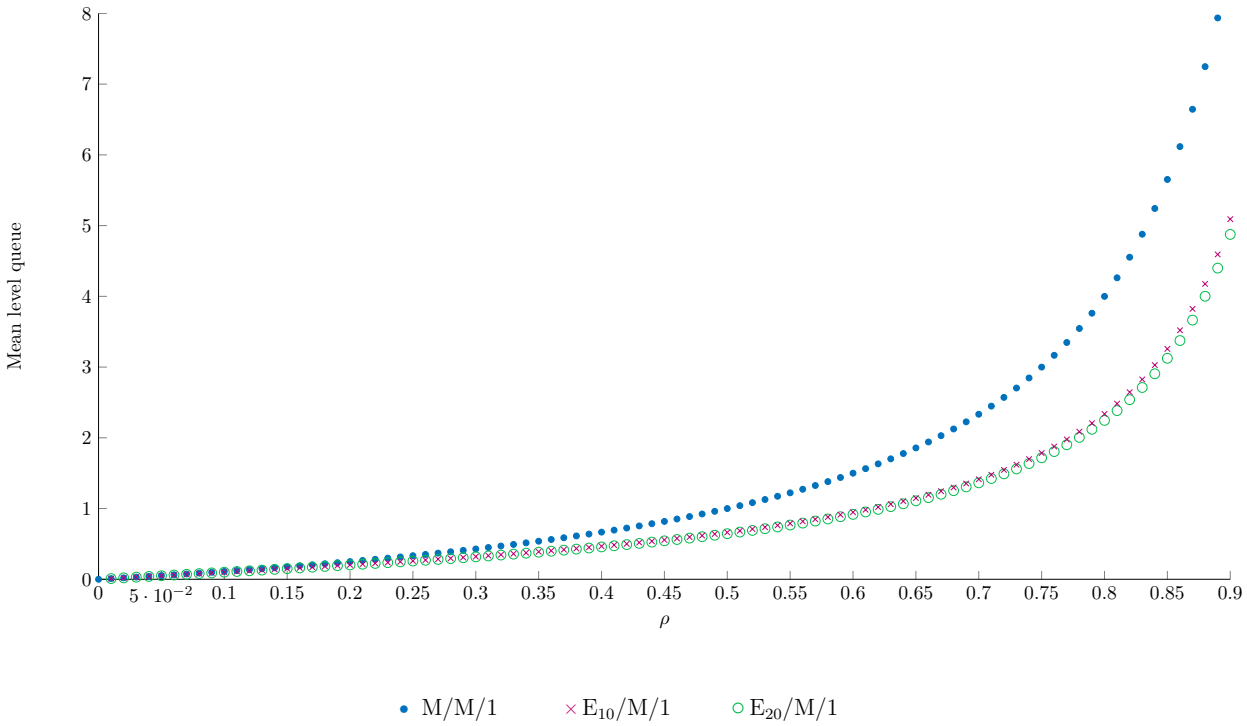


Figure 2.9: Mean level queue of the E_p/M/1 in terms of ρ for different number of phases. Here, $p = 1, 10, 20$.

Figure 2.8 represents the probability of number of customers/levels for an E_p/M/1 queueing system, where $\rho = 0.9$ and for different number of phases $p = 1, 2, 3$. Note that for the cases of $p = 2$ and $p = 3$, as we have more than one phase, the resulted probability of each level is the sum of probabilities of phases of that level. As we expected, by increasing the number of customers/levels, the probability of levels decreases. Moreover, as Figure 2.8 shows, although at first levels, distribution probability of E_p/M/1 (for $p > 1$) increases, but after a while, this trend will be reversed.

Figure 2.9 presents the changes of the mean number of customers/levels in terms of ρ for the E_p/M/1 queue. Here, the mean queue level is obtained for different values of $p = 1, 10, 20$, where ρ varies from 0 to 0.9. As the figure shows, by increasing the number of phases in E_p, the mean level of queue decreases. Further, as expected, by growing ρ towards 1, the mean

level of queue also grows.

2.4.3 Estimation Methods for SMMs

In general, there are two common approaches for parameter estimation of SMMs: a moment-based approach and a likelihood-based approach. Here, after a brief review of these methods, the pros and cons of both approaches are explained. Then, the likelihood-based EM algorithm, as a very common algorithm for parameter estimation of SMMs, is briefly reviewed.

Moment-based approach Assume that $X_1, \cdots, X_n$ form a sample from a joint distribution $f(x_1, \cdots, x_n, \theta)$ with identical marginal distributions $f(x_i, \theta)$, for $i = 1, \cdots, n$, where θ is the vector of model parameters.

On the other hand, we know for a random variable X, where $X \sim f(x, \theta)$, the r-th moment of X (assuming it exists) is given by:

$$M_r(\theta) = \mathbb{E}_\theta[X^r],$$

which can be estimated through the sample r-th moment for observations $X_1, \cdots, X_n$:

$$m_r = \frac{1}{n} \sum_{i=1}^{n} x_i^r.$$

The *method of moments estimator* for θ is denoted by $\widehat{\theta}$ and defined such that each of the first k sample moments is matched with the true moments, that is $\widehat{\theta}$ is the solution of the system of k non-linear equations

$$M_r(\theta) = m_r, \qquad r = 1, 2, \cdots, k.$$

In a more general setting, the marginal distributions may perhaps not be identical. In this case, instead of using moments, we can use the expectations of any sensible real-valued functions of the sample.

The most important benefit of the method of moments is that it is usually fast and often non-iterative. On the other hand, the method of moments estimators are not necessarily unique and applying this method when the number of parameters is large (high-dimensional context) is difficult. See [179] for more details.

Moment-based fitting for PH distributions with a low number of parameters is quite suitable. However, when the number of parameters increases, the shape of the density is not adequately described by the first few moments. Therefore, for an accurate fitting, higher order moments should be considered. Computing higher order moments causes two kinds of problems. First, matching all moments is usually hard and needs applying approximation methods [43]. Moreover, the estimators of higher order moments are unreliable. For instance, when network data traces with more than a million entries are considered, the confidence intervals of moments of order greater than three are extremely wide [111]. Applying the moment-based approach for fitting MAPs has the same problems. Approximating higher order joint moments is hard and often lower order joint moments do not capture the correlation structure [112].

Likelihood-based approach The basis of this method is the likelihood function of the observed data under a certain model and certain model parameters as realizations of the stochastic process. The most common likelihood method is the *maximum likelihood estimation* (MLE). The MLE provides estimates for model parameters which yield the largest likelihood of the observed data.

The widespread use of maximum likelihood is due to the convenient asymptotic properties of the MLE. For instance, model estimates coming from the MLE are optimal in terms of asymptotic variance. Another advantage of using the MLE is that many model selection procedures are based on MLE, such as AIC (Akaike's Information Criterion) and BIC (Bayesian Information Criterion). The computational effort required for the MLE is its major drawback.

The *EM algorithm* is a likelihood-based parameter estimation method that is quite effective for estimating parameters of models with some hidden or incomplete data. This effectiveness is due to its excellent properties in terms of numerical computation. An example of incomplete data is a random variable assumed from a PH distribution where the only obser-

vation is the absorption time and the underlying CTMC is unobserved. In such a situation, the EM algorithm replaces the sufficient statistics in the likelihood function by their conditional expectations given the observed data. So, the EM algorithm essentially maximises the incomplete likelihood function. The name "EM algorithm" stems from the alternating application of an *expectation* step (E-step) that replaces the sufficient statistics with their expectations (applying the given sample) in the likelihood function, and a *maximization* step (M-step) that yields a successively higher likelihood of the estimated parameters. More precisely, consider that there is a complete set of data $\mathbf{x}$ but we observe the incomplete data $\mathbf{y}$. The EM algorithm approaches the problem of finding an MLE by starting with an initial guess θ_0 for the parameter set $\theta = (\theta_1, \cdots, \theta_n)$ and then finds the expectation of sufficient statistics and maximizes the (incomplete) likelihood function based on the steps summarises as below, for more details see Chapter 12 of [32].

Algorithm 2 EM algorithm.

1: (Initial Guess) Set an initial guess θ_0 for the MLE. Iterate the following steps for $k = 0, 1, \cdots$.
2: (**Expectation step**) Calculate $l_k(\theta) = \mathbb{E}\left[\log \mathcal{L}(\theta; \mathbf{x}) | \mathbf{y}; \theta_k\right]$.
3: (**Maximization step**) Find the values of θ that maximise $l_k(\theta)$ and set them as θ_{k+1}.
4: Set k=k+1 and go back to step 2.

It is well known that starting from any initial value for the parameters and updating them by repeating the steps of the EM algorithm increases the likelihood in each step or equivalently $\mathcal{L}(\theta_{k+1}; \mathbf{y}) \geq \mathcal{L}(\theta_k; \mathbf{y})$ for all k and as $k \to \infty$ the algorithm converges (to a local or global maximum or a saddle point for $\mathcal{L}$), see Chapter 12 of [32] or [19]. More notes on the EM algorithm and its stopping criteria from the numerical point of view, can be found in [115]. The EM algorithm often converges at a very slow rate in a neighbourhood of the maximum point. This rate directly reflects the amount of missing data in a problem. The EM algorithm is a broadly applicable approach to the iterative computation of the MLE and has a variety of applications in incomplete-data problems. For instance, Hidden Markov Models (HMMs), MAPs, and PH distributions. For the last two models, the hidden data is comprised of transitions of the background CTMC. As one of the first papers about the EM algorithm, the reader is referred to [65] and for the advanced and complete theory to [137]. Although compared to the method of moments, the EM algorithm has the significant advantage of regarding all the available information (which is especially important when the sample size is small), the EM algorithm experience computational restrictions when either the sample size

is big, or there are a large number of parameters.

Estimation for PH Distributions

The research on PH fitting methods has a very long history. The method of phases proposed by A. K. Erlang at the beginning of the 20th century [67] can be considered as the starting point. Jensen [101] made an important contribution to translate Erlang's idea into the modern language. But it was not until the late 1970s that Neuts and his coworkers [150, 152, 153] established much of the theory of PH distributions in the modern form.

The first work on statistical methods for PH distributions was done by Bux and Herzog in 1977 [46]. They tried to minimize the maximum absolute value of the difference between the empirical distribution and Coxian distributions over a finite set of points. Coxian distributions are a versatile sub-class of PH distributions. More details are in [19, 46]. Applying the basic approach of the EM algorithm to arbitrary PH distributions turns out to be extremely costly regarding computation time, see [13]. Also, the fitted distribution depends heavily on initial values. For instance, see [114], where different methods (including EM algorithm) for fitting PH distributions were applied. Moreover, fitting general PH distributions becomes difficult when the number of phases increases. To overcome these problems, most of the research in this area carried out on subclasses of PH distributions or considered numerical MLE methods for Coxian distributions, see for instance [33, 34]. From an applied probability point of view, these restrictions are often not suitable. An alternative is, for example, using a mixture of Erlang distributions, see [102, 103, 181]. The mixture of Erlang distributions is versatile and can effectively approximate any distribution. However, applying a mixture of Erlang distributions leads to a large number of phases and therefore more complexity in algorithms.

In 1996, one of the most significant contributions in this area was published by Asmussen et. al. [19], where for the first time the EM algorithm was used for fitting PH distributions with an arbitrary structure. They considered the observed data y (time until absorption) from a $PH_p(\eta, T)$ distribution. This is a case of incomplete data and the complete observation of the jump process on the interval $(0, y]$ is given by $\mathbf{x} = (J_0, \cdots, J_{M-1}, \tau_0, \cdots, \tau_{M-1})$ which contains the states of the jump process J_i and the sojourn times satisfying $y = \tau_0 + \cdots + \tau_{M-1}$.

Let θ denotes a vector containing the parameters (η, T). The Likelihood function for an object with a complete observation $\mathbf{x}$ is given by:

$$\mathcal{L}(\theta; \mathbf{x}) = \eta_{J_0} \Lambda_{J_0} e^{-\Lambda_{J_0} \tau_0} p_{J_0 J_1} \cdots \Lambda_{J_{M-1}} e^{-\Lambda_{J_{M-1}} \tau_{M-1}} p_{J_{M-1} 0},$$

where we consider that M is the total number of jumps. Here,

$$p_{ij} = \mathbb{P}(J_{n+1} = j \mid J_n = i) = \begin{cases} \frac{\lambda_{ij}}{\Lambda_i} & i, j = 1, 2, \cdots, p\,, \\ \\ \frac{t_i}{\Lambda_i} & i = 1, 2, \cdots, p \quad and \quad j = 0, \end{cases}$$

where $\Lambda_i = -(t_i + \sum_j \lambda_{ij})$ is the intensity of the sojourn time in state i of the background CTMC. Therefore, the above likelihood can be written as:

$$\mathcal{L}(\theta; \mathbf{x}) = \eta_{J_0} e^{-\Lambda_{J_0} \tau_0} \lambda_{J_0 J_1} \cdots e^{-\Lambda_{J_{M-1}} \tau_{M-1}} t_{J_{M-1}},$$

where τ_i is the realization of exponentially distributed random variables with parameter Λ_i. We can rewrite the above function as:

$$\mathcal{L}(\theta; \mathbf{x}) = \eta_{J_0} (\prod_{i=1}^{p} \prod_{j=1, j \neq i}^{p} \lambda_{ij}^{N_{ij}} e^{-\lambda_{ij} Z_i}) \prod_{i=1}^{p} t_i^{N_i} e^{-t_i Z_i},$$

where

- N_{ij} is the number of jumps from state i to state j,
- Z_i is the total time spent in state i,
- N_i is the total number of jumps from state i to state 0.

Now suppose that we have n independent replication of the process. The complete data is $\mathbf{X} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$ where each $\mathbf{x}_i$ consists of states of jump process and sojourn times of i-th observation. So, the likelihood function is:

$$\mathcal{L}(\theta; \mathbf{X}) = \prod_{i=1}^{p} \eta_i^{B_i} \prod_{i=1}^{p} \prod_{j=1, j \neq i}^{p} \lambda_{ij}^{N_{ij}} e^{-\lambda_{ij} Z_i} \prod_{i=1}^{p} b_i^{N_i} e^{-t_i Z_i}, \tag{2.4.6}$$

where the sufficient statistic consists of

- B_i is the number of Markov jump processes starting in state i,
- Z_i is the total time spent in state i,
- N_{ij} is the total number of jumps from state i to state j, for $j \neq i$ and $j = 0, 1, \cdots, p$,
- N_i is the total number of processes jumping from state i to the absorbing state (state 0).

The resulted estimates (based on complete data) are, see [19]:

$$\widehat{\eta}_i = \frac{B_i}{n}, \qquad \widehat{T}_{ij} = \frac{N_{ij}}{Z_i}, \qquad \widehat{t}_i = \frac{N_i}{Z_i}, \qquad \widehat{T}_{ii} = -(\widehat{t}_i + \sum_{j \neq i} \widehat{T}_{ij}). \tag{2.4.7}$$

Then in the EM algorithm, the E-step consists of calculating the conditional expectation of the above measures given the (incomplete) observed data y and the current estimates of (η, T), say $(\eta, T)^{(k)}$. Applying the conditional expectations of the sufficient statistic and its observes value, in the M-step, the likelihood (2.4.6) is maximised. Then the resulted new estimates of (η, T) obtained by replacing the statistics in (2.4.7) with their conditional expectations evaluated in the E-step. See [19] for more details.

Moreover, Asmussen et. al. [19], showed the effectiveness of the EM algorithm through several numerical examples. Asmussen's work followed the censored case presented by Olsson [160] and is a further step of numerical MLE algorithms given by Ruhe [176], Render and Walker [171], and Harris and Sykes [80] for the class of mixed exponential distributions. For a survey of research on the analysis of PH distributions before 2000, see [14].

All the above PH fitting methods are time-consuming and depend on the size of the data trace. Therefore, following Asmussen's paper [19], the research on PH fitting methods focused on finding simpler and faster methods. For instance, in 2007, Telek and Horváth [190] introduced a moment matching algorithm for PH distributions (as well as MAPs). To use this method the representation for the PH distribution (alt. MAP) with the minimal number of parameters has to be found. Although finding the minimal representation of such objects is very important for developing effective fitting methods, this is in general still an open problem. In 2007, Panchenko and Thümmler [164] improved the EM algorithm for parameter estimation of PH distributions. In contrast to previous approaches, elements in the data trace are first aggregated and then the EM algorithm is applied. Their aggregation method is to divide the

empirical distribution of the data trace into a predefined small number of intervals. Then, the trace elements of each interval are represented by their mean values and a weight corresponding to the portions of the elements in that interval. They showed that this approach is much faster than the previous ones and is efficient for large data sets.

The Markov Chain Monte Carlo method is another option for estimating the PH related functionals, see for instance [31].

Estimation for MAPs

In spite of the wide range of applications of MAPs, their usage in practical system modelling is limited due to difficulties in potential MAP fitting methods. Nevertheless, some progress has been made in this direction:

1. *Heuristic fitting methods for special MAP structures.* In [5], the authors use a superposition of four two-state MMPPs for modelling a variable packet teletraffic with long-range dependency and self-similarity. Another example is [92], where the authors consider the behaviour of high-speed packet switched data networks and use MAM for analysis of these kinds of systems. In both of the above examples, the authors apply several heuristic fitting methods to obtain the Markovian approximation for the typical non-Markovian behaviours of data traces in telecommunication networks.

2. *Moment-based fitting for a MAP$_2$.* For instance, in [87], the method of moments is used to find elements of matrices C and D, in (3.4.3), for a MAP$_2$.

3. *The EM algorithm for MMPPs and MAPs.* For instance, in [94], a 2-step fitting process for MAPs is applied: the PH fitting of the inter-arrival time distribution and the lag-k correlation fitting. Another approach that is presented in [112] is applying the EM algorithm for MAPs with an aggregated version of the data trace which eliminates the dependency between the elements of the data trace.

To evaluate the properties of these approaches and to compile a general and robust fitting procedure, further research is needed. The limitations of the first and second method are obvious, since they handle special structures. The third method is general enough, but

high computational requirements and numerical instability are the negative aspects of this method. The EM algorithm usually works well for short data traces, but for long ones, it converges very slowly and the computation time is often extremely long. For a survey on parameter estimation of MAPs, see [179]. A good literature review about fitting methods of MMPPs can be found in [70]. In the following, a brief review of the efforts that have been made to improve these kinds of methods is presented.

In general, a MAP has a large number of parameters, and so, the direct approach to compute the MLE for a MAP requires large scale matrix computations, and it is generally hard to find the maxima of the likelihood. In 1987, Hellstern [141] made one of the first efforts in this area. She discussed an MLE algorithm for an MMPP_2, a Markov modulated Poisson process of order 2.

In 1994, Rydén [177] carried out a survey on parameter estimation of MMPPs and later applied the EM algorithm for general MMPPs [178]. In this paper, an experimental comparison between the given EM algorithm and the Nelder-Mead downhill simplex algorithm showed that the number of iterations in the EM algorithm is less than the required likelihood evaluation of the downhill simplex algorithm. However, Rydén did not give a complete answer to the MMPP fitting problem in [178]. Long data traces cannot be fit efficiently, and due to identifiability issues, there does not seem to be an illustration of a consistent estimation procedure.

It seems that with a focus on special cases of MAPs there is a good chance to be able to make significant progress. For example, by fitting 2-state MAPs, a good approximation of higher order MAPs is achievable, see [66]. Therefore the most recent papers on MAP fitting methods concentrate on special kinds of MAPs.

In 2002, Breuer [37] used the EM algorithm for fitting Batch Markovian arrival process (BMAP). BMAP is a generalisation of the MAP, by allowing more than one arrival at a time. And then, in 2003, Buchholz [42] used the EM algorithm for MAP fitting of real teletraffic data. His approach was a combination of the EM algorithm for BMAPs [109] and the EM algorithm for hidden Markov models [195]. After this paper, parameter estimation of MAPs with a forward-backward version of the EM algorithm was considered [5]. The forward-backward

[5] This method is usually used for HMMs. For more details about this method or HMMs, the reader is directed to [47] or [133]

version of the EM algorithm is used in the case of transient MAPs (tMAPs). A tMAP is a combination of a MAP with a PH distribution (the matrices C and D of MAP have the transition rate matrix structure of a PH distribution) which was introduced in 2003 by Latouche et. al. [120]. In 2003, Breuer et. al. [40] applied the EM algorithm for fitting tMAPs and showed that larger sample sizes do not necessarily lead to more precise parameter estimation. This is due to the lack of consistency of estimators coming out of the EM algorithm.

Later, Casale et. al. [48, 51] proposed the Kronecker Product Composition (KPC) method which finds the smallest MAP that can be fitted the data trace based on the superposition of MAPs of lower orders, applying sensitive analysis, and using the BIC for model selection. The KPC-Toolbox, a library of MATLAB scripts for fitting workload traces into MAPs (or PH distributions) in an automatic way, were presented in 2010 [52] and developed in 2012 [50]. Further, in 2010, Breuer and Kume [39] presented an EM algorithm for the case the arrival process is observed on a grid of discrete times only.

In 2014, Hautphenne and Fackrell [81] found an EM algorithm for Markovian Binary Trees (MBTs), which are branching processes that utilize both tMAPs and PH distributions and have applications in biology. Then a new version of the EM algorithm was presented by Kriege and Buchholz [112] which is applied to the case of an aggregated data trace. Their numerical experiments show that this new version of the EM algorithm is more computationally efficient than the previous ones. For a recent reference about parameter estimation of PH distributions and MAPs, the reader is directed to [44].

2.5 Markov Decision Processes

In general, the goal of a decision process is to present a mathematical model to automate the problem of decision making in a way that maximises the average achieved rewards (or minimises the total cost) by making a right trade-off between the immediate reward and the long-term gains. If the states of a decision process have Markov property, we call it a *Markov decision process* (MDP). Markov decision processes have a variety of applications in different areas such as game theory, finance, machine learning, queuing systems, and artificial intelligence. For a survey on the application of MDP for control of the network of

queues, see [187]. More applications can be found in [196] or [169].

Definition 2.5.1. *Consider a Markov process $X(t)$ with a finite state space $\mathcal{S}$ and the probability transition matrix P. The tuple $(\mathcal{S}, P, R, \beta)$ is said to be a* ***Markov reward process****, where R is a reward function given by $R(s) = \mathbb{E}[R(t+1) \mid X(t) = s]$ and $\beta \in [0, 1)$ is the discount rate.*

The *discount rate* β guarantees that the rewards in the future get discounted and hence modelling the fact that future rewards are generally less lucrative than present rewards. A reward of r received after t steps in the future, counts as $\beta^t r$. A consequence of this is that if rewards per time are bounded then the infinite horizon total reward is always finite. Note that when $\beta = 0$, only immediate rewards matter and when β tends to 0, only very near-term rewards are considered and we have a myopic evaluation of the reward. On the other extreme case, when β goes to 1, the importance of future rewards decays exponentially and we have a far-sighted evaluation. The value β can also be thought of as the probability that a trial will be allowed to continue after each step and has connections to interest rates in economic theory.

A Markov decision process is a discrete-time finite horizon stochastic controlled process which consists of a *controller* and a *Markov reward process*. At each time step, as Figure 2.10 illustrates, the controller chooses an action. Then, based on controller's action, the environment, which is a finite state Markov process, gets the controller a corresponding reward and moves randomly into a new state. Here, we consider that the set of actions is discrete and has a finite number of states.

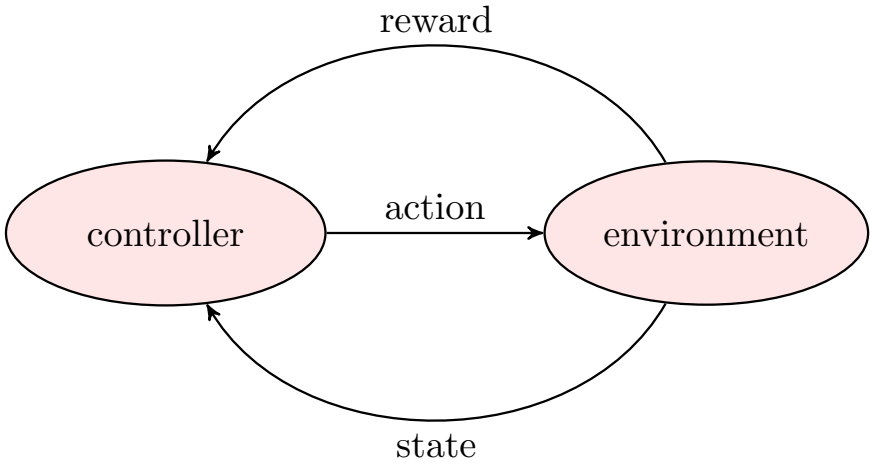


Figure 2.10: Demonstration of the MDP.

Definition 2.5.2. *A **Markov decision process** (MDP) is a tuple $(\mathcal{S}, A, P, R, \beta)$, where $\mathcal{S}$ is the finite state space of a Markov chain which is called the environment of the MDP, A is the actions set, P is the probability transition matrix of the MDP given by*

$$P(s' \mid s, a) = \mathbb{P}\Big(X(t+1) = s' \mid X(t) = s, A(t) = a\Big), \tag{2.5.1}$$

$R : \mathcal{K} \to \mathbb{R}$ is the reward function, where $\mathcal{K} = \big\{(s, a) \mid s \in \mathcal{S}, a \in A_s\big\}$, and A_s is the set of possible actions when the system is in state s. The $\beta \in [0, 1)$ is the discount rate.

Here, we are interested in finding the expected value of rewards:

$$R(s, a) = \mathbb{E}[R(t+1) \mid X(t) = s, A(t) = a]. \tag{2.5.2}$$

Note that the next state depends on the current state and the selected action. further, the next state is independent of the prior history (Markov property).

2.5.1 The Optimal Policy

Solving an MDP means finding a controller policy that collects the maximum reward results by making a right trade-off between the immediate reward and the long-term gains. Such a policy is called an *optimal policy*. A policy can be considered as a function $\pi : \mathcal{S} \to A$ that relates what has happened in the past and what has to be done at the current state. If we assume that $a_s = \pi\big(X(t) = s\big)$, then we can define the *state value function of the policy* π as the mean of total discounted rewards collected from state s under the control action a_s:

$$V_s^\pi = \mathbb{E}_\pi\big[R(t+1) + \beta\, R(t+2) + \beta^2\, R(t+3) + \cdots \mid X(t) = s\big], \tag{2.5.3}$$

Now the goal is to find the optimal policy $\pi^\star$ that maximises V_s^π for all states s. Consider a policy that at each time slot t is just a function of the current state (Markov property) does not depend on time (Stationarity), and its choice of the control action is non-randomizing (deterministic). Such a policy is said to be a *deterministic stationary Markov policy*. It can be shown that there exists a deterministic stationary Markov policy $\pi^\star$ that is optimal, that is such a policy maximises V_s^π for all states s, see Chapter 6 of [169].

Finding the Optimal Policy

From Eq. (2.5.3), we have $V_s^\pi = \mathbb{E}_\pi\big[R(t+1) + \beta\, V_{s'}^\pi \;\big|\; X(t) = s\big]$, where $s' = X(t+1)$. Using (2.5.1) and (2.5.2) results in:

$$V_s^\pi = R(s,a) + \beta \sum_{s'} P(s' \;|\; s,a)\, V_{s'}^\pi.$$

The above equation has a unique solution for the p unknowns, where p is the number of states of the Markov chain. In fact, this equation implies that the current value function is the expected value of the immediate reward plus the sum of the expected values of the discounted reward at next step. Now assume that $V^\star := V^{\pi^\star}$ and consider the *optimal action value function* $Q^\star(s,a)$ as the expected value of reward where the process starts at state s, performs the action a and applies the optimal policy $\pi^\star$ from there on. Then, we have the *Bellman equation* for MDPs :

$$V_s^\star = \max_a \Big\{ Q^\star(s,a) \Big\} = \max_a \Big\{ R(s,a) + \beta \sum_{s'} P(s' \;|\; s,a)\, V_{s'}^\star \Big\}. \tag{2.5.4}$$

Note that the Bellman equation is non-linear and in general, has no closed form solution. However, there are many iterative algorithms for solving this equation, such as value iteration and policy iteration. This equation says if the controller is already reached the goal, it does not need to execute any further action. Otherwise, the controller needs to choose an action that maximises the expected long-term rewards to reach the goal.

The Average-Reward Optimality Equation

The average reward criterion is applied for systems where decisions are made frequently based on system observations such as the system throughput. Practically, all approaches to solve the optimal control problem (find the policy that maximises the total reward) for these systems are related to find solutions of the *average reward optimality equation* for a stationary Markov policy π, see Chapter 8 of [169]:

$$\rho_s^\star = \sup_\pi \lim_{t\to\infty} \inf \frac{1}{t} \mathbb{E}_\pi \Big[\sum_{\tau=0}^{t-1} R(\tau) \;\Big|\; X(0) = s \Big]. \tag{2.5.5}$$

Moreover, we consider that the MDP is *unichain*, that is the probability transition matrix P consists of a single recurrent class plus a possibly empty set of transient states, see [169]. This implies that $\rho_s^\star$ does not depend on s and can be considered as a scalar $\rho^\star$. The reasonable guess for the solution of the above equation is

$$V_{s,k}^\star = k\,\rho^\star + h(s) + o(1),$$

where $o(1)$ denotes a vector that pointwise goes to 0 when $s \to \infty$. Therefore, we can put $V_{s,k}^\star \sim k\,\rho^\star + h(s)$. If we consider a finite horizon MDP and assume that $V_0 = 0$, after applying k iterations of the Bellman equation (2.5.4), we have:

$$V_{s,k}^\star = \max_a \Big\{ R(s,a) + \mathbb{E}[V_{s',k-1}^\star \mid s, a] \Big\},$$

or equivalently:

$$k\rho^\star + h(s) \sim \max_a \Big\{ R(s,a) + \mathbb{E}[(k-1)\rho^\star + h(s') \mid s, a] \Big\}.$$

Using the properties of expectation function, the above equation can be rewritten as:

$$\rho^\star + h(s) = \max_a \Big\{ R(s,a) + \mathbb{E}[h(s') \mid s, a] \Big\}.$$

2.5.2 Partially Observable Markov Decision Processes

A *partially observable Markov decision process* (POMDP) is an MDP where the controller does not monitor the states of the environment completely. The idea of POMDP first introduced in 1965 by Aström [22] where he studied an MDP with no complete estimation of the states. Then, in 1973, Smallwood and Sondik [184]. For more details on solving a POMDP and related technical algorithms, see [53], [125], or [169] and references therein.

Since for POMDP there is no direct information about the current state, we need a set of observations that give hints about the environment state. Further, as this observation set could be probabilistic, we need to specify the probability of each observation. So, we have the following definition.

Definition 2.5.3. *A **partially observable Markov decision process (POMDP)** is a tuple $(\mathcal{S}, A, O, P, R, Z, \beta)$, where S is the finite state space of the environment, A is the actions set, O is the observations set, P is the states' probability transition matrix, R is the reward function, Z is the observation probability matrix given by:*

$$Z(o \mid s', a) = \mathbb{P}\Big(O(t+1) = o \mid X(t+1) = s', A(t) = a\Big),$$

and $\beta \in [0, 1)$ is the discount rate.

Since for a POMDP, the policy depends on the history of observations (as opposed to the actual current state for MDP), we need to determine the actions in terms of the probability distribution of the states or in terms of *belief states*:

Definition 2.5.4. *For a POMDP, the states probability distribution conditioned on the history till time t is said to be the **belief state** at time t:*

$$b(t) := \Big(\mathbb{P}\big(X(t) = s_1 \mid \mathcal{H}(t)\big), \cdots, \mathbb{P}\big(X(t) = s_p \mid \mathcal{H}(t)\big)\Big) = \Big(b(s_1), \cdots, b(s_p)\Big),$$

where s_i denotes the environment state for $i = 1, \cdots, p$, and $\mathcal{H}(t)$ is prior history to time t.

The probability of observing state s'' after taking action a from belief state $b(t)$ is given by:

$$P(s'') = \sum_s b(s) \sum_{s'} P(s' \mid s, a)\, Z(s'' \mid s', a),$$

where we consider that the current state is $X(t) = s$ with probability $b(s)$. Therefore, we can update the belief state from state $b(s)$ at time t to state $b'_{s'}$ at time $t + 1$ by:

$$b'_{s'} = \frac{\sum_s b(s)\, P(s' \mid s, a)\, Z(s'' \mid s', a)}{P(s'')}.$$

This shows that the next belief state only depends on the current belief state and not on the whole history of the process. Therefore, the belief state process is Markovian.

Moreover, having the belief states, we can find the expected value of rewards after taking action a, $R(a)$, by:

$$R(a) = \sum_s \sum_{s'} b(s)\, R(s' \mid s, a),$$

where $R(s' \mid s, a)$ is the expected value of reward where the action a is taken in state s and the result is that the process moves to state s'.

Chapter 3

Bursty Stationary MAPs and Their Application in Queueing Theory

3.1 Introduction

Point processes on the line, generated by transitions of CTMCs have been studied intensely by the applied probability community over the past few decades under the umbrella of matrix analytic methods (MAM), see for instance [119]. These have been applied to Teletraffic [3], business networks [88], social operations research [199], and biological systems [159]. The typical model referred to as the Markovian arrival process (MAP) is comprised of a finite state irreducible CTMC which generates arrivals at selected instances of state change and according to Poisson processes modulated by the CTMC. MAPs have been shown to be dense in the class of point processes so that they can essentially approximate any point process, [18]. At the same time, they are analytically tractable and may often be incorporated effectively within more complex stochastic models [149]. Some notable descriptions of MAPs are in [15] (Chapter XI), [83](Chapter 2), [119] (Chapter 3), [130], and [149].

For a given model of a point process, one may often consider either the event-stationary or the time-stationary case. The probability laws of both cases agree in the case of the Poisson process. However, this is not true in general. For MAPs, time-stationarity and event-stationarity are easily characterized by the initial distribution of the background CTMC. Starting it at its stationary distribution yields time-stationarity and starting at the stationary

distribution of the embedded Markov chain (jump chain) yields event-stationarity. Details follow.

In describing and measuring stationary point processes, first and second order quantities are often very useful. The first order measure is the *rate* of the point process, λ^*, which specifies the mean number of event occurrences during a unit time. For the time-stationary version, $\mathbb{E}[N(t)] = \lambda^* t$ and for the event-stationary version, $\mathbb{E}[T_n]^{-1} = \lambda^*$. Then, the second order measures of interest, measured in the event-stationary case, is the *squared coefficient of variation* given by:

$$c^2 = \frac{\mathrm{Var}(T_n)}{\mathbb{E}^2\left[T_n\right]}. \tag{3.1.1}$$

A low c^2 indicates nearly deterministic processes and a high c^2 indicates that the process is bursty: Namely, there are many intervals with small T_n and occasional intervals with large T_n.

An additional second order measure, taking dependence into account is the sequence of lag-j autocorrelations computed for the event-stationary case, $\{\varrho_j\}_{j=1}^{\infty}$, also summarized by $R := \sum_{j=1}^{\infty} \varrho_j$. Here,

$$\varrho_j = \frac{\mathrm{Cov}(T_0, T_j)}{\mathrm{Var}(T_0)}.$$

In seeking a summary descriptors of a point process, the pair c^2 and R are prime choices that come to mind after the rate λ^*. This is because c^2 is a basic measure of the "burstiness" of the point process and R is a measure of the correlation structure. In addition to these quantities, a further second order measure of variability is the long-term asymptotic variance of the number of items normalized by the mean:

$$d^2 := \lim_{t\to\infty} \frac{\mathrm{Var}(N(t))}{\mathbb{E}[N(t)]}. \tag{3.1.2}$$

In certain applications, d^2 is referred to as the Fano-Factor, see [182]. It is well known (see [77]) that when this limit exists and is finite, we have: $d^2 = c^2(1 + 2R)$. The Poisson process, as the canonical point process, features $c^2 = d^2 = 1$ with $R = 0$. More generally, renewal processes have $c^2 = d^2$ (not necessarily 1) with $R = 0$. But in general, R may be positive or negative in which case $d^2 \neq c^2$.

In this chapter, we focus on MAPs satisfying,

$$c^2 \geq 1, \quad \text{and} \quad d^2 \geq 1, \tag{3.1.3}$$

referring to such processes as *bursty MAPs*, where if both inequalities are strict, we call the processes a *strictly bursty MAP*. This treats the Poisson process as a reference. We note that there have been other ways of describing burstiness (see [98] and [121]), yet there is not a single precise definition. Still, in applications estimating c^2 is often one of the first measurements taken by practitioners and if for example $c^2 \approx 0.2$ (as may be the case in manufacturing applications), the process is called non-bursty and if c^2 is greater than unity as may be the case in computing and telecommunications applications, the process is called bursty. Note that in the statistical analysis of counts, $d^2 > 1$ is often referred to as over-dispersion, see for instance [136]. Considering the autocorrelation structure and R is often the next measurement step, as in [77].

Our main contribution is in identifying the following classes of MAPs (characterized here based on their C and D matrices) as being bursty and further in finding relationships between them:

1. **The (well-known) *Markov Modulated Poisson Process* (MMPP)** – This is a MAP with a diagonal D, that is only "self-transitions" generate events. MMPPs correspond to doubly-stochastic Poisson processes where the modulating process is driven by a CTMC. MMPPs have been used extensively in modelling and analysis, see for instance [70].

