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Marginally trapped submanifolds in Lorentzian space forms and in the Lorentzian product of a space form by the real line

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We give local, explicit representation formulas for n-dimensional spacelike submanifolds which are marginally trapped in the Minkowski space $\mathbb{R}^{n+2}_1$, the de Sitter space $d\mathbb{S}^{n+2}$, the anti-de Sitter space $Ad\mathbb{S}^{n+2}$ and the Lorentzian products $\mathbb{S}^{n+1}\times\mathbb{R}$ and $\mathbb{H}^{n+1}\times\mathbb{R}$ of the sphere and the hyperbolic space by the real line. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4906936]

I. INTRODUCTION

Let $\mathcal{S}$ be a submanifold of a pseudo-Riemannian manifold $(\mathcal{N},g)$. If the induced metric on $\mathcal{S}$ is non-degenerate, one may define its mean curvature $\vec{H}$, a normal vector field along $\mathcal{S}$. We shall say that $\mathcal{S}$ is *marginally trapped* if $\vec{H}$ is a null vector, i.e., $g(\vec{H},\vec{H})$ vanishes identically (some authors call such submanifolds *quasi-minimal* or *pseudo-minimal*). Of course this may happen only if $\mathcal{S}$ has codimension greater than two and if the induced metric on the normal bundle is indefinite.

The case of a spacelike surface $\mathcal{S}$ of a four-dimensional Lorentzian manifold $(\mathcal{N},g)$ is the most interesting because of its physical interpretation in the setting of general relativity: marginally outer trapped surfaces (MOTS) play a fundamental role in the study of black holes and spacetime singularities (Refs. 10 and 14). Despite their physical relevance and the fact that marginally trapped is the most natural curvature condition which is purely pseudo-Riemannian, these submanifolds are still not very well understood. After a seminal paper (Ref. 15) where marginally trapped submanifolds are called *semi-minimal*, there has been recent work on the classification of marginally trapped surfaces satisfying several additional properties, such as being Lagrangian (Ref. 8), isotropic (Ref. 6), having flat normal bundle (Ref. 1), constant curvature (Ref. 7), or positive relative nullity (Refs. 9 and 16). On the other hand, in Ref. 13 (see also Ref. 2), a very interesting minimization property has been discovered concerning marginally trapped surfaces: a minimal spacelike surface of Minkowski space $\mathbb{R}^4_1$, although it is unstable, minimizes the area among marginally trapped surfaces satisfying a natural boundary data.

The purpose of this paper is to give local, explicit representation formulas for n-dimensional marginally trapped submanifolds in some of the simplest Lorentzian manifolds: the Minkowski space $\mathbb{R}^{n+2}_1$, the non-flat Lorentzian spaces forms, i.e., the de Sitter space $d\mathbb{S}^{n+2}$ and the anti-de Sitter space $Ad\mathbb{S}^{n+2}$, and finally, the Lorentzian products $\mathbb{S}^{n+1}\times\mathbb{R}$ and $\mathbb{H}^{n+1}\times\mathbb{R}$ of the sphere and the hyperbolic space by the real line.

Our construction is inspired by Ref. 3 and is based, although not explicitly, on the contact structure enjoyed by the space of null geodesics of a pseudo-Riemannian manifold (Refs. 11 and 12). For example, in the case of Minkowski space $\mathbb{R}^{n+2}_1$, a spacelike, n-dimensional submanifold $\bar{\mathcal{S}}$ is locally described in terms of its height function (i.e., its timelike coordinate) and a hypersurface $\mathcal{S}$ of $\mathbb{R}^{n+1}$. Then, the marginally trapped condition amounts to a simple algebraic relation between the

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height function of $\bar{\mathcal{S}}$ and the second fundamental form of $\mathcal{S}$. The interpretation in terms of contact geometry is the following: the set of null geodesics normal to $\mathcal{S}$ is a Legendrian submanifold in the set of null geodesics of $\mathbb{R}^{n+2}_1$, which is contactomorphic to the unit tangent bundle of $\mathbb{R}^{n+1}$; then, the hypersurface $\mathcal{S}$ is nothing but the projection on the basis $\mathbb{R}^{n+1}$ of this Legendrian submanifold.

This idea works in the same way in the other simple Lorentzian spaces $d\mathbb{S}^{n+2}, Ad\mathbb{S}^{n+2}$, $\mathbb{S}^{n+1}\times\mathbb{R}$, and $\mathbb{H}^{n+1}\times\mathbb{R}$. The construction can be performed in Robertson-Walker spaces as well, but the analysis becomes quite more involved since the equation relating the height function of $\bar{\mathcal{S}}$ and the second fundamental form of $\mathcal{S}$ is not any more polynomial, but remains algebraic. This case is discussed in Ref. 4.

II. STATEMENT OF RESULTS

Let $(\mathbb{R}^{n+1}, \langle .,. \rangle_0)$ be the Euclidean space endowed with its canonical Riemannian metric

$$\langle .,. \rangle_0 := dx_1^2 + \cdots + dx_{n+1}^2,$$

and denote by

$$\langle .,. \rangle_1 := \langle .,. \rangle_0 - dx_{n+2}^2$$

the flat Lorentzian metric of the Cartesian product $\mathbb{R}^{n+2}_1 = \mathbb{R}^{n+1}\times\mathbb{R}$.

We denote by $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ the canonical totally geodesic embedding $\iota(x) = (x, 0)$ and denote by $\nu_0 = (0, \ldots, 0, 1)$ its (constant) unit normal vector.

We recall that the second fundamental form h of an immersion $\mathcal{M} \to (\mathcal{N}, g)$ with non degenerate first fundamental form is the symmetric tensor $h : T\mathcal{M}\times T\mathcal{M} \to N\mathcal{M}$ defined by $h(X,Y) := (D_X Y)^\perp$, where $(.)^\perp$ denotes the projection onto the normal space $N\mathcal{M}$ and D is the Levi-Civita connection of g. If ν is a normal vector field along $\mathcal{M}$, we have the following important relation: $g(h(X,Y),\nu) = -g(D_X\nu, Y)$. The mean curvature vector of the immersion is the trace of h with respect to the induced metric.

