PRESENTATION COMPLEXES WITH THE FIXED POINT PROPERTY

IVÁN SADOFSCHI COSTA

Abstract. We prove that there exists a compact two-dimensional polyhedron with the fixed point property and even Euler characteristic. This answers a question posed by R.H. Bing in 1969. We also settle another of Bing's questions.

1. Introduction

In his influential article "The elusive fixed point property" [3], R.H. Bing stated twelve questions. Since then eight of these questions have been answered [6]. In this paper we answer Questions 1 and 8.

Recall that a space X is said to have the *fixed point property* if every map $f : X \to X$ has a fixed point. Motivated by an example of W. Lopez [8], Bing stated in [3] the following question.

Question 1.1 (Bing's Question 1). *Is there a compact two-dimensional polyhedron with the fixed point property which has even Euler characteristic?*

This question was studied in [10]. In [2] it was shown that such a space cannot have abelian fundamental group. In Corollary 2.4 we show that the answer to Question 1.1 is affirmative. Bing's Question 8 [3] may be rephrased as follows.

Question 1.2 (Bing's Question 8). *What is the lowest dimension for a compact polyhedron X with the fixed point property and such that a space Y without the fixed point property can be obtained by attaching a disk D to X along an arc?*

The answer to this question is clearly greater than 1. A one-dimensional polyhedron X with the fixed point property is a tree, and then any space Y obtained by attaching a disk along an arc is a contractible polyhedron. According to C.L. Hagopian [6], Bing conjectured that the answer to Question 1.2 is 2. This is the content of Theorem 2.8.

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2. Bing groups

If $\mathcal{P}$ is a presentation, the presentation complex of $\mathcal{P}$ will be denoted by $X_{\mathcal{P}}$. Presentation complexes are in fact polyhedra. If a finite group G is presented by a presentation $\mathcal{P}$ with g generators and r relators, then $r - g$ is at least the number of invariant factors of $H_2(G)$. If this lower bound is attained for $\mathcal{P}$, then the presentation is said to be *efficient*.

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Definition 2.1. Let G be a finite group and $d_1 \mid \dots \mid d_k$ be the invariant factors of $H_2(G)$. We say that G is a *Bing group* if for every endomorphism $\phi : G \to G$ we have $\mathrm{tr}(H_2(\phi) \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) \neq -1$ in $\mathbb{Z}_{d_1}$.

Theorem 2.2. *If $\mathcal{P}$ is an efficient presentation of a Bing group G then $X_{\mathcal{P}}$ has the fixed point property.*

Proof. Let $X = X_{\mathcal{P}}$ and $f : X \to X$ be a map. There is a $K(G,1)$ space Y with $X = Y^2$. Now f extends to a map $\overline{f} : Y \to Y$. In the following commutative diagram, the horizontal arrows, induced by the inclusion $i : X \hookrightarrow Y$, are epimorphisms:

$$\begin{array}{ccc} H_2(X) & \xrightarrow{i_*} & H_2(Y) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \overline{f}_*} \\ H_2(X) & \xrightarrow[i_*]{} & H_2(Y) \end{array}$$

Let $d_1 \mid \dots \mid d_k$ be the invariant factors of $H_2(G)$. Since $\mathcal{P}$ is efficient, the rank of $H_2(X)$ equals the number of invariant factors of $H_2(Y)$. Therefore the horizontal arrows in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc} H_2(X) \otimes \mathbb{Z}_{d_1} & \xrightarrow[\approx]{i_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}} & H_2(Y) \otimes \mathbb{Z}_{d_1} \\ {\scriptstyle f_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}}\downarrow & & \downarrow{\scriptstyle \overline{f}_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}} \\ H_2(X) \otimes \mathbb{Z}_{d_1} & \xrightarrow[i_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}]{\approx} & H_2(Y) \otimes \mathbb{Z}_{d_1} \end{array}$$

Now $\mathrm{tr}(f_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) = \mathrm{tr}(\overline{f}_* \otimes \mathbb{1}_{\mathbb{Z}_{d_1}}) \neq -1$ in $\mathbb{Z}_{d_1}$ since G is a Bing group. Here we are using the natural isomorphism $H_2(BG) \approx H_2(G)$ of [9, Theorem 5.1.27]. Recall that every map $BG \to BG$ is induced, up to homotopy, by an endomorphism $G \to G$.

Finally we obtain $\mathrm{tr}(f_*) \neq -1$ in $\mathbb{Z}$, since tensoring with $\mathbb{Z}_{d_1}$ reduces the trace modulo d_1. So $L(f) \neq 0$ and, by the Lefschetz fixed point theorem, f has a fixed point. □

Proposition 2.3. *The group G presented by*

$$\mathcal{P} = \langle x, y \mid x^3,\, xyx^{-1}yxy^{-1}x^{-1}y^{-1},\, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1} \rangle$$

is a finite group of order 243. We have $H_2(G) = \mathbb{Z}_3$, so $\mathcal{P}$ is efficient. Moreover G is a Bing group.