2. **The *Markov Transition Counting Process* (MTCP)** – This is a MAP with diagonal C where the diagonal elements of D are all 0. It is a process that exactly counts all ordinary transitions of a CTMC. From a path-wise perspective, MMPPs and MTCPs are in a sense the exact opposites. In MTCPs only ordinary events are counted while in MMPPs only self-transitions are counted. MTCPs have been analysed in classic works (for example in [175]) as well as [147] under different names.

3. ***Hyperexponential Renewal Process* (H-Renewal Process)** – When seeking a phase type distribution with $c^2 > 1$, using a hyperexponential is typically a prime candidate, [45]. In this respect, the H-renewal process is a basic bursty MAP. See [119] (Chapter 3) for details about PH-renewal processes. As a MAP, H-renewal processes are described by the diagonal matrix C and rank-one matrix D. In fact, $D = -C\mathbf{1}\eta$, where $\mathbf{1}$ is a column vector of 1's and η is the initial distribution of the hyperexponential distribution.

4. **The *Markov Switched Point Process* (MSPP)** – This is a MAP with a diagonal C: that is all transitions are counted (both ordinary and self). Therefore, both MTCP and H-renewal process are sub-classes of MSPP. Every Markov Renewal process (MRP) with exponential sojourn times is an MSPP, see [83]. MSPPs have received much less attention in the literature (in comparison to MMPPs). This class of MAPs was introduced and analysed in [63].

As we show in the sequel, establishing (3.1.3) for these processes (showing they are bursty) is straightforward in certain cases, but is more challenging in other cases. For example, from a modelling perspective, it is well accepted that MMPPs exhibit some sort of bursty behaviour. Showing this based on d^2, as in (3.1.2), is straightforward (for instance see Problem 1.1 Chapter XI in [15]); but surprisingly (to the best of our knowledge), it has not been shown previously that $c^2 > 1$, even though this is well accepted in influential stochastic modelling papers, such as [85]. We do so in this chapter.

Comparison of different stochastic processes to find a versatile model for describing observed data in an accurate manner is a fundamental objective in stochastic modelling. This motivates us to find relationships between these classes and primarily focus on MMPPs and MTCPs. The former is a natural generalization of the Poisson process and has been used in **thousands** of applied probability and stochastic modelling papers. As opposed to that, to date, MTCPs have not been employed widely in applications. But, is MMPP superior to MTCP in any manner?

When observing their related point processes, $N(t)$, in the MMPP setting there is no indication of jump times in the background CTMC. But, as opposed to that, in the MTCP setting it is known that the background CTMC jumps exactly every time t when $N(t)$ is incremented. This potentially makes MTCPs more attractive. Further, for MTCPs all inter-arrival times are exponentially distributed.

If we consider parameter estimation of them, in the literature, there are a number of methods for parameter estimation of MMPPs based on the method of moments and likelihood-based methods. However, most of these methods are applied to the special case of a two-state MMPP, the *interrupted Poisson process*, see for instance [177, 183]. The parameter estimation for the alternative model, MTCP, contains just the parameters of the underlying CTMC

and having a diagonal matrix C in this case, making certain algebraic quantities easier to compute and reducing the computational time in both likelihood base and moment-matching methods. Indeed, as we review in the sequel, typical quantities that one needs to compute for MAPs are C^{-k} or e^{Cx} for some integer $k > 0$ and real $x > 0$. We thus believe that considering MTCP as an alternative model for bursty traffic may be sensible.

As a stochastic modeller chooses a suitable queueing model for a given situation, there is typically more than one choice. Knowing that MTCP/PH/1 is similar to MMPP/PH/1 allows the modeller to have more freedom in model choice. So, we investigate the behaviour of the MTCP/PH/1 queue as an alternative to the MMPP/PH/1 queue through extensive numerical experiments. We show that the basic steady state characteristics (mean and variance of the queue) of a given MMPP/PH/1 queue can be emulated by an MTCP/PH/1 queue almost without relative error in most cases, and through numerical experiments relative errors are bounded in the worst case by 9$\%$. These preliminary results are significant for the emerging body of research dealing with finding alternative (but similar) queueing models.

The remainder of the chapter is structured as follows: in Section 3.2, we overview and summarize MAP results used in this chapter. In Section 3.3, we consider MMPP, MSPP, MTCP, and H-renewal process and present their moment results to establish that all of these classes are bursty. In Section 3.4, we focus on the relationships between MTCP and MMPP. Specifically, we show that for many instances of MMPP, one can find an associated MTCP with the same first and second moments of the distribution of the number of counts up to time t. In Section 3.5 numerical results for approximating a given MMPP$_2$/PH$_2$/1 with an MTCP$_4$/PH$_2$/1 are presented. Further, we consider a nonlinear optimisation procedure for matching inter-arrival process of an MTCP$_4$/M/1 queue with the inter-arrival process of a given MMPP$_2$/M/1 queue.

3.2 Markov Point Processes and Their Moments

Markovian Arrival Processes (MAPs) (see 2.3.2) are attractive due to many matrix-analytical formulas describing distribution functions, generating functions, and moments of both $N(\cdot)$ and the sequence $\{T_n\}$ of enter-arrival times. For example, for a MAP with parameters

(η, C, D) the joint density of inter-arrival times $T_1, \cdots, T_n$ at $t_1, \cdots, t_n$ is given by

$$f(t_1, \cdots, t_n) = \eta e^{Ct_1} D e^{Ct_2} D \cdots e^{Ct_n} D\mathbf{1}, \tag{3.2.1}$$

where $\mathbf{1}$ is a column vector of 1's. Therefore, the density of the time until the first event is $f(t) = \eta e^{Ct} D\mathbf{1}$. This is in the form of a Phase-type (PH) distribution $PH(\eta, C)$. Here, η is the so-called initial distribution and the sub-generator C describes state's transitions of the background CTMC until absorption (see 2.3.1. Note that here since $Q\mathbf{1} = \mathbf{0}$, the exit vector $-C\mathbf{1}$ can be represented by $D\mathbf{1}$ as well.

Note that we have the following relations between π and α, the stationary distribution of the background CTMC $X(t)$ ($\pi Q = 0,\ \pi\mathbf{1} = 1$) and the stationary distribution of the embedded Markov chain $J_n = X(T_n)$ ($\alpha P = \alpha,\ \alpha\mathbf{1} = 1$), respectively:

$$\alpha = \frac{\pi D}{\pi D\mathbf{1}} \qquad \text{and} \qquad \pi = \frac{\alpha(-C)^{-1}}{\alpha(-C)^{-1}\mathbf{1}} = \lambda^* \alpha(-C)^{-1}, \tag{3.2.2}$$

where $\lambda^* = \pi D\mathbf{1} = \pi(-C)\mathbf{1}$. Here, the following lines of calculations are applied. First we have:

$$\pi Q = 0 \quad \Rightarrow \quad \pi(C + D) = 0 \quad \Rightarrow \quad \pi D = \pi(-C).$$

Further, from the first equation, we can substitute πD by $\alpha(\pi D\mathbf{1})$. Hence, for finding the second equation, we set

$$\pi = \pi(-C)(-C)^{-1} = \pi D(-C)^{-1} = \alpha(-C)^{-1}(\pi D\mathbf{1}).$$

On the other hand, from definition of α (see 2.3.2), we have $\alpha P = \alpha(-C)^{-1} D = \alpha$. So,

$$\alpha(-C)^{-1} = \alpha D^{-1} \quad \Rightarrow \quad \alpha(-C)^{-1}\mathbf{1} = \alpha D^{-1}\mathbf{1} = \frac{\pi D}{\pi D\mathbf{1}} D^{-1}\mathbf{1} = \frac{1}{\pi D\mathbf{1}}.$$

Setting the initial distribution of the phase, η to be either α or π, makes the MAP *event-stationary* or *time-stationary* respectively, see [15] (Chapter XI- propositions 1.2 and 1.4). We now describe second order properties associated with each case.

Event-Stationary Case: The MAP is event-stationary[1] if there is an arrival at time $t = 0$ and $\eta = \alpha$. In this case, the (generic) inter-arrival time is a $PH(\alpha, C)$ distributed random variable and thus its k-th moment is:

$$M_k = \mathbb{E}[T_n^k] = k!\alpha(-C)^{-k}\mathbf{1} = k!\,\frac{1}{\lambda^*}(-\pi C)(-C)^{-k}\mathbf{1} = (-1)^{k+1}\,k!\,\frac{1}{\lambda^*}\pi\big(C^{-1}\big)^{k-1}\mathbf{1}, \qquad (3.2.3)$$

with the first and second moments (here represented in terms of π and C):

$$M_1 = \frac{1}{\lambda^*}\pi D(-C)^{-1}\mathbf{1} = \frac{1}{\lambda^*}, \qquad M_2 = 2\frac{1}{\lambda^*}\pi(-C)^{-1}\mathbf{1}. \qquad (3.2.4)$$

The squared coefficient of variation (SCV) of intervals has a simple formula.

$$c^2 + 1 = \frac{M_2}{M_1^2} = \frac{2\,(1/\lambda^*)\,\pi\,(-C)^{-1}\mathbf{1}}{(1/\lambda^*)^2} = 2\pi(-C)\mathbf{1}\pi(-C)^{-1}\mathbf{1}. \qquad (3.2.5)$$

For correlations we have (see [91] or [93]):

$$\mathbb{E}[T_i^k\,T_j^l] = \alpha\,k!\,(-C)^{-k}P^{j-i}\,l!\,(-C)^{-l}\mathbf{1}, \qquad j > i,$$

and when the covariance is normalised, we have the lag-j autocorrelation function:

$$\varrho_j = \frac{\mathrm{Cov}(T_0, T_j)}{\mathrm{Var}(T_0)} = \frac{\mathbb{E}[T_0\,T_j] - M_1^2}{M_2 - M_1^2}. \qquad (3.2.6)$$

Using the previous equations, we can write the lag-j autocorrelation in terms of parameters of the MAP (α, C, D) as (see [94]):

$$\varrho_j = \frac{\alpha(-C)^{-1}\big((-C)^{-1}D\big)^j(-C)^{-1}\mathbf{1} - (\frac{1}{\lambda^*})^2}{2\alpha(-C)^{-2}\mathbf{1} - (\frac{1}{\lambda^*})^2} = \frac{\lambda^*\pi P^j(-C)^{-1}\mathbf{1} - 1}{2\lambda^*\pi(-C)^{-1}\mathbf{1} - 1}.$$

Further, for an event stationary MAP, from (2.4.1) and (3.1.2), we have:

$$d^2 = c^2\left(1 + 2\sum_{j=1}^{\infty}\varrho_j\right) = c^2(1 + 2R). \qquad (3.2.7)$$

Time-stationary Case: A MAP with parameters (η, C, D) is time-stationary if $\eta = \pi$. It turns out that in understanding the transient behaviour of MAPs, the transient deviation

[1] Sometimes an event-stationary MAP is referred to as an interval-stationary MAP, see for instance [70].

matrix, $D_Q^\sharp(t)$, plays a key role:

$$D_Q^\sharp(t) = \int_0^t (e^{Qu} - \mathbf{1}\pi)\, du. \tag{3.2.8}$$

To find out the relation between $D_Q^\sharp(t)$ and functionals of the counting process of a MAP, first, consider the generating function of counting process of a MAP which is given by (see Chapter XI of [15]):

$$F_{ij}(s,t) = \sum_{n\geq 0} s^n \mathbb{P}\Big(N(t) = n, X(t) = j | X(0) = i\Big).$$

Moreover, $F(1,t) = e^{Qt}$ and $F(s,t) = e^{(C+Ds)t}$. Therefore, we have:

$$\frac{\partial}{\partial t} F(s,t) = (C + Ds)F(s,t). \tag{3.2.9}$$

Further, if we put $M_{ij}(t) = \mathbb{E}\left[N(t)\, \mathbb{1}_{X(t)=j} | X(0) = i\right] = \frac{\partial}{\partial s} F_{ij}(s,t)|_{s=1}$, then $M(t) = \frac{\partial}{\partial s} F(s,t)|_{s=1}$ and:

$$\mathbb{E}_\eta\left[N(t)\right] = \eta M(t)\mathbf{1}.$$

From (3.2.9) and $M(t) = \frac{\partial}{\partial s} F(s,t)|_{s=1}$, we have:

$$\frac{d}{dt} M(t) = De^{Qt} + QM(t), \qquad M(0) = 0.$$

The solution of this differential equation is:

$$M(t) = \int_0^t e^{Qu} D e^{Q(t-u)} du, \tag{3.2.10}$$

that results in:

$$\mathbb{E}_\eta\left[N(t)\right] = \eta M(t)\mathbf{1} = \eta D_Q^\sharp(t) D\mathbf{1} + \pi D\mathbf{1}t, \tag{3.2.11}$$

where $D_Q^\sharp(t)$ is the *transient deviation matrix* associated with Q and defined by (3.2.8). Note that the integral (3.2.10) does actually have an explicit solution obtained by taking exponential at time t of double dimension matrix $\begin{pmatrix} Q & D \\ 0 & Q \end{pmatrix}$, see [193]:

$$\exp\left(\begin{pmatrix} Q & D \\ 0 & Q \end{pmatrix} t\right) = \begin{pmatrix} e^{Qt} & M(t) \\ 0 & e^{Qt} \end{pmatrix}.$$

In the same way, by finding the second and third derivatives of (3.2.9) with respect to s, we can find the second and third moments of the counting process of a MAP. In the time-stationary case ($\eta = \pi$), we have (see [15]):

$$\mathbb{E}[N(t)] = \pi D\mathbf{1}\, t, \tag{3.2.12}$$

$$\mathrm{Var}\big(N(t)\big) = \{\pi D\mathbf{1} + 2\,\pi D D_Q^\sharp D\mathbf{1}\}\, t - 2\pi D D_Q^\sharp D_Q^\sharp(t) D\mathbf{1}, \tag{3.2.13}$$

$$\mathbb{E}\left[N^3(t)\right] = 6\pi D\Big(\int_0^t \int_0^u e^{Q(u-s)} D\big(\mathbf{1}\pi D\mathbf{1}s + D_Q^\sharp(s) D\mathbf{1}\big)\Big) ds\, du, \tag{3.2.14}$$

where $D_Q^\sharp$ is the *deviation matrix* associated with Q defined by the following formula.

$$D_Q^\sharp = \int_0^\infty (e^{Qu} - \mathbf{1}\pi)\, du, \tag{3.2.15}$$

Note that in some sources, for instance [15] and [145], the variance formula (3.2.13) is presented in terms of the matrix $Q^- := (\mathbf{1}\pi - Q)^{-1}$. The relation between these two matrices are given by $Q^- = D_Q^\sharp + \mathbf{1}\pi$, see [59].

The deviation matrix has the following properties:

$$D_Q^\sharp \mathbf{1} = \mathbf{0}, \quad \pi D_Q^\sharp = \mathbf{0}, \quad \text{and} \quad D_Q^\sharp Q = Q D_Q^\sharp = (\mathbf{1}\pi - I), \tag{3.2.16}$$

where the last one follows the fact that $\lim_{t\to\infty} e^{Qt} = \mathbf{1}\pi$. The relation between the deviation matrix and the transient deviation matrix of a Q-matrix is given by:

$$D_Q^\sharp(t) = D_Q^\sharp(I - e^{Qt}). \tag{3.2.17}$$

Note that from (3.2.16), we have $\pi D D_Q^\sharp D\mathbf{1} = \pi C D_Q^\sharp C\mathbf{1}$ and therefore, (3.2.7) can be written as:

$$1 + \frac{2}{\lambda^*}\,\pi C D_Q^\sharp C\mathbf{1} = \Big(2\pi C\mathbf{1}\pi C^{-1}\mathbf{1} - 1\Big)\Big(1 + 2\sum_{j=1}^{\infty} \frac{\lambda^*\pi P^j C^{-1}\mathbf{1} + 1}{2\lambda^*\pi C^{-1}\mathbf{1} + 1}\Big), \tag{3.2.18}$$

This relationship can also be obtained in an algebraic manner using the properties of the deviation matrix and the matrix exponential.

3.3 Bursty MAPs

In this section, we analyse the four classes of MAPs: MMPP, MTCP, H-renewal process, and MSPP. We show that they are *bursty*, as defined in (3.1.3), and establishing some relationships between these classes [2].

Theorem 3.3.1. *The classes of processes, MMPP, H-renewal process, MTCP, and MSPP are all bursty MAPs.*

Proof. (outline):
MMPP: $d^2 \geq 1$ is a well-known result for all doubly stochastic Poisson processes (Cox processes). The proof can be found for instance in Chapter 6 of [108]. $c^2 \geq 1$ is more complicated and is established in the Proposition 3.3.3 below.
MTCP: $c^2 \geq 1$ by using the Cauchy-Schwarz inequality as in Lemma 3.3.5 below. Then $d^2 \geq 1$ through the Lemma 3.3.6.
H-renewal process: $c^2 = d^2 \geq 1$ through the explicit formula (3.3.7) below.
MSPP: As for MTCP, $c^2 \geq 1$ through Lemma 3.3.5 and $d^2 \geq 1$ through Lemma 3.3.6. □

First, note that from (3.2.5), showing $c^2 \geq 1$ is equivalent to show that:

$$\pi C\mathbf{1}\pi C^{-1}\mathbf{1} \geq 1 \;\Leftrightarrow\; \pi D\mathbf{1}\pi(-C)^{-1}\mathbf{1} \geq 1 \;\Leftrightarrow\; \pi(-C)^{-1}\mathbf{1} \geq \frac{1}{\pi D\mathbf{1}}.$$

Or

$$\pi(-C)^{-1}\mathbf{1} \geq \alpha(-C)^{-1}\mathbf{1}. \tag{3.3.1}$$

Note that (3.3.1) is not true for all MAPs. One example of it is a PH-renewal process with SCV of the PH distribution less than 1 (for example Erlang-renewal). Here, is another (non-renewal) example. Consider a MAP_2 with

$$C = \begin{pmatrix} -\lambda-\epsilon & \lambda \\ \epsilon & -\lambda-\epsilon \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \epsilon & 0 \\ \lambda & 0 \end{pmatrix}.$$

Then, its stationary distribution is $\pi = (\frac{\epsilon+\lambda}{2\lambda+\epsilon} \;\; \frac{\lambda}{2\lambda+\epsilon})$ and $\pi C\mathbf{1}\pi C^{-1}\mathbf{1} = \frac{\epsilon^2+6\epsilon\lambda+2\lambda^2}{(2\lambda+\epsilon)^2}$ which gives

[2] We refer the reader to arXiv:1802.08400 for updates of this section

values in the interval $[0, 1.125]$ where we consider $\lambda = 1$ and change ϵ in $[0, 2]$.

Furthermore, note that from (3.3.1), a process with $\alpha = \pi$ or $(-C)^{-1}\mathbf{1} = \mathbf{1}$ satisfies $c^2 = 1$. One example of such a process is a Poisson process. However, note that the Poisson process is not the only example of having a MAP with $c^2 = d^2 = 1$. For example take a PH-renewal with inter-arrival density that is, $f(x) = \frac{3}{4}\lambda e^{-\lambda x} + \frac{1}{4}\frac{\lambda^3 x^2 e^{-\lambda x}}{2}$. This PH distribution is obtained by a mixture of an exponential distribution and an Erlang-3 (E_3) distribution with the same rate. Using Equation (2.3.2), it is easy to check that this PH-distribution has $c^2 = 1$ (and so when used to construct a renewal processes has $d^2 = 1$).

In general, we know that for point processes under the Palm distribution α (see [62]), for all $n > 1$, $T_n \overset{d}{=} T_1$, but this is not true under the equilibrium distribution π, for instance, $T_2 \overset{d}{\neq} T_1$. However, if a point process satisfying (3.3.1), then the mean of inter-arrival times under equilibrium distribution is greater than or equal to the mean of inter-arrival times under the Palm distribution, that is $\mathbb{E}_\pi[T_1] \geq \mathbb{E}_\alpha[T_1]$. More generally, we have:

Proposition 3.3.2. *For stationary point processes*

$$c^2 \geq 1 \quad \Leftrightarrow \quad T_1^\pi \underset{st}{\geq} T_1^\alpha,$$

where $\underset{st}{\geq}$ *shows the stochastic order,* α *is the Palm distribution and* π *is the equilibrium distribution of the point process.*

Proof. For a stationary point process $\{N(t), t \geq 0\}$ with points occurring one at a time and $\mathbb{P}\big(N([0,\infty)) = \infty\big) = 1$, we have (see Eq. (3.4.17) of [62]):

$$\mathbb{E}_\pi[T_1] = \frac{1}{2}\lambda^* \mathbb{E}_\alpha[T_1^2].$$

Since from (3.2.4) we have $\mathbb{E}_\alpha[T_1] = \frac{1}{\lambda^*}$, the above equation implies that:

$$c^2 = \frac{\mathbb{E}_\alpha[T_1^2]}{\big(\mathbb{E}_\alpha[T_1]\big)^2} - 1 \geq 1 \quad \Leftrightarrow \quad \mathbb{E}_\pi[T_1] \geq \mathbb{E}_\alpha[T_1].$$

Further, from (3.3.1) by using the fact that $\int_0^\infty e^{Ct}dt = (-C)^{-1}$, we have:

$$c^2 \geq 1 \quad \Leftrightarrow \quad \pi\Big(\int_0^\infty e^{Ct}dt\Big)\mathbf{1} \geq \alpha\Big(\int_0^\infty e^{Ct}dt\Big)\mathbf{1} \quad \Leftrightarrow \quad \pi e^{Ct}\mathbf{1} \geq \alpha e^{Ct}\mathbf{1}.$$

Here, we applied the proof by contradiction for second $\Rightarrow$. Equivalently, we use the fact that:

$$\pi e^{Ct}\mathbf{1} < \alpha e^{Ct}\mathbf{1} \quad \Rightarrow \quad \pi\Big(\int_0^\infty e^{Ct}dt\Big)\mathbf{1} < \alpha\Big(\int_0^\infty e^{Ct}dt\Big)\mathbf{1}.$$

On the other hand, if $T_1^\pi \underset{st}{\geq} T_1^\alpha$, by definition of stochastic order we have $\mathbb{P}(T_1^\pi > x) \geq \mathbb{P}(T_1^\alpha > x)$ which is equal to $\pi e^{Ct}\mathbf{1} \geq \alpha e^{Ct}\mathbf{1}$.

Therefore, $c^2 \geq 1 \quad \Leftrightarrow \quad T_1^\pi \underset{st}{\geq} T_1^\alpha$. □

MMPP: The parameters of an MMPP$_p$ are $D = \text{diag}(\lambda_i)$, where $\lambda_i \geq 0$ for $i = 1, \cdots, p$, and $C = Q - D$. Here Q is the transition rate matrix of a CTMC. For MMPPs, (3.2.12) and (3.2.13) are simplified by using the following relations:

$$\pi D\mathbf{1} = \sum_{i=1}^{p} \pi_i\lambda_i, \qquad D\mathbf{1} = \lambda = (\lambda_1, \cdots, \lambda_p)', \qquad \pi D = (\pi_1\lambda_1, \cdots, \pi_p\lambda_p)\,.$$

We first have the following:

Proposition 3.3.3. *For MMPPs,* $c^2 \geq 1$.

Proof. We need to show that (3.3.1) is true for MMPPs or equivalently $(\pi - \alpha)(-C)^{-1}\mathbf{1} \geq 0$. For the sake of simplicity, denote the row vector $(\pi - \alpha)(-C)^{-1}$ by ω, then, the claim of the proposition is equivalent to show that $\omega\mathbf{1} \geq 0$.

First note that by applying the definition of the probability transition matrix $P = (-C)^{-1}D$, using the fact that P is a stochastic matrix, and applying $\alpha\mathbf{1} = \pi\mathbf{1} = 1$, we have

$$0 = (\pi - \alpha)\mathbf{1} = (\pi - \alpha)P\mathbf{1} = (\pi - \alpha)(-C)^{-1}D\mathbf{1}.$$

Equivalently, we have:

$$0 = \sum_{i=1}^{p} \omega_i\lambda_i, \tag{3.3.2}$$

where $\{\lambda_i\}$ are elements of the matrix $D = \text{diag}(\lambda)$ of the MMPP.

Without loss of generality, we can assume that there is an order $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$. On

the other hand, since $\{\lambda_i\}_{i=1,\cdots,p}$ is increasing, $\{\lambda^* - \lambda_i\}_{i=1,\cdots,p}$ is a decreasing sequence and therefore in the sequence $\{\pi_i - \alpha_i\} = \{\frac{\pi_i}{\lambda^*}(\lambda^* - \lambda_i)\}$ when an element $\pi_k - \alpha_k$ is negative, all the elements $\pi_i - \alpha_i$ for $i \geq k$ are remain negative. Moreover, both π and α are probability distributions, so $(\pi - \alpha)\mathbf{1} = \sum_i(\pi_i - \alpha_i) = 0$. Therefore, at least the first element in the sequence $\{\pi_i - \alpha_i\} = \{\frac{\pi_i}{\lambda^*}(\lambda^* - \lambda_i)\}$ is positive. So, in general, we can assume that there exists an index $1 < k \leq p$ such that $\pi_i - \alpha_i$ for $i = 1, \cdots, k-1$ is non-negative and for $i = k, \cdots, p$ is negative. Therefore, since elements of the matrix $(-C)^{-1}$ are all non-negative, we have:

$$(\pi - \alpha)(-C)^{-1}\mathbf{1} = \omega\mathbf{1} = \underbrace{\sum_{i=1}^{k-1}\omega_i}_{\text{non-negative}} + \underbrace{\sum_{i=k}^{p}\omega_i}_{\text{negative}}.$$

Now if we consider that $\omega\mathbf{1} < 0$, then from the above equation we have:

$$\underbrace{\sum_{i=1}^{k-1}\omega_i}_{\text{non-negative}} < \underbrace{-\sum_{i=k}^{p}\omega_i}_{\text{non-negative}}.$$

Then, since $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$, the above equation results in (note that we multiply the left hand side with values of λ_i that are all less than the values of λ_i multiplied in the right hand side):

$$\sum_{i=1}^{k-1}\omega_i\lambda_i < -\sum_{i=k}^{p}\omega_i\lambda_i.$$

But, from (3.3.2), we have:

$$\sum_{i=1}^{k-1}\omega_i\lambda_i = -\sum_{i=k}^{p}\omega_i\lambda_i. \tag{3.3.3}$$

Consequently, the assumption $\omega\mathbf{1} < 0$ is not true, and we have $\omega\mathbf{1} \geq 0$.

□

In the case $p = 2$ there are exact explicit formulas for d^2 and c^2:

Example 3.3.4. *Consider an MMPP$_2$ with matrices* $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, C = \begin{pmatrix} -\sigma_1 - \lambda_1 & \sigma_1 \\ \sigma_2 & -\sigma_2 - \lambda_2 \end{pmatrix}$. *As in [84], evaluation of the transient deviation matrix through (for example) Laplace transform inversion yields a neat expression for the IDC function (see Section 2.4.1):*

$$\mathcal{I}_t = 1 + \frac{2\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)^2(\lambda_1\sigma_2 + \lambda_2\sigma_1)} - \frac{2\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)^3(\lambda_1\sigma_2 + \lambda_2\sigma_1)t}(1 - e^{-(\sigma_1+\sigma_2)t}).$$

Therefore:

$$d^2 = 1 + \frac{2\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)^2(\lambda_1\sigma_2 + \lambda_2\sigma_1)}.$$

Further, explicit computation yields,

$$c^2 = 1 + \frac{2\sigma_1\sigma_2(\lambda_1 - \lambda_2)^2}{(\sigma_1 + \sigma_2)^2(\lambda_2\sigma_1 + \lambda_1(\lambda_2 + \sigma_2))}.$$

This is in agreement with Theorem 3.3.1 and indicates that the MMPP$_2$ is strictly bursty as long as $\lambda_1 \neq \lambda_2$.

Furthermore, note that when we set $\lambda_1 = 0$, we have:

$$d^2 = c^2 = 1 + \frac{2\sigma_2\lambda_2}{(\sigma_1 + \sigma_2)^2},$$

which implies that where $\lambda_2 \to \infty$ then, $c^2 \to \infty$ and $d^2 \to \infty$. The same happens when we set $\lambda_2 = 0$ and assume that $\lambda_1 \to \infty$. So, there is no upper bound for c^2 or d^2 of an MMPP$_2$.

MTCP: Another example of a MAP is the Markov Transition Counting Process (MTCP) which is a point process counting every transition of an irreducible finite state CTMC. One can consider MMPP and MTCP as two extreme examples of MAPs. For MMPP when there is an event (arrival) with probability 1 we are sure that there is no transition in the phase process (background Markov chain) but in the case of MTCP when there is an event (arrival) with probability 1 we know that there is a transition in the phase process. Therefore, the matrix C of an MTCP, which records the phase transitions with no arrival, is diagonal and the diagonal elements of matrix D are all zero. For an early reference that analyses both the MMPP and the MTCP (without using these terms, as such), see [175].

Lemma 3.3.5. *Any MAP$_p$ with diagonal matrix $C = -diag(c_i)$ for $i = 1, \cdots, p$ [3], satisfies*

$$1 \leq c^2 \leq 2\frac{\kappa^2}{\gamma^2} - 1,$$

where $\kappa = \frac{\min c_i + \max c_i}{2}$ and $\gamma = \sqrt{(\min c_i)(\max c_i)}$.

Proof. From Eq. (3.2.5), we have $c^2 + 1 = 2(\pi C \mathbf{1} \pi C^{-1} \mathbf{1})$. For $C = -\text{diag}(c_i)$, we have

[3] This is the case for MTCP, H-renewal process, and MSPP.

$C^{-1} = -\text{diag}(\frac{1}{c_i})$ and so,

$$\pi C \mathbf{1} \pi C^{-1} \mathbf{1} = \big(\sum_{i=1}^{p} \pi_i c_i\big)\big(\sum_{i=1}^{p} \pi_i \frac{1}{c_i}\big).$$

On the other hand, from the Cauchy-Schwarz inequality, we have:

$$\Big[\sum_{i=1}^{p} \pi_i \, (c_i)^{\frac{1}{2}} \, (\frac{1}{c_i})^{\frac{1}{2}}\Big]^2 \leq \big(\sum_{i=1}^{p} \pi_i c_i\big)\big(\sum_{i=1}^{p} \pi_i \frac{1}{c_i}\big).$$

Since for all $i = 1, \cdots, p$, we have $c_i \, \frac{1}{c_i} = 1$ and $\sum_{i=1}^{p} \pi_i = 1$, we can conclude:

$$1 \leq \big(\sum_{i=1}^{p} \pi_i c_i\big)\big(\sum_{i=1}^{p} \pi_i \frac{1}{c_i}\big). \tag{3.3.4}$$

Now, consider that $m := \min c_i$ and $M := \max c_i$. Without loss of generality and by multiplying m, M, and all c_i, $i = 1, \cdots, p$ by a positive constant, we can consider that $M = \frac{1}{m}$ (or $\gamma = 1$). So, we have for $i = 1, \cdots, p$: $c_i + \frac{1}{c_i} \leq m + \frac{1}{m} = 2\kappa$ which results in (using the fact that $\sum_{i=1}^{p} \pi_i = 1$):

$$\Big(\sum_{i=1}^{p} \pi_i c_i\Big) + \Big(\sum_{i=1}^{p} \pi_i \frac{1}{c_i}\Big) = \sum_{i=1}^{p} \pi_i \big(c_i + \frac{1}{c_i}\big) \leq 2\kappa.$$

So, since the arithmetic mean of non-negative real valued numbers is always greater than their geometric mean, we have:

$$\Big[\Big(\sum_{i=1}^{p} \pi_i c_i\Big)\Big(\sum_{i=1}^{p} \pi_i \frac{1}{c_i}\Big)\Big]^{\frac{1}{2}} \leq \frac{\Big(\sum_{i=1}^{p} \pi_i c_i\Big) + \Big(\sum_{i=1}^{p} \pi_i \frac{1}{c_i}\Big)}{2} \leq \kappa.$$

Therefore,

$$\Big(\sum_{i=1}^{p} \pi_i c_i\Big)\Big(\sum_{i=1}^{p} \pi_i \frac{1}{c_i}\Big) \leq \kappa^2.$$

The above inequality is known as the Kantorovich's Inequality (see [186]) and is written for $\gamma \neq 1$ as:

$$\big(\sum_{i=1}^{p} \pi_i c_i\big)\big(\sum_{i=1}^{p} \pi_i \frac{1}{c_i}\big) \leq \frac{\kappa^2}{\gamma^2}. \tag{3.3.5}$$

Now, from (3.3.4) and (3.3.5), we have:

$$1 \leq \big(\sum_{i=1}^{p} \pi_i c_i\big)\big(\sum_{i=1}^{p} \pi_i \frac{1}{c_i}\big) \leq \frac{\kappa^2}{\gamma^2}, \qquad \text{or} \qquad 1 \leq \frac{c^2+1}{2} \leq \frac{\kappa^2}{\gamma^2},$$

which gives the result. □

Lemma 3.3.6. *For a MAP with diagonal matrix C, $d^2 \geq 1$.*

Proof. From (3.2.13), the asymptotic variance of the counting process of a MAP is given by

$$\overline{V} = \lim_{t\to\infty} \frac{\mathrm{Var}\big(N(t)\big)}{t} = \pi D\mathbf{1} + 2\,\pi D D_Q^\sharp D\mathbf{1}.$$

The above formula and the fact that for a given MMPP, we have $d^2 \geq 1$, result in:

$$\pi D D_Q^\sharp D\mathbf{1} \geq 0, \quad \text{for any diagonal matrix } D. \tag{3.3.6}$$

On the other hand, since $Q = C + D$ and $D_Q^\sharp Q\mathbf{1} = 0$, we have $D_Q^\sharp D\mathbf{1} = -D_Q^\sharp C\mathbf{1}$. Further, $\pi Q D_Q^\sharp = 0$ or equivalently, $\pi D D_Q^\sharp = -\pi C D_Q^\sharp$. So, all MAPs satisfy:

$$\pi D D_Q^\sharp D\mathbf{1} = \pi C D_Q^\sharp C\mathbf{1}.$$

Therefore, since C is a diagonal matrix, from (3.3.6) we have:

$$d^2 = \frac{\overline{V}}{\pi D\mathbf{1}} = 1 + 2\frac{\pi D D_Q^\sharp D\mathbf{1}}{\pi D\mathbf{1}} \geq 1.$$

□

Example 3.3.7. *For a given MTCP$_2$ with parameters* $D = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$, *from* (3.2.5)*, we have:*

$$c^2 = 1 + 2\frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}.$$

Moreover, using (3.2.12) *and* (3.2.13) *results in:*

$$d^2 = 1 + \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}.$$

H-Renewal Process: Remind that any PH-renewal process is characterised by the parameters (η, T), where η and T are the initial distribution and transition rate matrix of the PH distribution, respectively. Moreover, any PH-renewal process can be considered as a MAP with parameters $C = T$ and $D = -T\mathbf{1}\eta$ (see Subsection 2.3.2). For PH-renewal processes there is no possibility to define correlated patterns and from (3.2.7), we have $d^2 = c^2$. This comes from the fact that whenever there is an arrival, the new initial phase is selected ac-

cording to the initial distribution of the PH distribution, η. Thus the new phase is independent of the past, and we can not define correlated patterns.

The hyperexponential distribution, as a PH distribution, is the distribution of the mixture of independent exponentially distributed random variables. One of the characterisations of the hyperexponential distribution is that its transition rate matrix T is diagonal. Now if we consider that the time between inter-arrival times of a point process has hyperexponential distribution, that is, whenever the point process goes to the absorption state, the process restarts from a state that is chosen according to the initial distribution of the hyperexponential distribution, then we have a H-renewal process. Using the formulas of mean and variance of H_p distribution, we have:

$$d^2 = c^2 = \frac{\left(\sum_{i=1}^{p} \frac{\eta_i}{\lambda_i}\right)^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} \eta_i\eta_j\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j}\right)^2}{\left(\sum_{i=1}^{p} \frac{\eta_i}{\lambda_i}\right)^2} = 1 + \frac{\sum_{i=1}^{p}\sum_{j=1}^{p} \eta_i\eta_j\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_j}\right)^2}{\left(\sum_{i=1}^{p} \frac{\eta_i}{\lambda_i}\right)^2}. \quad (3.3.7)$$

While this formula is simple, to the best of our knowledge it has not appeared elsewhere. Its virtue is that it immediately shows that $d^2 = c^2 \geq 1$. Here, λ_i's are rates of exponential distributions constructing the H distribution. Formulas of moments of the general case of Markov renewal process can be found in [97].

Example 3.3.8. *Consider a H_2-renewal process with λ_1, λ_2 and η_1 as the parameters of a H_2 and*

$$f_{H_2}(t) = \eta_1\lambda_1 e^{-\lambda_1 t} + (1-\eta_1)\lambda_2 e^{-\lambda_2 t},$$

as its density function, where we set $\eta = (\eta_1, 1-\eta_1)$. Then, the matrices of the H_2-renewal process are $C = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1\eta_1 & \lambda_1(1-\eta_1) \\ \lambda_2\eta_1 & \lambda_2(1-\eta_1) \end{pmatrix}$ and we have:

$$d^2 = c^2 = 1 + \frac{2\eta_1(1-\eta_1)(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})^2}{(\frac{\eta_1}{\lambda_1} + \frac{1-\eta_1}{\lambda_2})^2}.$$

As we can see, c^2 and d^2 are always greater than 1 and by assuming that $\lambda_1 = \lambda_2$ or $\eta_1 = 0$ or $\eta_1 = 1$, we have $c^2 = d^2 = 1$. Further, if we consider that $\lambda_1 \to \infty$, then the above value goes to a constant $1 + \frac{2\eta_1}{1-\eta_1}$. Now, if η_1 increases to 1, $c^2 \to \infty$. The same happens when $\lambda_2 \to \infty$ and then $\eta_1 \to 0$. Therefore, the lower bound for c^2 or d^2 is 1 but there is no upper bound for these values.