Theorem 1. *Let Ω be an open domain of $\mathbb{R}^n$ and $\tau \in C^2(\Omega)$. Then, the immersion $\bar{\varphi} : \Omega \to \mathbb{R}^{n+2}_1$ defined by*

$$\bar{\varphi}(x) = \big(\iota(x), 0\big) + \tau(x)\big(\nu_0, 1\big)$$

is flat and its second fundamental form is given by

$$\bar{h}(X,Y) = Hess_\tau(X,Y)(\nu_0, 1).$$

In particular, $\bar{\varphi}$ has null second fundamental form and is therefore marginally trapped. Conversely, any n-dimensional spacelike submanifold with null second fundamental form is locally congruent to the image of such an immersion.

Let φ be an immersion of class C^4 of an n-dimensional manifold $\mathcal{M}$ into $\mathbb{R}^{n+1}$ and denote by ν the Gauss map of φ, which is therefore $\mathbb{S}^n$-valued. Assume that φ admits p distinct, non-vanishing principal curvatures $\kappa_1, \ldots, \kappa_p$, $p \geq 2$ with multiplicity m_i and denote by τ_i the $p-1$ roots of the polynomial

$$P(\tau) := \sum_{i=1}^p m_i \prod_{j\neq i}^p (\kappa_j^{-1} - \tau).$$

Then, the $p-1$ immersions $\bar{\varphi}_i : \mathcal{M} \to \mathbb{R}^{n+2}_1$ defined by

$$\bar{\varphi}_i = (\varphi + \tau_i \nu, \tau_i)$$

are marginally trapped.

Conversely, any n-dimensional marginally trapped submanifold of $\mathbb{R}^{n+2}_1$ whose second fundamental form is not null is locally congruent to the image of such an immersion.

In particular, in the $n = 2$ case, let φ be a non-flat C^4-immersion of a surface $\mathcal{M}$ into $\mathbb{R}^3$ which is free of umbilic points. Denote by ν its Gauss map, by H and K the mean curvature with respect to

ν and the Gaussian curvature of φ. Then, immersion $\bar{\varphi} : \mathcal{M} \to \mathbb{R}^4_1$ defined by

$$\bar{\varphi} := \left(\varphi + \frac{H}{K}\nu, \frac{H}{K}\right)$$

is marginally trapped. As a corollary, an immersed surface which is contained in a time slice $\{x_4 = const.\}$ is marginally trapped if and only if $\frac{H}{K}$ is constant, i.e., φ is linear Weingarten.

It may be interesting to relate the latter formula to one found by Palmer: it is proved in Ref. 13 that, given Ω an open subset of $\mathbb{S}^2$ and $f \in C^4(\Omega)$, the immersion $\varphi : \Omega \to \mathbb{R}^4_1$ defined by

$$\bar{\varphi}(x) = \left(\nabla f(x) + f(x)x, -f(x) - \frac{1}{2}\Delta f(x)\right), \tag{1}$$

where ∇ and Δ denote the gradient and Laplace operators with respect to the round metric, which is marginally trapped.

We first observe that a convex surface of Euclidean space may be constructed from its support function: let $\varphi : \Omega \to \mathbb{R}^3$ parametrized by its unit normal vector ν and introduce $f(\nu) = \langle \varphi(\nu), \nu\rangle_0$, i.e., f is the support function of φ. Hence, we have (see Ref. 5)

$$\varphi(\nu) = f(\nu)\nu + \nabla f(\nu). \tag{2}$$

It will be seen along the proof of Theorem 1 that a spacelike immersion $\bar{\varphi} : \Omega \to \mathbb{R}^4_1$ such that the null geodesic $\{(\varphi(\nu), 0) + t(\nu, 1) | \, t \in \mathbb{R}\}$ crosses orthogonally the surface $\bar{\varphi}(\Omega)$ at $\bar{\varphi}(\nu)$ takes the form

$$\bar{\varphi}(\nu) = (\varphi(\nu) + \tau(\nu)\nu, \tau(\nu)),$$

where τ is a smooth real map on Ω. Moreover, the mean curvature vector of $\bar{\varphi}$ is collinear to the null vector field $\bar{\nu} = (\nu, 1)$ if and only if $\tau = \frac{H}{K}$. On the other hand, the following formula holds (see Ref. 5):

$$\frac{H}{K} = -f - \frac{1}{2}\Delta f, \tag{3}$$

so using formulas (2) and (3) together with the formula $\bar{\varphi} := \left(\varphi + \frac{H}{K}\nu, \frac{H}{K}\right)$ of Theorem 1, we recover Palmer's formula (formula (1)).

The same construction may be applied in the case of the Lorentzian space forms. In order to state the result, we set

$$\mathbb{S}^{n+1} := \{x \in \mathbb{R}^{n+2} | \, \langle x, x\rangle_0 = 1\} \quad \text{and} \quad \mathbb{H}^{n+1} := \{x \in \mathbb{R}^{n+2} | \, \langle x, x\rangle_1 = -1\}.$$

These are the hyperquadric models of the $n+1$-dimensional Riemannian space forms. Similarly, the $(n+2)$-dimensional Lorentzian space forms are defined as follows:

$$d\mathbb{S}^{n+2} := \{x \in \mathbb{R}^{n+3} | \, \langle x, x\rangle_1 = 1\} \quad \text{and} \quad Ad\mathbb{S}^{n+2} := \{x \in \mathbb{R}^{n+3} | \, \langle x, x\rangle_2 = -1\},$$

where

$$\langle ., . \rangle_2 := dx_1^2 + \cdots + dx_{n+1}^2 - dx_{n+2}^2 - dx_{n+3}^2.$$

Theorem 2. *Let Ω be an open domain of $\mathbb{S}^n$ (respectively, $\mathbb{H}^n$) and $\tau \in C^2(\Omega)$. Denote by $\iota : \mathbb{S}^n \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ (respectively, $\mathbb{H}^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$) the canonical totally geodesic embedding $\iota(x) = (x, 0)$ and denote by $\nu_0 = (0, \ldots, 0, 1)$ the corresponding (constant) unit normal vector. Then, the immersion $\bar{\varphi} : \mathbb{S}^n \to d\mathbb{S}^{n+2}$ (respectively, $\mathbb{H}^n \to Ad\mathbb{S}^{n+2}$) defined by*

$$\bar{\varphi}(x) = \big(\iota(x), 0\big) + \tau(x)\big(\nu_0, 1\big)$$

is flat and its second fundamental form is given by

$$\bar{h}(X, Y) = Hess_\tau(X, Y)(\nu_0, 1).$$

In particular, $\bar{\varphi}$ has null second fundamental form and is therefore marginally trapped. Conversely, any n-dimensional spacelike submanifold with null second fundamental form is locally congruent to the image of such an immersion.