Proof. We will need the following GAP [5] program, that uses the packages HAP [4] and SONATA [1].

```
LoadPackage("HAP");;
LoadPackage("SONATA");;
F:=FreeGroup(2);;
G:= F/[F.1^3, F.1*F.2*F.1^-1*F.2*F.1*F.2^-1*F.1^-1*F.2^-1,
F.1^-1*F.2^-4*F.1^-1*F.2^2*F.1^-1*F.2^-1];;
Order(G);
G:=SmallGroup(IdGroup(G));;
R:=ResolutionFiniteGroup(G,3);;
Homology(TensorWithIntegers(R),2);
Set(List(Endomorphisms(G),
```

```
f->Homology(TensorWithIntegers(EquivariantChainMap(R,R,f)),2)));
```

The program prints the order of G, a list with the invariant factors of $H_2(G)$ and a list with the endomorphisms of $H_2(G)$ that are induced by an endomorphism of G. The output is:

```
243
[ 3 ]
[ [ f1 ] -> [ <identity ...> ], [ f1 ] -> [ f1 ] ]
```

Therefore $|G| = 243$ and $H_2(G) = \mathbb{Z}_3$. Since for every endomorphism $\phi : G \to G$ we have that $H_2(\phi)$ is either the zero map or the identity, G is a Bing group. □

By Theorem 2.2 and Proposition 2.3 we have:

Corollary 2.4. *The complex $X_{\mathcal{P}}$ associated to the presentation*

$$\mathcal{P} = \langle x, y \mid x^3,\, xyx^{-1}yxy^{-1}x^{-1}y^{-1},\, x^{-1}y^{-4}x^{-1}y^2x^{-1}y^{-1}\rangle$$

has the fixed point property. Moreover $\chi(X_{\mathcal{P}}) = 2$.

Corollary 2.5. *There are compact 2-dimensional polyhedra with the fixed point property and Euler characteristic equal to any positive integer n.*

Proof. For $n = 1$ this is immediate. For $n > 1$ take a wedge of $n - 1$ copies of the space $X_{\mathcal{P}}$ of Corollary 2.4. □

To prove Theorem 2.8 we will need another efficient Bing group:

Proposition 2.6. *The group H presented by $\mathcal{Q} = \langle x, y \mid x^4, y^4, (xy)^2, (x^{-1}y)^2\rangle$ is a finite group of order 16. We have $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, so $\mathcal{Q}$ is efficient. Moreover H is a Bing group.*

Proof. As above we will use a GAP program.

```
LoadPackage("HAP");;
LoadPackage("SONATA");;
F:=FreeGroup(2);;
H:= F/[F.1^4, F.2^4, (F.1*F.2)^2, (F.1^-1*F.2)^2];;
Order(H);
H:=SmallGroup(IdGroup(H));;
R:=ResolutionFiniteGroup(H,3);;
Homology(TensorWithIntegers(R),2);
Set(List(Endomorphisms(H),
f->Homology(TensorWithIntegers(EquivariantChainMap(R,R,f)),2)));
```

The program produces the following output:

```
16
[ 2, 2 ]
[ [ f1, f2 ] -> [ <identity ...>, <identity ...> ],
[ f1, f2 ] -> [ f1, f2 ],[ f1, f2 ] -> [ f1^-1*f2^-1, f2^-1 ] ]
```

This proves that $|H| = 16$, $H_2(H) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and that H is a Bing group. □

We recall the following theorem:

Theorem 2.7 (Jiang,[7, Theorem 7.1]). *In the category of compact connected polyhedra without global separating points, the fixed point property is a homotopy type invariant.*

Moreover, if $X \simeq Y$ are compact connected polyhedra such that Y lacks the fixed point property and X does not have global separating points, then X lacks the fixed point property.

The following shows that the answer to Question 1.2 is 2:

Theorem 2.8. *There is a compact 2-dimensional polyhedron Y without the fixed point property and such that the polyhedron X, obtained from Y by an elementary collapse of dimension 2, has the fixed point property.*

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ be the presentations of Propositions 2.3 and 2.6. By Theorem 2.2, $X_{\mathcal{P}}$ and $X_{\mathcal{Q}}$ have the fixed point property, so $X = X_{\mathcal{P}} \vee X_{\mathcal{Q}}$ also has the fixed point property. Since neither $X_{\mathcal{P}}$ nor $X_{\mathcal{Q}}$ have global separating points, by adding a 2-simplex, we can turn X into a polyhedron Y, without global separating points and such that, by collapsing that 2-simplex, we obtain X. We have $H_2(\pi_1(Y)) = H_2(\pi_1(X_{\mathcal{P}}) * \pi_1(X_{\mathcal{Q}})) = H_2(\pi_1(X_{\mathcal{P}})) \oplus H_2(\pi_1(X_{\mathcal{Q}})) = \mathbb{Z}_2 \oplus \mathbb{Z}_6$ and $\mathrm{rk}(H_2(Y)) = 3$. By [2, Proposition 3.3] and Theorem 2.7, Y does not have the fixed point property. □

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Departamento de Matemática, FCEyN-Universidad de Buenos Aires, Buenos Aires, Argentina

E-mail address: `ivansadofschi@gmail.com`