It is known that, see [70], an MMPP$_2$ is a renewal process if and only if

- the arrival rate takes on alternatively the values $\lambda > 0$ and 0,
- its inter-arrival times have exponential distribution (related to the times during which $\lambda > 0$).

The simplest example of an MMPP that can be considered as a renewal process is the interrupted Poisson process (IPP). The IPP switches between On and Off phases and whenever the process is in phase On (Off) it remains there for an exponentially distributed time. When the process is in phase On, the arrival occurs with Poisson rate λ and when the process goes to the phase Off, there is no arrival. Therefore matrix D has either λ or 0 in its diagonal:

$$C = \begin{pmatrix} -\sigma_1 - \lambda & \sigma_1 \\ \sigma_2 & -\sigma_2 \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix},$$

where σ_i for $i = 1, 2$ is the transition rate of background CTMC. and the relations between parameters of IPP and H$_2$ are as follows.
IPP $\longrightarrow$ H$_2$:

$$\eta_1 = \frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2}, \qquad \lambda_i = \frac{1}{2}\big(\lambda + \sigma_1 + \sigma_2 \pm \sqrt{(\lambda + \sigma_1 + \sigma_2)^2 - 4\lambda\sigma_2}\big) \text{ for } i = 1, 2.$$

H$_2$ $\longrightarrow$ IPP:

$$\lambda = \eta_1\lambda_1 + (1 - \eta_1)\lambda_2, \qquad \sigma_1 = \frac{\eta_1(1 - \eta_1)(\lambda_1 - \lambda_2)^2}{\lambda}, \qquad \sigma_2 = \frac{\lambda_1\lambda_2}{\lambda}.$$

In general, we have:

Proposition 3.3.9. *Any MAP$_p$ with a rank one matrix D can be represented as a H$_p$-renewal process.*

Proof. Whenever the matrix D of a MAP$_p$ is of rank one (like IPP case), we can decompose it as $D = \mathbf{vu}$ where $\mathbf{v}$ is a column vector of dimension p and $\mathbf{u}$ is a row vector of dimension p. On the other hand, for a H$_p$-renewal process with parameters (η, C, D), we have $D = -C\mathbf{1}\,\eta$. Now, if we set $\eta = \frac{\mathbf{u}}{\|\mathbf{u}\|}$ and $C = \text{diag}\big(-\mathbf{v}\|\mathbf{u}\|\big)$, then $D = \mathbf{vu} = -C\mathbf{1}\,\eta$. So, we have a H$_p$-renewal process representation of the given MAP$_p$. □

It is easy to verify that the MTCP$_p$ is a natural generalisation of the H$_p$-renewal process. Consider an MTCP$_p$ where the sojourn time in each phase i (of the background CTMC) has an exponential distribution with parameter λ_i for $i = 1, \cdots, p$. The representation of an MTCP$_p$ as a MAP$_p$ includes $\pi = (\pi_1, \cdots, \pi_p)$, $C = \text{diag}(-\lambda_i)$, and the matrix D has non-negative elements such that $D\mathbf{1} = (\lambda_1, \cdots, \lambda_p)'$. Therefore, the distribution of inter-arrival times is $f(t) = \pi e^{Ct} D\mathbf{1} = \sum_{i=1}^{p} \pi_i \lambda_i e^{-\lambda_i t}$ which is the density function of H$_p$. Here, note that the matrix D of MTCP$_p$ is not necessarily of rank one. In fact, in the MTCP$_p$, times between arrivals are H$_P$-distributed. But, they are not necessarily independent.

MSPP: Another special case of MAPs that we discuss here as a bursty MAP is the Markov switched Poisson process (MSPP) which is a MAP with diagonal matrix C. For MSPP$_p$, arrivals switch between p Poisson processes with arrival rates $\lambda_1, \cdots, \lambda_p$. After each arrival, the MAP may switch to another Poisson process (where the phase is changed) or stay in the same Poisson process (where the phase remains the same). We also remark that the modulation in the MSPP is of a discrete nature and it occurs at arrival epochs, whereas the modulation of the MMPP is performed in continuous time. See [7] and [83].

As a result of Lemma 3.3.5 and Lemma 3.3.6, we have:

Corollary 3.3.10. *An MSPP is a bursty MAP, that is for an MSPP $c^2 \geq 1$ and $d^2 \geq 1$.*

Using the moments' formulas for the MSPP results in the following.

Proposition 3.3.11. *Moments and autocorrelation functions of the inter-arrival process of an event-stationary MSPP$_p$ are given by:*

$$M_1 = \sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i}, \qquad M_2 = 2 \sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i^2},$$

$$\text{Var}(T) = 2 \sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i^2} - \Big(\sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i}\Big)^2, \quad \varrho_j = \frac{\sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i^{j+2}} d_i^j - \Big(\sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i}\Big)^2}{2 \sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i^2} - \Big(\sum_{i=1}^{p} \frac{\alpha_i}{\lambda_i}\Big)^2}.$$

Here, the initial distribution of the event-stationary MSPP$_p$ is α (where $\alpha P = \alpha$), $C =$ diag$(-\lambda_i)$ for $i = 1, \cdots, p$ and in the last equation d_i^j is the sum of $i-$th row elements of the matrix D^j.

Note that as MTCP and H-renewal process are special cases of MSPP, all of the above mo-

ments formulas are true for them as well.

Furthermore, we can write the matrix D of an MSPP as the sum of two matrices corresponding to an MMPP and an MTCP, that is $D = D_1 + D_2$, where D_1 is diagonal and has diagonal elements of D (like an MMPP) and D_2 has off-diagonal elements of D and zero on its diagonal (like an MTCP). Then, define $C_1 = Q - D_1$ and $C_2 = Q - D_2$. Note that from the construction of D_2, C_2 is a diagonal matrix, and we have:

$$d^2 = \frac{\overline{V}_1 + \overline{V}_2 + \mathcal{V}}{\lambda_1^* + \lambda_2^*},$$

where $\overline{V}_1$ and $\overline{V}_2$ are asymptotic variances of counting process for associated MMPP (with matrices D_1 and C_1) and MTCP (with matrices D_2 and C_2), respectively. Here, we set $\mathcal{V} = 2\pi\big(D_1 D_Q^\sharp D_2 + D_2 D_Q^\sharp D_1\big)\mathbf{1}$. Using the fact that for MSPP, MMPP and MTCP $d^2 = \frac{\overline{V}}{\lambda^*} \geq 1$, we have

$$\frac{\frac{d_1^2}{\lambda_2^*} + \frac{d_2^2}{\lambda_1^*} + \frac{\mathcal{V}}{\lambda_1^*\lambda_2^*}}{\frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}} \geq \frac{\frac{1}{\lambda_2^*} + \frac{1}{\lambda_1^*} + \frac{\mathcal{V}}{\lambda_1^*\lambda_2^*}}{\frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}} = 1 + \frac{\frac{\mathcal{V}}{\lambda_1^*\lambda_2^*}}{\frac{1}{\lambda_1^*} + \frac{1}{\lambda_2^*}} \geq 1.$$

The above relation implies that $\mathcal{V} \geq 0$ or $\pi\big(D_1 D_Q^\sharp D_2 + D_2 D_Q^\sharp D_1\big)\mathbf{1} \geq 0$, where D_1 and D_2 are event intensity matrices of the associated MMPP and MTCP, respectively. In the next section, we will present more relations between the matrices and moments of these processes.

3.4 Moment Relationships Between MTCP and MMPP

Proposition 3.2 of [147], implies that every MTCP has an associated MMPP with the same first two moments. We present this proposition in an alternative form here:

Proposition 3.4.1. *[147] Let $\overline{N}(t)$ be the counting process of a time-stationary MTCP$_p$. Then there is an MMPP$_p$, with the counting process $\tilde{N}(t)$, such that their first and second moments are matched. That is, for all $t \geq 0$,*

$$\mathbb{E}[\tilde{N}^k(t)] = \mathbb{E}[\overline{N}^k(t)]\,, \qquad \textit{for} \quad k = 1, 2\,.$$

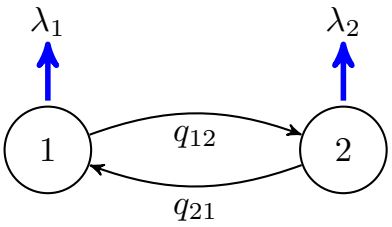
Proof. Assume that the matrix D of the MTCP$_p$ is given by $\overline{D} = Q - \text{diag}(Q)$, where Q is the Q-matrix of the background CTMC. We can construct an MMPP$_p$ with the same background CTMC by setting $\tilde{D} = -\text{diag}(Q)$. From (3.2.12) and (3.2.13), if we show that $\overline{D}\mathbf{1} = \tilde{D}\mathbf{1}$ and

$\pi\overline{D} = \pi\tilde{D}$, these processes have the same first two moments and proof is completed. Since $Q\mathbf{1} = 0$ and $\pi Q = \mathbf{0}'$ the result follows. □

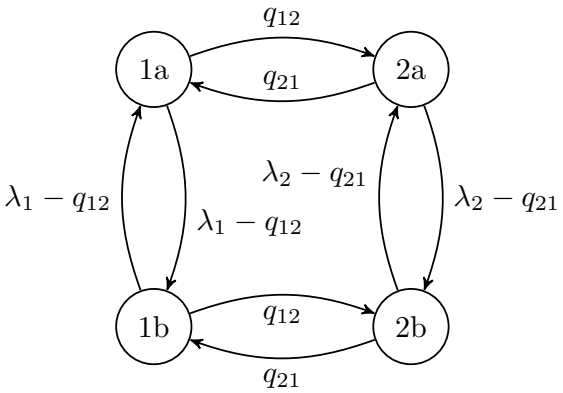
The proof shows that in order to construct an MMPP matching the first two moments with an MTCP with the same Q-matrix Q, we need to set $\tilde{D} = \text{diag}(\lambda)$ and $\tilde{C} = Q - \tilde{D}$, where $\lambda := (\lambda_1, \cdots, \lambda_p)' = -\text{diag}(Q)$.

Now the question is that can we construct an MTCP matching the first two moments with a given MMPP? Based on the above proposition, the answer is given for the special case of MMPPs where $\lambda = -\text{diag}(\tilde{Q})$, that is $\lambda_i = \sum_{j \neq i} \tilde{q}_{ij}$. But this is a very restricted case since it does not leave any freedom with λ_i. We now show that for each instance of a class of MMPPs, where $\lambda_i > \sum_{j \neq i} \tilde{q}_{ij}$ which we call "slow MMPPs", there is an associated MTCP that exhibits the same first and second moments for the counting process $N(t)$.

Definition 3.4.2. *A **slow Markov modulated Poisson process (slow MMPP)** is an MMPP where the arrival rate in any phase i is greater than the total rate of leaving that phase, that is $\lambda_i > \sum_{j \neq i} q_{ij}$.*



(a) Transition diagram of the phase process of an MMPP_2.



(b) Transition diagram of the phase process of related MTCP_4.

Figure 3.1: An MMPP_2 and its associated MTCP_4.

We can associate an MTCP_{2p} to any slow MMPP_p as illustrated in Figure 3.1 for the case of $p = 2$. As the figure shows, if the transition rate matrix and the event intensity matrix of the slow MMPP_2 are given by:

$$\tilde{Q} = \begin{pmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Then, the corresponding matrices for associated MTCP$_4$ are:

$$\overline{Q} = \begin{array}{c} \\ 1_a \\ 1_b \\ \\ 2_a \\ 2_b \end{array} \begin{array}{c} \begin{array}{cccc} 1_a & 1_b & 2_a & 2_b \end{array} \\ \left(\begin{array}{cc|cc} -\lambda_1 & \lambda_1 - q_{12} & q_{12} & 0 \\ \lambda_1 - q_{12} & -\lambda_1 & 0 & q_{12} \\ \hline q_{21} & 0 & -\lambda_2 & \lambda_2 - q_{21} \\ 0 & q_{21} & \lambda_2 - q_{21} & -\lambda_2 \end{array}\right) \end{array}, \quad \overline{D} = \overline{Q} - \text{diag}(\overline{Q}). \tag{3.4.1}$$

We can generalise this construction from $p = 2$ to an arbitrary p. Here, given an MMPP$_p$, we construct an MTCP$_{2p}$ with the transition rate and the event intensity matrices:

$$\overline{Q} = \begin{pmatrix} \overline{\Lambda}_1 & H_{12} & \cdots & H_{1p} \\ H_{21} & \overline{\Lambda}_2 & \cdots & H_{2p} \\ \cdots & \cdots & \ddots & \cdots \\ H_{p1} & H_{p2} & \cdots & \overline{\Lambda}_p \end{pmatrix}, \qquad \overline{D} = \begin{pmatrix} \overline{D}_1 & H_{12} & \cdots & H_{1p} \\ H_{21} & \overline{D}_2 & \cdots & H_{2p} \\ \cdots & \cdots & \ddots & \cdots \\ H_{p1} & H_{p2} & \cdots & \overline{D}_p \end{pmatrix}, \tag{3.4.2}$$

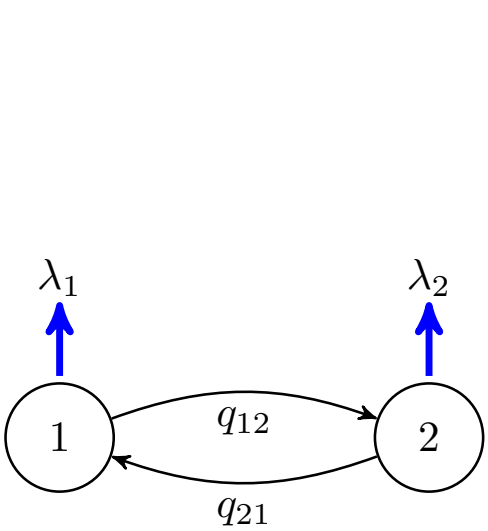
where $\overline{\Lambda}_i$, $\overline{D}_i$, and H_{ij} for $i \neq j$ and $i, j = 1, \cdots, p$ are 2×2 matrices given by

$$\overline{\Lambda}_i = \begin{pmatrix} -\lambda_i & \lambda_i - S_i \\ \lambda_i - S_i & -\lambda_i \end{pmatrix}, \qquad \overline{D}_i = \begin{pmatrix} 0 & \lambda_i - S_i \\ \lambda_i - S_i & 0 \end{pmatrix}, \qquad H_{ij} = \begin{pmatrix} q_{ij} & 0 \\ 0 & q_{ij} \end{pmatrix}.$$

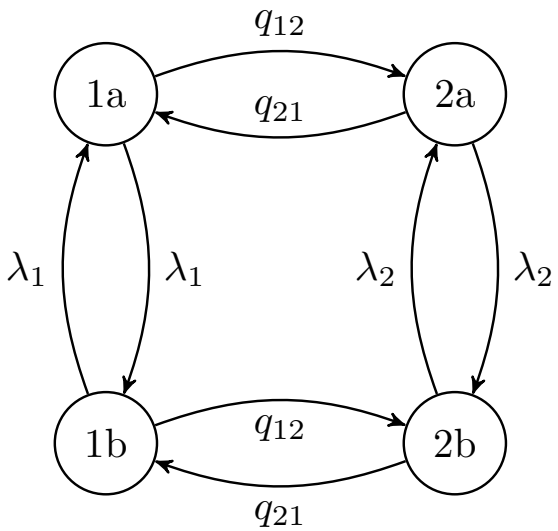
Notice that $\overline{D} = \overline{Q} - \text{diag}(\overline{Q})$. Moreover, we use the notation $S_i = \sum_{j \neq i} q_{ij}$. We prove that for an arbitrary order p (≥ 2), the counting processes of the initial time-stationary slow MMPP$_p$ and its associated MTCP$_{2p}$ exhibit the same first and second moments. Specifically, for any initial phase distribution, their IDCs are the same.

To compare properties of the MMPP$_p$ and the MTCP$_{2p}$, we construct a MAP$_{2p}$ with the same counting process as the MMPP$_p$. This is for comparing processes with the same number of phases and is done by coupling the events of the phase process of the MMPP$_p$. When the process is in phase k, coupling events results in a transition from phase k_a to k_b or vice versa. Figure 3.2 shows this for the case of $p = 2$.

This is nothing but a common modelling way to describe “self-transitions” in a CTMC. Note though that the MAP$_4$ is not an MTCP. We denote the phase transition matrix of the resulting



(a) Transition diagram of the phase process of an MMPP$_2$.



(b) Transition diagram of the phase process of the coupled MAP$_4$.

Figure 3.2: Construction of a MAP$_4$ from a given MMPP$_2$ by coupling.

MAP$_{2p}$ by $\tilde{Q}$ and its event intensity matrix by $\tilde{D}$. Then, for the case of $p = 2$, we have:

$$\tilde{Q} = \begin{array}{c} \\ 1_a \\ 1_b \\ \\ 2_a \\ 2_b \end{array} \begin{array}{c} \begin{array}{cccc} 1_a & 1_b & 2_a & 2_b \end{array} \\ \left(\begin{array}{cc|cc} -(\lambda_1+q_{12}) & \lambda_1 & q_{12} & 0 \\ \lambda_1 & -(\lambda_1+q_{12}) & 0 & q_{12} \\ \hline q_{21} & 0 & -(\lambda_2+q_{21}) & \lambda_2 \\ 0 & q_{21} & \lambda_2 & -(\lambda_2+q_{21}) \end{array}\right) \end{array}, \tag{3.4.3}$$

and

$$\tilde{D} = \begin{array}{c} \\ 1_a \\ 1_b \\ \\ 2_a \\ 2_b \end{array} \begin{array}{c} \begin{array}{cccc} 1_a & 1_b & 2_a & 2_b \end{array} \\ \left(\begin{array}{cc|cc} 0 & \lambda_1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & \lambda_2 & 0 \end{array}\right) \end{array}. \tag{3.4.4}$$

Carrying out this process for an arbitrary value of p, the transition rate matrix and the event

intensity matrix are given by:

$$\tilde{Q} = \begin{pmatrix} \tilde{\Lambda}_1 & H_{12} & \cdots & H_{1p} \\ H_{21} & \tilde{\Lambda}_2 & \cdots & H_{2p} \\ \cdots & \cdots & \ddots & \cdots \\ H_{p1} & H_{p2} & \cdots & \tilde{\Lambda}_p \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} \tilde{D}_1 & 0 & \cdots & 0 \\ 0 & \tilde{D}_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \tilde{D}_p \end{pmatrix}, \tag{3.4.5}$$

where $\tilde{\Lambda}_i$ and $\tilde{D}_i$ are 2×2 matrices given by

$$\tilde{\Lambda}_i = \begin{pmatrix} -(\lambda_i + S_i) & \lambda_i \\ \lambda_i & -(\lambda_i + S_i) \end{pmatrix}, \qquad \tilde{D}_i = \begin{pmatrix} 0 & \lambda_i \\ \lambda_i & 0 \end{pmatrix}.$$

Now, we can compare these two processes with the same dimension. Define the transition rate matrix G with

$$G_{ij} = q_{ij} \quad \text{for} \quad i \neq j, \qquad \text{and} \quad G_{ii} = -\sum_{j \neq i} q_{ij},$$

and let $\mathbf{1} = (1\,,\,1)'$. Then, we have the following lemma.

Lemma 3.4.3. *For all $p \geq 2$ the following relations hold:*

(i) For any $k \geq 0$: $\quad (I_p \otimes \mathbf{1}')\,\tilde{Q}^k = (I_p \otimes \mathbf{1}')\,\overline{Q}^k = G^k\,(I_p \otimes \mathbf{1}')$,

and $\quad \tilde{Q}^k\,(I_p \otimes \mathbf{1}) = \overline{Q}^k\,(I_p \otimes \mathbf{1}) = (I_p \otimes \mathbf{1})\,G^k$,

where $\otimes$ denotes the Kronecker product[4].

(i)′ For any $t \geq 0$: $\quad (I_p \otimes \mathbf{1}')\,e^{\tilde{Q}t} = (I_p \otimes \mathbf{1}')\,e^{\overline{Q}t} = e^{Gt}\,(I_p \otimes \mathbf{1}')$,

and $\quad e^{\tilde{Q}t}\,(I_p \otimes \mathbf{1}) = e^{\overline{Q}t}\,(I_p \otimes \mathbf{1}) = (I_p \otimes \mathbf{1})\,e^{Gt}$,

(ii) Both $\tilde{Q}$ and $\overline{Q}$ have the same stationary distribution π which can be written as:

$$\pi = \frac{1}{2}\vartheta\,(I_p \otimes \mathbf{1}')\,,$$

where ϑ is the stationary distribution of G.

[4] The Kronecker product of an $m \times n$ matrix A with a $p \times q$ matrix B is the $mp \times nq$ block matrix defined by:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \cdots & \ddots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

(iii) $\pi\tilde{D} = \pi\overline{D} = \frac{1}{2}\,\vartheta$ *diag*$(\lambda)\,(I_p \otimes \mathbf{1}')$, *where* $\lambda = (\lambda_1, \cdots, \lambda_p)'$.

(iv) $\tilde{D}\mathbf{1} = \overline{D}\mathbf{1} = (I_p \otimes \mathbf{1})\,\lambda$.

(v) $(I_p \otimes \mathbf{1}')\,D^{\sharp}_{\tilde{Q}}\,(I_p \otimes \mathbf{1}) = (I_p \otimes \mathbf{1}')\,D^{\sharp}_{\overline{Q}}\,(I_p \otimes \mathbf{1}) = 2D^{\sharp}_{G}$.

(v)′ $D^{\sharp}_{\overline{Q}}\,(I_p \otimes \mathbf{1}) = D^{\sharp}_{\tilde{Q}}\,(I_p \otimes \mathbf{1})$ *and* $(I_p \otimes \mathbf{1}')\,D^{\sharp}_{\overline{Q}} = (I_p \otimes \mathbf{1}')\,D^{\sharp}_{\tilde{Q}}$.

(vi) $D^{\sharp}_{\overline{Q}}\,\overline{D}\mathbf{1} = D^{\sharp}_{\tilde{Q}}\,\tilde{D}\mathbf{1}$, *and* $D^{\sharp}_{\overline{Q}}(t)\overline{D}\mathbf{1} = D^{\sharp}_{\tilde{Q}}(t)\tilde{D}\mathbf{1}$.

(vii) $\pi\overline{D}D^{\sharp}_{\overline{Q}} = \pi\tilde{D}D^{\sharp}_{\tilde{Q}}$.

(viii) $(I_p \otimes \mathbf{1}')\,D^{\sharp}_{\tilde{Q}}\,D^{\sharp}_{\tilde{Q}}(t)\,(I_p \otimes \mathbf{1}) = (I_p \otimes \mathbf{1}')\,D^{\sharp}_{\overline{Q}}\,D^{\sharp}_{\overline{Q}}(t)\,(I_p \otimes \mathbf{1}) = 2D^{\sharp}_{G}\,D^{\sharp}_{G}(t)$.

Proof. The proof of (i) follows from the mathematical induction and the structure of matrices in (3.4.2) and (3.4.5). For $k = 0$, the relations are obvious. For $k = 1$, it is easy to check that multiplying both $\tilde{Q}$ and $\overline{Q}$ from the left gives the same result. Further, for instance, if we consider that $(I_p \otimes \mathbf{1}')\tilde{Q}^k = G^k(I_p \otimes \mathbf{1}')$, then, for $k+1$ we have:

$$\begin{aligned}(I_p \otimes \mathbf{1}')\tilde{Q}^{k+1} &= \Big((I_p \otimes \mathbf{1}')\,\tilde{Q}^k\Big)\tilde{Q}\\ \text{(by induction assumption)} \quad &= \Big(G^k(I_p \otimes \mathbf{1}')\Big)\tilde{Q} = G^k\Big((I_p \otimes \mathbf{1}')\tilde{Q}\Big)\\ \text{(by induction assumption)} \quad &= G^k\Big(G(I_p \otimes \mathbf{1}')\Big) = G^{k+1}(I_p \otimes \mathbf{1}').\end{aligned}$$

The other relations can be proved similarly.

The proof of (i)′ is a consequence of (i) and the fact that for any matrix Q: $e^{Qt} = \sum_{k=0}^{\infty} \frac{(Qt)^k}{k!}$. For part (ii), we need to show that $\pi\tilde{Q} = \pi\overline{Q} = 0$ and $\pi\mathbf{1}_p = 1$, where $\mathbf{1}_p$ is a p-dimensional column vector of ones. First, from part (i), we have

$$\pi\tilde{Q} = \frac{1}{2}\vartheta\,(I_p \otimes \mathbf{1}')\,\tilde{Q} = \frac{1}{2}\vartheta\,G\,(I_p \otimes \mathbf{1}') = 0.$$

The last equality comes from the fact that ϑ is the stationary distribution of G. Moreover, we have

$$\pi\mathbf{1}_p = \tfrac{1}{2}\vartheta\,(I_p \otimes \mathbf{1}')\,\mathbf{1}_p = \tfrac{1}{2}\vartheta \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix} = \vartheta \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 1.$$

Again, the last equality comes from the fact that ϑ is a distribution. The proof for the case of $\overline{Q}$ follows the same line.

By considering part (ii) and the structure of matrices $\overline{D}$ and $\tilde{D}$ in (3.4.2) and (3.4.5), proofs of (iii) and (iv) are obvious.
For part (v), we use the definition of the deviation matrix in (3.2.15):

$$
\begin{aligned}
& (I_p \otimes \mathbf{1}')\, D^{\sharp}_{\tilde{Q}}\, (I_p \otimes \mathbf{1}) \\
&= \int_0^{\infty} (I_p \otimes \mathbf{1}')\, (e^{\tilde{Q}u} - \mathbf{1}_p \pi)\, (I_p \otimes \mathbf{1})\; du \\
&= \int_0^{\infty} \Big((I_p \otimes \mathbf{1}')\, e^{\tilde{Q}u}\, (I_p \otimes \mathbf{1}) - (I_p \otimes \mathbf{1}')\, \mathbf{1}_p \pi\, (I_p \otimes \mathbf{1}) \Big)\; du \\
\big(\text{by part } (i)' \text{and part } (ii)\big) \quad &= \int_0^{\infty} \left(e^{Gu}\, (I_p \otimes \mathbf{1}')\, (I_p \otimes \mathbf{1}) - (I_p \otimes \mathbf{1}')\, \mathbf{1}_p\, \frac{1}{2} \vartheta\, (I_p \otimes \mathbf{1}')\, (I_p \otimes \mathbf{1}) \right) du \\
&= 2 \int_0^{\infty} (e^{Gu} - \mathbf{1}_p \vartheta)\; du \\
&= 2 D^{\sharp}_{G}\,.
\end{aligned}
$$

In the proof, we used the formula $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ (when matrix dimensions agree for the multiplication) to show that:

$$(I_p \otimes \mathbf{1}')(I_p \otimes \mathbf{1}) = (I_p \otimes 2) = 2 I_p.$$

Then, since $(I_p \otimes \mathbf{1}')\mathbf{1}_p = 2\,\mathbf{1}_p$, we have the result. The proof for the case of $\overline{Q}$ follows the same lines and the proof of (v)′ is a corollary of the proof of part (v). The proofs of parts (vi) and (vii) are the result of parts (iii), (iv) and (v)′.

For part (viii), from (3.2.15) and (3.2.8), we have:

$$
\begin{aligned}
(I_p \otimes \mathbf{1}')\, D^{\sharp}_{\tilde{Q}} D^{\sharp}_{\tilde{Q}}(t)\, (I_p \otimes \mathbf{1}) &= \textstyle\int_0^{\infty} \int_0^{t} (I_p \otimes \mathbf{1}')\, (e^{\tilde{Q}(u+v)} - \mathbf{1}_p \pi)\, (I_p \otimes \mathbf{1})\; dv\, du \\
&= 2 \textstyle\int_0^{\infty} \int_0^{t} (e^{G(u+v)} - \mathbf{1}_p \vartheta)\, dv\, du \\
&= 2 D^{\sharp}_{G} D^{\sharp}_{G}(t),
\end{aligned}
$$

where in addition to the previous relations, in the first step we use the fact that for any transition rate matrix Q with stationary distribution π, we have:

$$e^{Qt}\mathbf{1}_p = \mathbf{1}_p \text{ (since } e^{Qt} \text{ is stochastic)}, \qquad \pi e^{Qt} = \pi.$$

Therefore, we have: $\mathbf{1}_p \vartheta\, \mathbf{1}_p \vartheta = \mathbf{1}_p \vartheta e^{Gv} = e^{Gu} \mathbf{1}_p \vartheta = \mathbf{1}_p \vartheta$, The proof for $\overline{Q}$ follows the same lines. □

Having the above lemma, we can prove that there exists an MTCP corresponding to a given slow MMPP with the same first two moments (but not necessarily the equivalent third moments). This generalises the result of Proposition 3.4.1 to a wider class of MMPPs. Further, for modelling a bursty process, gives the practitioner freedom to apply MTCPs instead of slow MMPPs.

Theorem 3.4.4. *Let $\tilde{N}(t)$ and $\overline{N}(t)$ be the counting processes of a time-stationary slow MMPP$_p$ and its associated MTCP$_{2p}$, respectively. Then, these processes have the same first and second moment. That is, $\forall p \geq 2$ and $\forall t \geq 0$,*

$$\mathbb{E}[\tilde{N}^k(t)] = \mathbb{E}[\overline{N}^k(t)]\,, \qquad \textit{for} \quad k = 1, 2\,.$$

Further, $\tilde{N}(t)$ and $\overline{N}(t)$ have different third moments.

A proof for the case $p = 2$ appears in [9] and here we present the general proof.

Proof. First, note that from (3.2.12) and part (iii) of Lemma 3.4.3 we have

$$\mathbb{E}[\tilde{N}(t)] = \mathbb{E}[\overline{N}(t)].$$

Then for the variance, the proof is straightforward by using Eq. (3.2.13) and parts (iii), (vi) and (viii) of Lemma 3.4.3.

For the second part, consider the MAP's third moment formula given by (3.2.14). From parts (iii), (iv) and (i)′ of Lemma 3.4.3, we see that the first term in integral (3.2.14) is the same for both processes. The second term in integral (3.2.14) for an MTCP, by applying parts (iii) and (iv) of Lemma 3.4.3 and the definition of the transient deviation matrix, can be written as:

$$\begin{aligned}
&\int_0^s \vartheta\, \mathrm{diag}(\lambda)\, (I_p \otimes \mathbf{1}')\, e^{\overline{Q}(u-s)}\overline{D}\big(e^{\overline{Q}v} - \mathbf{1}\pi\big)\, (I_p \otimes \mathbf{1})\, \lambda\, dv. \\
&= \int_0^s \vartheta\, \mathrm{diag}(\lambda)\, \Big((I_p \otimes \mathbf{1}')\, e^{\overline{Q}(u-s)}\overline{D}e^{\overline{Q}v}\, (I_p \otimes \mathbf{1}) - (I_p \otimes \mathbf{1}')\, e^{\overline{Q}(u-s)}\overline{D}\mathbf{1}\pi\, (I_p \otimes \mathbf{1})\Big)\, \lambda\, dv.
\end{aligned}$$

By using parts (i)′ and (iv) of Lemma 3.4.3, we see that second term of this integral is the same for MMPP and its associated MTCP. For the first term, if we write $e^{\overline{Q}k}$ in terms of series

and put aside the common terms or scalars, we have the following term.

$$(I_p \otimes \mathbf{1}')\,\overline{Q}^k \overline{D}\,\overline{Q}^l\,(I_p \otimes \mathbf{1})\,, \quad \forall k, l \geq 0. \tag{3.4.6}$$

Using part (i) of Lemma 3.4.3, the above term can be written as:

$$G^k\,(I_p \otimes \mathbf{1}')\,\overline{D}\,(I_p \otimes \mathbf{1})\,G^l.$$

If we rewrite $\overline{D}$ and $\tilde{D}$ from (3.4.2) and (3.4.5) in terms of Kronecker products, we have:

$$\overline{D} = \overline{Q} + (\mathsf{diag}(\lambda) \otimes I_2)\,, \qquad \tilde{D} = \mathsf{diag}(\lambda) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.4.7}$$

It is easy to verify that $(I_p \otimes \mathbf{1}')\tilde{D} = (I_p \otimes \mathbf{1}')\big(\mathsf{diag}(\lambda) \otimes I_2\big)$. Therefore:

$$(I_p \otimes \mathbf{1}')\tilde{D}(I_p \otimes \mathbf{1}) = (I_p \otimes \mathbf{1}')\big(\mathsf{diag}(\lambda) \otimes I_2\big)(I_p \otimes \mathbf{1}). \tag{3.4.8}$$

Comparing the above equation with (3.4.7) implies that the moments of these processes are not the same and the MTCP has the following extra term:

$$\int_0^s \vartheta\,\mathsf{diag}(\lambda)G^k(I_p \otimes \mathbf{1}')\overline{Q}\,(I_p \otimes \mathbf{1})\,G^l\lambda dv, \tag{3.4.9}$$

which cannot be zero unless the process is a Poisson process (see Remark 3.4.7 below). □

Corollary 3.4.5. *The above proposition implies that the index of dispersion for counts (IDC), $\mathcal{I}_t = \frac{\mathrm{Var}(N(t))}{\mathbb{E}[N(t)]}$, for a time-stationary slow MMPP$_p$ and its associated MTCP$_{2p}$ is the same.*

Remark 3.4.6. *Proposition 3.4.4 only holds for slow MMPPs. Otherwise the construction of MAP$_{2p}$ from MMPP$_p$ does not hold due to some non-positive elements $\lambda_i - S_i$ in the matrices $\overline{\Lambda}_i$ and $\overline{D}_i$.*

Remark 3.4.7. *Only for the case of Poisson process, the third moments of the slow MMPP$_p$ and its associated MTCP$_{2p}$ are the same. This result comes from the fact that (3.4.9) is equal to zero just for Poisson processes:*

$$\begin{aligned} & \vartheta\,\mathit{diag}(\lambda)G^k(I_p \otimes \mathbf{1}')\overline{Q}\,(I_p \otimes \mathbf{1})\,G^l\lambda = 0 \\ \Leftrightarrow \quad & \vartheta\,\mathit{diag}(\lambda)G^{k+1+l}\lambda = 0, \quad \forall\, k, l \geq 0, \\ \Leftrightarrow \quad & \vartheta\,\mathit{diag}(\lambda)G^n\lambda = 0, \quad \forall\, n \geq 1, \end{aligned}$$

where the first step holds by using part (i) of Lemma 3.4.3 and the fact that $(I_p \otimes \mathbf{1}')(I_p \otimes \mathbf{1}) = 2I_p$. *Since* ϑ *is the stationary distribution of* G, *the last equality holds when* $\lambda = \lambda \mathbf{1}_p$ *or* $\vartheta\, \mathrm{diag}(\lambda) = \lambda\, \vartheta$ *(which implies again* $\lambda = \lambda \mathbf{1}_p$*) which is the case for the Poisson process. For more details on when a general MAP is Poisson, see [27].*

Further, for any initial distribution (not restricting to the time-stationary case), we have the following proposition.

Proposition 3.4.8. *Let* $\tilde{N}(t)$ *and* $\overline{N}(t)$ *be the counting processes of a slow MMPP*$_p$ *and its associated MTCP*$_{2p}$*, respectively. Then, these processes have the same first moment but not necessarily the same second moment.*

Proof. The first moment of a non-stationary MAP is given in terms of the transient deviation matrix by (3.2.11), where η is the initial distribution. By using parts (iii) and (vi) of Lemma 3.4.3, we see that the first moments of both processes in the non-stationary case are the same.

To show that the second moments are different, we show that the y-intercepts of their asymptotic variance are different. For a non-stationary MAP, $\mathrm{Var}(N(t))$ has a linear y-intercept given by [82]:

$$\bar{b}_\eta = -2\pi D D_Q^\sharp D_Q^\sharp D\mathbf{1} - 2\pi D\mathbf{1}\eta \left(D_Q^\sharp\right)^2 D\mathbf{1} - \left(\eta D_Q^\sharp D\mathbf{1}\right)^2 + 2\eta D_Q^\sharp D D_Q^\sharp D\mathbf{1}.$$

From parts (v′), (vi) and (vii) of Lemma 3.4.3, we see that the first three terms of the above y-intercept are the same for an MMPP and its associated MTCP. For the last term, by considering an initial distribution in the form of $\eta = (1, 0, \cdots, 0)(I_p \otimes \mathbf{1}')$ and by using the definition of deviation matrix and part (i′) of Lemma 3.4.3, we see that this term for both processes equals to:

$$\int_0^\infty (2, 0, \cdots, 0) e^{Gt} (I_p \otimes \mathbf{1}')\, D D_Q^\sharp D\mathbf{1}\, dt - \int_0^\infty (2, 0, \cdots, 0)(I_p \otimes \mathbf{1}')\, \mathbf{1}_p \vartheta D_Q^\sharp D\mathbf{1}\, dt.$$

Note that from part (vi) of Lemma 3.4.3 we have $D_Q^\sharp D\mathbf{1}$ is the same for both processes and thus the second integral is the same for both processes. However, different event matrix D for slow MMPPs and their corresponding MTCPs results in different left integral. The explicit

calculation shows that the left term in $D_Q^\sharp D\mathbf{1}$ is $(I_p \otimes \mathbf{1})$, so from (3.4.7) and (3.4.8), we can conclude that this term is different for a slow MMPP and its associated MTCP. □

Remark 3.4.9. *Note that in the proof of the last part of the above proposition, for having the same second moment, we need that the initial distribution either satisfies* $\eta e^{\overline{Q}t} = 0$ *(which is the case for the time-stationary distribution) or* $(I_p \otimes \mathbf{1}')\,\overline{Q}\,(I_p \otimes \mathbf{1}) = 0$ *which by considering the structure of* $\overline{Q}$*, results in* $\forall i, j$*:* $q_{ij} = 0$*. Since the latter is impossible, we conclude that for having the same second moment, we must have* $\eta = \pi$*.*

3.5 The Steady-State Queue Approximation

The use of Markov models (like MAP) in queueing analysis benefits from established theoretical results and efficient solution algorithms. Because of this primacy, Markov models provide a convenient way to evaluate the performance measures of network traffic and system workloads. However, since the autocorrelation sequence of a MAP always converges, MAPs can not directly generate long-range dependent behaviour. This made them inaccurate for approximating systems with long-range dependency. But, this problem can be sufficiently solved in several ways such as superposition of MAPs, see [5] and [50].