Let φ be an immersion of class C^4 of an n-dimensional manifold $\mathcal{M}$ into $\mathbb{S}^{n+1}$ (respectively, $\mathbb{H}^{n+1}$) and denote by ν the Gauss map of φ, which is therefore $\mathbb{S}^{n+1}$-valued (respectively, $d\mathbb{S}^{n+1}$-valued). Assume that φ admits p distinct, non-vanishing principal curvatures $\kappa_1, \dots, \kappa_p$, $p \geq 2$ with multiplicity m_i and denote by τ_i the $p-1$ roots of the polynomial

$$P(\tau) := \sum_{i=1}^{p} m_i \prod_{j \neq i}^{p} (\kappa_j^{-1} - \tau).$$

Then, the $p-1$ immersions $\bar{\varphi}_i : \mathcal{M} \to d\mathbb{S}^{n+2}$ (respectively, $Ad\mathbb{S}^{n+2}$) defined by

$$\bar{\varphi}_i = (\varphi + \tau_i \nu, \tau_i)$$

are marginally trapped.

Conversely, any n-dimensional marginally trapped submanifold of $d\mathbb{S}^{n+2}$ (respectively, of $Ad\mathbb{S}^{n+2}$) whose second fundamental form is not null is locally congruent to the image of such an immersion.

We observe that all the examples found in Refs. 6 and 9 have null fundamental form (see Sec. VI).

The construction works as well in the case of the Lorentzian product of a space form by the real line. We endow $\mathbb{S}^{n+1} \times \mathbb{R}$ with the Lorentzian metric $\langle .,. \rangle_0 - dx_{n+3}^2$, where $\langle .,. \rangle_0$ is the round metric of $\mathbb{S}^{n+1}$ and x_{n+3} denotes the canonical coordinate of the real line $\mathbb{R}$.

Theorem 3. *There is no non-totally geodesic n-dimensional submanifold of $\mathbb{S}^{n+1} \times \mathbb{R}$ with null second fundamental form.*

Let φ be an immersion of class C^4 of an n-dimensional manifold $\mathcal{M}$ into $\mathbb{S}^{n+1}$. Denote by ν the Gauss map of φ and by $\kappa_1, \dots, \kappa_p$ its p distinct curvatures with multiplicity m_i. Then, the polynomial

$$P(s) := \sum_{i=1}^{p} m_i (\kappa_i s + 1) \prod_{j \neq i}^{p} (s - \kappa_j)$$

admits exactly $p-1$ roots s_i if φ is minimal and p roots otherwise. Moreover, the $p-1$ or p immersions $\bar{\varphi}_i : \mathcal{M} \to \mathbb{S}^{n+1} \times \mathbb{R}$ defined by

$$\bar{\varphi}_i := \left(\frac{s_i \varphi + \nu}{\sqrt{1 + s_i^2}}, \cot^{-1} s_i \right), \ 1 \leq i \leq p-1 \text{ or } p$$

are marginally trapped.

Conversely, any n-dimensional marginally trapped submanifold of $\mathbb{S}^{n+1} \times \mathbb{R}$ is locally congruent to the image of such an immersion.

In particular, in the $n = 2$ case, given a non minimal C^4-immersion φ of a surface into $\mathbb{S}^3$ and

$$a := \frac{\kappa_1 \kappa_2 - 1}{\kappa_1 + \kappa_2},$$

the two immersions into $\mathbb{S}^3 \times \mathbb{R}$ defined by

$$\bar{\varphi}_{\pm} := \left(\frac{(a \pm \sqrt{a^2+1})\varphi + \nu}{\sqrt{2}\sqrt{a^2 + 1 \pm a\sqrt{a^2+1}}}, \cot^{-1}(a \pm \sqrt{a^2+1}) \right)$$

are marginally trapped.

Analogously, $\mathbb{H}^{n+1} \times \mathbb{R}$ is endowed with the metric $\langle .,. \rangle_1 - dx_{n+3}^2$, where $\langle .,. \rangle_1$ is the standard metric of $\mathbb{H}^{n+1}$ and x_{n+3} denotes the canonical coordinate of the real line $\mathbb{R}$.

Theorem 4. *There is no non-totally geodesic n-dimensional submanifold of $\mathbb{H}^{n+1} \times \mathbb{R}$ with null second fundamental form.*

Let φ be an immersion of class C^4 of an n-dimensional manifold $\mathcal{M}$ into $\mathbb{H}^{n+1}$. Denote by ν the Gauss map of φ and by $\kappa_1, \dots, \kappa_p$ its p distinct curvatures with multiplicity m_i. Denote by s_i, $1 \leq i \leq q \leq p$, the q roots of the polynomial

$$P(s) := \sum_{i=1}^{p} m_i(\kappa_i s - 1) \prod_{j \neq i}^{p} (s - \kappa_j)$$

satisfying $|s_i| > 1$. Then, the q immersions $\bar{\varphi}_i : \mathcal{M} \to \mathbb{H}^{n+1} \times \mathbb{R}$ defined by

$$\bar{\varphi}_i := \left(\frac{s_i \varphi + \nu}{\sqrt{s_i^2 - 1}}, \coth^{-1} s_i \right), \; 1 \leq i \leq q$$

are marginally trapped.

Conversely, any n-dimensional marginally trapped submanifold of $\mathbb{H}^{n+1} \times \mathbb{R}$ is locally congruent to the image of such an immersion.

In particular, in the $n = 2$ case, given a non minimal C^4-immersion φ of a surface into $\mathbb{H}^3$ such that $a := \frac{\kappa_1 \kappa_2 + 1}{\kappa_1 + \kappa_2} \in (1, \infty)$, the immersion into $\mathbb{H}^3 \times \mathbb{R}$ defined by

$$\bar{\varphi} := \left(\frac{(a + \sqrt{a^2 - 1})\varphi + \nu}{\sqrt{2}\sqrt{a^2 - 1 + a\sqrt{a^2 - 1}}}, \frac{1}{2} \coth^{-1}(a) \right)$$

is marginally trapped.

III. THE MINKOWSKI CASE: PROOF OF THEOREM 1

Let $\bar{\varphi} = (\psi, \tau)$ be an immersion of a n-dimensional manifold $\mathcal{M}$ into $\mathbb{R}^{n+2}_1$ which is spacelike, i.e., the induced metric $\bar{g} := \bar{\varphi}^* \langle ., . \rangle_1$ is definite positive. In particular, the induced metric on the normal space of $\bar{\varphi}$ is Lorentzian, and we may define locally two null, non-vanishing normal vector fields. Moreover, a null vector field $\bar{\nu}$ may be normalized in the following form: $\bar{\nu} = (\nu, 1)$, with $\nu : \mathcal{M} \to \mathbb{S}^n$. From now on, we consider a null normal vector field $\bar{\nu} := (\nu, 1)$ and we set $\varphi := \psi - \tau \nu$.

Lemma 1. The map $(\varphi, \nu) : \mathcal{M} \to \mathbb{R}^{n+1} \times \mathbb{S}^n$ is an immersion.

Proof. Suppose (φ, ν) is not an immersion, so that there exists a non-vanishing vector $v \in T\mathcal{M}$ such that $(d\varphi(v), d\nu(v)) = (0, 0)$. Since we have $d\psi = d\varphi + \tau d\nu + d\tau \nu$, it follows that

$$d\bar{\varphi}(v) = (d\psi(v), d\tau(v)) = (d\tau(v)\nu, d\tau(v)) = d\tau(v)\bar{\nu},$$

which is a null vector. This contradicts the assumption that $\bar{\varphi}$ is spacelike. □

Lemma 2. We have the following relation (this corresponds to the fact that the immersion (φ, ν) is Legendrian with respect to the canonical contact structure of the unit bundle of $\mathbb{R}^{n+1}$):

$$\langle d\varphi, \nu \rangle_0 = 0.$$

Proof. Using again that $d\psi = d\varphi + \tau d\nu + d\tau \nu$ and observing that $\langle \nu, d\nu \rangle_0 = 0$, we have

$$0 = \langle d\bar{\varphi}, \bar{\nu} \rangle_1 = \langle (d\psi, d\tau), (\nu, 1) \rangle_1 = \langle d\psi, \nu \rangle_0 - d\tau = \langle d\varphi, \nu \rangle_0.$$

□

Lemma 3. Given $x \in \mathcal{M}$ and $\epsilon > 0$, there exists a neighbourhood U of x and $t_0 \in (-\epsilon, \epsilon)$ such that $\varphi + t_0 \nu$ is an immersion of U and $\nu\big|_U$ is its Gauss map.

Proof. The claim follows from the fact that, $\forall x \in \mathcal{M}$, the set

$$\{t \in \mathbb{R} |\; d\varphi_x + t\, d\nu_x \text{ has not maximal rank}\}$$

contains at most n elements. To see this, observe that given a pair of distinct real numbers (t,t'), we have

$$Ker(d\varphi_x + t\, d\nu_x) \cap Ker(d\varphi_x + t'\, d\nu_x) = \{0\}$$

(otherwise, we would have a contradiction with the fact that (φ,ν) is an immersion). Hence, there cannot be more than n distinct values t such that $Ker(d\varphi_x + t\, d\nu_x) \neq \{0\}$. Moreover, such real numbers t depend continuously on the point $x \in \mathcal{M}$, so we may choose a neighbourhood U of x such that $\{t \in \mathbb{R} \mid \varphi + t\nu$ is an immersion of $U\}$ contains a neighbourhood of 0, which implies the first part of the claim.

The fact that ν is the Gauss map of $\varphi + t_0\nu$ comes from Lemma 2

$$\langle d(\varphi + t_0\nu), \nu\rangle_0 = \langle d\varphi, \nu\rangle_0 + t_0\langle d\nu, \nu\rangle_0 = 0.$$

□

Since the whole discussion is local, Lemma 3 shows that there is no loss of generality in assuming that φ is an immersion: if it is not the case, we may translate the immersion $\bar{\varphi}$ along the vertical direction, setting $\bar{\varphi}_{t_0} := \bar{\varphi} - (0,t_0)$. Of course $\bar{\varphi}$ is marginally trapped if and only if $\bar{\varphi}_{t_0}$ is so, and moreover, the vector field $\bar{\nu}$ is still normal to $\bar{\varphi}_{t_0}$. Finally, observe that the map $\varphi_{t_0} : \mathcal{M} \to \mathbb{R}^{n+1}$ associated to $\bar{\varphi}_{t_0}$ is

$$\varphi_{t_0} = \psi - (\tau - t_0)\nu = \psi - \tau\nu + t_0\nu = \varphi + t_0\nu,$$

hence an immersion.

We now describe the first fundamental form of $\bar{\varphi}$ and its second fundamental form with respect to $\bar{\nu}$, both in terms of the geometry of the immersion φ.