There are several papers regarding fitting the inter-arrival time process of a MAP with real data traces instead of considering the counting process of a MAP, see for instance [86] and [94]. Although evaluating the performance measures of inter-arrival time process is harder than measuring the counts, this approach is popular due to the existence of closed-form analytical expression for the moments and lag correlations of inter-arrival process, as in formulas (3.2.3) and (3.2.6). Note that for the counting process there are expressions only for the first moments, as we see in Section 3.2.

As we mentioned before, in a high dimensional context which often requires several states for the underlying Markov chain, the model parametrisation is the main obstacle of both the method of moments and the EM algorithm. Therefore, the significant majority of papers consider only MAPs with 2 or 3 states, see for instance [19] and [158] for using the EM algorithm and [66] and [86] for applying the method of moments. Another way for fitting data traces with long-range dependency with higher order MAPs is the Kronecker product

composition (KPC) operator method, see [50] and references therein.

Here, first, we consider the counting process and apply MAM and the results of Section 3.4 to approximate a given (slow) $\text{MMPP}_2/\text{PH}_2/1$ with an $\text{MTCP}_4/\text{PH}_2/1$. In general, our computations are for $\text{MAP}_p/\text{PH}_q/1$ queues where the service time distributions are parametrized by their workloads, ρ and their SCVs, c^2.

Then, considering the inter-arrival process and using a simple version of the optimization algorithm described in [50], we approximate a slow $\text{MMPP}_2/\text{M}/1$ with an $\text{MTCP}_4/\text{M}/1$ and show that this approximation works well.

3.5.1 QBD Representation of the MMPP/PH/1 Queues

The MMPP/PH/1 queue is a special case of the general single-server queue MAP/G/1, where the stream of arrivals and service mechanism are modelled by MMPP and PH distribution, respectively. Figure 3.3 illustrates an example of an $\text{MMPP}_2/\text{PH}_2/1$ queue. Methods of analysing the MMPP/PH/1 queueing models can be found in [76] and [131]. Here, we use the uniform framework of QBD processes which is an efficient way to analyse more general models using matrix-analytic methods, see [119].

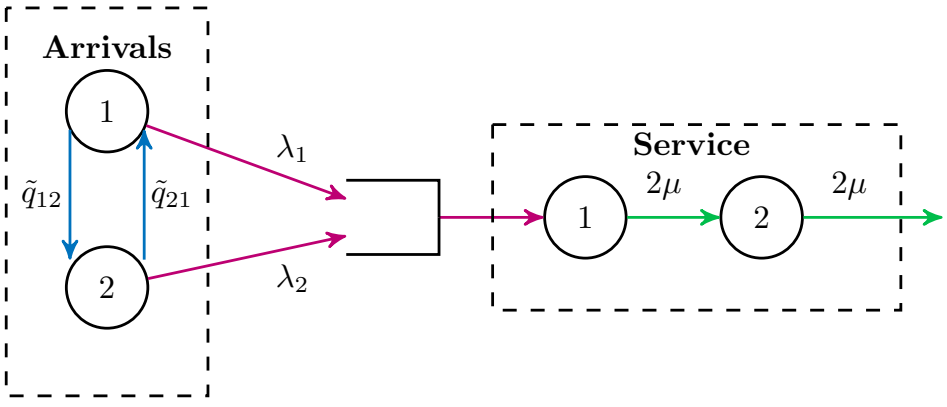


Figure 3.3: A schematic illustration of the $\text{MMPP}_2/\text{E}_2/1$ queue (here, mean of E_2 is μ^{-1}). The circles illustrate phases of the arrival and service mechanism.

In representing the $\text{MMPP}_p/\text{PH}_q/1$ queue as a QBD_r, where $r = p \times q$, the phase records (in lexicographic order) both the background state of the MMPP (arrival) and the current phase of the service (see Figure 3.3 for illustration of the phases in the special case of

MMPP$_2$/E$_2$/1). The level represents the number of items in the system.

Modelled as a QBD, we have (see Eq. (2.3.5)):

$$B_{-1} = I_p \otimes \mathbf{t}, \quad B_0 = C, \quad B_1 = \mathsf{diag}(\lambda) \otimes \eta,$$

where we assume that C and $D = \mathsf{diag}(\lambda)$ are matrices of MMPP$_p$. Moreover, we assume that the parameters of PH$_q$ are (η, T). Here, $\mathbf{t} = -T\mathbf{1}$, where $\mathbf{1}$ is a column vector of 1's with appropriate dimension. Further,

$$A_{-1} = I_p \otimes \mathbf{t}\eta, \quad A_0 = I_p \otimes T + C \otimes I_q, \quad A_1 = \mathsf{diag}(\lambda) \otimes I_q.$$

As is well known in the theory of QBDs, the stationary distribution of a positive-recurrent QBD$_r$, π, admits a matrix-geometric form $\pi_n = \pi_{n-1}R$, (see Eq. (2.4.3)) where R is the solution of a quadratic fixed-point matrix equation $R = A_1 + RA_0 + R^2A_{-1}$ and π_n are row vectors of dimension r, see [119]. We use the-state-of-the-art SMC solver to find the matrix R and the stationary distribution of a given QBD$_r$, see [29]. It is easy to show that A is irreducible due to the properties of the building blocks and irreducibility of Q. Moreover, characterizing the positive-recurrence can be done as follows[5].

Lemma 3.5.1. *The QBD representing a MAP$_p$/PH$_q$/1 queue is positive-recurrent if and only if,*

$$\rho := \frac{\beta A_1 \mathbf{1}}{\beta A_{-1}\mathbf{1}} = \frac{\lambda^*}{\frac{1}{-\eta T^{-1}\mathbf{1}}} < 1,$$

where $-\eta T^{-1}\mathbf{1}$ is the first moment of PH$_q$ and $\lambda^* = \pi D\mathbf{1}$ is the first moment of a time-stationary MAP$_p$ with parameters (π, C, D), and β is the stationary distribution of $A_{-1} + A_0 + A_1$.

Proof. From Theorem 7.2.4 in [119], we know that a necessary and sufficient condition for a QBD to be positive recurrence is that $\beta A_1\mathbf{1} < \beta A_{-1}\mathbf{1}$. Therefore, it remains to show that both representations of ρ agree. First, we show that $\beta = \pi \otimes \gamma$, where γ is the unique solution of

[5] To the best of our knowledge, the algebra behind this intuitive lemma has not appeared elsewhere.

$\gamma(T + \mathbf{t}\eta) = \mathbf{0}'$ and $\gamma\mathbf{1} = 1$[6]. It is immediate that $(\pi \otimes \gamma)\mathbf{1} = 1$. Further, we have

$$\begin{aligned}(\pi \otimes \gamma)(A_{-1} + A_0 + A_1) &= (\pi \otimes \gamma)\Big(I_p \otimes \mathbf{t}\eta + (I_p \otimes T + C \otimes I_q) + D \otimes I_q\Big) \\ &= (\pi \otimes \gamma)\Big(I_p \otimes (\mathbf{t}\eta + T) + (C + D) \otimes I_q\Big) \\ &= (\pi \otimes \gamma)\Big((C + D) \otimes (\mathbf{t}\eta + T)\Big) \\ &= \mathbf{0}',\end{aligned}$$

where the last two steps follow the formula $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.

Now we need to show that $\frac{\beta A_1 \mathbf{1}}{\beta A_{-1}\mathbf{1}} = \frac{\lambda^*}{\frac{1}{-\eta T^{-1}\mathbf{1}}}$ or equivalently:

$$\beta A_1 \mathbf{1} = (\beta A_{-1}\mathbf{1})\, \lambda^*\, (-\eta T^{-1}\mathbf{1}),$$

which for the MAP$_p$/PH$_q$/1 queue is written as:

$$(\pi \otimes \gamma)(D \otimes I_q)\mathbf{1} = (\pi \otimes \gamma)(I_p \otimes \mathbf{t}\eta)\mathbf{1}(\pi D\mathbf{1})(-\eta T^{-1}\mathbf{1}). \tag{3.5.1}$$

For the left-hand side, we have $(\pi \otimes \gamma)(D \otimes I_q)\mathbf{1} = (\pi D \otimes \gamma)\mathbf{1} = \pi D\mathbf{1}$. Therefore, we need to show that the right-hand side of (3.5.1) is also equal to $\lambda^* = \pi D\mathbf{1}$, or equivalently:

$$(\beta A_{-1}\mathbf{1})\,(-\eta T^{-1}\mathbf{1}) = (\pi \otimes \gamma)(I_p \otimes \mathbf{t}\eta)\mathbf{1}(-\eta T^{-1}\mathbf{1}) = 1.$$

Since $\pi\mathbf{1} = \eta\mathbf{1} = 1$, we have $(\pi \otimes \gamma)(I_p \otimes \mathbf{t}\eta)\mathbf{1} = (\pi \otimes \gamma\mathbf{t}\eta)\mathbf{1} = \gamma\mathbf{t}$. Moreover, from $\gamma(T + \mathbf{t}\eta) = \mathbf{0}'$ we have $\gamma\mathbf{t}\eta = -\gamma T$ which results in $\gamma\mathbf{t}(-\eta T^{-1}\mathbf{1}) = 1$. □

3.5.2 Illustration on Matching the Counting Process

If the PH distribution is an Erlang-2 (E$_2$), we have $c^2 = \frac{1}{2}$, where we consider E$_2$ as the sum of two i.i.d. exponential random variables with rate $\frac{2\lambda^*}{\rho}$. Here, λ^* is the arrival rate as in (3.2.12) and ρ is the workload. In the case of $c^2 = 1$, we use exponentially distributed random variables with rate $\mu = \frac{\lambda^*}{\rho}$. For the case of $c^2 > 1$, we use the hyperexponential-2 (H$_2$) distribution which is a mixture of two independent exponential random variables. With

[6]Note that γ is the limiting distribution of the phase in a PH$_q$-renewal process.

probability $p = \frac{1}{2c^2-1}$ we take an exponential distribution with rate $\frac{\lambda^*}{\rho\, c^2}$ and with probability $1-p$ we take an exponential distribution with rate $\frac{2\lambda^*}{\rho}$. It is easy to verify that this H_2 random variable has mean 1 and the desired c^2. We compute the matrix R and the stationary distri-

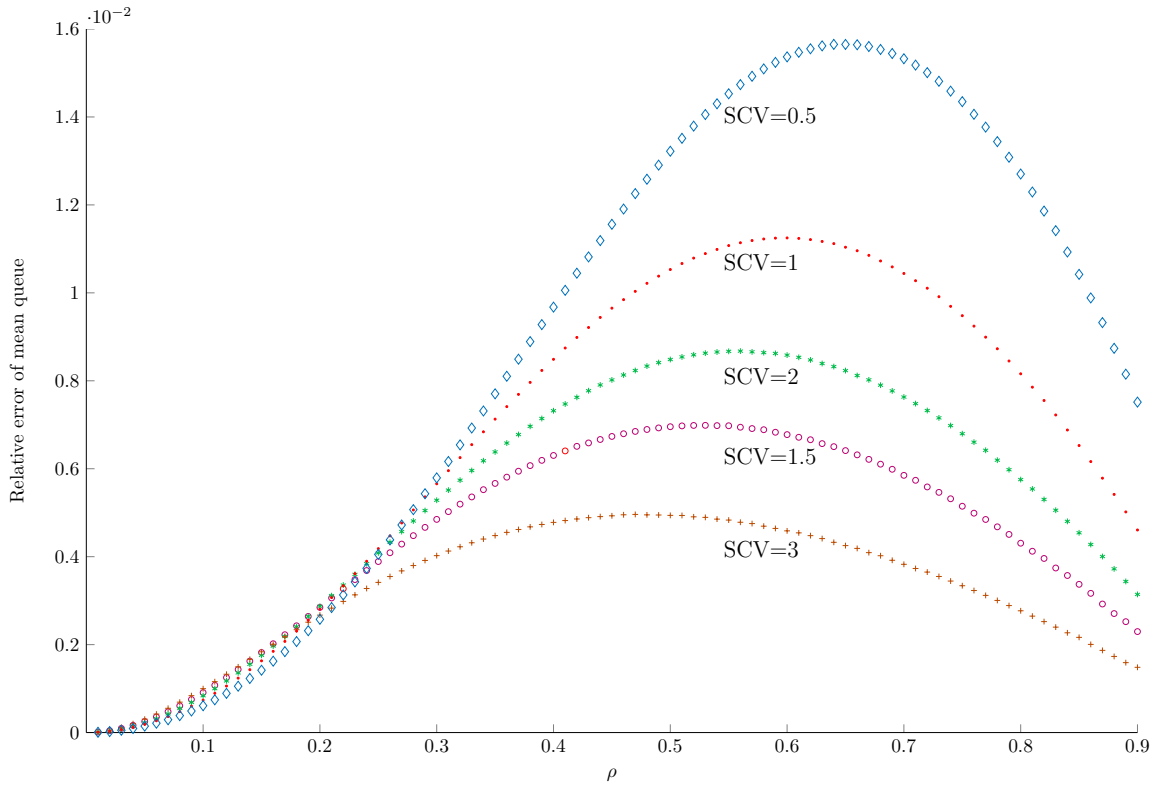


Figure 3.4: The relative mean error of approximating an $MMPP_2/PH_2/1$ steady state queue with an $MTCP_4/PH_2/1$ steady state queue. The parameters of $MMPP_2$ are $\tilde{q}_{12} = \tilde{q}_{21} = 5$, $\lambda_1 = 10$, $\lambda_2 = 20$. The mean service time is varied to accommodate for the desired ρ.

bution of $MMPP_2/PH_2/1$ and $MTCP_4/PH_2/1$ as QBDs by using the SMC solver, see [29]. The numerical computation for finding the relative errors,

$$\text{relative error} = \frac{\text{true value - approximate value}}{\text{true value}},$$

shows the same properties for the curves of the relative error of mean and SCV of steady state queue when we consider all of the above-mentioned PH distributions.

Figure 3.4 shows different relative errors of the steady state mean for various service time SCVs. The bigger the SCV of service time, the less relative error of the mean. Figure 3.5 (right) shows different relative errors of the steady state SCV for various service time SCVs. The minimum absolute value of the relative error is again for the case that the service dis-

tribution is hyperexponential, that is the bigger the SCV of service time, the less absolute value of the relative error of SCV of steady state queue.

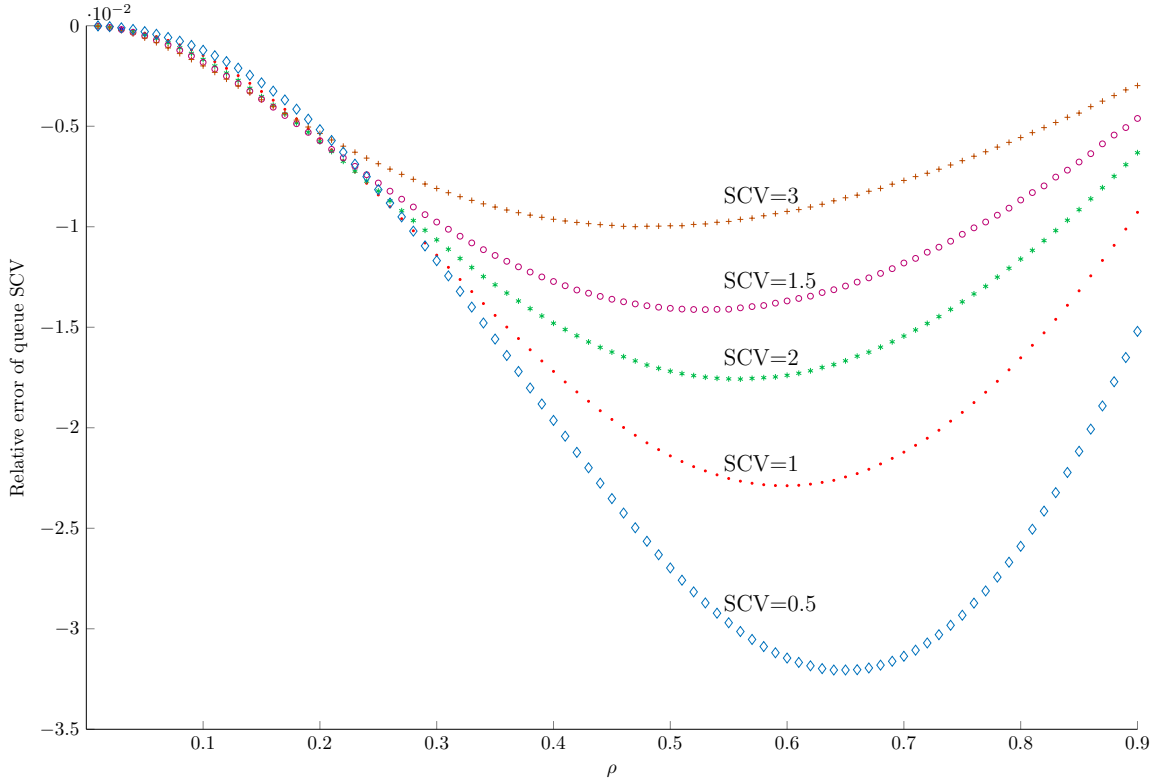


Figure 3.5: The relative SCV error of approximating an $MMPP_2/PH_2/1$ steady state queue with an $MTCP_4/PH_2/1$ steady state queue. The parameters of $MMPP_2$ are $\tilde{q}_{12} = \tilde{q}_{21} = 5$, $\lambda_1 = 10$, $\lambda_2 = 20$. The mean service time is varied to accommodate for the desired ρ.

A key contradiction between the errors in mean and SCV is that in contrast to the relative error of means which have positive values, the relative error of SCV of the steady state queue has negative values. This shows that the true value for mean is always greater than the approximate one and the opposite holds for SCV. From a further investigation of the variance (not appearing in the figures), it also holds that the true variance is less than or equal to the approximated variance.

Table 3.1: Maximum relative error of mean queue in approximation of $MMPP_2/PH_2/1$ queue by $MTCP_4/PH_2/1$ queue where $\lambda_1 = 10$. Note that the H_2 case corresponds to $c^2 = 1.1$.

Model	λ_2	$\tilde{q}_{12}$	$\tilde{q}_{21}$	Max Relative Error of Mean Queue
$MMPP_2/E_2/1$	500	8	70	0.0893
$MMPP_2/M/1$	300	9	70	0.0725
$MMPP_2/H_2/1$	400	5	70	0.0715

As is evident from the figures, in any case, the relative error is negligible. Note though, that for more bursty arrival processes we may have bigger relative errors than those in the figure,

yet we carried out an extensive computational study to find an empirical boundary for relative error. Assuming that λ_1 is constant (=10) and varying the values of λ_2, $\tilde{q}_{12}$ and $\tilde{q}_{21}$ gives the results in Table 3.1 for the maximum relative error.

In summary, these empirical results indeed suggest that the MTCP/PH/1 is a very sensible alternative model to MMPP/PH/1.

3.5.3 Illustration on Matching the Inter-Arrival Time Process

With the results above at hand, we now wish to compare our approximation method (based on Theorem 3.4.4) to the method of matching arrival process characteristics explained in [52]. The latter method considers the inter-arrival times and matches their moments and auto-correlations. As we show in the comparative results below, while carrying out such matching schemes somewhat outperforms our simple approximation, it still holds that the error of our simple approximation is tiny.

Here, we consider an MMPP$_2$ with parameters $\tilde{q}_{12} = \tilde{q}_{21} = 5$, $\lambda_1 = 10$, $\lambda_2 = 40$. Applying the formulas in Section 3.2 results in $M_1 = 0.04$, $c^2 = 1.6923$, $M_3 = 0.0014$, $\varrho_1 = 0.2546$, $\varrho_2 = 0.1972$, and $\varrho_3 = 0.1618$.

For matching an MTCP$_4$ with similar moments and lag-k autocorrelations to this MMPP$_2$, we calculate the first 50 autocorrelations of the MMPP$_2$, ϱ_k, from (3.2.6) and minimise the function $f(\mathbf{x}) = \sum_{k=1}^{50} \left(\varrho_k - \widehat{\varrho}_k(\mathbf{x})\right)^2$, where $\widehat{\varrho}_k(\mathbf{x})$ is the lag-k autocorrelation of an MTCP$_4$ specified by $\mathbf{x} = (x_1, \cdots, x_{12})$ and having matrices

$$D = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ x_4 & 0 & x_5 & x_6 \\ x_7 & x_8 & 0 & x_9 \\ x_{10} & x_{11} & x_{12} & 0 \end{pmatrix}, \qquad C = \mathsf{diag}(-D\mathbf{1}).$$

Then, applying a constrained nonlinear version of "fmincon" solver in MATLAB, we find the vector $\mathbf{x}$ that minimizes $f(\mathbf{x})$ with tolerance 10^{-6}. This result is subject to $\widehat{M}_1(\mathbf{x}) = M_1$ and $\widehat{c}_J^2(\mathbf{x}), \widehat{M}_3(\mathbf{x}), \widehat{\varrho}_1(\mathbf{x}), \widehat{\varrho}_2(\mathbf{x})$, and $\widehat{\varrho}_3(\mathbf{x})$ being constrained to their corresponding values for

MMPP$_2$, c_J^2, M_3, ϱ_1, ϱ_2, and ϱ_3, with a tolerance of 10^{-6}. Since this is generally a non-convex optimization problem, we experiment with randomly selected initial values for the "fmincon" solver and then seek the optimal one. The parameters of the resulting MTCP$_4$ are

$$\mathbf{x} = \big(12.3519, 1.0078, 0.9110, 9.4306, 1.2491, 3.4953, 3.3339, 2.9924, 39.2952, 3.4383, 3.5185, 39.4102\big).$$

Comparison of the above values for an MMPP$_2$ and the MTCP$_4$ obtained through optimisation method based on inter-arrival process and the MTCP$_4$ obtained from our approximation method based on counting process is presented in Table 3.2.

Table 3.2: Comparison of moments and autocorrelations in moments matching of MMPP$_2$ with an MTCP$_4$. The parameters of MMPP$_2$ are $q_{12} = q_{21} = 5$, $\lambda_1 = 10$, and $\lambda_2 = 40$.

Model	M_1	d^2	c^2	M_3	ϱ_1	ϱ_2	ϱ_3
MMPP$_2$ (taken as ground truth)	0.0400	2.8000	1.6923	0.0008	0.1259	0.0775	0.0477
MTCP$_4$ (optimisation method)	0.0400	2.7933	1.6923	0.0008	0.1256	0.0771	0.0473
MTCP$_4$ (our approximation method)	0.0400	2.8000	2.1250	0.0013	0.0993	0.0372	0.0140

As the table shows, the corresponding MTCP$_4$ obtained from our approximation method in Section 3.4 has the same first moment as the MTCP$_4$ resulting from the optimisation procedure. This comes from the fact that the first moments for both time-stationary counting process and event-stationary inter-arrival process are the same:

$$M_1 = \frac{1}{\pi D \mathbf{1}} = \alpha(-C)^{-1}\mathbf{1}.$$

As we expected, the value of d^2 which is related to the counting processes are exactly the same for the given MMPP$_2$ and the corresponding MTCP$_4$ that resulted from our approximation method. Based on Table 3.2, the MMPP$_2$ and the MTCP$_4$ resulting from optimisation process are of comparable moments and autocorrelations (their difference is less than 10^{-3}). But, as it is expected, the MTCP$_4$ obtained from our method (based on matching first two moments of the counting process) has some values that differ. We now compare both approximated arrival models through a queueing simulation.

Figure 3.6 demonstrates the difference between the mean queue length curves of the MMPP$_2$ /M/1 queue (green curve) with the MTCP$_4$/M/1 queue. The MTCP$_4$ obtained either from the optimisation method (red curve) or from our approximation method as described in Section 3.4 (blue curve). The parameters of the MMPP$_2$ are the same as before, that is $q_{12} = q_{21} = 5$, $\lambda_1 = 10$, and $\lambda_2 = 40$. Note that the value of workload ρ varies from 0 to 0.9 by

changing the rate of the service process.

Furthermore, the resulting mean queue lengths in Figure 3.6 are calculated in the same way as Section 3.5.1, that is by considering the MAP/M/1 queue as a QBD and then applying the SMC solver. The same method is used to obtain the corresponding measures for figures 3.7 and 3.8.

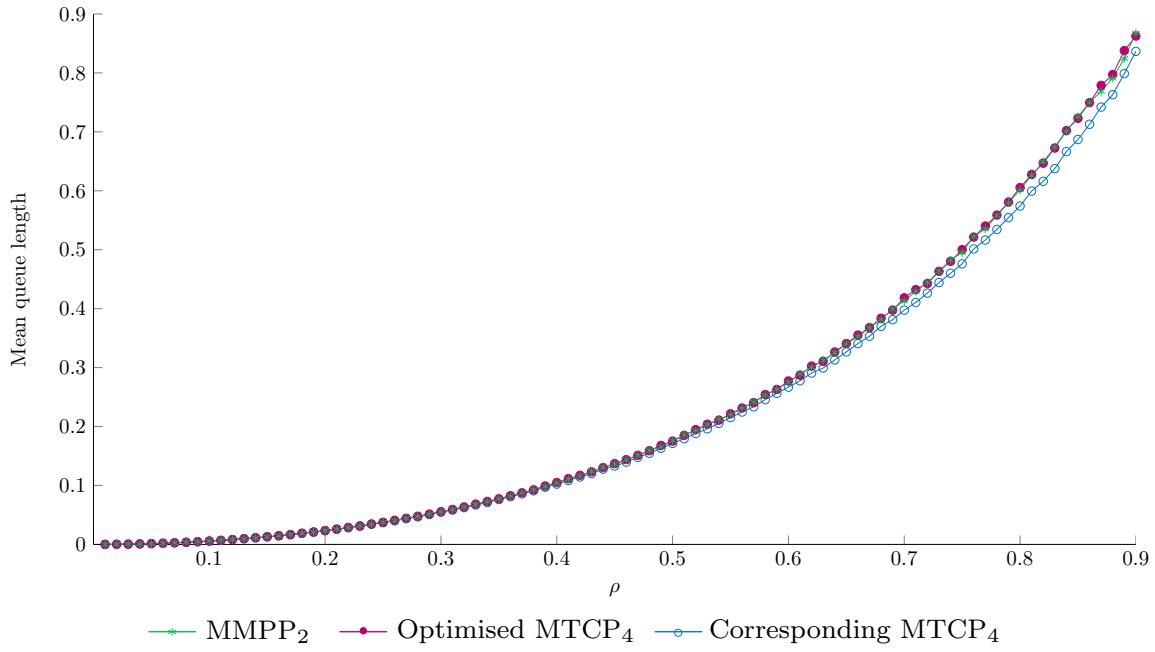


Figure 3.6: Comparison of mean queue length for approximating an $MMPP_2$/M/1 queue with an $MTCP_4$/M/1 queue. The red curve represents the mean queue length where $MTCP_4$ is resulting from an optimisation procedure and the blue curve is for the corresponding $MTCP_4$ obtained by the method of Section 3.4. The green curve is the mean queue length of the $MMPP_2$/M/1 queue. Here, the workload varies from 0 to 0.9.

In Figure 3.7, comparison of the proportional error of mean queue length of a steady-state $MMPP_2$/M/1 queue, where the $MMPP_2$ approximates with an $MTCP_4$ resulting from either the above optimization method (red curve) or the corresponding $MTCP_4$ as in Section 3.4 (blue curve) is presented. As the figure shows, the $MTCP_4$ resulting from optimisation procedure gives a better approximation in comparison to the $MTCP_4$ resulting from the analytical method in Section 3.4. However, numerical experiments show that the proportional error of mean queue for an $MMPP_2$/M/1 where approximating the $MMPP_2$ with the corresponding $MTCP_4$ suggested in section 3.4 is negligible (less than 3.5×10^{-2} in the above example and less than 7.3×10^{-2} when checking the various examples, see Table 3.1).

Therefore, regarding the simplicity of finding the associated $MTCP_4$ suggested in Section 3.4 and negligibility of the proportional error of mean queue, we can suggest that this method

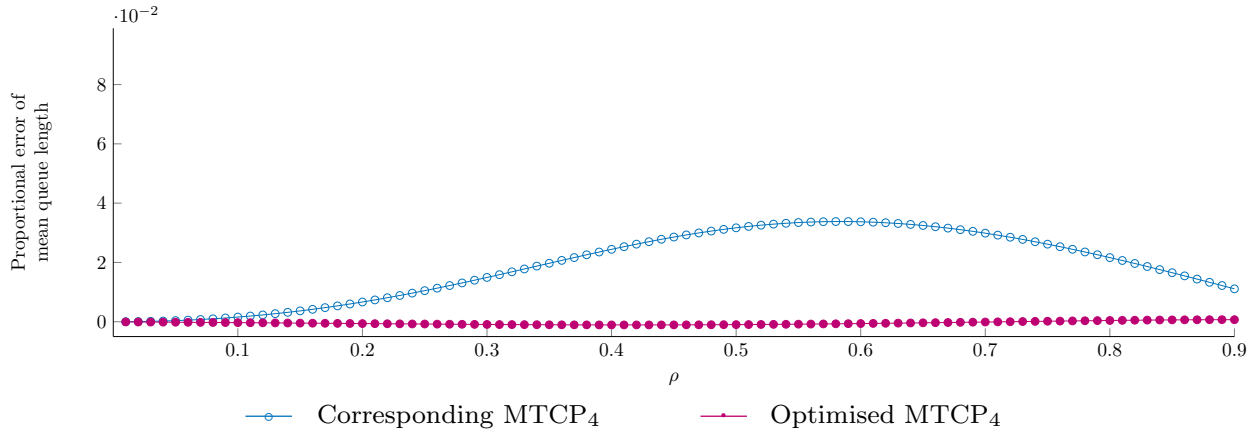


Figure 3.7: Comparison of proportional error of mean queue length for approximating an MMPP$_2$/M/1 queue with an MTCP$_4$/M/1 queue. The blue curve represents the proportional error of optimisation procedure and the red curve represents the case of associated MTCP$_4$ obtaining from approximation method of Section 3.4. Here, the workload varies from 0 to 0.9.

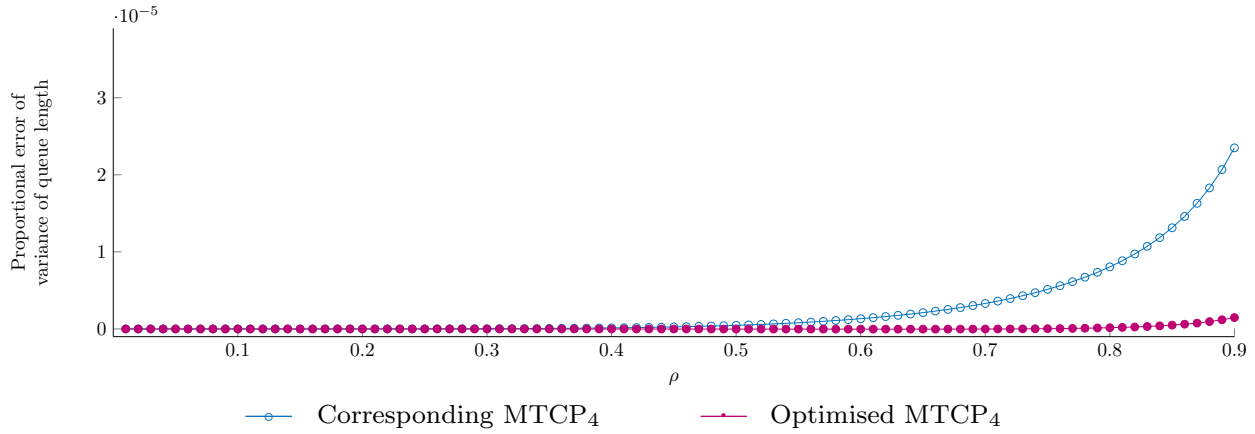


Figure 3.8: Comparison of proportional error of variance of queue length for approximating an MMPP$_2$/M/1 queue with an MTCP$_4$/M/1. The blue curve represents the proportional error of applying the optimisation procedure and the red curve is resulted by considering the associated MTCP$_4$ suggested in Section 3.4. Here, the workload varies from 0 to 0.9.

is suitable for approximating an MMPP$_2$ with an MTCP$_4$ in queueing applications. To be more sure about this suggestion, we consider the proportional error in measuring another important performance measure, the variance of queue length where approximating an MMPP$_2$/M/1 queue with an MTCP$_4$/M/1 queue. Figure 3.8 demonstrates the proportional error of variance of queue length when approximating an MMPP$_2$/M/1 queue with an MTCP$_4$/M/1 queue. The blue curve obtained by considering the approximation method in Section 3.4 and the red curve obtained by considering the optimisation procedure for finding the corresponding MTCP$_4$. For both cases by changing the service rate, the value of workload ρ varies from 0 to 0.9. As the Figure 3.8 shows the proportional error of variance is less than 3×10^{-5} where the parameters of the MMPP$_2$ are $q_{12} = q_{21} = 5$, $\lambda_1 = 10$, and $\lambda_2 = 40$.

Chapter 4

Control of Queueing Systems

4.1 Introduction

Performance evaluation and control of queueing systems subject to randomly varying environments is an area of research that has received much attention during the past few decades (see for example, [54], [104], [180] and references therein). This is because numerous situations arise in practice where a controller needs to decide how to utilise resources best, and these are often subject to changing conditions. Examples of such situations arise in wireless communication, supply chain logistics, health care, manufacturing, and transportation. In all these situations, it is very common for service rates to vary in a not fully predictable manner. Using Markovian random environments has often been a natural modelling choice due to the tractability and general applicability of Markov models. See for example Section A.1 in [144] for a general discussion on the ubiquity of Markov models.

The bulk of the literature dealing with performance evaluation and control of these types of problems, has considered the situation where the state of the underlying random environment is observable. In such a situation, it is already a non-trivial task to carry out explicit system performance analysis (see [107] as an example). Further, finding optimal or even merely stabilizing control is typically a formidable achievement (see for example [23], [188] or the more recent [78]). But in practice, the actual environment state is often not a directly observed quantity or is at best only partially observable. The situation is further complicated when control decisions do affect not only immediate rewards but also the observation made.

In standard linear-quadratic optimal control settings (for example Part III of [197]), the certainty equivalence principle allows decoupling state estimation based on observations and control decisions. However, in more complicated settings such as what we consider here, certainty equivalence almost certainly does not hold.

In this chapter, we augment the body of literature dealing with exploration vs. exploitation trade-offs in systems where a controller needs to choose a server (channel/resource/bandit) at any given time, and the choice influences both the immediate reward (service success) and the information obtained. A general class of such problems, denoted *reward observing restless multi armed bandits* (RORMAB), is outlined in [113], where much previous literature is surveyed. Key contributions in this area are [110] and the more recent [126]. The former finds the structure of *optimal policies* from first principles. The latter generalizes the setting and utilizes the celebrated *Whittle index*, [198] for such a partially observable case. Related recent results dealing with RORMAB problems are in [116] and [117]. Of further interest is the latest rigorous account on the asymptotic optimality of the Whittle index, [194], as well as in the context of partially observable two state Markov chains [163].

Our focus in this chapter is on a controlled queueing systems, where server environments vary and the controller (choosing servers) only observes partial information. Our aim is to explore the role of information in system stability. For this, we devise what is perhaps the simplest non-trivial model possible: a single discrete time queue is served by either Server 1 or Server 2 where each server environment is an independent two-state Markov chain. A controller is having (potentially) only partial state information, selects one of the two servers at each time instance.

The role of information is explored by considering different observation schemes. At one extreme, the controller has full information of the servers' environment states. At the other extreme, the controller is completely unaware of the servers' environment states. Obviously, the stability region of the system in the latter situation is a subset of the former. Our contribution is in considering additional more realistic observation schemes. One such scheme is a situation where the controller only observes the state of the server currently chosen. This type of situation has been widely studied in some of the references mentioned above and surveyed in [113], but most of the literature dealing with this situation does not consider a queue. A more constrained scenario is one where the controller only observes the success/-

failure of service (from the server chosen) at every time slot. Such a partial observability situation was recently introduced in [146] in the context of stability and analysed in [142] with respect to the Whittle index. In [123], the stability of a related multi-server system was analysed.

An additional observation scheme that we consider is one where the server is only aware of the queue size process. In (non-degenerate) continuous time systems, such an observation scheme is identical to the former scheme. But an artefact of our discrete time model is that such a scheme reveals less information to the controller (this is due to the fact that both an arrival and a departure may coincide, going unnoticed by the controller).