Lemma 4. Denote by $g := \varphi^*\langle .,.\rangle_0$ the metric induced on $\mathcal{M}$ by φ and A the shape operator associated to ν, i.e., $A(v) := -d\nu(v)$, $\forall v \in T\mathcal{M}$. Then, the metric $\bar{g} := \bar{\varphi}^*\langle .,.\rangle_1$ induced on $\mathcal{M}$ by $\bar{\varphi}$ is given by the formula

$$\bar{g} = g(.,.) - 2\tau g(A.,.) + \tau^2 g(A.,A.).$$

In particular, the non-degeneracy assumption on $\bar{g}$ implies that τ^{-1} is not equal to any principal curvature of φ. Moreover, the second fundamental form of $\bar{\varphi}$ with respect to $\bar{\nu}$ is given by

$$\bar{h}_{\bar{\nu}} := \langle \bar{h}(.,.), \bar{\nu}\rangle_1 = g(.,A.) - \tau g(A.,A.)$$

and

$$\langle \vec{H}_{\bar{\varphi}}, \bar{\nu}\rangle_1 = \frac{1}{n}\sum_{i=1}^{n} \frac{\kappa_i}{(1-\tau\kappa_i)}, \tag{4}$$

where the κ_i are the principal curvatures of φ.

Proof. Since $\langle d\varphi, \nu\rangle_0 = \langle d\nu, \nu\rangle_0 = 0$, we have given $v_1, v_2 \in T_x\mathcal{M}^n$,

$$\begin{aligned}
\bar{g}(v_1,v_2) &= \langle d\bar{\varphi}(v_1), d\bar{\varphi}(v_2)\rangle_1 \\
&= \langle d\varphi(v_1), d\varphi(v_2)\rangle_0 + \tau\langle d\varphi(v_1), d\nu(v_2)\rangle_0 + \tau\langle d\nu(v_1), d\varphi(v_2)\rangle_0 \\
&\quad + \tau^2\langle d\nu(v_1), d\nu(v_2)\rangle_0 + d\tau(v_1)d\tau(v_2)\langle \nu,\nu\rangle_0 - d\tau(v_1)d\tau(v_2) \\
&= g(v_1,v_2) - \tau(g(v_1,Av_2) + g(Av_1,v_2)) + \tau^2 g(Av_1,Av_2) \\
&= g(v_1,v_2) - 2\tau g(Av_1,v_2) + \tau^2 g(Av_1,Av_2).
\end{aligned}$$

We calculate the second fundamental form of $\bar{\varphi}$ with respect to $\bar{\nu} := (\nu, 1)$,

$$\begin{aligned}
\bar{h}_{\bar{\nu}} &= -\langle d\bar{\varphi}, d\bar{\nu}\rangle_1 \\
&= -\langle d\varphi + \tau d\nu + d\tau\nu, d\nu\rangle_0 \\
&= -\langle d\varphi, d\nu\rangle_0 - \tau\langle d\nu, d\nu\rangle_0 \\
&= g(.,A.) - \tau g(A.,A.).
\end{aligned}$$

To complete the proof, observe that in the totally umbilic case $A = \kappa Id$, we obviously have

$$\langle \vec{H}_{\bar\varphi}, \bar\nu \rangle_1 = \frac{\kappa}{1-\tau\kappa}.$$

If φ is not totally umbilic, we introduce, away from isolated umbilic points, a principal orthonormal frame $(e_1,\dots,e_n)$ along $\mathcal{M}$, i.e., such that $g(e_i,e_j) = \delta_{ij}$ and $Ae_i = \kappa_i e_i$. Hence,

$$\bar g(e_i,e_j) = (1 - 2\tau\kappa_i + \tau^2\kappa_i^2)\delta_{ij},$$
$$\bar h(e_i,e_j) = \kappa_i(1-\tau\kappa_i)\delta_{ij}$$

and the proof follows. □

We are now in position to complete the proof of Theorem 1. We first assume that φ is totally geodesic, i.e., A vanishes. This is locally equivalent to assume that ν is constant, and without loss of generality, we may assume that $\nu = \nu_0 := (0,\dots,0,1)$.

From Eq. (4), it is immediately seen $\bar h_{\bar\nu}$ vanishes, so the second fundamental form $\bar h$ of $\bar\varphi$ takes value in the null line directed by $\bar\nu$. It is then straightforward to check that $\bar h(X,Y) = Hess_\tau(X,Y)(\nu,1)$. We therefore recover the first part of Theorem 1.

In order to complete the proof, we order the non-vanishing principal curvatures κ_i, taking into account their multiplicity m_i, in such a way that the corresponding radii of curvature are increasing $r_1 := \kappa_1^{-1} < \dots < r_p := \kappa_p^{-1}$. Hence,

$$\begin{aligned} &\langle \vec{H}_{\bar\varphi}, \bar\nu\rangle_1 = 0 \\ \iff &\sum_{i=1}^p \frac{m_i\kappa_i}{(1-\tau\kappa_i)} = 0 \\ \iff &\sum_{i=1}^p \frac{m_i}{r_i - \tau} = 0 \\ \iff &P(\tau) := \sum_{i=1}^p m_i \prod_{j\neq i}^p (r_j - \tau) = 0. \end{aligned}$$

We have

$$P(r_i) = \sum_{k=1}^{p-1} m_k \prod_{j\neq k}^{p-1}(r_j - r_i) = m_i \prod_{j\neq i}^{p-1}(r_j - r_i).$$

It follows that $P(r_p) > 0, P(r_{p-1}) < 0$ and that more generally, the signs of $P(r_i)$, $i = 1,\dots,p$ are alternate. We deduce that $P(\tau)$ admits at least $p-1$ distinct roots τ_i, $i = 1,\dots,p-1$, satisfying $r_i < \tau_i < r_{i+1}$. Since $P(\tau)$ has degree $p-1$, it have no other roots.

Remark 1. If φ is minimal, $\tau = 0$ is a root of $P(\tau)$. The corresponding immersion $\bar\varphi = (\varphi, 0)$ is not only marginally trapped but also minimal.