With the introduction of the five observation schemes mentioned above, this work takes first steps to analyse the effect of information on the achievable stability region. A controller of such a system makes use of a belief state implementation. We put forward (simple) explicit belief state update recursions for each of the observation schemes. These are then embedded in Bellman equations describing optimal solutions of associated *partially observable Markov decision processes* (POMDP). Numerical solution of the POMDPs then yields insight on structural properties and achievable stability regions. By construction, two-state Markov server environments are more predictable when the mixing times of the Markov chains increase. We quantify this use of channel-memory, through numerical and analytic results.

It is often the case that MDPs (or POMDPs) associated with queueing models, can be cast as QBDs once a class of control policies is found. See for example [143]. We follow this paradigm in the current work and present a detailed QBD model of the system. The virtue of our QBD based model is that we are able to quantify the effect of a finite state controller on the achievable stability region whose upper bound is given by an elegant matrix analytic expression.

The remainder of this chapter is structured as follows. In Section 4.2, we introduce the system model and different observation schemes. In Section 4.3, we develop recursions for belief state updates for the non-trivial observation schemes. In Section 4.4, we present the myopic policy and the Bellman equations for different observation schemes and present findings from a numerical investigation. In Section 4.5, we construct a QBD representation of the system, find the stability criterion and put forward the numerical results which are

matched with the results of Bellman equations of Section 4.4.

4.2 System Model

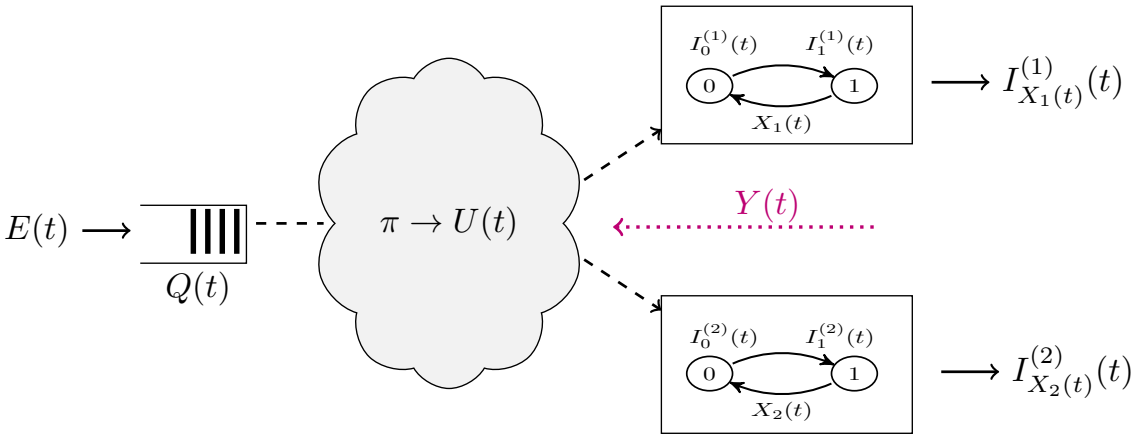


Figure 4.1: A controller operating under a decision rule, π, decides at each time step, t, if to use server $U(t) = 1$ or $U(t) = 2$ based on previously observed information, $Y(t-1), Y(t-2), \ldots$. The server environments, $X_i(t)$ are Markov modulated.

Consider a situation as depicted in Figure 4.1. Jobs arrive into the queue, $Q(t)$, and are potentially served by one of two servers $j = 1$ or $j = 2$, according to some control policy. The system is operating in discrete time steps $t = 0, 1, \ldots$, where in each time step, the following sequence of events occur:

(1) An arrival occurs as indicated through $E(t)$: $E(t) = 1$ indicates an arrival and otherwise $E(t) = 0$.

(2) The environments of the servers update from $\big(X_1(t-1),\, X_2(t-1)\big)$ to $\big(X_1(t),\, X_2(t)\big)$, autonomously. That is, the updating of the environment is not influenced by arrivals, queue length and controller choice.

(3) A control decision $u = U(t)$ of which server to select is made based on observations in previous time steps, denoted by $Y(t-1), Y(t-2), \ldots$. This is through a decision rule, π. We consider different observation schemes as described below.

(4) The control action is executed, and the queue length is updated as follows:

$$Q(t+1) = Q(t) + E(t) - I(t), \qquad \text{with} \qquad I(t) = I^{(u)}_{X_u(t)}(t).$$

Here, $I(t)$ indicates whether there was a service success or not. It is constructed from the primitive sequences,

$$\big\{I_i^{(j)}(t),\ t=0,1,2,\dots\big\},$$

for servers $j=1,2$ and environment states $i=0,1$. Note that when $Q(t)=0$, we notionally assume that $U(t)=u=0$, indicating "no action" and in this case denote $I_i^{(0)}\equiv 0$ for $i=0,1$.

(5) The observation of $Y(t)$ is made and is used in subsequent time steps.

The sequence of events (1)–(5) as above repeats in every time step and fully defines the evolution law of the system. We consider the following distinct observation schemes:

(I) Full observation: The controller knows the state of both servers all the time. In this case

$$Y(t)=\big(X_1(t),\,X_2(t)\big),$$

and further, the sequence of steps above is slightly modified with step (5) taking place between steps (2) and (3) and the policy at step 3 being

$$u=U(t)=\pi\big(Y(t)\big).$$

(II) State observation: The controller observes the state of the selected server at time t, but does not observe the other server at that time. Hence

$$Y(t)=\begin{cases}\big(X_1(t),\emptyset\big) & \text{if } u=1,\\ \big(\emptyset,X_2(t)\big) & \text{if } u=2.\end{cases}$$

(III) Output observation: The controller observes the success or failure of outputs of the server selected (but gains no information about the other server at that time). Hence

$$Y(t)=\begin{cases}\big(I^{(1)}_{X_1(t)}(t),\emptyset\big) & \text{if } u=1,\\ \big(\emptyset,I^{(2)}_{X_2(t)}(t)\big) & \text{if } u=2.\end{cases}$$

(IV) Queue observation: The controller only observes the queue length, $Q(t)$, and can thus utilize the differences

$$\Delta Q(t) = Q(t+1) - Q(t) = E(t) - I(t).$$

Note that since the system is operating in discrete time, there is some loss of information compared to case III : If $\Delta Q(t) = 1$ or $\Delta Q(t) = -1$, then it is clear that $I(t) = 0$ or $I(t) = 1$, respectively. But if $\Delta Q(t) = 0$, then since the controller does not observe $E(t)$, there is not a definitive indication of $I(t)$.

(V) No observation: We assume the controller does not observe anything. Nonetheless, as with the other cases, the controller knows the system parameters as described below.

We consider the simplest non-trivial probably model for the primitives. These are $E(t)$, $I_i^{(j)}(t)$ and the environment processes, $X_j(t)$, all assumed mutually independent. The arrivals, $E(t)$, are an i.i.d. sequence of Bernoulli random variables, each with the probability of success λ. The service success indicators, for each server $j = 1, 2$ and state $i = 0, 1$, denoted by $\{I_i^{(j)}(t),\ t = 0, 1, 2, \ldots\}$, are each an i.i.d. sequence with

$$I_i^{(j)}(t) \ \sim\ \text{Bernoulli}\big(\mu_i^{(j)}\big).$$

Moreover, we assume

$$\mu_0^{(2)} \le \mu_0^{(1)} < \mu_1^{(1)} \le \mu_1^{(2)}. \tag{4.2.1}$$

Hence states $i = 1$ for both servers are better than states $i = 0$. Further, the spread of the chance of success for Server 2 is greater or equal to that for Server 1.

For the environment processes, we restrict attention to a two-state Markov chain, sometimes referred to as a Gilbert–Elliot channel [180]. We denote the probability transition matrix for server j as:

$$P^{(j)} = \begin{bmatrix} \overline{p}_j & p_j \\ q_j & \overline{q}_j \end{bmatrix} = \begin{bmatrix} 1 - \gamma_j\,\overline{\rho}_j & \gamma_j\,\overline{\rho}_j \\ \overline{\gamma}_j\,\overline{\rho}_j & 1 - \overline{\gamma}_j\,\overline{\rho}_j \end{bmatrix}, \tag{4.2.2}$$

with $\overline{x} := 1 - x$. In the sequel, we omit the server index j from the individual parameters of $P^{(j)}$. A standard parametrisation of this Markov chain uses transition probabilities $p, q \in [0, 1]$. Alternatively, we may specify the stationary probability of being in state 1, denoted by

$\gamma \in [0,1]$, together with the second eigenvalue of P, denoted by $\rho \in \left[1 - \min(\gamma^{-1}, \overline{\gamma}^{-1}),\ 1\right]$.

Using this parametrization, ρ quantifies the time-dependence of the chain; when $\rho = 0$ the chain is i.i.d., otherwise there is a memory. If $\rho > 0$ then environment states are positively correlated, otherwise they are negatively correlated. Our numerical examples in this paper deal with positive correlation as it is often the more reasonable model for channel memory. The relationship between the (γ, ρ) parametrisation and the (p, q) parametrization is given by $p = \gamma\,\overline{\rho}$, $q = \overline{\gamma}\,\overline{\rho}$, $\gamma = p/(p+q)$, and $\rho = 1 - p - q$. See Figure 4.2. Moreover, for each server, we can consider the *spread of* μ as $\mu_0^{(j)} = \gamma_j - \varepsilon_j$ and $\mu_1^{(j)} = \gamma_j + \varepsilon_j$, where ε_j shows the spread. So from above assumptions, the spread of Server 2 is greater than spread of Service 1: $\varepsilon_2 > \varepsilon_1$.

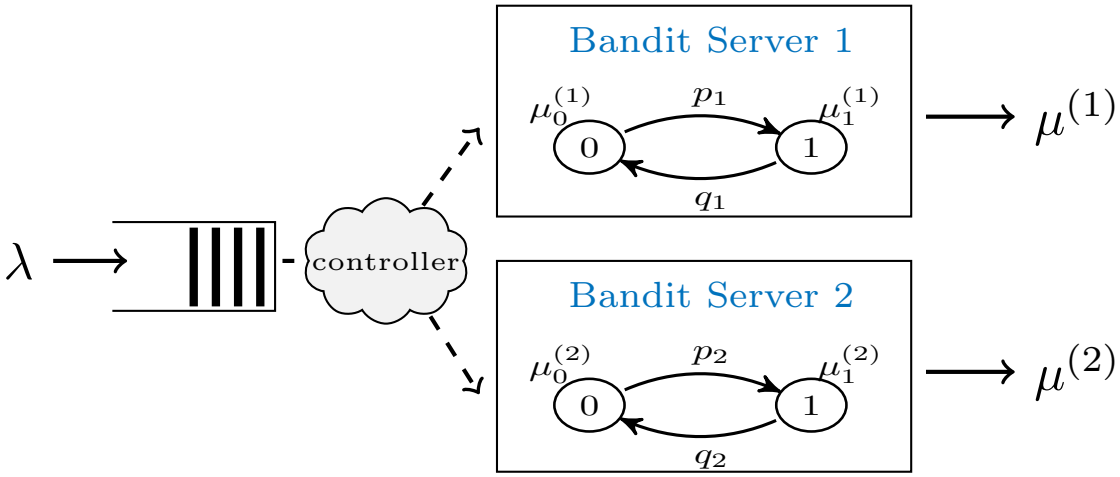


Figure 4.2: Parameters of the system.

It is instructive to consider the long term behaviour of $I^{(j)}_{X_j(t)}(t)$ for $j = 1, 2$ by assuming the sequence $\{X_j(t),\ t = 0, 1, 2, \ldots\}$ is stationary and thus each $X_j(t)$ is Bernoulli distributed with parameter γ_j. In this case

$$\begin{aligned}
\mathbb{E}[I^{(j)}_{X_j(t)}(t)] &= \overline{\gamma}_j\,\mu_0^{(j)} + \gamma_j\,\mu_1^{(j)}\,, \\
\mathrm{Var}[I^{(j)}_{X_j(t)}(t)] &= \overline{\gamma}_j\,\mu_0^{(j)}\,\overline{\mu}_0^{(j)} + \gamma_j\,\mu_1^{(j)}\,\overline{\mu}_1^{(j)} + \gamma_j\,\overline{\gamma}_j\left(\mu_1^{(j)} - \mu_0^{(j)}\right)^2.
\end{aligned}$$

These quantities are useful for obtaining a rough handle on the performance of the system.

The *stability region* of the system is defined as the set of arrival rates for which there exist a decision rule, π, under which the Markov chain that is describing the system (see Section 2.5.2) is positive recurrent. Note that this Markov chain, encapsulates $Q(t)$, but may

also include additional state variables[1]. We postulate that stability regions are of the form

$$\{\lambda \,:\, \lambda < \mu^*\},$$

where the *stability bound* μ^* varies according to the observation schemes I–V above and is identical to the maximal throughput rate that may be obtained in a system without a queue (but rather with an infinite supply of jobs). Further, by the construction of the information observation schemes, we have

$$\mu^*_{\text{no}} \leq \mu^*_{\text{queue}} \leq \mu^*_{\text{output}} \leq \mu^*_{\text{state}} \leq \mu^*_{\text{full}}\,, \tag{4.2.3}$$

where μ^*_{no} corresponds to case V, μ^*_{queue} corresponds to case IV and so forth.

The lower and upper bounds, μ^*_{no} and μ^*_{full}, are easily obtained as we describe now. However, the other cases are more complicated and are analysed in the sections that follow. For the lower and upper bounds we have

$$\begin{aligned}
\mu^*_{\text{no}} &= \max\left\{\mathbb{E}[I^{(1)}_{X_1(t)}(t)],\ \mathbb{E}[I^{(2)}_{X_2(t)}(t)]\right\},\\
\mu^*_{\text{full}} &= \overline{\gamma}_1\overline{\gamma}_2\mu_0^{(1)} + \gamma_2\mu_1^{(2)} + \gamma_1\overline{\gamma}_2\mu_1^{(1)}.
\end{aligned}$$

The lower bound, μ^*_{no} is trivially achieved with a control policy that always uses the server with the higher mean throughput. The upper bound is achieved with a control policy that uses the best server at any given time. Under the ordering in (4.2.1), the throughput in this case is calculated as follows: If both servers are in state 0, then since $\mu_0^{(2)} \leq \mu_0^{(1)}$, the controller selects Server 1. This situation occurs at a long term proportion, $\overline{\gamma}_1\overline{\gamma}_2$, hence we obtain the first term of μ^*_{full}. The other terms of μ^*_{full} are obtained with a similar argument. Note that when $\gamma_1 = \gamma_2 = \gamma$ and $\mu_i^{(1)} = \mu_i^{(2)} = \mu_i$ for $i = 0, 1$, the expression is reduced to

$$\mu^*_{\text{full}} = \overline{\gamma}^2\mu_0 + (1 - \overline{\gamma}^2)\mu_1, \tag{4.2.4}$$

and can be obtained by a Binomial argument.

[1]There may be different Markov chain representations of this system. One concrete description for a POMDP where each state consists of the number of jobs in the queue, server states, and controller states is presented in Section 4.5.

As a benchmark numerical case, all the examples we present use

$$\gamma = 0.5\,,\ \mu_0 = 0.2\,,\ \mu_1 = 0.8\,, \tag{4.2.5}$$

for both servers. Under these parameters $\mu^*_{\text{no}} = 0.5$ and $\mu^*_{\text{full}} = 0.65$. Hence, in the examples that follow, we explore how μ^*_{queue}, μ^*_{output} and μ^*_{state} vary within the interval $[0.5, 0.65]$ as ρ_j, $j = 1, 2$ varies.

4.3 Belief States

In implementing a controller for each of the observation schemes, the use of *belief states* reduces both the complexity of the controller and the related analysis. The idea is to summarize the history of observations, $Y(t-1), Y(t-2), \ldots$, into *sufficient statistics* that are updated by the controller. For our model, a natural choice for the *belief state* of server j is

$$\omega_j(t) = \mathbb{P}\big(X_j(t) = 1 \,|\, \text{Prior knowledge to time } t\big).$$

As we describe now, it is a simple matter to recursively update this sequence in a Bayesian manner. Denoting $\omega_j(t)$ by ω, the believed chance of success is

$$r(\omega) := \overline{\omega}\mu_0 + \omega\mu_1.$$

The updating algorithms (different for each observation scheme) make use of the following:

$$\begin{aligned} &\tau_n(\omega) := \omega\rho + \gamma\overline{\rho}\,, \qquad \tau_f(\omega) := \tfrac{\overline{q}\,\overline{\mu}_1\,\omega + p\,\overline{\mu}_0\,\overline{\omega}}{\overline{r}(\omega)}\,, \\ &\tau_s(\omega) := \tfrac{\overline{q}\,\mu_1\,\omega + p\,\mu_0\,\overline{\omega}}{r(\omega)}, \quad \tau_c(\omega) := \lambda\tau_s(\omega) + \overline{\lambda}\tau_f(\omega). \end{aligned} \tag{4.3.1}$$

Note that in the above, superscripts j are omitted for clarity. The probabilistic meaning of these functions is described in the sequel. These are used to define recursions for updating the belief state. Each observation scheme entails a different type of recursion:

(II) State observation:

$$\big(\omega_1(t+1),\omega_2(t+1)\big)=\begin{cases}\big(X_1(t),\tau_n^{(2)}\big(\omega_2(t)\big), & U(t)=1,\\ \big(\tau_n^{(1)}\big(\omega_1(t)\big),X_2(t)\big), & U(t)=2.\end{cases}$$

(III) Output observation:

$$\big(\omega_1(t+1),\omega_2(t+1)\big)=\begin{cases}\big(\tau_f^{(1)}(\omega_1(t)),\tau_n^{(2)}\big(\omega_2(t)\big)\big), & I^{(1)}_{X_1(t)}(t)=0,\\ & \qquad U(t)=1,\\ \big(\tau_s^{(1)}(\omega_1(t)),\tau_n^{(2)}\big(\omega_2(t)\big)\big), & I^{(1)}_{X_1(t)}(t)=1,\\ \big(\tau_n^{(1)}\big(\omega_1(t)\big),\tau_f^{(2)}(\omega_2(t))\big), & I^{(2)}_{X_2(t)}(t)=0,\\ & \qquad U(t)=2.\\ \big(\tau_n^{(1)}\big(\omega_1(t)\big),\tau_s^{(2)}(\omega_2(t))\big), & I^{(2)}_{X_2(t)}(t)=1,\end{cases}$$

(IV) Queue observation:

$$\big(\omega_1(t+1),\omega_2(t+1)\big)=\begin{cases}\big(\tau_f^{(1)}(\omega_1(t)),\tau_n^{(2)}\big(\omega_2(t)\big)\big), & \Delta Q(t)=1,\\ \big(\tau_c^{(1)}(\omega_1(t)),\tau_n^{(2)}\big(\omega_2(t)\big)\big), & \Delta Q(t)=0, \qquad U(t)=1,\\ \big(\tau_s^{(1)}(\omega_1(t)),\tau_n^{(2)}\big(\omega_2(t)\big)\big), & \Delta Q(t)=-1,\\ \big(\tau_n^{(1)}\big(\omega_1(t)\big),\tau_f^{(2)}(\omega_2(t))\big), & \Delta Q(t)=1,\\ \big(\tau_n^{(1)}\big(\omega_1(t)\big),\tau_c^{(2)}(\omega_2(t))\big), & \Delta Q(t)=0, \qquad U(t)=2.\\ \big(\tau_n^{(1)}\big(\omega_1(t)\big),\tau_s^{(2)}(\omega_2(t))\big), & \Delta Q(t)=-1,\end{cases}$$

Upon applying the recursions above, we indeed track the belief state as needed.

Proposition 4.3.1. *For each of the observation schemes, assume that at* $t=0$, $\omega_j(0)=\mathbb{P}\big(X_j(0)=1\big)$. *Then upon implementing the recursion above, based on the observations, it holds that*

$$\omega_j(t)=\mathbb{P}\big(X_j(t)=1\,|Y(t),Y(t-1),\dots,Y(0)\big),\qquad t=1,2,\dots.$$

Proof. The proof and derivation of the operators in (4.3.1) follows from elementary conditional probabilities and induction. We illustrate this for the output observation case here. It

holds that

$$\mathbb{P}\Big(X(t)=1 \;\Big|\; I(t-1)=0\Big) = \frac{\mathbb{P}\left(X(t)=1, I(t-1)=0\right)}{\mathbb{P}\left(I(t-1)=0\right)}$$
$$= \frac{\mathbb{P}\left(X(t)=1, I=0 \;\middle|\; X=1\right)\mathbb{P}\left(X=1\right) + \mathbb{P}\left(X(t)=1, I=0 \;\middle|\; X=0\right)\mathbb{P}\left(X=0\right)}{\mathbb{P}\left(I=0 \;\middle|\; X=1\right)\mathbb{P}\left(X=1\right) + \mathbb{P}\left(I=0 \;\middle|\; X=0\right)\mathbb{P}\left(X=0\right)},$$

where we denote $X = X(t-1)$ and $I = I(t-1)$. Since $X(t)$ and $I(t-1)$ are conditionally independent given $X(t-1)$, the above numerator can be written as:

$$\mathbb{P}\left(X(t)=1 \;\middle|\; X=1\right)\mathbb{P}\left(I=0 \;\middle|\; X=1\right)\mathbb{P}\left(X=1\right)$$
$$+\,\mathbb{P}\left(X(t)=1 \;\middle|\; X=0\right)\mathbb{P}\left(I=0 \;\middle|\; X=0\right)\mathbb{P}\left(X=0\right) = \overline{q}\,\overline{\mu}_1\,\omega + p\,\overline{\mu}_0\,\overline{\omega}.$$

Similarly, for the denominator we have:

$$\mathbb{P}\left(I=0 \;\middle|\; X=1\right)\mathbb{P}\left(X=1\right) + \mathbb{P}\left(I=0 \;\middle|\; X=0\right)\mathbb{P}\left(X=0\right) = \overline{\mu}_1\omega + \overline{\mu}_0\overline{\omega},$$

which is equal to $\overline{r}(\omega)$. Hence as expected, we find that

$$\mathbb{P}\Big(X(t)=1 \;\Big|\; I(t-1)=0\Big) = \tau_f(\omega).$$

The derivation of $\tau_n(\omega)$ and $\tau_s(\omega)$ follows similar lines. For the queue observation case, notice that

$$\mathbb{P}\Big(X(t)=1 \;\Big|\; \Delta Q(t)=1\Big) = \tau_f(\omega)\,, \qquad \mathbb{P}\Big(X(t)=1 \;\Big|\; \Delta Q(t)=-1\Big) = \tau_s(\omega)\,,$$
$$\mathbb{P}\Big(X(t)=1 \;\Big|\; \Delta Q(t)=0\Big) = \lambda\tau_s(\omega) + \overline{\lambda}\tau_f(\omega) = \tau_c(\omega).$$

The state observation case follows similar lines. □

Note that the fixed point of τ_n is the stationary probability γ. The fixed points of τ_f and τ_s are also of interest. When $\rho \neq 0$ and $\mu_0 \neq \mu_1$, τ_f and τ_s are (real) hyperbolic Möbius transformations of the form $(a\omega+b)/(c\omega+d)$ for $\omega \in [0,1]$. As such, they each have two distinct fixed points, one stable and one unstable. Here, excluding trivialities where $p, q \in \{0,1\}$, the stable fixed point of each lies in $(0,1)$ and is of the form $\left(a-d+\sqrt{(a-d)^2+4bc}\right)/2c$ (see also Lemma 2 and 3 of [134]). For τ_f, we have $a = \overline{q}\,\overline{\mu}_1 - p\,\overline{\mu}_0$, $b = p\,\overline{\mu}_0$, $c = \overline{\mu}_1 - \overline{\mu}_0$, and $d = \overline{\mu}_0$. Fixed point of τ_s comes from the same formula by replacing $\overline{\mu}_i$ by μ_i.

Denote by $\omega_s^{(j)}$ and $\omega_f^{(j)}$, the stable fixed point of $\tau_s^{(j)}$ and $\tau_f^{(j)}$, respectively for $j = 1, 2$. Then

$$\Omega = \Omega_1 \times \Omega_2 \subset [0,1] \times [0,1],$$

where we put

$$\Omega_j = [\min(\omega_f^{(j)}, \omega_s^{(j)}),\ \max(\omega_f^{(j)}, \omega_s^{(j)})],$$

is the *belief state space* and the limit of any infinite subsequence of the mappings τ_n, τ_f, τ_s and τ_c (for $\omega_1, \omega_2 \in [0,1]$) lies within Ω, see [146] for more details.

4.4 Maximal Throughput

Having defined sufficient statistics for the belief state and their evolution, the problem of finding a maximally stabilizing control can be posed as a POMDP, see for example [26] or the historical reference [184]. The objective for the POMDP is

$$\mu^* = \sup_{\pi} \liminf_{T\to\infty} \frac{1}{T} \mathbb{E}_\pi \Big[\sum_{t=0}^{T-1} I(t) \Big],$$

where $U(t) = \pi\big(\omega_1(t), \omega_2(t)\big)$ influences the $I(t)$ as outlined in Section 2. We note that a formal treatment of the POMDP in the *average reward* case, including the validity of the *Bellman equations* below is a mathematically delicate matter, requiring investigation in its own right. See [169] for an account of average reward criteria. It is not the focus of the current research. Nevertheless, using the Bellman equations as we do below, makes sense from a practical perspective.

The Myopic Policy

One specific policy is the *myopic policy* given by:

$$\pi(\omega_1, \omega_2) = \begin{cases} \text{Server 2} & \text{if} \quad \omega_2 \geq \frac{\mu_1^{(1)} - \mu_0^{(1)}}{\mu_1^{(2)} - \mu_0^{(2)}}\, \omega_1 + \frac{\mu_0^{(1)} - \mu_0^{(2)}}{\mu_1^{(2)} - \mu_0^{(2)}}, \\ \text{Server 1} & \text{if} \quad \text{otherwise.} \end{cases} \tag{4.4.1}$$

The affine threshold in this policy is obtained by comparing the immediate expected mean

throughput for any given pair (ω_1, ω_2) and choosing the server that maximizes it. Such a policy is attractive in that it is easy to implement. Further, when the servers are symmetric (all parameters are identical), it holds from symmetry that it is optimal. In this case, it can be represented as

$$\pi(\omega_1, \omega_2) = \text{argmax}_{i=1,2}\, \omega_i,$$

and we refer to it as the *symmetric myopic policy*.

Simulation Result

Figure 4.3 demonstrates results obtained through a Monte Carlo simulation[2] of the model for observation schemes (I)–(V). We use the parameters in (4.2.5) and vary ρ with $\rho_1 = \rho_2 = \rho$ in the range $[0, 1]$ with steps of 0.01. The policy used is the symmetric myopic policy and is optimal since the servers are identical.

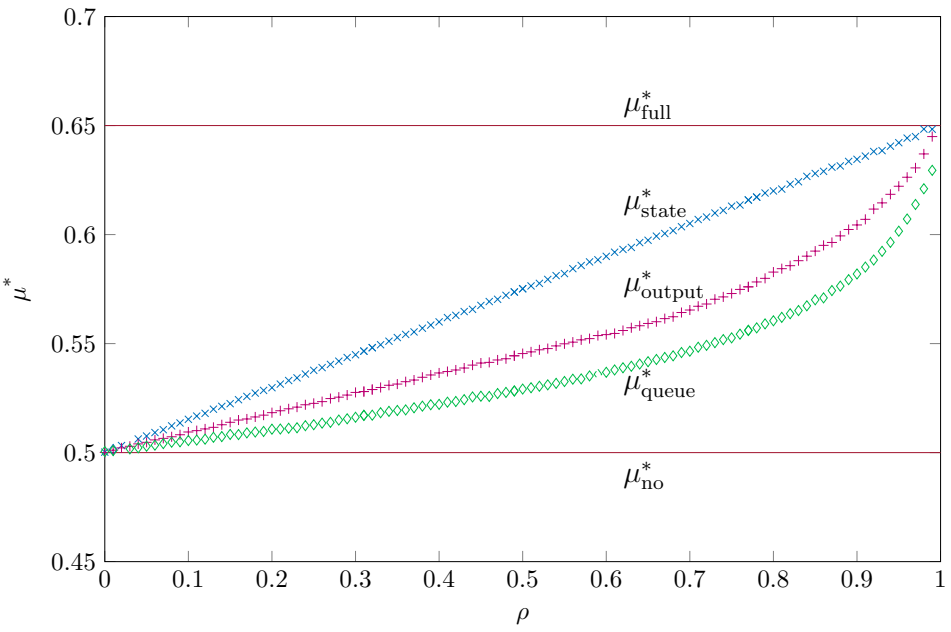


Figure 4.3: The stability bound is displayed as a function of ρ for the various observation schemes. This plot is based on simulation results using the parameters in (4.2.5) and $\rho_1 = \rho_2 = \rho$.

As we see from the figure, the ordering (4.2.3) appears to hold. Further, at the i.i.d. case $\rho = 0$, historical observations are not useful and the throughput of all observation schemes, except for full observation, is at 0.5. At the other extreme, when $\rho \to 1$, we have that the

[2]Simulation details: We run the process for $t = 5,000,000$ time units, using common random numbers for each run and recording average throughput.

state observation scheme converges to a throughput identical to that of the full observation scheme. This is because, at that regime, the server state environments rarely change. Thus, from a throughput perspective, the controller behaves as though it has full information. On the other hand, even at $\rho = 1$, the output observation scheme and queue observation scheme still perform at a lower throughput. Finally, for $\rho \in (0, 1)$ it is evident that there is a gap in performance for each observation scheme. This gap quantifies the value of information in controlling our model and motivates further analysis.

Bellman Equations

It is well-known that the optimal policy that maximizes the throughput follows from the *average reward Bellman equations*. See [169] for background on Markov decision processes (MDP) or [89] for a discussion of average reward optimality with such state spaces. The *Bellman equation* is then

$$\mu^* + h(\omega_1, \omega_2) = \max\left\{h^{(1)}(\omega_1, \omega_2),\, h^{(2)}(\omega_1, \omega_2)\right\},$$

where h is the relative value function and the individual components $h^{(j)}(\cdot, \cdot)$, vary as follows:

(II) State observation:

$$\begin{aligned} h^{(1)}(\omega_1, \omega_2) &:= r^{(1)}(\omega_1) + \left[\overline{\omega}_1\, h\big(p_1, \tau_n^{(2)}(\omega_2)\big) + \omega_1\, h\big(\overline{q}_1, \tau_n^{(2)}(\omega_2)\big)\right], \\ h^{(2)}(\omega_1, \omega_2) &:= r^{(2)}(\omega_2) + \left[\overline{\omega}_2\, h\big(\tau_n^{(1)}(\omega_1), p_2\big) + \omega_2\, h\big(\tau_n^{(1)}(\omega_1), \overline{q}_2\big)\right]. \end{aligned}$$

(III) Output observation:

$$\begin{aligned} h^{(1)}(\omega_1, \omega_2) &:= r^{(1)}(\omega_1) + \left[\overline{r}^{(1)}(\omega_1)\, h\big(\tau_f^{(1)}(\omega_1), \tau_n^{(2)}(\omega_2)\big) + r^{(1)}(\omega_1)\, h\big(\tau_s^{(1)}(\omega_1), \tau_n^{(2)}(\omega_2)\big)\right], \\ h^{(2)}(\omega_1, \omega_2) &:= r^{(2)}(\omega_2) + \left[\overline{r}^{(2)}(\omega_2)\, h\big(\tau_n^{(1)}(\omega_1), \tau_f^{(2)}(\omega_2)\big) + r^{(2)}(\omega_2)\, h\big(\tau_n^{(1)}(\omega_1), \tau_s^{(2)}(\omega_2)\big)\right]. \end{aligned}$$

(IV) Queue observation:

$$\begin{aligned} h^{(1)}(\omega_1, \omega_2) := r^{(1)}(\omega_1) + \Big[&\lambda \overline{r}^{(1)}(\omega_1)\, h\big(\tau_f^{(1)}(\omega_1), \tau_n^{(2)}(\omega_2)\big) + \overline{\lambda} r^{(1)}(\omega_1)\, h\big(\tau_s^{(1)}(\omega_1), \tau_n^{(2)}(\omega_2)\big) \\ &+ \Big(\overline{\lambda}\overline{r}^{(1)}(\omega_1) + \lambda r^{(1)}(\omega_1)\Big)\Big(h(\tau_c^{(1)}(\omega_1), \tau_n^{(2)}(\omega_2))\Big)\Big], \end{aligned}$$

$$h^{(2)}(\omega_1,\omega_2) := r^{(2)}(\omega_2) + \Big[\lambda\overline{r}^{(2)}(\omega_2)\, h\big(\tau_n^{(1)}(\omega_1),\tau_f^{(2)}(\omega_2)\big) + \overline{\lambda} r^{(2)}(\omega_2)\, h\big(\tau_n^{(1)}(\omega_1),\tau_s^{(2)}(\omega_2)\big) + \Big(\overline{\lambda}\overline{r}^{(1)}(\omega_2) + \lambda r^{(1)}(\omega_2)\Big)\Big(h(\tau_n^{(1)}(\omega_1),\tau_c^{(2)}(\omega_2))\Big)\Big].$$

The optimal decision is then to choose Server 1 if and only if $h^{(1)}(\omega_1,\omega_2) \geq h^{(2)}(\omega_1,\omega_2)$, breaking ties arbitrarily. Since $\tau_n^{(j)}(0) = \tau_s^{(j)}(0) = \tau_f^{(j)}(0) = p_j$ and $\tau_n^{(j)}(1) = \tau_s^{(j)}(1) = \tau_f^{(j)}(1) = \overline{q}_j$, for all three aforementioned cases we have:

$$\begin{aligned} \mu^* + h(0,0) &= \max\big\{\mu_0^{(1)} + h(p_1,p_2), \mu_0^{(2)} + h(p_1,p_2)\big\}, \\ \mu^* + h(1,1) &= \max\big\{\mu_1^{(1)} + h(\overline{q}_1,\overline{q}_2), \mu_1^{(2)} + h(\overline{q}_1,\overline{q}_2)\big\}. \end{aligned} \tag{4.4.2}$$

From the ordering in (4.2.1), the above equations imply that at point $(\omega_1,\omega_2) = (0,0)$, choosing Server 1 is optimal and at point $(\omega_1,\omega_2) = (1,1)$ choosing Server 2 is optimal. This observation gives some initial insight into the structure of optimal policies. We now purse these further numerically.

Numerical Investigation of Optimal Policies

A solution to the above Bellman equations can be obtained numerically using relative value iteration and discretization of the belief state space, Ω. Here, we applied the MDP toolbox in Matlab (see [55]) by considering that each interval $[0,1]$ for ω_1 and ω_2 is partitioned to 1000 equal sub-intervals. We then run relative value iteration with an accepted error set to 10^{-4}.

Our various numerical experiments indicate the following:

1. The ordering in (4.2.3) holds.
2. Increasing (positive) ρ_j always yields an increase in μ^*.
3. Though the myopic policy does not appear to be generally optimal, when both servers are identical, the optimal policy is the symmetric myopic policy. See Figure 4.4.
4. In all cases, the optimal policy is given by a non-decreasing switching curve within Ω.

That is, there exists a function $\omega_2^*(\omega_1)$ where the optimal policy is

$$\pi(\omega_1, \omega_2) = \begin{cases} \text{Server 2} & \text{if} \quad \omega_2 \geq \omega_2^*(\omega_1), \\ \text{Server 1} & \text{if} \quad \text{otherwise.} \end{cases}$$

5. When the ordering in (4.2.1) has strict inequalities, $\omega_2^*(0) > 0$ and $\omega_2^*(1) < 1$.

6. For identical servers, it holds that the switching curve for the output observation case is sandwiched between the switching curve of the state observation case and the myopic switching line (4.4.1).

7. The switching curve for the queue observation case depends on λ. Further, when λ is at either of the extreme points ($\lambda = 0$ or $\lambda = 1$), the queue observation case agrees with the output observation case.

8. Increasing spread of servers ε_j, always yields an increase in μ^*. Figure 4.4 illustrates the switching curve derived by considering $\varepsilon_1 = 0.3$, $\gamma = \rho = 0.5$ for both servers, and increasing values of ε_2.

9. As expected, when one of the servers is considered with $\rho = 0.5$, $\gamma = 0.5$, and $\mu_0 = \mu_1$ that server acts like a safe server, and the results are the same as the results in [146].

10. For identical servers with $\gamma = 0.5$, when $\rho \to 1$ and $\varepsilon \to 0.5$, as expected from Binomial distribution (see Eq. (4.2.4)), $\mu^* \to 0.75$.

As one illustration of some of the above properties, consider Table 4.1 based on the parameters of (4.2.5) and various values of ρ_1 and ρ_2. The results in the table further affirm comments 1 and 3 above and contains values that agree with Monte Carlo simulation results, similar to those of Figure 4.3.

ρ_1	ρ_2	μ^*_{state}	μ^*_{output}	μ^*_{queue}
0.2	0.5	0.5543	0.5314	0.5190
0.4	0.5	0.5673	0.5400	0.5231
0.6	0.5	0.5823	0.5489	0.5289
0.8	0.5	0.6009	0.5647	0.5360

Table 4.1: Stability region bounds for observation schemes (II)-(IV) for various ρ_1 and ρ_2 values. Note the queue observation case is with $\lambda = 0.5$.