IV. THE DE SITTER AND ANTI-DE SITTER CASES: PROOF OF THEOREM 2

A. The de Sitter case

Let $\bar\varphi = (\psi,\tau) : \mathcal{M} \to d\mathbb{S}^{n+2}$ an immersion such that the induced metric $\bar g := \bar\varphi^*\langle .,.\rangle_1$ is space-like. Let $\bar\nu = (\nu, 1)$ be one of the two normalized, null normal field to $\bar\varphi$. We define the *null projection* of $\bar\varphi$ to be $\varphi := \psi - \tau\nu$. The fact that $(\nu,1) \in T_{\bar\varphi} d\mathbb{S}^{n+2}$, i.e., $0 = \langle(\psi,\tau),(\nu,1)\rangle_1 = \langle\psi,\nu\rangle_0 - \tau$, implies that $\langle\psi,\nu\rangle_0 = \tau$. Hence,

$$\begin{aligned} \langle\varphi,\varphi\rangle_0 &= \langle\psi,\psi\rangle_0 - 2\tau\langle\psi,\nu\rangle_0 + \tau^2\langle\nu,\nu\rangle_0 \\ &= \langle\psi,\psi\rangle_0 - \tau^2 \\ &= \langle\bar\varphi,\bar\varphi\rangle_1 \\ &= 1, \end{aligned}$$

which shows that φ is $\mathbb{S}^{n+1}$-valued. The proofs of the next two lemmas are omitted, since they are similar to the Minkowski case.

Lemma 5. The map $(\varphi, \nu) : \mathcal{M} \to \mathbb{S}^{n+1} \times \mathbb{S}^{n+1}$ *is an immersion.*

Lemma 6. We have the following relations (this corresponds to the fact that the immersion (φ, ν) *is Legendrian with respect to the canonical contact structure of the unit bundle of* $\mathbb{S}^{n+1}$*):*

$$\langle \varphi, \nu \rangle_0 = 0 \quad \text{and} \quad \langle d\varphi, \nu \rangle_0 = 0.$$

Unlike in the Minkowski case, there is no vertical translation in $d\mathbb{S}^{n+2}$. We may however, up to a arbitrarily small, linear perturbation, assume that φ is an immersion.

Lemma 7. Given $x \in \mathcal{M}$ *and* $\epsilon > 0$*, there exists a neighbourhood* U *of* x*,* $\alpha \in (-\epsilon, \epsilon)$ *and a hyperbolic rotation* R^α *of angle* α *such that the null projection* φ^α *of* $\bar{\varphi}^\alpha := R^\alpha \bar{\varphi}$ *is an immersion.*

Proof. Set

$$R^\alpha = \begin{pmatrix} \cosh\alpha & & \sinh\alpha \\ & Id & \\ \sinh\alpha & & \cosh\alpha \end{pmatrix} \in SO(n+2, 1)$$

and $\bar{\varphi}^\alpha := R^\alpha \bar{\varphi}, \bar{\nu}^\alpha := R^\alpha \bar{\nu}$. Observe that $\bar{\nu}^\alpha := (\nu^\alpha, \sigma^\alpha)$ is not anymore normalized, *a priori*, since its last component $\sigma^\alpha := \bar{\nu}^\alpha_{n+3}$ is equal to $\cosh(\alpha) + \sinh(\alpha)\nu_1$, where ν_1 is the first component of the vector ν.

Nevertheless, the null geodesic passing through the point $\bar{\varphi}^\alpha$ and directed by the vector $\bar{\nu}^\alpha$ crosses the slice $d\mathbb{S}^{n+2} \cap \{x_{n+3} = 0\}$ at the point

$$(\varphi^\alpha, 0) := \left(\psi^\alpha - \frac{\tau^\alpha}{\sigma^\alpha} \nu^\alpha, 0 \right).$$

Clearly, φ^α is an immersion if and only if $R^{-\alpha}\varphi^\alpha = \psi - \frac{\tau^\alpha}{\sigma^\alpha}\nu = \varphi + \left(\tau - \frac{\tau^\alpha}{\sigma^\alpha}\right)\nu$ is so. Observe that

$$\begin{aligned} \tau - \frac{\tau^\alpha}{\sigma^\alpha} &= \tau - \frac{\cosh(\alpha)\tau + \sinh(\alpha)\psi_1}{\cosh(\alpha) + \sinh(\alpha)\nu_1} \\ &= \tau - \frac{\tau + \tanh(\alpha)\psi_1}{1 + \tanh(\alpha)\nu_1} \\ &= \tanh(\alpha)(-\psi_1 + \tau\nu_1) + o(\alpha) \\ &= -\alpha\, \varphi_1 + o(\alpha). \end{aligned}$$

Now, assume that $R^{-\alpha}\varphi^\alpha$ fails to be an immersion in any compact neighbourhood U of x, $\forall \alpha \in (-\epsilon, \epsilon)$. Hence, there exists a sequence $(x_n, v_n) \in T^1\mathcal{M}$ (the unit tangent bundle of $\mathcal{M}$) such that $x_n \to x$ and $d(R^{-1/n}\varphi^{1/n})_{x_n}(v_n) = 0$. We have

$$\begin{aligned} d(R^{-1/n}\varphi^{1/n})_{x_n}(v_n) &= d\Big(\varphi - \frac{1}{n}\varphi_1\nu + o(1/n)\Big)_{x_n}(v_n) \\ &= d\varphi_{x_n}(v_n) - \frac{1}{n}\Big((d\varphi_1)_\nu + \varphi_1 d\nu\Big)_{x_n}(v_n) + o(1/n). \end{aligned}$$

Thus, there exists a non vanishing v_0 such that a subsequence of v_n tends to v_0, and we obtain

$$\begin{cases} d\varphi_x(v_0) = 0 \\ (d\varphi_1)_x(v_0)\nu(x) + \varphi_1(x) d\nu_x(v_0) = 0 \end{cases}.$$

Remembering that φ_1 is the first coordinate of φ, this system implies the vanishing of $\varphi_1(x)d\nu_x(v_0)$. By Lemma 5, $d\nu_x(v_0)$ and $d\varphi_x(v_0)$ cannot vanish simultaneously, therefore $\varphi_1(x)$ vanishes. Repeating the argument with suitable rotations yields that all the other coordinates of $\varphi(x)$ vanish, a contradiction since $\varphi \in \mathbb{S}^{n+1}$. □

By the previous lemma, since the discussion is local, we may assume that φ is an immersion. The remainder of the proof of Theorem 2 follows the lines of Theorem 1, in particular, Lemma 4 still holds here.