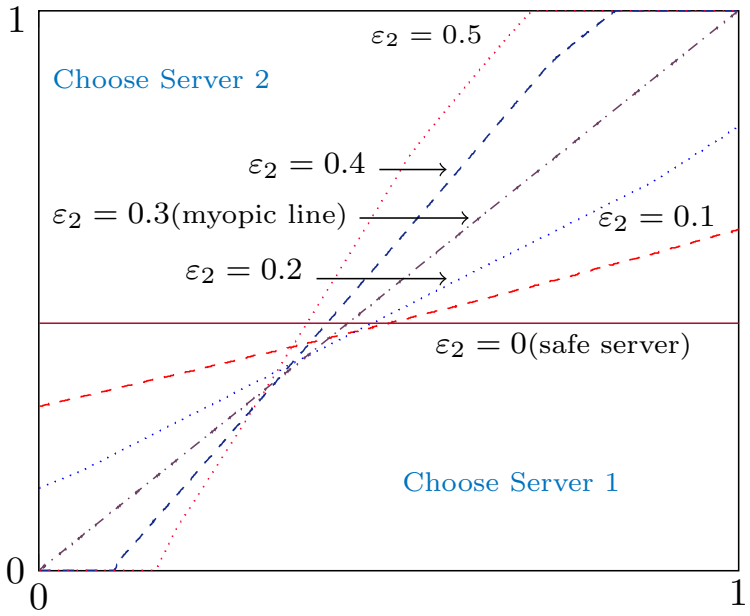


Figure 4.4: Switching curve for increasing spread.

As a further illustration, Figure 4.5 shows switching curves, $w_2^*(\cdot)$ for the parameters of (4.2.5) with $\rho_1 = 0.5$ and $\rho_2 = 0.7$. In the figure, the red dotted line is the myopic policy line (suboptimal). The solid blue curve is the switching curve for the output observation case. The green loosely dashed line is related to the queue observation case and the orange densely dashed curve is the switching curve for the state observation case. These curves were obtained by finding the optimal decision for every (discretized) element of Ω and then observing the switching curve structure.

4.5 QBD Structured Models for Finite State Controllers

Here we illustrate how matrix-analytic methods (MAM) can be used to analyse a finite state controller that approximates an optimal controller. Our analysis is for the output observation scheme (Case III). A similar analysis can be applied to the other observation schemes as well.

A finite state controller operates by using a finite discrete belief state $\tilde{\Omega}$, representing a discrete grid in Ω. With such a controller, we consider the whole system as a quasi-birth-and-death (QBD) process (for more details about QBD process see for example [119]). Using the QBD structure, we find a matrix analytic expression for μ^*_{output} (denoted by μ^* in this

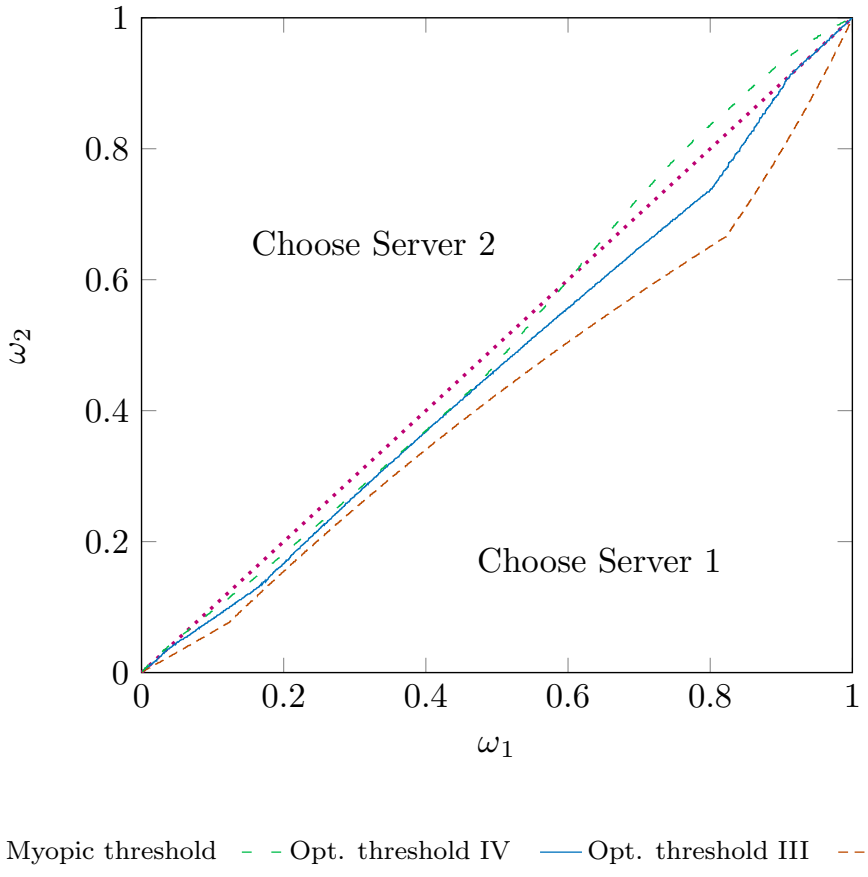


Figure 4.5: Myopic and optimal policies for the state observation Case II, output observation Case III and queue observation Case IV ($\lambda = 0.5$). This is for a system with $\rho_1 = 0.5$ and $\rho_2 = 0.7$.

section).

Take $\tilde{\Omega} = \{1, \ldots, M\}^2$ and define the controller state at time t by $(\psi_1(t), \psi_2(t)) \in \tilde{\Omega}$. In doing so, we treat $\psi_j(t)$ as $\lceil M\omega_j(t) \rceil$. The controller action is (potentially) randomized based on a matrix of probabilities C so that Server 2 is chosen with probability $C_{(\psi_1(t),\psi_2(t))}$ and otherwise the choice is Server 1. That is, the matrix C is a randomized control policy. Such a policy encodes information as in Figure 4.5.

The controller state is updated in a (potentially) randomized manner based on the $M \times M$ stochastic matrices $N^{(j)}, S^{(j)}, F^{(j)}$ for $j = 1, 2$. The rows of matrices $N^{(j)}, S^{(j)}$ and $F^{(j)}$ indicate how to (potentially randomly) choose the next controller state. Here, N stands for No service, S for Success, and F for Failure as follows: if Server 1 was not selected (no service either because there were no jobs in the queue, or because the other server was selected), the distribution of the new state is $\left(N^{(1)}_{\psi_1(t),1}, \ldots, N^{(1)}_{\psi_1(t),M}\right)$; that is taken from the row indexed by $\psi_1(t)$. Similarly, if Server 1 was chosen and service was successful ($I = 1$),

the distribution of the new state is $\big(S^{(1)}_{\psi_1(t),1}, \ldots, S^{(1)}_{\psi_1(t),M}\big)$. Finally, if Server 1 was chosen and the service failed ($I = 0$), the distribution of the new state is $\big(F^{(1)}_{\psi_1(t),1}, \ldots, F^{(1)}_{\psi_1(t),M}\big)$. Similarly, for Server 2, we have $\big(N^{(2)}_{1,\psi_2(t)}, \ldots, N^{(2)}_{M,\psi_2(t)}\big)$, $\big(S^{(2)}_{1,\psi_2(t)}, \ldots, S^{(2)}_{M,\psi_2(t)}\big)$ and $\big(F^{(2)}_{1,\psi_2(t)}, \ldots, F^{(2)}_{M,\psi_2(t)}\big)$, respectively.

We construct the matrices $N^{(j)}, S^{(j)}, F^{(j)}$ based on a discretization of $\tau_n^{(j)}, \tau_s^{(j)}$ and $\tau_f^{(j)}$, respectively. For example, construction of S from τ_s is as follows: construct the row elements of S by putting $j = M\tau_s\big(\frac{i-1}{M}\big)$ for each $i = 1, \cdots, M$ and then put

$$\begin{cases} S_{i,j} = 1 & j \text{ is an integer}, \\ S_{i,\lfloor j \rfloor} = 1 & 1 \leq \lfloor j \rfloor \leq M, \\ S_{i,\lceil j \rceil} = 1 & \text{otherwise}, \end{cases}$$

and $S_{i,k} = 0$ for all other row elements with $k \neq j$ and $k = 1, \ldots, M$. After this, we ensure irreducibility of this matrix by fixing $\epsilon > 0$ (for instance $\epsilon = 0.001$ in our numerical examples) and adding ϵ/M to each element of the matrix and then renormalizing it.

The matrices F and N are constructed similarly based on τ_f and τ_n, respectively. This is simply a mechanism to encode the transition operators over the finite grid. Hence the matrices $N^{(j)}, S^{(j)}, F^{(j)}$ describe the propagation of ψ_j through the belief operators, similarly to the propagation of ω through their continuous counterparts.

Now, given such a controller with

$$\text{Controller parameters} = \Big(N^{(1)}, S^{(1)}, F^{(1)}, N^{(2)}, S^{(2)}, F^{(2)}, C\Big),$$

we construct a Markov chain, $Z(t)$ for the system. The state of this model at time t is given by the queue length, server environment state, and controller state as follows:

$$Z(t) = \Big(\underbrace{Q(t)}_{\text{Level}}, \underbrace{\big(\overbrace{\big(X_1(t), X_2(t)\big)}^{\text{Servers}}, \overbrace{\big(\psi_1(t), \psi_2(t)\big)}^{\text{Controller}}\big)}_{\text{Phase}}\Big) \in \{0, 1, \ldots\} \times \{1, 2\}^2 \times \{1, \ldots, M\}^2.$$

4.5.1 Explicit QBD Construction

When the states of $Z(t)$ are lexicographically ordered, with first component countably infinite (levels) and the other components finite (phases), the resulting (infinite) probability transition matrix is of the QBD form:

$$A = \begin{bmatrix} B_0 & B_1 & & & 0 \\ B_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & A_{-1} & A_0 & A_1 \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{4.5.1}$$

where B_{-1}, B_0, B_1, A_{-1}, A_0, and A_1 are block-matrices. The matrix A_{-1} represents the phase transition where there is a one level decrease. Similarly, the matrix A_1 represents phase transition where there is a one level increase, and A_0 represents the phase transition where the level remains the same. The corresponding matrices for level 0 are B_i for $i = -1, 0, 1$, as in Section 2.3.3.

QBD Construction for a Model with a Safe Server

For the sake of simplicity, let us start with the toy model where one of the servers has only one state as depicted in Figure 4.6. Such a server is called a *safe server*.

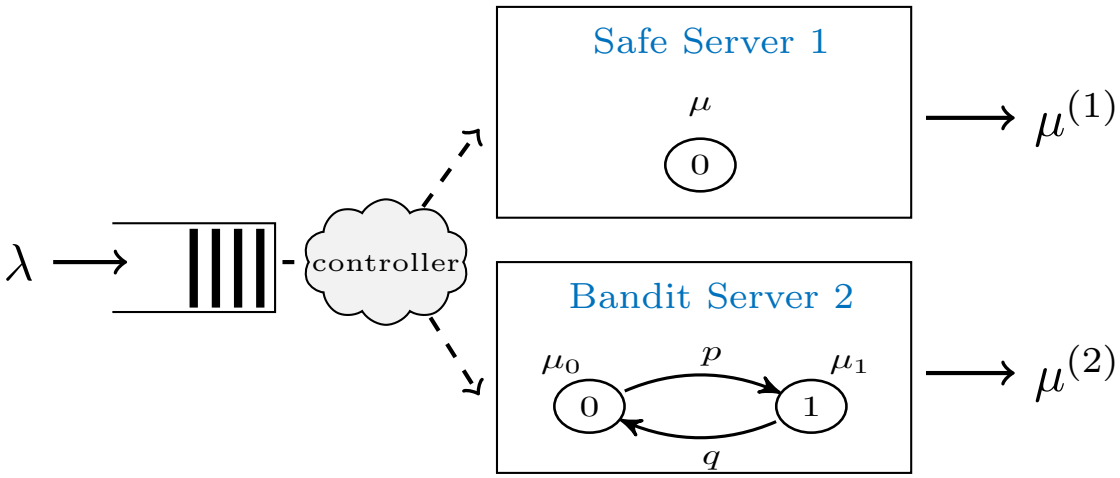


Figure 4.6: Parameters of the toy model.

For such a simple system, the controller parameters are

$$\text{Controller parameters} = \Big(S, S^{(2)}, F^{(2)}, \mathbf{c}\Big),$$

where $\mathbf{c}$ is the vector of probabilities representing the controller action. The bandit server is chosen for service with probability c_i, otherwise (with probability $1 - c_i$) the safe server is chosen for service. The matrix S shows the probability transitions of the controller's states when the safe server is chosen. The matrices $S^{(2)}$ and $F^{(2)}$ are the probability transition matrices of the controller's states where the second server is chosen, and there is a success or failure, respectively. The constructed Markov chain $Z(t)$ for this system is given by:

$$Z(t) = \Big(\underbrace{Q(t)}_{\text{Level}}, \underbrace{(\overbrace{X(t)}^{\text{Bandit}}, \overbrace{\psi(t)}^{\text{Controller}})}_{\text{Phase}} \Big) \in \{0, 1, \ldots\} \times \{1, 2\} \times \{1, \ldots, M\}.$$

The block matrices in (4.5.1) are of size $2M \times 2M$ as we construct now. For $\ell \in \{1, 2\}$, denote

$$\tilde{S}_\ell = \mu_\ell \,\mathsf{diag}(\mathbf{c}) S^{(2)} + \mu \,\mathsf{diag}(\overline{\mathbf{c}}) S\,,$$
$$\tilde{F}_\ell = \overline{\mu}_\ell \,\mathsf{diag}(\mathbf{c}) F^{(2)} + \overline{\mu}_s \,\mathsf{diag}(\overline{\mathbf{c}}) S\,.$$

Here $\tilde{S}_\ell(i, j)$ is the probability of updating controller state from i to j when the server state is ℓ and the transmission was successful. Similarly, $\tilde{F}_\ell(i, j)$ is the probability of doing so when transmission was unsuccessful.

Now define the $2M \times 2M$ matrices,

$$\tilde{S} = \begin{bmatrix} \overline{p}\,\tilde{S}_1 & p\,\tilde{S}_1 \\ q\,\tilde{S}_2 & \overline{q}\,\tilde{S}_2 \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} \overline{p}\,\tilde{F}_1 & p\,\tilde{F}_1 \\ q\,\tilde{F}_2 & \overline{q}\,\tilde{F}_2 \end{bmatrix}, \quad \overline{S} = \begin{bmatrix} \overline{p}\,S & p\,S \\ q\,S & \overline{q}\,S \end{bmatrix}.$$

These are combined to form the basic elements of the QBD transition probability matrix (4.5.1) of the (complicated) Markov chain $Z(t)$ as:

$$B_0 = \overline{\lambda}\,\overline{S}, \quad B_1 = \lambda\overline{S}, \quad B_{-1} = A_{-1} = \overline{\lambda}\tilde{S}, \quad A_0 = \overline{\lambda}\tilde{F} + \lambda\tilde{S}, \quad A_1 = \lambda\tilde{F}.$$

QBD Construction for a Two-State Model

For our model with two bandit server, the block matrices of (4.5.1) are $4M^2 \times 4M^2$ matrices that constructed as follows:

$$B_0 = \overline{\lambda}\tilde{N}, \quad B_1 = \lambda\tilde{N},$$
$$A_{-1} = \overline{\lambda}\tilde{S}, \quad A_0 = \overline{\lambda}\tilde{F} + \lambda\tilde{S}, \quad A_1 = \lambda\tilde{F}, \tag{4.5.2}$$

where the matrices, $\tilde{S}, \tilde{F}, \tilde{N}$ (each of order $4M^2$) denote the change of phase together with a service success, service failure or no service attempt, respectively. For instance, the (i,j)-th entry of $\tilde{S}$ is the chance of a service success together with a change of phase from i to j (note that i and j are each 4-tuples). The sum $\tilde{S} + \tilde{F}$ is a stochastic matrix (as is evident from the construction below). Similarly, $\tilde{N}$ is a stochastic matrix. Hence the overall transition probability (infinite) matrix A is stochastic as well.

To construct $\tilde{S}, \tilde{F}$ and $\tilde{N}$, we define $M^2 \times M^2$ matrices $\tilde{S}_{\_k\ell}$, $\tilde{F}_{\_k\ell}$ and $\tilde{N}_{\_k\ell}$ for $k, \ell = 0, 1$. Taking $\tilde{S}_{\_k\ell}$ as an example, its (i,j)-th entry (each represented as a 2-tuple), describes the chance of success together with a transition of belief state from i to j, when the environment of the first server is in state k and that of the second server is in state ℓ. Here i and j, each represent the overall system belief state in lexicographic order. That is, we should refer to i as (i_1, i_2) and similarly to j. A similar interpretation holds for $\tilde{F}_{\_k\ell}$ and $\tilde{N}_{\_k\ell}$. These aforementioned matrices are constructed (for $k, \ell = 0, 1$) as follows:

$$\begin{aligned}
\tilde{S}_{_k\ell} &= \mu_\ell^{(2)} \left(\mathsf{diag}\big(vec(C')\big)\right)(N^{(1)} \otimes S^{(2)}) + \mu_k^{(1)} \left(\mathsf{diag}\big(vec(\overline{C}')\big)\right)(S^{(1)} \otimes N^{(2)}), \\
\tilde{F}_{_k\ell} &= \overline{\mu}_\ell^{(2)} \left(\mathsf{diag}\big(vec(C')\big)\right)(N^{(1)} \otimes F^{(2)}) + \overline{\mu}_k^{(1)} \left(\mathsf{diag}\big(vec(\overline{C}')\big)\right)(F^{(1)} \otimes N^{(2)}), \\
\tilde{N}_{_k\ell} &= (N^{(1)} \otimes N^{(2)}),
\end{aligned}$$

where $vec(\cdot)$ is an operation taking a matrix and resulting in a vector with the columns of the matrix stacked up one by one and $\otimes$ is the standard Kronecker product.

To see the above, let us consider (for example) an element of the matrix $\tilde{S}_{\_k\ell}$ at coordinate

$i = (i_1, i_2)$ and $j = (j_1, j_2)$. This describes the probability of the event

$$W = \{\text{Success of service together with a transition to belief state } (j_1, j_2)\},$$

where $X_1 = k$, $X_2 = \ell$, $\psi_1 = i_1$, and $\psi_2 = i_2$. The event W can be partitioned into W_1 (service attempt was on 1) and W_2 (service attempt was on 2). The chance of W_2 is C_{i_1,i_2}. With choosing Server 2 the success probability is $\mu_\ell^{(2)}$. Then under the event W_2, the belief state of Server 1 will be updated according to $N^{(1)}$ and the belief state of Server 2 with $S^{(2)}$. The $M^2 \times M^2$ matrix diag$(vec(C'))$ is a diagonal matrix where its diagonal elements are the rows of the matrix C, each represent the chance of $U = 2$.

With the matrices $\tilde{S}_{\_k\ell}$, $\tilde{F}_{\_k\ell}$ and $\tilde{N}_{\_k\ell}$ (for $k, \ell = 0, 1$) in hand, we construct the matrices $\tilde{S}$, $\tilde{F}$ and $\tilde{N}$ as:

$$\tilde{S} = (P^{(1)} \otimes P^{(2)}) \circledast \begin{bmatrix} \tilde{S}_{\_00} & 0 & 0 & 0 \\ 0 & \tilde{S}_{\_01} & 0 & 0 \\ 0 & 0 & \tilde{S}_{\_10} & 0 \\ 0 & 0 & 0 & \tilde{S}_{\_11} \end{bmatrix},$$

$$\tilde{F} = (P^{(1)} \otimes P^{(2)}) \circledast \begin{bmatrix} \tilde{F}_{\_00} & 0 & 0 & 0 \\ 0 & \tilde{F}_{\_01} & 0 & 0 \\ 0 & 0 & \tilde{F}_{\_10} & 0 \\ 0 & 0 & 0 & \tilde{F}_{\_11} \end{bmatrix},$$

$$\tilde{N} = (P^{(1)} \otimes P^{(2)}) \circledast \begin{bmatrix} \tilde{N}_{\_00} & 0 & 0 & 0 \\ 0 & \tilde{N}_{\_01} & 0 & 0 \\ 0 & 0 & \tilde{N}_{\_10} & 0 \\ 0 & 0 & 0 & \tilde{N}_{\_11} \end{bmatrix},$$

where $P^{(j)}$ for $j = 1, 2$ are the 2×2 probability transition matrices of the servers given by (4.2.2). The operation $\circledast$, for a given $n \times n$ matrix U and $n \times n$ diagonal block matrix V is

defined as follows.

$$U \circledast V = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \cdots & \ldots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix} \circledast \begin{bmatrix} V_1 & 0 & \cdots & \\ 0 & V_2 & \cdots & 0 \\ \vdots & \cdots & \ldots & \vdots \\ 0 & 0 & \cdots & V_n \end{bmatrix} = \begin{bmatrix} u_{11}V_1 & u_{12}V_1 & \cdots & u_{1n}V_1 \\ u_{21}V_2 & u_{22}V_2 & \cdots & u_{2n}V_2 \\ \vdots & \cdots & \ldots & \vdots \\ u_{n1}V_n & u_{n2}V_n & \cdots & u_{nn}V_n \end{bmatrix}.$$

Applying $\circledast$ multiplies each element of the i-th row of matrix U (which is a scalar) by the i-th diagonal-element of matrix V (which is a matrix) and set the resulting matrices as the i-th row of a new block matrix

Putting all of the above components together yields the probability transition matrix of $Z(t), A$.

Stability Criterion

A well-known sufficient condition for positive recurrence (stability) of QBDs such as $Z(t)$ is

$$\pi_\infty (A_1 - A_{-1})\mathbf{1} < 0,$$

where π_∞ is the stationary distribution of the (finite) stochastic matrix $A_{-1} + A_0 + A_1$ and $\mathbf{1}$ is a column vector of ones. From (4.5.2), we see that this is also the stationary distribution of $\tilde{S} + \tilde{F}$ which does not depend on λ. This property of our QBD allows us to represent the stability criterion as

$$\lambda < \mu^* = \pi_\infty \tilde{S}\mathbf{1}, \tag{4.5.3}$$

with μ^* depending on the controller and system parameters but not depending on λ.

In addition to the stability criteria, a further virtue of modelling the system as a QBD is that we can use the vast body of MAM knowledge and algorithms for analysing the system and ultimately optimizing controllers. Nevertheless, our focus in this paper is on stability.

Numerical Illustration

We now use our QBD model and the stability criterion (4.5.3) to explore the performance of finite state controllers.

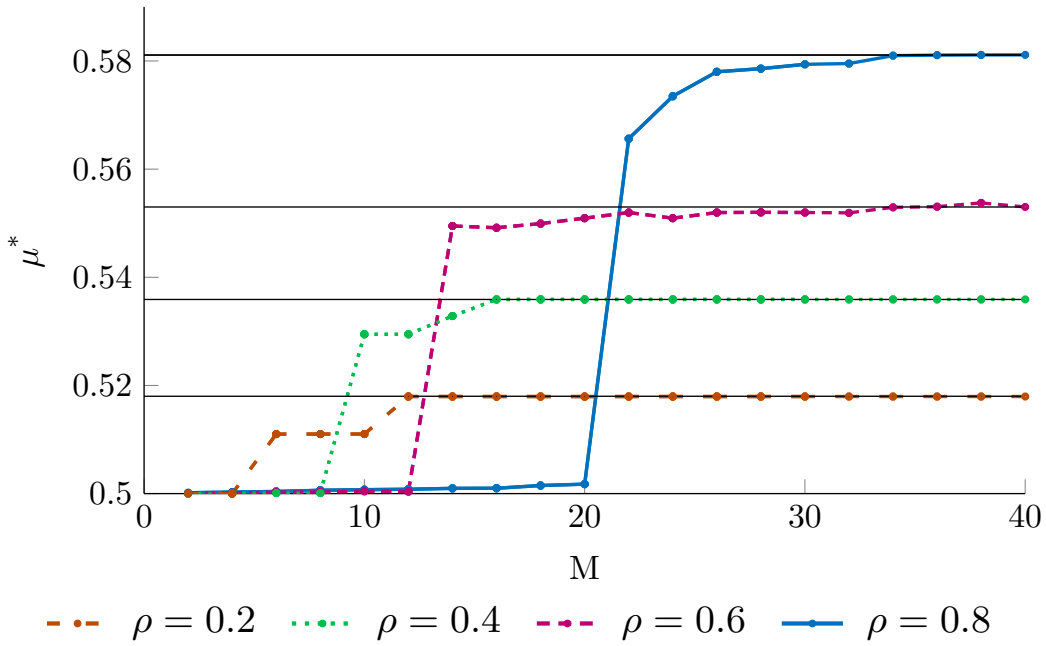


Figure 4.7: Stability bound achieved by finite state controllers for observation scheme III with increasing M, computed by (4.5.3). The limiting horizontal lines are at μ^* computed by means of relative value iteration of Bellman equations.

In doing so, we consider the parameters as in (4.2.5) with $\rho_1 = \rho_2 = \rho$. Since in this situation, the servers are identical, the symmetric myopic policy is optimal, and we thus restrict attention to a matrix C with

$$C_{i,j} = \begin{cases} 1, & i < j, \\ 0.5, & i = j, \\ 0, & i > j. \end{cases}$$

Using these parameters, we evaluated (4.5.3) for increasing M and various values of ρ. The results are in Figure 4.7. As expected, the performance of the finite state controller converges to that found by numerical solution of the Bellman equations as in the previous section. The sudden increase in performance (for instance, at $M = 20$ for $\rho = 0.8$) can be attributed to discretization phenomena. Discretization Phenomena is the error resulting from approximating a continuous function by finite number (here M) of evaluations and reduces when $M \to \infty$. For reference, the values of μ^* obtained by Bellman equation (as well as the QBD when $M \to \infty$) are 0.5179, 0.5359, 0.5539 and 0.5815 for $\rho = 0.2$, 0.4, 0.6 and 0.8, respectively.

Chapter 5

Semi-Markov Models

5.1 Introduction

In biostatistics, many of models for survival and reliability analysis are two-state stochastic processes which lead to a particular event such as death or outcome of a particular treatment or drug. However, applying a multi-state stochastic process allows the modeller to provide a richer and more accurate model by adding more details. These details are, for instance, some alternative paths to the event of interest, specify all the intermediate events, or partial failure modes in a progressive disease.

A multi-state stochastic process is a process $X(t)$ for $t \geq 0$, where $X(t)$ can take a finite number of values $1, 2, \cdots, p$. This process can be considered as a family of random variables $X(t)$ indexed by t. The quantities of interest are often the probability of being in a state at a given time and the distribution of first passage time (the time until the process reach a given state for the first time from a particular starting state).

In some applications of multi-state stochastic processes, the dependence on the history of the process is negligible. Therefore, for the sake of mathematical tractability, assuming the Markov property (where future transitions between states depend only upon the current state) is convenient. For instance, Markov chains are widely used in modelling the movements of patients between units of a hospital or between stages of a disease, see for instance [41, 189], or in the prediction of the incidence rate of diseases, see [2].

However, in certain cases the Markov assumption is unrealistic. For instance, mechanical or electronic component failure time, usually do not follow an exponential distribution and often have a heavy-tailed distribution such as Weibull or lognormal distribution, see for instance Chapter 2 of [25]. Further, some aspects of systems' behaviour can not be captured by ordinary Markov chains. For instance, the risk of chronic disease such as AIDS essentially depends on the time since infection, see [105]. For these cases, applying a *semi-Markov process* (SMP), as an extension of an ordinary Markov process, where future probability transitions depend on the sojourn time (the time spent in the current state), and the clock is reset to zero after each transition into a new state, seems a suitable choice. In fact, in semi-Markov processes states of the process follow the Markov property but the distribution of sojourn times are not necessarily exponential. SMPs have a variety of applications in healthcare. For instance, for predicting a disease progression [72], health care manpower supply prediction [192], and recovery progress of patients with a specific type of disease [167].

For biomedical applications, especially those concerned with characterizing an individual's progression through various stages of a disease, a three-state semi-Markov process known as the illness-death model is very popular, see for instance [36,105]. The illness-death model is applied for modelling the trajectory of patients in intensive care units (ICUs) of hospitals. Here, our main focus is on the statistical methodology of semi-Markov processes for this model.

We compare and contrast two approaches for defining SMPs. The first approach is based on knowing sojourn times, and the other one is based on knowing the transition rates. When it comes to parameter estimation for these two approaches, we see that the second approach has some advantages over the first approach. The second approach can be expressed by using fewer parameters. On the other hand, specifically for the popular illness-death model, we can show that the likelihood function of the second approach SMP can be written as the product of likelihoods of two-state models. This is very helpful for reducing the computational efforts for likelihood-based parameter estimations.

Further, the comparison of these approaches helps to construct a prediction model that predicts risks and chances of expected trajectories of patients through ICUs. Here, we present relations between some quantities of interest in these two approaches. These relations, for

instance, are used to derive the distribution of transition rates of an illness-death model when we just have information on the distribution of sojourn times.

Another problem with modelling and analysis of survival, is that observations are often very coarse, see for instance [1, 95]. For solving this problem, applying PH distributions seems quite useful. For instance, a PH distribution can explain the stages of a disease happening behind a visible stage. This is through a background (Markov) process going through a set of stages which are not observable, see [135]. Moreover, PH distributions are a versatile class of distributions that are dense in the class of all distributions defined on the non-negative real line. So, applying them for approximating an unknown survival (or failure) time distribution seems quite suitable. In addition to the above reasons, applying MAM for analysing PH distributions, makes them an interesting tool in medical statistics, see for instance [24, 64, 129]. We find the related formulas to extend the illness-death model to the case of having, what we call, a multi-absorption phase-type (MAPH) distribution as the sojourn time distribution or intensity distribution of the semi-Markov model.

The remainder of this chapter is structured as follows. In Section 5.2, we introduce the semi-Markov process (SMP) and represent two different approaches to find the survival time of an SMP based on known parameters. The first approach is based on sojourn times and the second approach is based on transition rates. Moreover, we show relations between parameters of these two models. Section 5.3 is about inference for the above-mentioned two approaches. We define the likelihood function of both methods. For the case of illness-death model, we show that the first approach has a privilege to the second approach. Section 5.4 compares the hazard functions of the two above-mentioned approaches. Section 5.5 is about applying PH distributions for survival analysis. In Section 5.6, we introduce the notion of an MAPH distribution through an example of a multi-state process and find out all the related formulas for a general MAPH distribution.

5.2 Semi-Markov Processes

Semi-Markov processes (SMPs) are a natural generalization of both renewal processes and Markov jump processes in continuous time. In many real-world applications, we need a

stochastic process where the states of the process come from a Markov jump process but with some dependence between jump times. For instance, in biostatistics, where the trajectory of patients in hospital is considered, a limitation of the exponential distribution as a model of the waiting (sojourn) time is that exponential distribution is non-ageing (memoryless). However, the future situation of patients depends only on their current situation (Markov property). For these kinds of applications, modelling with an SMP removes the restriction of memoryless sojourn times and at the same time, preserves the usefulness of applying Markov jump processes in continuous time. SMPs are applied for modelling a variety of processes in different areas such as economics, reliability, and health care, see [99].

Consider a homogeneous continuous-time Markov chain (CTMC) $X(t)$ on states $\mathcal{S} = \{1, 2, \cdots, p\}$. Denote the increasing sequence of jump times by $T_0 = 0 < T_1 < T_2 < T_3 < \cdots$ and its associated embedded Markov chain with $J_n = X(T_n)$. Consider the *semi-Markov process* (SMP) $J(t) := J_{N(t)}$ for $t \geq 0$, where $N(t) = \max\{n : T_n \leq t\}$ with (see Definition 2.2.6):

$$p_{ij} = \mathbb{P}(J_{n+1} = j \mid J_n = i),$$

and

$$F_{ij}(t) = \mathbb{P}(\tau_n \leq t \mid J_n = i, J_{n+1} = j) \quad t \geq 0.$$

Note that for a Markov process, the sequence of sojourn times possess the Markov property and follows either a geometric (discrete case) or an exponential (continuous case) distribution. But, in real application, this distribution is found by comparing different distributions to the data to find the best-fitted distribution. Therefore, often applying a semi-Markov process, which allows arbitrarily distributed sojourn times in any state but retains the Markov property for the chain of states, is more suitable.

In most applications of SMPs in healthcare, a very popular three state semi-Markov process known as the *illness-death model* is applied, see for instance [167]. We describe this model in the following.

Illness-death model. The illness-death model is the most common model in epidemiology and describes in Figure 5.1. The illness-death model is often applied in studying chronic

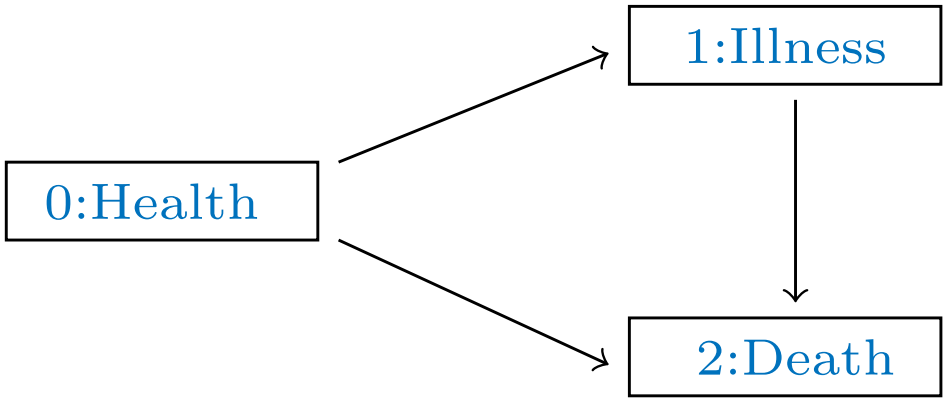


Figure 5.1: The Illness-Death Model.

diseases.

In this model, we have three states “Health”, “Illness” and “Death” which are denoted by 0,1 and 2, respectively. There are three kinds of transitions: $0 \to 1$, $1 \to 2$ and $0 \to 2$ and state 2 is absorbing. Since this model is often used to describe server illnesses, there is no possibility of recovery, and therefore the model is irreversible. If the remissions can be obtained by treatments, it is more appropriate to construct a model with an additional state “Remission” rather than to consider that there is a possibility of moving back to the “Health” state. The time until the process enters the final state (state 2) is a positive valued random variable T that is called the *survival time* and we are interested to find $\mathbb{P}(T \geq t)$.

Here, we present two approaches for defining a semi-Markov process and find the relations between them and then will apply them for determining the likelihood of an illness-death model. The first approach is based on sojourn times within states and the second approach is based on the transition intensities.

5.2.1 First Approach to the SMP (in terms of sojourn times)

Consider (J_n, T_n) as the sequence of states and jump times of a Markov jump process $X(t)$ on state space $\mathcal{S} = \{1, \cdots, p\}$ where $J_n = X(T_n)$. The stochastic process $X(t)$ is an SMP if by assuming that the process is in state i, the next state is j with probability

$$p_{ij} = \mathbb{P}(J_{n+1} = j \mid J_n = i),$$

and consider that the current state, J_n, and the next state, J_{n+1}, are known, the cumulative distribution function of the sojourn time is given by:

$$F_{ij}(t) = \mathbb{P}(\tau_n \leq t \mid J_n = i, J_{n+1} = j) \qquad t \geq 0.$$

Note that this approach is compatible with Definition 2.2.6. For this definition of an SMP, the quantities of interest (parametrized by the transition $i \to j$) are:

- The probability density function of the sojourn time:

$$f_{ij}(t) = \lim_{\triangle t \to 0} \frac{1}{\triangle t} \mathbb{P}(\tau_n \in (t, t + \triangle t] \mid J_n = i, J_{n+1} = j).$$

- The **survival function** that shows the probability that the subject is still in the current state after time t:

$$S_{ij}(t) = \mathbb{P}(\tau_n > t \mid J_n = i, J_{n+1} = j) = 1 - F_{ij}(t)$$

  Note that $S_{ij}(t)$ is a decreasing function, that is $S_{ij}(0) = 1$ and $\lim_{t \to +\infty} S_{ij}(t) = 0$.

- The **hazard function** which is often thought of as the probability that a jump occurs in a specified interval $(t, t + \triangle t]$ given no jump before time t:

$$\alpha_{ij}(t) = \lim_{\triangle t \to 0} \frac{1}{\triangle t} \mathbb{P}\big(\tau_n \in (t, t + \triangle t] \mid J_n = i, J_{n+1} = j, \tau_n > t\big).$$

Here, note that by definition of conditional probability we have:

$$\alpha_{ij}(t) = \frac{f_{ij}(t)}{S_{ij}(t)}. \tag{5.2.1}$$

On the other hand, $F'_{ij}(t) = -S'_{ij}(t)$ and $\frac{f_{ij}(t)}{S_{ij}(t)} = \frac{F'_{ij}(t)}{1 - F_{ij}(t)} = -(\ln S_{ij}(t))'$. So, we have $\int_0^t \alpha_{ij}(u)du = -\ln S_{ij}(t)$ or,

$$S_{ij}(t) = e^{-\int_0^t \alpha_{ij}(u)du}.$$

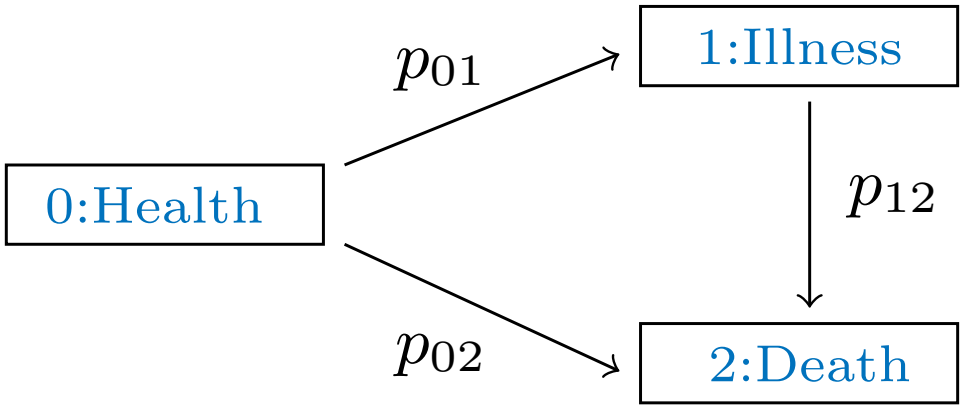


Figure 5.2: The first approach for the illness-death model. Here, p_{ij} denotes transition probability of going from state i to state j.

Moreover, we can write $f_{ij}(t)$ in terms of $\alpha_{ij}(t)$ as:

$$f_{ij}(t) = \alpha_{ij}(t) e^{-\int_0^t \alpha_{ij}(u)du}. \tag{5.2.2}$$

For the case that just the current state is known, we have:

$$S_i(t) = \mathbb{P}(\tau_n > t \mid J_n = i) = \sum_{j \neq i} p_{ij} S_{ij}(t).$$

5.2.2 Second Approach to the SMP (in terms of transition rates)

The first approach required specification of the parameters regarding two types of objects: (i) Transition probabilities of the embedded chain (p_{ij}). (ii) The distribution of sojourn times given a transition $i \to j$. The second approach which we present now is more succinct in that it only requires one type of object: transition rates.

The stochastic process $X(t)$ on state space $\mathcal{S} = \{1, \cdots, p\}$ is an SMP if by assuming that the process is in state i, it goes to the next state, j, during the time interval $(t, t + \Delta t]$ with probability

$$P_{ij}(t, t + \Delta t) = \mathbb{P}\big(X(t + \Delta t) = j \mid X(t) = i, \mathcal{H}(t^-)\big),$$

and with the following transition rate:

$$\tilde{\alpha}_{ij}\big(t, \mathcal{H}(t^-)\big) = \lim_{\Delta t \to 0^+} \frac{P_{ij}(t, t + \Delta t)}{\Delta t},$$

where $\mathcal{H}(t^-)$ denotes the history of the process just before time t (t^-)[1]. Here, the transitions intensities depend only on the time spent in the current state. This comes from the strong Markov property, for more details see Chapter 2 of [38].

Taking the illness-death model as an example, it implies that t is the time since ICU entry. So, we have:

$$\tilde{\alpha}_{0j}\big(t, \mathcal{H}(t^-)\big) = \tilde{\alpha}_{0j}(t), \qquad j = 1, 2,$$

and

$$\tilde{\alpha}_{12}\big(t, \mathcal{H}(t^-)\big) = \tilde{\alpha}_{12}(t - T_1),$$

where T_1 is the time at which the process enters state 1.

By defining SMPs using this approach, the quantities of interest are:

- $\tilde{f}_{ij}(t) = \lim_{\triangle t \to 0} \frac{1}{\triangle t} \mathbb{P}(\tau_n \in (t, t + \triangle t], J_{n+1} = j \mid J_n = i)$. Note that this is (in general) not a density, that is

$$\begin{aligned}
\int_0^\infty \tilde{f}_{ij}(t)dt = \lim_{t\to\infty} \int_0^t \tilde{f}_{ij}(u)du &= \lim_{t\to\infty} \mathbb{P}(\tau_n \le t, J_{n+1} = j \mid J_n = i) \\
&= \lim_{t\to\infty} \frac{\mathbb{P}(\tau_n \le t, J_{n+1}=j, J_n=i)}{\mathbb{P}(J_n=i)} \\
&= \lim_{t\to\infty} \frac{\mathbb{P}(\tau_n \le t \mid J_{n+1}=j, J_n=i)\, \mathbb{P}(J_{n+1}=j, J_n=i)}{\mathbb{P}(J_n=i)} \\
&= \lim_{t\to\infty} F_{ij}(t)\, \mathbb{P}(J_{n+1} = j \mid J_n = i) = p_{ij} \le 1\,.
\end{aligned}$$

- The elements $c_{ij}(t) = \mathbb{P}(\tau_n \le t, J_{n+1} = j \mid J_n = i)$, where $c_{ij}(0) = 0$ and

$$\lim_{t\to\infty} c_{ij}(t) = \mathbb{P}(J_{n+1} = j \mid J_n = i) = p_{ij}.$$

- $\tilde{S}_{ij}(t) = \mathbb{P}(\tau_n > t, J_{n+1} = j \mid J_n = i)$.

- $\tilde{\alpha}_{ij}(t) = \lim_{\triangle t \to 0} \frac{1}{\triangle t} \mathbb{P}(\tau_n \in (t, t + \triangle t], J_{n+1} = j \mid J_n = i, \tau_n > t)$.

Then we have these relations:

[1] Note that in some text books, $\mathcal{H}(t^-)$ refers to a σ-algebra, see for instance, [38]

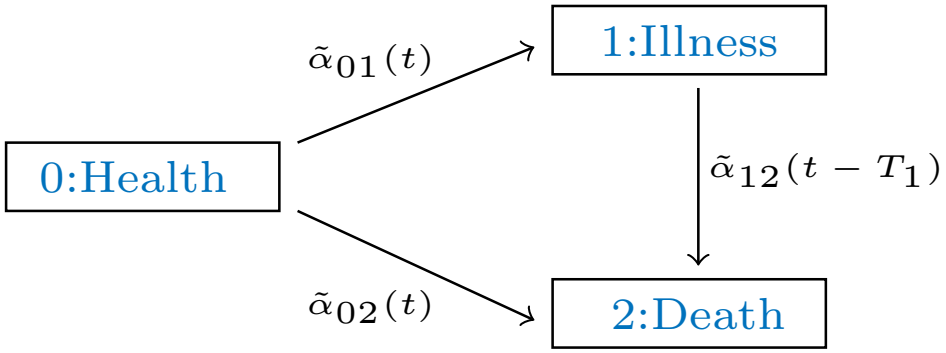


Figure 5.3: The Illness-Death Model - Second Approach. Here, $\tilde{\alpha}_{ij}(t)$ is the transition rates of going from state i to state j and T_1 is the time that process jumped in state 1.

- $\frac{d}{dt}\tilde{S}_{ij}(t) = -\tilde{f}_{ij}(t)$, that comes from the following lines.

$$\begin{aligned}\tfrac{d}{dt}\tilde{S}_{ij}(t) &= \lim_{\triangle t\to 0}\tfrac{\mathbb{P}(\tau_n>t+\triangle t, J_{n+1}=j \mid J_n=i)-\mathbb{P}(\tau_n>t, J_{n+1}=j \mid J_n=i)}{\triangle t}\\ &= \lim_{\triangle t\to 0}\tfrac{-\mathbb{P}(\tau_n\in(t,t+\triangle t], J_{n+1}=j \mid J_n=i)}{\triangle t}\\ &= -\lim_{\triangle t\to 0}\tfrac{1}{\triangle t}\mathbb{P}(\tau_n\in(t,t+\triangle t], J_{n+1}=j \mid J_n=i) = -\tilde{f}_{ij}(t).\end{aligned}$$

- $\tilde{\alpha}_{ij}(t)\cdot\tilde{S}_{ij}(t) \neq \tilde{f}_{ij}(t)$. In fact, we have:

$$\begin{aligned}&\tilde{\alpha}_{ij}(t)\cdot\tilde{S}_{ij}(t)\\ &= \lim_{\triangle t\to 0}\mathbb{P}(\tau_n\in(t,t+\triangle t], J_{n+1}=j \mid J_n=i, \tau_n>t)\cdot\mathbb{P}(\tau_n>t, J_{n+1}=j \mid J_n=i)\\ &\neq \lim_{\triangle t\to 0}\mathbb{P}(\tau_n\in(t,t+\triangle t], J_{n+1}=j \mid J_n=i) = \tilde{f}_{ij}(t).\end{aligned}$$

5.2.3 Relations Between the Two Approaches

The above two approaches can be related with the following relations. First note that using the conditional probability argument $\mathbb{P}(A,B|C) = \mathbb{P}(B|C)\mathbb{P}(A|B,C)$, we have:

$$c_{ij}(t) = p_{ij}\, F_{ij}(t), \tag{5.2.3}$$

where $p_{ij} \neq 0$ and $c_{ij}(t) = \mathbb{P}(\tau_n \leq t, J_{n+1}=j \mid J_n=i)$. Moreover,

$$\tilde{f}_{ij}(t) = p_{ij} f_{ij}(t), \tag{5.2.4}$$

and

$$\tilde{S}_{ij}(t) = p_{ij} S_{ij}(t). \tag{5.2.5}$$

Further,

$$\tilde{\alpha}_{ij}(t) = \mathbb{P}(J_{n+1} = j \mid \tau_n > t\,, J_n = i)\,\alpha_{ij}(t).$$

The term $\mathbb{P}(J_{n+1} = j \mid \tau_n > t, J_n = i)$ can be represented as

$$\mathbb{P}(J_{n+1} = j \mid \tau_n > t, J_n = i) = \frac{\tilde{S}_{ij}(t)}{S_i(t)} = \frac{p_{ij}S_{ij}(t)}{S_i(t)}.$$

Hence,

$$\tilde{\alpha}_{ij}(t) = \frac{p_{ij}S_{ij}(t)}{S_i(t)}\ \alpha_{ij}(t), \tag{5.2.6}$$

which, by using (5.2.1), can be written as:

$$\tilde{\alpha}_{ij}(t) = \frac{p_{ij}f_{ij}(t)}{S_i(t)}.$$

Using the relations in (5.2.4), we can write the above equation as:

$$\tilde{\alpha}_{ij}(t) = \frac{\tilde{f}_{ij}(t)}{\sum_{i\neq j}\tilde{S}_{ij}(t)}.$$

Note that in the CTMC case Eq. (5.2.6) is given by:

$$\tilde{\alpha}_{ij}(t) = p_{ij}\alpha_{ij}(t).$$

More relations in the CTMC case, are presented in the next proposition.

Proposition 5.2.1. *Consider a finite state CTMC, where the sojourn time in state i has the exponential distribution with rate λ_i and the transition rate from state i to state j has exponential distribution with rate λ_{ij}. In this case, $\lambda_i = \sum_{j\neq i}\lambda_{ij}$ and the above-mentioned quantities and their relations are given by:*

- $f_{ij}(t) = \lambda_i e^{-\lambda_i t}$.
- $\tilde{\alpha}_{ij}(t) = \lambda_{ij}$ *($\tilde{\alpha}_{ij}(t)$ is a constant that is denoted by λ_{ij}).*
- $F_{ij}(t) = 1 - e^{-\lambda_i t}$*(does not depend on j).*
- $S_{ij}(t) = 1 - F_{ij} = e^{-\lambda_i t}$.
- $\alpha_{ij}(t) = \frac{f_{ij}(t)}{S_{ij}(t)} = \lambda_i$.

- $S_i(t) = \sum_{j \neq i} p_{ij}\, S_{ij}(t) = \sum_{j \neq i} p_{ij}\, e^{-\lambda_i t} = e^{-\lambda_i t}$ *(note that* $\sum_{j \neq i} p_{ij} = 1$*).*
- $p_{ij} = \frac{\lambda_{ij}}{\lambda_i}$ *(Using* (5.2.6)*).*
- $c_{ij}(t) = p_{ij}(1 - e^{-\lambda_i t}) = \frac{\lambda_{ij}}{\lambda_i}(1 - e^{-\lambda_i t})$.
- $\tilde{f}_{ij}(t) = p_{ij} f_{ij}(t) = \frac{\lambda_{ij}}{\lambda_i} \lambda_i e^{-\lambda_i t}$.
- $\tilde{S}_{ij}(t) = p_{ij} S_{ij}(t) = \frac{\lambda_{ij}}{\lambda_i} e^{-\lambda_i t}$.

5.3 Inference

Survival analysis usually applies for cohort or clinical studies. So, the data is usually gathered from the same subjects repeatedly during a time interval $[0, \mathcal{T}]$. The quantity of interest is the expected time until the specific event (the event of study) happens. The time until the event of study happens is a random variable T and is called the *survival time* . As we mentioned before, the main issue in survival analysis would be having incomplete or sparse observations. For instance, in the case of chronic diseases, when the event of study is death, time of occurrence of this event is not observed for the subjects still alive at the end of observation time. This type of incomplete observation is called *right-censoring* . There are other kinds of censoring, like left-censoring, which are not of our interest, see [58].

5.3.1 Likelihood Function

In this section, we derive the likelihood function of an SMP from two different approaches explained in the previous section.

First Approach Likelihood Here, we consider the common case of incomplete data or the right censored data. Further, we assume that our model has an absorbing state (for instance the illness-death model).

In this context, two different paths could happen for each subject h:

1. Subject h gradually enters the absorbing state. Here, there is no censored data and the history of this subject till time t is given by:

$$\mathcal{H}_h(t) = \big(J_{h;0}, J_{h;1}, \cdots, J_{h;N_h}, \tau_{h;0}, \tau_{h;1}, \cdots, \tau_{h;N_h}\big),$$

   where $J_{h;0}, J_{h;1}, \cdots, J_{h;N_h}$, assuming that $J_{h;j} \neq J_{h;j+1}$, are consecutive jumps (states) at the consecutive times $0 = T_0 < \cdots < T_{N_h}$. Further, $\tau_{h;0}, \tau_{h;1}, \cdots, \tau_{h;N_h}$ are the corresponding sojourn times and $N_h \geq 0$ is the total number of jumps of the subject h.

2. Subject h does not enter in the absorbing state. In this case, the right censoring happens either because the subject h leaves the study before going to absorbing state (the sojourn time of $(\tau_{h;N_n})$ is censored) or since the observation time ends before this subject enters the absorbing state. The history of this subject is given by:

$$\mathcal{H}_h(t) = \big(J_{h;0}, J_{h;1}, \cdots, J_{h;N_h}, \tau_{h;0}, \tau_{h;1}, \cdots, \tau_{h;N_h}, U_h\big),$$

   where U_h is the time between last jump time of this subject $T_{h;N_h}$ and the end of the study.

For each subject h, the likelihood function can be obtained corresponding to the two above-mentioned paths:

1. The subject h stays at state i for time t and then goes to state j. Therefore the likelihood for this subject is:

$$S_i(t)\tilde{\alpha}_{ij}(t) = p_{ij} f_{ij}(d) = c_{ij}(t),$$

   where c_{ij} are as in Eq. (5.2.3).

2. The subject h is in state i for a time t and the sojourn time is censored (right-censoring). For this case, the likelihood is given by:

$$S_i.(t) = \sum_j p_{ij} S_{ij}(t). \tag{5.3.1}$$

Therefore, if we consider that for the subject h, δ_h is 1 where the subject is censored and

otherwise is 0, then the likelihood of this subject is given by:

$$\mathcal{L}_h = \prod_{k=1}^{N_h} p_{J_{h;k-1}\,J_{h;k}} f_{J_{h;k-1}\,J_{h;k}}(\tau_{h;k}) \Big(S_{J_{h;N_h}}(U_h)\Big)^{\delta_h},$$

where U_h represents the time between $T_{h;N_h}$ and the end of observation/study. The total likelihood when there are n different subjects ($h = 1, \cdots, n$) is the product of likelihoods of all subjects:

$$\mathcal{L} = \prod_{h=1}^{n} \mathcal{L}_h.$$

Second Approach Likelihood Having the second approach for defining an SMP, we can derive the likelihood function as below.

Consider that the number of jumps is E and the process starts at time $T_0 = 0$ from state J_0 and the other states of the jump process are labelled as J_j where $j = 1, \cdots E$. Denote the transition times by T_j and consider that the observation happens during the time interval $[0, \mathcal{T}]$. At a given time t, the process either stays in state J_{j-1} (with probability $P_{j-1\,,j-1}(T_{j-1}, t)$) or jumps from state $J_{j-1} = m$ to state $J_j = l$ (with transition rate $\tilde{\alpha}_{ml}(T_j)$ if it happens at j-th jump). Therefore the likelihood of a given subject can be written as:

$$\mathcal{L} = \Big(\prod_{j=1}^{E} P_{j-1,\,j-1}(T_{j-1}, T_j^-)\, \tilde{\alpha}_{j-1,\,j}(T_j) \Big)\, P_{E,\,E}(T_E, \mathcal{T}). \tag{5.3.2}$$

On the other hand, by using the solutions of the forward Kolmogorov equation, see Chapter 7 of [58], we have:

$$P_{j-1,\,j-1}(T_{j-1}, T_j^-) = \exp\big(\int_{T_{j-1}}^{T_j} \tilde{\alpha}_{j-1}(u) du \big),$$

where $\tilde{\alpha}_{j-1}(u) = -\sum_{i \neq j-1} \tilde{\alpha}_{j-1,\,i}(u)$. Therefore, the likelihood function in (5.3.2) can be written as:

$$\mathcal{L} = \prod_{j=1}^{E} \exp\Big(\int_{T_{j-1}}^{T_j} \tilde{\alpha}_{j-1}(u) du \Big)\, \tilde{\alpha}_{j-1,\,j}(T_j)\, \exp\Big(\int_{T_E}^{\mathcal{T}} \tilde{\alpha}_E(u) du \Big), \tag{5.3.3}$$

which can be simplified to:

$$\mathcal{L} = \prod_{j=1}^{K} \Big(\exp\big(\int_{T_{j-1}}^{T_j} \tilde{\alpha}_{j-1}(u) du \big) \Big)\, \tilde{\alpha}_{j-1,\,j}(T_j)^{\mathbb{1}_{\{j \neq K\}}}, \tag{5.3.4}$$

where $K = E + 1$ and we assume that $T_K = \mathcal{T}$.

For writing the complete likelihood, let us assume that at time t the process is in state m, then the following two factors are essential in writing the likelihood function:

$$D_{mlj} = \mathbb{1}_{\{J_{j-1}=m,\, J_j=l\}} \quad \text{and} \quad R_m(t) = \sum_{j=1}^{K} \mathbb{1}_{\{J_{j-1}=m,\, T_{j-1}<t\leq T_j\}}.$$

Note that the first indicator is 1 where the j-th jump is from state $J_{j-1} = m$ to state $J_j = l$ and otherwise, it is equal to zero. The second indicator is 1 where at time t the process is in state $J_{j-1} = m$. Therefore, by having the number of jumps and their orders, the likelihood can be formulated as:

$$\mathcal{L} = \prod_{m=1}^{p} \prod_{l=1}^{p} \Big[\big(\prod_{j=1}^{K} \tilde{\alpha}_{ml}(T_j)^{D_{mlj}} \big) \exp\big(- \int_0^{\mathcal{T}} R_m(u) \tilde{\alpha}_{ml}(u) du \big) \Big], \tag{5.3.5}$$

where p is the number of states of the SMP, see [96].

5.3.2 Likelihood Function of the Illness-Death Model

Here we formulate the likelihood function of the above mentioned two approaches for illness-death model and discuss that the inference for the second approach is easier.

The likelihood of illness-death model – First approach The first approach to illness-death model is described in Figure 5.2. Here, by assuming that the observation time is $\mathcal{T}$, we derive the likelihood for the four different possible paths:

1. During time interval $(0, \mathcal{T})$, the subject h remains in state 0 (type 1 subject):

$$\mathcal{L}_{0\to 0} = S_{0\cdot}(\mathcal{T}) = p_{01}\, S_{01}(\mathcal{T}) + p_{02}\, S_{02}(\mathcal{T}).$$

2. During time interval $(0, \mathcal{T})$, the subject h goes from state 0 to state 1 (type 2 subject):

$$\mathcal{L}_{0\to 1} = p_{01}\, f_{01}(T_1)\, S_{1\cdot}(\mathcal{T} - T_1) = p_{01}\, f_{01}(T_1)\, p_{12}\, S_{12}(\mathcal{T} - T_1),$$

   where the second equality comes from (5.3.1).

3. During time interval $(0, \mathcal{T})$, the subject h goes from state 0 to state 2 (type 3 subject):

$$\mathcal{L}_{0\to 2} = p_{02} f_{02}(T_2).$$

4. During time interval $(0, \mathcal{T})$, the subject h goes from state 0 to state 1 and then to state 2 (type 4 subject):

$$\mathcal{L}_{0\to 1\to 2} = p_{01} f_{01}(T_1)\, p_{12}\, f_{12}\,(T_2 - T_1).$$

Then, the likelihood function is given by:

$$\mathcal{L} = \mathcal{L}_{0\to 0} \times \mathcal{L}_{0\to 1} \times \mathcal{L}_{0\to 2} \times \mathcal{L}_{0\to 1\to 2} \tag{5.3.6}$$

Note that from Eq. (5.2.2), $f_{ij}(t) = \alpha_{i,j}(t)\, e^{-\int_0^t \alpha_{i,j}(u)du}$. Further we have: $p_{01} = 1 - p_{02}$ and $p_{12} = 1$. So, for this approach, we have four unknowns to estimate. These unknowns are p_{01} (or equivalently p_{02}) and three density functions $f_{01}(t)$, $f_{02}(t)$, and $f_{12}(t)$.

The likelihood of illness-death model – second approach The likelihood of the illness-death model illustrated in Figure 5.3, can be expressed regarding the intensity rates $\tilde{\alpha}_{ij}(t)$.

Here, again for a given subject h, we obtain the likelihood of four possible paths:

1. During time interval $(0, \mathcal{T})$, the subject h remains in state 0 (type 1 subject):

$$\mathcal{L}_{0\to 0} = P_{00}(0, \mathcal{T}).$$

2. During time interval $(0, \mathcal{T})$, the subject h goes from state 0 to state 1 (type 2 subject):

$$\mathcal{L}_{0\to 1} = P_{00}(0, T_1)\, \tilde{\alpha}_{01}(T_1)\, P_{11}(T_1, \mathcal{T}).$$

3. During time interval $(0, \mathcal{T})$, the subject h goes from state 0 to state 2 (type 3 subject):

$$\mathcal{L}_{0\to 2} = P_{00}(0, T_2)\, \tilde{\alpha}_{02}(T_2).$$

4. During time interval $(0, \mathcal{T})$, the subject h goes from state 0 to state 1 and then goes to

state 2 (type 4 subject):

$$\mathcal{L}_{0\to1\to2} = P_{00}(0,T_1)\,\tilde{\alpha}_{01}(T_1)\,P_{11}(T_1,T_2)\,\tilde{\alpha}_{12}(T_2).$$

Here, note that from the forward Kolmogorov equation for the illness-death model, the $P_{00}(t)$ and $P_{11}(t)$ can be written in terms of $\tilde{\alpha}_{ij}(\cdot)$ as

$$\begin{aligned} P_{00}(t) &= \exp\Big(-\int_0^t \big(\tilde{\alpha}_{01}(u)+\tilde{\alpha}_{02}(u)\big)du\Big),\\ P_{11}(t) &= \exp\Big(-\int_0^t \tilde{\alpha}_{12}(u)du\Big). \end{aligned}$$

Therefore, for this approach, we only have three unknowns: $\tilde{\alpha}_{01}(t)$, $\tilde{\alpha}_{02}(t)$, and $\tilde{\alpha}_{12}(t)$.

Having fewer unknowns for the same model, make it sensible to write the full likelihood of illness-death model based on second approach (since for the first approach, we had 4 unknowns and for the second approach we have 3 unknowns). Moreover, from Eq. (5.3.5) we can see that each transition can be considered separately if it has its own set of parameters which could facilitate the estimation. However, in the case that there are common parameters for different transitions, they must be considered together.

For the special example of illness-death model since the process will not back to its previous states, for the sake of simplicity we can denote state J_j by just j. Moreover, note that for this example, a given subject only departs states $m = 0, 1$ and only enters states $l = 1, 2$. Therefore Eq. (5.3.5), for this example can be written as:

$$\begin{aligned} \mathcal{L} = \prod_{j=1}^{3}\Big(\tilde{\alpha}_{01}(T_j)\Big)^{D_{01j}} \times \exp\big(-\int_0^{\mathcal{T}}\tilde{\alpha}_{01}(u)du\big) \times \Big(\tilde{\alpha}_{02}(T_j)\Big)^{D_{02j}}\\ \times \exp\big(-\int_0^{\mathcal{T}}\tilde{\alpha}_{02}(u)du\big) \times \Big(\tilde{\alpha}_{12}(T_j)\Big)^{D_{12j}} \times \exp\big(-\int_0^{\mathcal{T}}\tilde{\alpha}_{12}(u-T_1)du\big), \end{aligned} \tag{5.3.7}$$

where

- $D_{01j} = 1$ just when $j = 1$ and first jump is from 0 to 1.
- $D_{02j} = 1$ just when $j = 1$ and first jump is from 0 to 2.
- $D_{12j} = 1$ just when $j = 2$ and second jump is from 1 to 2.

Now consider that there are N different subjects and $N = N_1 + N_2 + N_3 + N_4$, where N_i shows the number of subjects of type i for $i = 1, \cdots, 4$. Using Eq. (5.3.7), the full likelihood for N subjects is given by:

$$
\begin{aligned}
&\mathcal{L} = \prod_{k_2=1}^{N_2} \prod_{k_3=1}^{N_3} \prod_{k_4=1}^{N_4} \\
&\Big(\{ \exp\big(- \int_0^{\mathcal{T}} \tilde{\alpha}_{01}(u)\, du \big) \}^{N_1} \times \exp\big(- \int_0^{T_1^{k_2}} \tilde{\alpha}_{01}(u)\, du \big) \times \tilde{\alpha}_{01}(T_1^{k_2}) \\
&\times \exp\big(- \int_0^{T_2^{k_3}} \tilde{\alpha}_{01}(u)\, du \big) \times \exp\big(- \int_0^{T_1^{k_4}} \tilde{\alpha}_{01}(u)\, du \big) \times \tilde{\alpha}_{01}(T_1^{k_4}) \\
&\times \{ \exp(- \int_0^{\mathcal{T}} \tilde{\alpha}_{02}(u)\, du) \}^{N_1} \times \exp\big(- \int_0^{T_1^{k_2}} \tilde{\alpha}_{02}(u)\, du \big) \\
&\times \exp\big(- \int_0^{T_2^{k_3}} \tilde{\alpha}_{02}(u)\, du \big) \times \tilde{\alpha}_{02}(T_2^{k_3}) \times \exp\big(- \int_0^{T_1^{k_4}} \tilde{\alpha}_{02}(u)\, du \big) \\
&\times \exp\big(- \int_{T_1^{k_2}}^{\mathcal{T}} \tilde{\alpha}_{12}(u - T_1^{k_2})\, du \big) \times \exp\big(- \int_{T_1^{k_4}}^{T_2^{k_4}} \tilde{\alpha}_{12}(u - T_1^{k_4})\, du \big) \times \tilde{\alpha}_{12}(T_2^{k_4} - T_1^{k_4}) \Big),
\end{aligned}
\tag{5.3.8}
$$

where $T_l^{k_i}$ for $i = 2, 3, 4$ and $l = 1, 2$ denotes the transition time of $k-$th subject of type i, to the state l. In the above likelihood, the end of the study, $\mathcal{T}$, has an important role. For instance, if $\mathcal{T}$ is less than both T_1^j and T_2^j, then as it can be seen from lines 2 and 3 of (5.3.8), the likelihoods of subjects of types 2, 3 and 4 include $\tilde{\alpha}_{01}(u)$ term. This point and the fact that (5.3.8) is just written in terms of $\tilde{\alpha}_{01}(u)$, $\tilde{\alpha}_{02}(u)$ and $\tilde{\alpha}_{12}(u)$ motivate us to split this likelihood into individual likelihoods of sub-models of illness-death model which are: sub-model $0 \to 1$, sub-model $0 \to 2$ and sub-model $1 \to 2$. The next proposition shows the likelihood of these sub-models are just parts of (5.3.8) which are written in terms of $\tilde{\alpha}_{01}(u)$, $\tilde{\alpha}_{02}(u)$, and $\tilde{\alpha}_{12}(u)$, respectively.

Proposition 5.3.1. *The full likelihood of the illness-death model can be written as the product of likelihoods of three two-state sub-models* $0 \to 1$*,* $0 \to 2$ *and* $1 \to 2$*.*

Proof. First consider the sub-model $0 \to 1$ including the subjects of full likelihood which stay in state 0 or tend to go from state 0 to state 1; that is parts of full likelihood (5.3.8) in terms of $\tilde{\alpha}_{01}$. Denote the censored time by T_C. First, assume that $T_C = \mathcal{T}$ (there is no censorship). Then, if $\mathcal{T} < T_1^j$, we have subjects of type 1 which are included in this sub-model. Otherwise, where $T_1^j < \mathcal{T}$, there are some subjects of type 2 which are included in this sub-model. If we assume that $T_C = T_2^j$, subjects of type 3 can be included in this sub-model. Finally, where there is no censorship and $T_1^j < T_2^j$ and $T_1^j < \mathcal{T}$, subjects of type 4 can be included in this

sub-model. Therefore, the first two lines of $\mathcal{L}$ can be written as:

$$\mathcal{L}_{0\to1} = \prod_{j=1}^{N_{01}} \exp\big(-\int_0^{\mathcal{T}=T_C} \tilde{\alpha}_{01}(u)\,du\big) \times \exp\big(-\int_0^{T_1^j} \tilde{\alpha}_{01}(u)\,du\big) \times \tilde{\alpha}_{01}(T_1^j)$$
$$\times \exp\big(-\int_0^{T_2^j=T_C} \tilde{\alpha}_{01}(u)\,du\big) \times \exp\big(-\int_0^{T_1^j} \tilde{\alpha}_{01}(u)\,du\big) \times \tilde{\alpha}_{01}(T_1^j).$$

Now if we put $\tilde{T}_1^j = \min\{T_1^j, T_C\}$, where T_C is either T_2^j or $\mathcal{T}$ and by using:

$$\delta_1^j = \begin{cases} 1 & \text{if } \; T_1^j < \mathcal{T} \text{ and } T_1^j < T_2^j \qquad \text{(for subjects of type 2 or 4)}, \\ 0 & \text{otherwise} \qquad \text{(for subjects of type 1 or 3)}. \end{cases}$$

Then, we have:

$$\mathcal{L}_{0\to1} = \prod_{j=1}^{N_{01}} \Big(\exp\big(-\int_0^{\tilde{T}_1^j} \tilde{\alpha}_{01}(u)du\big)\,\tilde{\alpha}_{01}(\tilde{T}_1^j)\Big)^{\delta_1^j} \Big(\exp\big(-\int_0^{\tilde{T}_1^j} \tilde{\alpha}_{01}(u)du\big)\Big)^{1-\delta_1^j},$$

where N_{01} denotes the number of subjects including in this sub-model.

Similarly, the parts of (5.3.8) which are written in terms of $\tilde{\alpha}_{02}(\cdot)$; that is the likelihood of sub-model $0 \to 2$ can be formulated as below:

$$\mathcal{L}_{0\to2} = \prod_{j=1}^{N_{02}} \Big(e^{-\int_0^{\tilde{T}_2^j} \tilde{\alpha}_{02}(u)du}\,\tilde{\alpha}_{02}(\tilde{T}_2^j)\Big)^{\delta_2^j} \Big(e^{-\int_0^{\tilde{T}_2^j} \tilde{\alpha}_{02}(u)du}\Big)^{1-\delta_2^j},$$

where N_{02} denotes the number of subjects including in the sub-model $0 \to 2$. Here, $\tilde{T}_2^j = \min\{T_2^j, T_C\}$, and T_C is either T_{01} (which denotes the time that a subject jumps from state 0 to state 1) or $\mathcal{T}$ (for the case that the subject censored). Further,

$$\delta_2^j = \begin{cases} 1 & \text{if } \; T_2^j < \mathcal{T} \text{ and } T_2^j < T_1^j \qquad \text{(for subjects of type 3 or 4)}, \\ 0 & \text{otherwise} \qquad \text{(for subjects of type 1 or 2)}. \end{cases}$$

Finally, for parts of the full likelihood (5.3.8) written in terms of $\tilde{\alpha}_{12}(\cdot)$, by changing the variables $v = u - T_1^{k_2}$ and $w = u - T_1^{k_4}$, we have:

$$\prod_{j=1}^{N_{12}} \exp\big(-\int_{T_1^j}^{\mathcal{T}} \tilde{\alpha}_{12}(u - T_1^j)\,du\big) \times \exp\big(-\int_{T_1^j}^{T_2^j} \tilde{\alpha}_{12}(u - T_1^j)\,du\big) \times \exp\big(\tilde{\alpha}_{12}(T_2^j - T_1^j)\big)$$
$$= \exp\big(-\int_0^{\mathcal{T}-T_1^j} \tilde{\alpha}_{12}(v)\,dv\big) \times \exp\big(-\int_0^{T_2^j-T_1^j} \tilde{\alpha}_{12}(w)\,dw\big) \times \exp\big(\tilde{\alpha}_{12}(T_2^j - T_1^j)\big).$$

Therefore, by considering $\tilde{T}_{12}^j = \min\{T_2^j, \mathcal{T}\}$ and

$$\delta_{12}^j = \begin{cases} 1 & \text{if } T_2^j < \mathcal{T} \qquad \text{(for subjects of type 4)}, \\ 0 & \text{otherwise} \qquad \text{(for subjects of type 1)}, \end{cases}$$

the above part of the full likelihood, the likelihood of sub-model $1 \to 2$, can be written as:

$$\mathcal{L}_{1\to 2} = \prod_{j=1}^{N_{12}} \left(e^{-\int_0^{\tilde{T}_{12}^j - T_1^j} \tilde{\alpha}_{12}(u)du} \, \tilde{\alpha}_{12}(\tilde{T}_{12}^j - T_1^j) \right)^{\delta_{12}^j} \left(e^{-\int_0^{\tilde{T}_{12}^j - T_1^j} \tilde{\alpha}_{12}(u)du} \right)^{1-\delta_{12}^j},$$

where N_{12} denotes the number of subjects including in the sub-model $1 \to 2$. Therefore, the full likelihood 5.3.8 can be written as:

$$\begin{aligned} &\mathcal{L} = \textstyle\prod_{j=1}^{N_{01}} \prod_{j=1}^{N_{02}} \prod_{j=1}^{N_{12}} \\ &\Big\{ \Big(\exp\big(-\int_0^{\tilde{T}_1^j} \tilde{\alpha}_{01}(u)du\big) \, \tilde{\alpha}_{01}(\tilde{T}_1^j) \Big)^{\delta_1^j} \times \Big(\exp\big(-\int_0^{\tilde{T}_1^j} \tilde{\alpha}_{01}(u)du\big) \Big)^{1-\delta_1^j} \times \Big(e^{-\int_0^{\tilde{T}_2^j} \tilde{\alpha}_{02}(u)du} \, \tilde{\alpha}_{02}(\tilde{T}_2^j) \Big)^{\delta_2^j} \\ &\times \Big(e^{-\int_0^{\tilde{T}_2^j} \tilde{\alpha}_{02}(u)du} \Big)^{1-\delta_2^j} \times \Big(e^{-\int_0^{\tilde{T}_{12}^j - T_1^j} \tilde{\alpha}_{12}(u)du} \, \tilde{\alpha}_{12}(\tilde{T}_{12}^j - T_1^j) \Big)^{\delta_{12}^j} \times \Big(e^{-\int_0^{\tilde{T}_{12}^j - T_1^j} \tilde{\alpha}_{12}(u)du} \Big)^{1-\delta_{12}^j} \Big\} \\ &= \mathcal{L}_{0\to 1} \times \mathcal{L}_{0\to 2} \times \mathcal{L}_{1\to 2} \end{aligned} \tag{5.3.9}$$

□

Note that in the above-mentioned formula, whenever there is an observation of data the related indicator (δ) is equal to 1 and whenever there is censorship, the related indicator is 0.

When it comes to comparison of the two above-mentioned approaches for SMP, the above proposition and the fact that the number of parameters that need to be estimated is less in the second approach, implying that the second approaches is the superior approach.

5.3.3 Numerical Experiments

Here, we present the results of some preliminary numerical experiments computed using R for estimating the parameters of the illness-death model via the two approaches of the SMP.

First, we set the observation time as $\mathcal{T} = 50$ and generate a sample of $n = 5000$ data paths from the first approach to SMPs. We assume that sojourn times are distributed exponentially

with parameters $\alpha_{01} = 0.5, \alpha_{02} = 0.33$, $\alpha_{12} = 1$. Further, we set $p_{01} = 0.5$. Note that since $\alpha_{01} \neq \alpha_{02}$, this is not a CTMC.

Table 5.1: Comparison of the estimated parameter values with true parameters for the first approach illness-death model.

	Estimated Value	True Value
p_{01}	0.4931994	0.5000000
α_{01}	0.5190281	0.5000000
α_{02}	0.3285477	0.3333333
α_{12}	1.0032200	1.0000000

Table 5.1 presents the estimated values of α_{ij} and the transition probability p_{01}. The estimated values are obtained by applying MLE method (see Section 2.4.3 for more details). Here, we write the likelihood function of the illness-death model through the first approach as in Eq. (5.3.6). For the parameter estimation, we apply the nonlinear minimization function (nlm) in "stat" package to find the maximum likelihood estimators (MLEs).