B. The anti-de Sitter case

Let $\bar{\varphi} = (\psi, \tau) : \mathcal{M} \to Ad\mathbb{S}^{n+2}$ an immersion such that the induced metric $\bar{g} := \bar{\varphi}^*\langle .,. \rangle_2$ is space-like. Let $\bar{\nu} = (\nu, 1)$ be a normalized, null vector field which is normal to $\bar{\varphi}$. We define the *null projection* of $\bar{\varphi}$ to be $\varphi := \psi - \tau\nu$.

The fact that $(\nu, 1) \in T_{\bar{\varphi}} Ad\mathbb{S}^{n+2}$, i.e., $0 = \langle (\psi, \tau), (\nu, 1) \rangle_2 = \langle \psi, \nu \rangle_1 - \tau$ implies that $\langle \psi, \nu \rangle_1 = \tau$. Hence,

$$\begin{aligned} \langle \varphi, \varphi \rangle_1 &= \langle \psi, \psi \rangle_1 - 2\tau \langle \psi, \nu \rangle_1 + \tau^2 \langle \nu, \nu \rangle_1 \\ &= \langle \psi, \psi \rangle_1 - \tau^2 \\ &= \langle \bar{\varphi}, \bar{\varphi} \rangle_2 \\ &= -1, \end{aligned}$$

which shows that φ is $\mathbb{H}^{n+1}$-valued.

The remainder of the proof is similar to the previous case (de Sitter case) and is therefore omitted.

V. THE CASE OF THE PRODUCT OF A SPACE FORM BY THE REAL LINE: PROOF OF THEOREMS 3 AND 4

A. The $\mathbb{S}^{n+1} \times \mathbb{R}$ case

Let $\bar{\varphi} = (\psi, \tau) : \mathcal{M} \to \mathbb{S}^{n+1} \times \mathbb{R}$ an immersion such that the induced metric $\bar{g} := \bar{\varphi}^*\langle .,. \rangle_1$ is space-like. Let $\bar{\nu} = (\nu, 1)$, where $\nu \in \mathbb{S}^{n+1}$ be a normalized, null normal field along $\bar{\varphi}$. We set $\varphi := \cos(\tau)\psi - \sin(\tau)\nu$ and $\nu_\varphi = \sin(\tau)\psi + \cos(\tau)\nu$.

Lemma 8. The map $(\varphi, \nu_\varphi) : \mathcal{M} \to \mathbb{S}^{n+1} \times \mathbb{S}^{n+1}$ *is an immersion.*

Lemma 9.

$$\langle d\varphi, \nu_\varphi \rangle_0 = 0.$$

Lemma 10. Given $x \in \mathcal{M}$ *and* $\epsilon > 0$, *there exists a neighbourhood* U *of* x *and* $t_0 \in (-\epsilon, \epsilon)$ *such that* $\cos(t_0)\varphi + \sin(t_0)\nu_\varphi$ *is an immersion of* U *and* $\cos(t_0)\nu_\varphi - \sin(t_0)\varphi$ *is its Gauss map.*

The proof of Lemmas 8, 9, and 10 is similar to that of Lemmas 1, 2, and 3 of Sec. III and is therefore omitted. Since we are working locally, Lemma 10 proves that, up to a vertical translation, we may assume that φ is an immersion.

Lemma 11. Denote by $g = \varphi^*\langle .,. \rangle_0$ *the metric induced on* $\mathcal{M}$ *by* φ *and* A *the shaped operator associated to* ν. *Then, the metric* $\bar{g} = \bar{\varphi}^*\langle .,. \rangle_1$ *induced on* $\mathcal{M}$ *by* $\bar{\varphi}$ *is given by the formula*

$$\bar{g} = \cos^2(\tau) g(.,.) - 2\sin(\tau)\cos(\tau) g(A.,.) + \sin^2(\tau) g(A.,A.).$$

In particular, the non-degeneracy assumption on $\bar{g}$ *implies to* $\cot(\tau)$ *is not equal to a principal curvature of* φ. *Moreover,*

$$\bar{h}_{\bar{\nu}} := \langle \bar{h}(.,.), \bar{\nu} \rangle_1 = (\cos^2(\tau) - \sin^2(\tau)) g(A.,.) + \sin(\tau)\cos(\tau)(g(.,.) - g(A.,A.))$$

and

$$\langle \vec{H}_{\bar{\varphi}}, \bar{\nu} \rangle_1 = \frac{1}{n} \sum_{i=1}^{n} \frac{\kappa_i + \tan(\tau)}{1 - \tan(\tau)\kappa_i},$$

where the κ_i *are the principal curvatures of* φ.

We first claim that if $\bar{h}_{\bar{\nu}}$ vanishes, it must be totally geodesic: according to the previous lemma, this implies $A = \pm\kappa Id$, (hence, κ is constant) and $\kappa = \cot(\tau + \pi/2)$. Then, a routine calculation shows that the shape operator of $\psi = \cos(\tau)\varphi + \sin(\tau)\nu$ vanishes, i.e., ψ is totally geodesic. Since the height function τ is constant, $\bar{\varphi}$ is totally geodesic itself.

We now label the principal curvatures $\kappa_1 < \cdots < \kappa_p$, taking into account their multiplicity m_i. Hence, $\langle \vec{H}_{\bar{\varphi}}, \bar{\nu}\rangle_1$ vanishes if and only if

$$\sum_{i=1}^{p} m_i \frac{\kappa_i + \tan(\tau)}{1 - \tan(\tau)\kappa_i} = 0.$$

Introducing $s := \cot(\tau)$, we see that $\bar{\varphi}$ is marginally trapped with respect to $\bar{\nu}$ if and only if the following polynomial vanishes:

$$P(s) := \sum_{i=1}^{p} m_i(\kappa_i s + 1) \prod_{j\neq i}^{p} (s - \kappa_j) = 0.$$

It is easy to check that signs of $P(\kappa_i)$ are alternate. Therefore, the polynomial $P(s)$ admits at least $p-1$ distinct roots s_i such that $\kappa_i < s_i < \kappa_{i+1}$. In particular, the degree of $P(s)$ is at least $p-1$. Since the term of degree p of $P(s)$ is $\sum_{i=1}^{p} m_i\kappa_i = nH$, it has degree $p-1$ when φ is minimal and degree p otherwise. In the first case, since we already found $p-1$ roots, we conclude that there are exactly $p-1$ roots. Observe moreover that if φ is minimal and $\tau = 0$ (which corresponds to $s = \pm\infty$), the immersion $\bar{\varphi} = (\varphi, 0)$ is not only marginally trapped but also minimal. In the non-minimal case, by looking at $\lim_{s\to\pm\infty} \frac{P(s)}{s^p}$, we check that there exists one more root s_p in $(-\infty, \kappa_1)$ or in (κ_p, ∞), depending on whether p is even or odd and H is positive or negative.