Then, we consider the CTMC case, where the sojourn times are distributed exponentially with parameters $\alpha_{01} = \alpha_{02} = 0.33$, $\alpha_{12} = 1$, and $p_{01} = 0.5$. Here, again we set $\mathcal{T} = 50$ and generate a sample of $n = 5000$ data paths from the first approach to SMPs. By assuming that the transition rates follow an exponential distribution, we write the full likelihood (5.3.8) and split likelihood (5.3.9) for the illness-death model. Then we find the resulting estimated values of transition rates by applying the "nlm" function for both likelihood functions. The estimated values of $\tilde{\alpha}_{ij}$ are presented in the first two columns of Table 5.2. As the table shows, the results for applying the full likelihood (5.3.8) and the split likelihood (5.3.9) are similar.

Table 5.2: Comparison of the estimated values of transition rates of a CTMC with true values for the illness-death model.

	Estimated Value Full Likelihood	Estimated Value Split Likelihood	Estimated Value from Estimated α_{ij}, p_{ij}	True Values
$\tilde{\alpha}_{01}$	0.1740820	0.1740821	0.1751180	0.1666667
$\tilde{\alpha}_{02}$	0.1663211	0.1663209	0.1652979	0.1666667
$\tilde{\alpha}_{12}$	1.0210028	1.0210027	1.0210030	1.0000000

In the case of CTMC (where the sojourn times are exponential and $\alpha_{01} = \alpha_{02}$), by having the estimated values of α_{ij} and p_{ij}, and applying $\tilde{\alpha}_{ij} = p_{ij}\alpha_{ij}$, we can find the estimated values of $\tilde{\alpha}_{ij}$. The third column of Table 5.2 presents the estimated values of $\tilde{\alpha}_{ij}$ by knowing the estimated values of α_{ij} and p_{ij}. Here, the estimated values for α_{ij} and p_{ij} are $\alpha_{01} = 0.3424290$, $\alpha_{02} = 0.3383088$, $\alpha_{12} = 1.0210030$, and $p_{01} = 0.5113995$. The last column of table presents the true values of $\tilde{\alpha}_{ij}$ derived by knowing the true parameters (α_{ij} and p_{ij}) and using $\tilde{\alpha}_{ij} = p_{ij}\alpha_{ij}$.

The numerical experiments were also carried out on different parameter sets, for both the exponential and the Weibull distribution. In both cases, they imply that the estimated values under the second approach by using the full likelihood (5.3.8) and the split likelihood (5.3.9) are the same and very close to the true values. Carrying out such a check is important in the preliminary analysis because the formulas in Subsection 5.3.2 are very complex.

Furthermore, we can check the model from data by considering the shape of the survival function. Here, we consider the data paths generated from the first approach by assuming the exponential distribution for sojourn times with parameters $\alpha_{01} = 0.5, \alpha_{02} = 0.33$, $\alpha_{12} = 1$, and $p_{01} = 0.5$. The observation time set as $\mathcal{T} = 50$ and we generate a sample of $n = 5000$ data path.

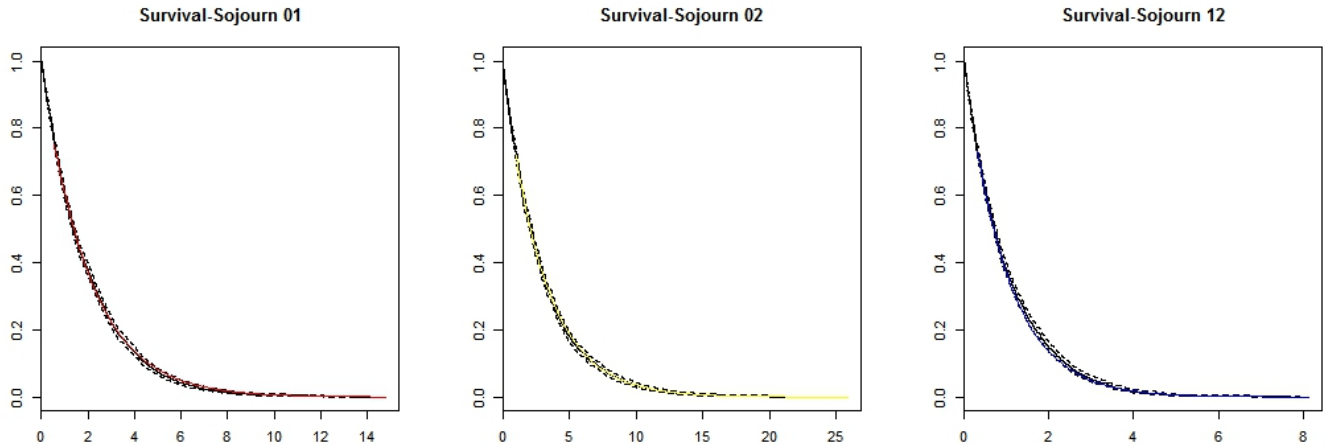


Figure 5.4: Comparison of survival function of data with true survival function. The black curves are the fitted survival functions of sojourn times derived from data, and the coloured curves are true exponential survival functions with parameters (left to right)$\alpha_{01} = 0.5, \alpha_{02} = 0.33$, and $\alpha_{12} = 1$, respectively.

Figure 5.4 plots the estimated survival functions of sojourn times derived from data and compares them with the shape of corresponding true survival functions. The black curves

are the fitted survival functions to the given data. These curves are drawn by applying the "survfit" function in the R "survival" package. The coloured curves are the corresponding exponential survival functions $S(t) = -\lambda t$ where instead of λ we consider the corresponding values of α_{ij}. As the Figure 5.4 demonstrates, the difference between the fitted and real survival function curves is negligible, and by knowing the shape of the fitted survival function, we can find the distribution of the sojourn times. For having a better view of the possible deviation between data sojourn times distribution and the true sojourn times distribution, especially in the tail area consider Figure 5.5. This figure presents comparison of data log-survival functions with the related (true) exponential log-survival functions.

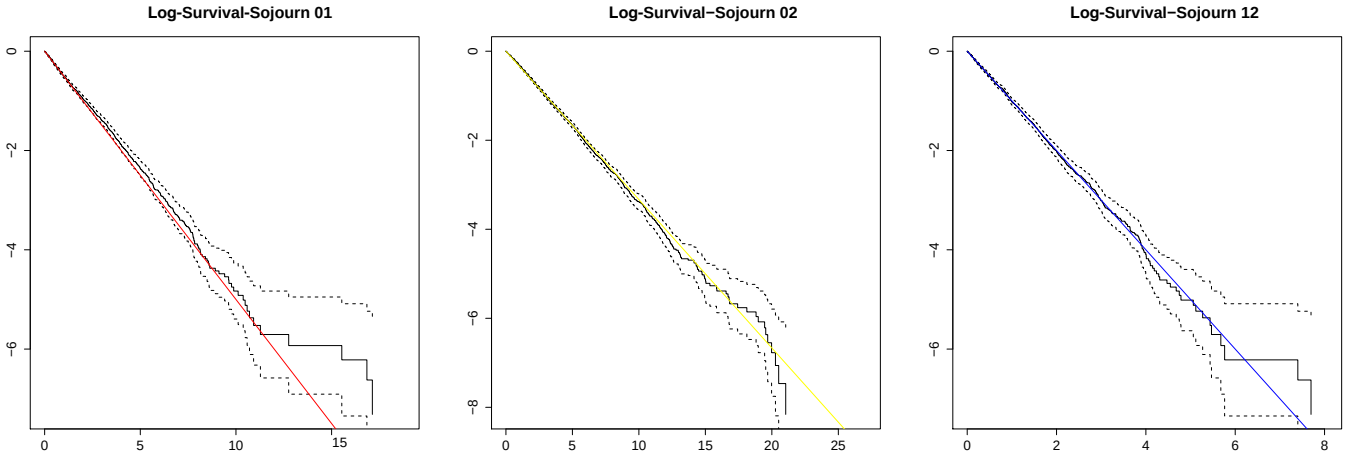


Figure 5.5: Comparison of log-survival function of data with true log-survival function. The black curves are the fitted log-survival functions of sojourn times derived from data, and the coloured lines are true exponential log-survival functions where parameters (from left to right) are $\alpha_{01} = 0.5, \alpha_{02} = 0.33$, and $\alpha_{12} = 1$, respectively.

5.4 Comparison of Distributions of Two Approaches

In this section, we apply different distributions for the illness-death model from the viewpoint of two different approaches of the SMP. The main goal is to find out the distribution of sojourn times where we know the distribution of transition times and vice versa.

For this purpose, first note that from (5.2.6), for the illness-death model, since $p_{12} = 1$ and $S_1(t) = S_{12}(t)$, always $\alpha_{12}(t) = \tilde{\alpha}_{12}(t)$. So, we just consider the other two cases in the following. In addition, for the sake of simplicity, we put $p_{01} = p$ and therefore we have

$p_{02} = 1 - p$. Consequently, the Eq. (5.2.6) for illness-death model is written as:

$$\tilde{\alpha}_{01}(t) = \frac{p\, S_{01}(t)}{p\, S_{01}(t) + (1-p)\, S_{02}(t)} \alpha_{01}(t)\,, \qquad \tilde{\alpha}_{02}(t) = \frac{(1-p)\, S_{02}(t)}{p\, S_{01}(t) + (1-p)\, S_{02}(t)} \alpha_{02}(t). \quad (5.4.1)$$

Since $S_{ij}(t) = e^{-\int_0^t \alpha_{ij}(u)du}$, if we have $\alpha_{01}(t) = \alpha_{02}(t)$, then $S_{01}(t) = S_{02}(t)$. Therefore, from Eq. (5.4.1), we have $\tilde{\alpha}_{02}(t) = \frac{1-p}{p}\, \alpha_{02}(t)$ or $\tilde{\alpha}_{01}(t) = \frac{p}{1-p}\, \alpha_{01}(t)$. Now we consider relations between distribution of the sojourn times and the transition rates for some popular distributions.

Exponential Distribution Under exponential distribution both sojourn times and transition rates are constants. Moreover, $f_{ij}(t) = \lambda_{ij} e^{-\lambda_{ij} t}$ and $S_{ij}(t) = e^{-\lambda_{ij} t}$. Therefore, $\alpha_{ij}(t) = \frac{f_{ij}(t)}{S_{ij}(t)} = \lambda_{ij}$. For the illness-death model we have:

- Consider that $\alpha_{01}(t)$ is the hazard function of an exponential distribution, then using the first part of Eq. (5.4.1), we have:

  $$\tilde{\alpha}_{01}(t) = \frac{p\, e^{-\lambda_{01} t}}{p\, e^{-\lambda_{01} t} + (1-p)\, e^{-\lambda_{02} t}} \lambda_{01}.$$

  Now, whenever $\lambda_{01} = \lambda_{02} = \lambda$ (for example the CTMC case) the transition rate $\tilde{\alpha}_{01}(t)$ is a constant given by:

  $$\tilde{\alpha}_{01}(t) = p\, \lambda,$$

  which means the transition time $0 \to 1$ has an exponential distribution with rate $\tilde{\alpha}_{01}(t)$. The same calculation, shows that the transition time $0 \to 2$ has an exponential distribution with rate $\tilde{\alpha}_{02}(t) = (1-p)\, \lambda$.

- Consider that transition time $0 \to 1$ has an exponential distribution with rate $\tilde{\alpha}_{01}(t)$, or equivalently: $\tilde{\alpha}_{01}(t) = \tilde{\alpha}_{01}$. Then, from definition of the SMP, we have a CTMC and so, $\alpha_{01}(t) = \alpha_{01}$ is a constant. Moreover, $\alpha_{01} = \alpha_{02}$.

Weibull Distribution A random variable has a Weibull distribution with parameters k, λ, Weibull(k, λ), if its density is given by:

$$f(t) = \frac{k}{\lambda}\, \left(\frac{t}{\lambda}\right)^{k-1} e^{-\left(\frac{t}{\lambda}\right)^k}.$$

Then, the hazard function and the survival function under Weibull(k, λ) have the following forms:

$$\alpha(t) = \frac{k}{\lambda}\,\big(\frac{t}{\lambda}\big)^{k-1}, \qquad\qquad S(t) = e^{-\left(\frac{t}{\lambda}\right)^{k}}.$$

- Consider that the sojourn time distribution for model $0 \to 1$ has a Weibull distribution with parameters (k_{01}, λ_{01}), then using Eq. (5.4.1), we have:

$$\tilde{\alpha}_{01}(t) = \frac{p\, e^{-\left(\frac{t}{\lambda_{01}}\right)^{k_{01}}}}{p\, e^{-\left(\frac{t}{\lambda_{01}}\right)^{k_{01}}} + (1-p)\, e^{-\left(\frac{t}{\lambda_{01}}\right)^{k_{01}}}} \frac{k_{01}}{\lambda_{01}}\,\big(\frac{t}{\lambda_{01}}\big)^{k_{01}-1}.$$

  Now, if $\alpha_{01}(t) = \alpha_{02}(t)$ or equivalently if $k_{01} = k_{02} = k$ and $\lambda_{01} = \lambda_{02} = \lambda$, the above formula reduces to:

$$\tilde{\alpha}_{01}(t) = p\,\frac{k}{\lambda}\,\big(\frac{t}{\lambda}\big)^{k-1} = k(\frac{\sqrt[k]{p}}{\lambda})^{k} t^{k-1},$$

  which means that the transition time for $0 \to 1$ has a Weibull distribution with parameters $(k, \frac{\sqrt[k]{p}}{\lambda})$. Similarly, the transition time for $0 \to 2$ has a Weibull distribution with parameters $(k, \frac{\sqrt[k]{1-p}}{\lambda})$.

5.5 Phase-type Distributions for illness-death Model

Denseness of phase-type (PH) distributions (in the sense of weak topology) in the class of distributions defined on the non-negative real numbers (see Section 2.3.1 for more details) make them suitable distributions for fitting data. Moreover, for the case of hidden or sparse data which usually happens in healthcare studies, PH distributions are quite useful for explaining the hidden stages of diseases or lost stages of patients trajectories in a hospital. Here, we present a quick review of PH distributions and obtain the corresponding SMP quantities of interest based on PH distributions.

Consider a *phase-type (PH) distribution* $PH_p(\eta, B)$ with density function $f(t) = \eta e^{Bt}\mathbf{b}$, where $\mathbf{b}$ is the exit vector. For more details on PH distributions and their parameter estimation methods, the reader is referred to Section 2.3.1 and Subsection 2.4.3, respectively.

The survival and hazard functions in terms of a PH distribution with parameters (η, B) are

given by:

$$S(t) = \eta e^{Bt}\mathbf{1}, \qquad \alpha(t) = \frac{\eta e^{Bt}\mathbf{b}}{\eta e^{Bt}\mathbf{1}},$$

where 1 is the column vector of ones.

Note that for applying PH distributions for the illness-death model, without loss of generality, we can consider that the PH distribution is a Coxian PH distribution. For a Coxian PH distribution: (i) the initial state is unique (and here non-absorbing), and (ii) for each state, the next non-absorbing state is unique (that is the next state is the unique non-absorbing state or the absorbing state). The illness-death model satisfies both of these conditions.

Example 5.5.1. *(illness-death model) In the illness-death model, if we consider that* $\alpha_{01}(t) = \alpha_{02}(t)$ *and come from an exponential distribution, then we can consider the model as a CTMC and therefore, the time until absorption to state* 2 *has a PH*$_2$ *distribution. For this model, we have:*

$$\eta = (1, 0), \qquad B = \begin{pmatrix} -(\lambda_{01} + \lambda_{02}) & \lambda_{01} \\ 0 & -\lambda_{12} \end{pmatrix}, \qquad \mathbf{b} = (\lambda_{02}, \lambda_{12}),$$

where for this model, $\tilde{\alpha}_{01}(t) = \lambda_{01}$*,* $\tilde{\alpha}_{02}(t) = \lambda_{02}$ *and* $\tilde{\alpha}_{12}(t) = \lambda_{12}$*. Therefore, since for this model the only transient states are* $0, 1$*, the absorption state is state* 2*, and the process starts with probability 1 from state* 0*, the likelihood* (2.4.6) *reduces to:*

$$\mathcal{L} = \lambda_{01}^{N_{01}} e^{-\lambda_{01} Z_0} \left(\lambda_{02}^{N_0} e^{-\lambda_{02} Z_0} \lambda_{12}^{N_1} e^{-\lambda_{12} Z_1} \right).$$

where

- Z_i *is the total time spent in state* i $(i = 0, 1)$*,*
- N_{01} *is the total number of jumps from state* 0 *to state* 1*,*
- N_i *is the total number of processes jumping from state* i *for* $i = 0, 1$ *to the absorbing state (state* 2*).*

From the above likelihood we can find 3 unknown parameters: λ_{01}*,* λ_{02}*, and* λ_{12} *by using the formulas in Subsection 2.4.3 related to the EM algorithm.*

Comparison distributions of α_{ij} and $\tilde{\alpha}_{ij}$. The density function, the survival function and the hazard function of PH distribution with parameters η and B are given respectively by:

$$f(t) = \eta e^{BT}\mathbf{b}, \qquad S(t) = \eta e^{Bt}\mathbf{1}, \qquad \alpha(t) = \frac{\eta e^{Bt}\mathbf{b}}{\eta e^{Bt}\mathbf{1}},$$

where $\mathbf{b} = -B\mathbf{1}$.

- Consider that the sojourn time of model $0 \to 1$ follows a PH distribution with parameters (η, B), and the sojourn time of model $0 \to 2$ follows a PH distribution with parameters $(\overline{\eta}, \overline{B})$. Then, applying Eq. (5.4.1) results in:

$$\tilde{\alpha}_{01}(t) = \frac{p\,\eta\, e^{Bt}\mathbf{1}}{p\,\eta\, e^{Bt}\mathbf{1} + (1-p)\,\overline{\eta}\, e^{\overline{B}t}\mathbf{1}} \left(\frac{\eta e^{Bt}\mathbf{b}}{\eta e^{Bt}\mathbf{1}}\right).$$

  Assuming that $(\eta, B) = (\overline{\eta}, \overline{B})$, the above formula reduces to $\tilde{\alpha}_{01}(t) = \left(\frac{p}{\eta e^{Bt}\mathbf{1}}\right)\eta e^{Bt}\mathbf{b}$. This means that the transition times of model $0 \to 1$, follows a PH distribution with parameters $(\left(\frac{p}{\eta e^{Bt}\mathbf{1}}\right)\eta, \left(\frac{p}{\eta e^{Bt}\mathbf{1}}\right)B)$. The proof for $\tilde{\alpha}_{02}(t)$ follows the same lines.

As above mentioned, for the illness-death model we can apply PH distributions by considering that there are either hidden stages in the middle of the illness-death model stages where the transition rates have PH distribution, or by assuming that there are some hidden stages in each stage of the illness-death model which means that the sojourn time have PH distribution. In both cases, when we need to consider the effectiveness of different treatments or distinguish between different causes of death, we need to consider multi absorption states. The analytical formulas related to this notion is presented in the following section.

5.6 Multi Absorption Phase-Type (MAPH) Distributions

In this section, we define and derive the distribution and the corresponding formulas for a multi absorption PH distribution.

Let $X(t)$ be a CTMC with p transient states and q absorption states. A bivariate random variable (U, J) where J is a discrete component shows the state in which the process absorbed

and U is the time until absorption to that state, has an MAPH$_{p,q}(\eta, B, B^0)$ distribution. Here,

$$U = \inf\{t \geq 0 \mid X(t) \in \{\text{absorbing states}\}\}, \qquad J = X(U).$$

The row vector η is the initial distribution of the phase process, and the matrix B is the transition rate matrix of the phase process. The matrix B^0 shows the transition rates of going from each phase to one of the absorption states.

Example 5.6.1. *Consider a PH$_{2,3}$ demonstrated in Figure 5.6, where there are two transient states 1, 2 and three absorbing states 3, 4 and 5.*

Then, the transition rate matrix Q is given by:

$$Q = \begin{pmatrix} B_{p\times p} & B^0_{p\times q} \\ 0_{q\times p} & 0_{q\times q} \end{pmatrix} = \begin{pmatrix} -\lambda_1 & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \lambda_{21} & -\lambda_2 & \lambda_{23} & \lambda_{24} & \lambda_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where,

$$B = \begin{pmatrix} -\lambda_1 & \lambda_{12} \\ \lambda_{21} & -\lambda_2 \end{pmatrix} \qquad \textit{and} \qquad B^0 = \begin{pmatrix} \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \lambda_{23} & \lambda_{24} & \lambda_{25} \end{pmatrix}. \tag{5.6.1}$$

The above matrices are the transition rate matrix of the phase process and the exit matrix, respectively.

By using the Laplace-Stieltjes transform of U, we can find the moments of the random vector (U, J). Define:

$$\phi_i(s, j) := \mathbb{E}[e^{-sU}\mathbb{1}\{J = j\} \mid X_0 = i].$$

A "subject of the form" $\mathbb{E}[e^{-sU}\mathbb{1}\{J = j\}]$ where U and J are some jointly distributed continuous and discrete random variables, can take on the following meaning:

$$\mathbb{E}[e^{-sU}\mathbb{1}\{J = j\}] = \int_0^\infty e^{-st} f(t, j)dt,$$

where,

$$f(t, j) = \frac{d}{dt}\mathbb{P}(U \leq t, J = j),$$

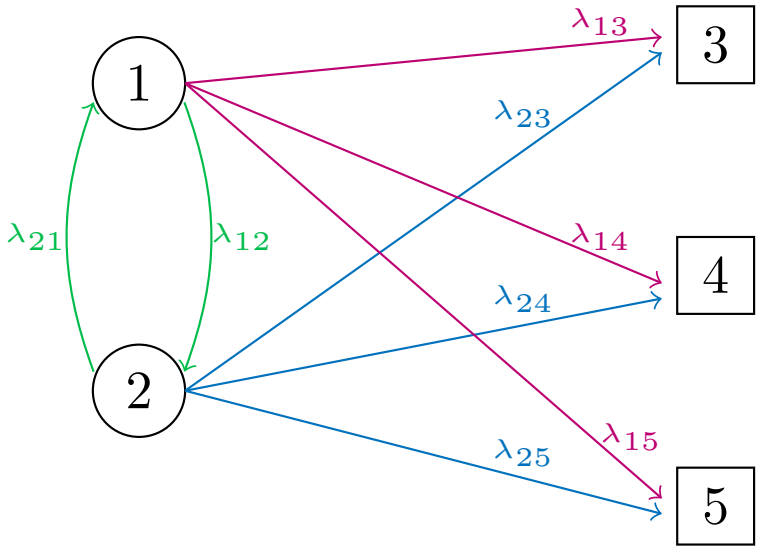


Figure 5.6: State transition diagram of an MAPH with 2 transient states (circles) and 3 absorption states (squares).

is the density at j. Hence having $\phi_i(s,j)$ implies that the inverse Laplace transform is the density at j. So, for example, if $\phi_i(s,j)$ is a rational function of s we know (essentially) how to represent $f(t,j)$ as a bilinear form of a matrix exponential.

Consider $\tau_i \sim exp(\lambda_i)$, where $\lambda_i = \sum_{j\neq i}\lambda_{ij}$, as the sojourn time in state i. Further, consider I as the chosen path after state i. Let $\overline{U}$ denote the time that it takes for the process to go to one of the absorption states after leaving the state i. Since τ_i is independent of $\overline{U}$, we have:

$$\begin{aligned}\phi_i(s,j) &= \mathbb{E}_{\tau_i,I}\mathbb{E}[e^{-s(\tau_i+\overline{U})}\mathbb{1}\{J=j\}|X(0)=i,\tau_i,I] \\ &= \mathbb{E}[e^{-s\tau_i}]\mathbb{E}_I\mathbb{E}[e^{-s\overline{U}}\mathbb{1}\{J=j\}|X(0)=i,I] \\ &= \tfrac{\lambda_i}{\lambda_i+s}\big(\textstyle\sum_{k\neq i,j} p_{ik}\phi_k(s,j)+p_{ij}\big).\end{aligned} \tag{5.6.2}$$

The first term of the above formula comes from:

$$\mathbb{E}[e^{-s\tau_i}] = \int_0^\infty e^{-st}\lambda_i e^{-\lambda_i t}dt = \frac{\lambda_i}{s+\lambda_i},$$

and in the case that the state after i is the absorption state $J=j$, we have $\tilde{U}=0$. Therefore,

$$\mathbb{E}_I\mathbb{E}[e^{-s\tilde{U}}\mathbb{1}\{J=j\}|X(0)=i,I] = \mathbb{E}_I\mathbb{E}[1|X(0)=i,I] = p_{ij}.$$

Note that here there is no self-transition (transition from a state to itself). Since $p_{ik} = \frac{\lambda_{ik}}{\lambda_i}$, we

can rewrite the system of equations in (5.6.2), in the matrix form:

$$(sI - B)\begin{bmatrix} \phi_1(s,j) \\ \phi_2(s,j) \\ \vdots \\ \phi_p(s,j) \end{bmatrix} = B_j^0. \tag{5.6.3}$$

For the above example, if we put $\phi_i = \phi_i(s,j)$, the following system of equations holds:

$$\begin{aligned} \phi_1 &= \tfrac{\lambda_1}{\lambda_1+s}\Big(\tfrac{\lambda_{12}}{\lambda_1}\phi_2 + \tfrac{\lambda_{1j}}{\lambda_1}1\Big), \\ \phi_2 &= \tfrac{\lambda_2}{\lambda_2+s}\Big(\tfrac{\lambda_{21}}{\lambda_2}\phi_1 + \tfrac{\lambda_{2j}}{\lambda_2}1\Big). \end{aligned}$$

The above equations can be written in the matrix form as:

$$\begin{bmatrix} \lambda_1+s & -\lambda_{12} \\ -\lambda_{21} & \lambda_2+s \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \lambda_{1j} \\ \lambda_{2j} \end{bmatrix} = (sI-B)\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = B_j^0.$$

If we put $\overline{\phi}(s,j)$ as the vector of $\phi_i(s,j)$, then from Eq. (5.6.3), we have:

$$(sI - B)\overline{\phi}(s,j) = B_j^0,$$

which results in

$$\overline{\phi}(s,j) = (sI - B)^{-1}B_j^0,$$

where s is greater than the spectral radius of B. Since B is an M-matrix (see Section 3.2), $-B^{-1}$ exists and is a positive matrix if and only if all of the eigenvalues of B are strictly within the left half of the complex plane. Therefore, we consider $s \in [0,\infty)$ (or for complex s: $Re(s) \geq 0$) which means that s is greater than the spectral radius of B and so $(sI-B)^{-1}$ exists. Hence, if we put $\phi(s,j) := \sum_i \eta_i \phi_i(s,j)$, the above formula can be represented as

$$\phi(s,j) = \eta(sI - B)^{-1}B_j^0. \tag{5.6.4}$$

Compare this with a phase type distribution (only one absorbing state). Then, B_j^0 is simply the exit vector $B^0 = -B\mathbf{1}$ and (5.6.4) is the Laplace-Stieltjes transform of the PH random variable U with the following density function $f(u) = \eta e^{Bu}B^0$. In the same way for an MAPH

random variable, we can conclude from (5.6.4) that the density function for $J = j$ is:

$$f(u, j) = \eta e^{Bu} B_j^0.$$

Note that $f(u) = \sum_j f(u, j)$ is indeed the density function for a PH random variable (all states J are joined into one absorbing state). So,

$$F(u, j) = \int_0^u f(x, j)dx = -\eta B^{-1}(I - e^{Bu})B_j^0$$

Note that in the above, if it was simply a PH distribution (that is $B_j^0 = B^0 = -B\mathbf{1}$) then, using the facts that $\eta\mathbf{1} = 1$ and e^{Bu} and B commute, we have:

$$F(u, j) = \eta B^{-1}(I - e^{Bu})B\mathbf{1} = 1 - \eta e^{Bu}\mathbf{1}.$$

Using the same type of calculations, and assuming that we can find a V_j such that $B_j^0 = -BV_j$ (since B is invertible, such a vector V_j exists: $V_j = -B^{-1}B_j^0$), results in:

$$F(u, j) = \int_0^u f(x, j)dx = \eta B^{-1}(I - e^{Bu})BV_j = \eta V_j - \eta e^{Bu} V_j = \eta(I - e^{Bu})V_j.$$

Now let $\tilde{P}$ denotes transition probability matrix of the phase process, then (since the phase process has a finite number of states) we have $\tilde{P}(u) = e^{Bu}$. Taking $u \to \infty$, clearly results in $\tilde{P}(u) \to 0$ (since the process eventually goes to one of the absorption states). Therefore,

$$\begin{aligned}\lim_{u\to\infty} F_i(u, j) &= \lim_{u\to\infty} \mathbb{P}\big(U \le u, J = j \;\big|\; X(0) = i\big) = \mathbb{P}\big(J = j \;\big|\; X(0) = i\big) \\ &= \mathbb{P}\big(X(0) = i\big)\ \mathbb{P}\big(J = j, X(0) = i\big) = \eta_i V_{ij}.\end{aligned}$$

Here, we can conclude that: $V_{ij} = \mathbb{P}(J = j \;|\; X(0) = i)$ (the probability that absorption happens in j given that the process starts at i). There is an analogous result from the canonical form of a CTMC $X(t)$, see Chapter 7 of [132].

Now, by multiplying both sides of $B^0 = -BV$ from left by diag$(\frac{1}{\lambda_i})$, we obtain:

$$\tilde{P}^0 = (I - \tilde{P})V, \tag{5.6.5}$$

where $\tilde{P^0}$ denotes the transition probability matrix that includes all transition probabilities from transient states to absorbing states. So, the transition probability matrix of the CTMC $X(t)$ can be written as:

$$P = \begin{pmatrix} \tilde{P}_{p\times p} & \tilde{P}^0_{p\times q} \\ 0_{q\times p} & I_{q\times q} \end{pmatrix}.$$

Rewriting (5.6.5) in terms of its elements results in:

$$\tilde{P}^0_{ij} = V_{ij} - \sum_k \tilde{P}_{ik}V_{kj}, \qquad \text{or} \qquad V_{ij} = \tilde{P^0}_{ij} + \sum_k \tilde{P}_{ik}V_{kj},$$

which means starting from a transient state i, the process either goes to the absorption state j (w.p. $\tilde{P}^0_{ij}$) or goes to another transient state and then eventually goes to the absorption state j (w.p. $\sum_k \tilde{P}_{ik}v_{kj}$).

Therefore, we have:

Proposition 5.6.2. *For an MAPH$_{p,q}$ with parameters (η, B, B^0), we have:*

- *The density:* $f(u,j) = \eta e^{Bu}B^0_j$
- *The distribution function :* $F(u,j) = \eta V_j - \eta e^{Bu}V_j$
- *Survival function:* $S(u,j) = 1 - F(u,j) = 1 - \eta V_j + \eta e^{Bu}V_j$
- *Hazard/Failure function:* $H(u,j) = \lim_{\Delta t\to 0} \frac{\mathbb{P}(t<U<t+\Delta t|U>t)}{\Delta t} = \frac{f(u,j)}{S(u,j)} = \frac{\eta e^{Bu}B^0_j}{1-\eta V_j+\eta e^{Bu}V_j}$
- *Laplace transform:* $\phi(s,j) = \eta(sI - B)^{-1}B^0_j$
- *The $n-$th moment:* $M_n = \int_0^\infty u^n f(u,j)du = (-1)^{n+1}\eta\, n!\, B^{-(n+1)}B^0_j$
- *The moment generating function:* $\mathbb{E}(e^{su}) = -\eta(sI + B)^{-1}B^0_j$

where $j = 1, \cdots, q$.

Chapter 6

Conclusion

In this thesis, we considered the role of structured Markov models in stochastic modelling. We analysed a class of stationary MAPs that we call bursty. We applied matrix analytic methods (MAM) to find relations between bursty MAPs. Further, we considered the problem of finding an optimal policy for a controlled queueing system. Here, in addition to considering POMDP, we presented a QBD structure for finding the maximum throughput. Then, we considered the application of semi-Markov processes (SMPs) in healthcare and presented two different approaches for SMPs. In addition to the comparison of these approaches, we introduced the multi-absorption PH (MAPH) distribution as an applicable distribution in the survival analysis of complex systems. We now summarize our results and highlight open questions.

We call a MAP *bursty* if both the squared coefficient of variation and the asymptotic index of dispersion of counts are greater than unity. As we know a MAP is characterised by (η, C, D) where η is the initial distribution of the underlying finite-state CTMC, the matrix C shows the transition between the states of CTMC with no arrival and the matrix D shows the transitions accompanied with an arrival (event). We see that for the special cases where the matrix D is diagonal (MMPP) or the matrix C is diagonal (MSPP, H_p-renewal, and MTCP) the given MAP is bursty. This thesis presented key relationships between these classes of MAPs. Specifically, we proved that the MTCP could be considered as an alternative model for a rich class of MMPPs that we call slow MMPPs.

Having analysed MMPPs in depth and establishing that the squared coefficient of variation,

c^2, is greater than one, we encountered a few related open questions dealing with MMPPs as well as the more general class of doubly stochastic Poisson processes (Cox processes). First and foremost, for a general Cox process on the line, [73], we conjecture that whenever $\mathbb{E}\left[(T_1)^2\right]$ is finite, it holds that $c^2 \geq 1$. Establishing such a result would clearly use different methods than the matrix analytic methods used in Theorem 3.3.1. Nevertheless, based on intuition, we believe that the result is true. This is because we believe that any Cox process can be approximated by an MMPP with a high number of phases.

The second branch of questions deals with characterizing the Poisson process via $c^2 = 1$ and considering when an MMPP is Poisson. For example, for the general class of MAPs, the authors of [27] provide a condition for determining if a given MAP is Poisson. It is not hard to construct a MAP with $c^2 = 1$ that is not Poisson. But, we believe that all MMPPs with $c^2 = 1$ are Poisson. Yet, we do not have a proof. Further, we believe that for an MMPP, if $c^2 = 1$ then all λ_i are equal (the converse is trivially true). We do not have a proof of this either. Related questions also hold for the more general Cox Processes.

Moreover, for the two extreme examples of MAPs, the MMPP (where there is no arrival at epochs of transitions of the underlying CTMC) and the MTCP (where all transitions of the underlying CTMC are accompanied with an arrival), we show that the first two moments of counts for large class of MMPPs, slow MMPPs, and MTCPs are the same. Therefore, from a modelling point of view, one can construct an MMPP from a given MTCP (see Proposition 3.4.1) and for a given slow MMPP we show that there is an MTCP with the same first two counting moments. Now an open question arises: Is there any one-to-one relationship between the class of MMPPs and the class of MTCPs? In other words, can we construct an MTCP for a given non-slow MMPP?

The next problem that we addressed is in the area of parameter estimation for bursty MAPs. We consider parameters of a slow MMPP_2 and use the method of moments and a nonlinear optimisation procedure to estimate parameters of an MTCP_4 with the same first moments and autocorrelations of the inter-arrival process. Comparing the results of this moment-based estimation method with parameters resulting from matching the first two moments of the counting process in Section 3.4 shows that the difference between these results is negligible. There are some papers related to fitting data with a MAP, see for instance [157] and references therein. As we know time-homogeneity of the data trace is a significant

property in available fitting methods. So, the question arises here for future work is that what is the best fitting method if the real data trace is non-homogeneous?

When it comes to controlling of stochastic systems, we described some results from a research effort attempting to handle control of stochastic systems with partial observations where the control decision influences the observation made. Explicit analysis of such systems is extremely challenging as is evident by both the complicated Bellman equations and the QBD structure that we put forward in Chapter 4 (even for a simple system as we consider). Nevertheless, insights obtained on the role of information, for example the effect of the observation scheme (I–V) on system stability are of interest.

Our model and numerical results, pave the way for explicit proofs of some structural properties that are apparent from the numerical results. Some of the future work in this project would be: comparing the performance of the optimal policies and the myopic policy, and comparing and understanding the difference between optimal curves, under different information settings for a range of parameters. In fact, the existence of optimal switching curves, as in figures 4.4 and 4.5, requires proof. Moreover, the analysis remains to be extended to more general server environment models, as well as systems with more queues and control decisions.

Related work is in [146], our first paper in this area of research. An aspect in [146] that remains to be further considered is the networked case where the authors investigated (through simulation) cases in which the relationship of stability and throughput is not as immediate as in our current paper. A further related (recent) paper, [142], deals with a situation similar to our output observation case (III). In that paper, the authors consider the Whittle index applied to a similar system (without considering a queue and stability). Relating the Whittle index and system stability is a further avenue that requires investigation, see [162] as an example of using an index policy (based on Whittle index) to compute the stability region of a queueing system.

Another objective of this dissertation was describing two different approaches for a semi-Markov process (SMP). The first approach considers the sojourn times and the second one considers transition rates. We compare the results of applying these approaches for formulating the likelihood of the illness-death model. This is a three-state semi-Markov process

that applies for modelling trajectories of patients in the ICUs of hospitals. Finding the relations between parameters of these two approaches leads us to find the corresponding distribution of transition rates by knowing the distribution of sojourn times. Furthermore, we found the corresponding formulas for the case that the hazard function has a PH distribution. Also, we obtained the related formulas for a generalisation of PH distributions, the multi-absorption PH (MAPH) distribution.

There are some theoretical aspects of this project which need to be considered for future work. For instance, adapting an EM algorithm for parameter estimation of MAPH distributions, use a model selection criterion to choose between the proposed models, investigate whether the class of MAPH distributions is dense in the class of non-negative distributions on the real line or not, and compare the results with a non-homogeneous Markov model. From numerical point of view, fitting an SMP with an MAPH distribution and illustration of the proposed models on a real data-set can be considered.

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Index