The conclusion of Theorem 3 comes from the formula

$$\left(\cos(\cot^{-1}s), \sin(\cot^{-1}s)\right) = \left(\frac{s}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}\right).$$

Finally, if $n = 2$, and φ is not minimal, setting $a := \frac{\kappa_1\kappa_2 - 1}{\kappa_1 + \kappa_2}$, the polynomial $P(s)$ is equivalent to $s^2 - 2as - 1 = 0$, whose two distinct roots are $s_\pm = a \pm \sqrt{a^2+1}$. Hence,

$$\tau_\pm = (\cot)^{-1}\left(a \pm \sqrt{a^2+1}\right),$$

so we get the required formula. Observe however that if $a = 0$, then τ is constant and moreover, $\psi = \frac{1}{\sqrt{2}}(\pm\varphi + \nu)$ is minimal, so again $\bar{\varphi}$ is minimal.

B. The $\mathbb{H}^{n+1} \times \mathbb{R}$ case

Let $\bar{\varphi} = (\psi, \tau) : \mathcal{M} \to \mathbb{H}^{n+1} \times \mathbb{R}$ an immersion whose induced metric is spacelike. Let $\bar{\nu} = (\nu, 1)$, where $\nu \in d\mathbb{S}^{n+1}$, be a normalized, null normal field along $\bar{\varphi}$. We set $\varphi := \cosh(\tau)\psi + \sinh(\tau)\nu$. Reasoning like in the previous cases, we easily prove that, up to a vertical translation and reasoning locally, we may assume that φ is an immersion and that $\nu_\varphi := \sinh(\tau)\psi + \cosh(\tau)\nu$ is its Gauss map. Moreover, the non-degeneracy assumption on the induced metric on $\bar{\varphi}$ implies that $\coth(\tau)$ is not equal to a principal curvature κ_i of φ. Finally, counting the principal curvatures with their multiplicity m_i, we have that $\langle \vec{H}_{\bar{\varphi}}, \bar{\nu}\rangle_1$ vanishes if and only if $\sum_{i=1}^{p} m_i \frac{\kappa_i - \tanh(\tau)}{1 - \tanh(\tau)\kappa_i}$ vanishes as well. Hence, if s_i is a root of the polynomial,

$$P(s) := \sum_{i=1}^{p} m_i(\kappa_i s - 1) \prod_{j\neq i}^{p} (s - \kappa_j)$$

satisfying in addition $|s_i| > 1$, the immersion

$$\begin{aligned}\bar{\varphi}_i &:= \left(\cosh(\coth^{-1}(s_i))\varphi + \sinh(\coth^{-1}(s_i))\nu, \coth^{-1}(s_i)\right)\\ &= \left(\frac{s_i\varphi + \nu}{\sqrt{s_i^2 - 1}}, \coth^{-1}(s_i)\right)\end{aligned}$$

is marginally trapped.

It seems difficult to determine exactly the number q of roots of $P(s)$ such that $|s| > 1$. However, observe that given a monotone sequence of κ_i such that $\kappa_i < -1, |\kappa_i| < 1$ or $\kappa_i > 1$, the signs of $P(\kappa_i) = m_i(\kappa_i^2 - 1)\prod_{j\neq i}^{p}(\kappa_i - \kappa_j)$ are alternate. Hence, introducing

$$\begin{aligned} \alpha &:= \#\{\kappa_i, 1 \leq i \leq p, \kappa_i < -1\},\\ \beta &:= \#\{\kappa_i, 1 \leq i \leq p, |\kappa_i| < 1\},\\ \gamma &:= \#\{\kappa_i, 1 \leq i \leq p, \kappa_i > 1\},\\ \delta &:= \#\{\kappa_i, 1 \leq i \leq p, |\kappa| = 1\}. \end{aligned}$$

We deduce that there exist $\alpha - 1$ roots s_i satisfying $\kappa_i < s_i < \kappa_{i+1} < -1$, giving rise to $\alpha - 1$ marginally trapped immersions $\bar{\varphi}_i$. Analogously, there exist $\gamma - 1$ solutions satisfying $1 < \kappa_i < s_i < \kappa_{i+1}$. Analysing the signs of $\lim_{s\to\pm\infty}\frac{P(s)}{s^p}$ as in the $\mathbb{S}^{n+1}\times\mathbb{R}$ case, we see that if $\sum_{i=1}^{p} m_i\kappa_i = nH$ does not vanish, the existence of one more solution $s \in (-\infty, \inf_i \kappa_i) \cup (\sup_i \kappa_i, +\infty)$ is granted. On the other hand, if $\beta \neq 0$, the $\beta - 1$ roots satisfying $-1 < \kappa_i < s_i < \kappa_{i+1} < 1$ lead to no marginally trapped immersion. Finally, if 1 or -1 is a principal curvature, it is also a root of $P(s)$, which again corresponds to no marginally trapped immersion. Finally, if φ is not minimal, we obtain the following inequalities:

$$\alpha + \gamma - 1 \leq q \leq p - (\beta - 1) - \delta = \alpha + \gamma + 1.$$

If $n = 2$ and φ is not minimal, the polynomial $P(s)$ is equivalent to $s^2 - 2as + 1$, where we set $a := \frac{\kappa_1\kappa_2+1}{\kappa_1+\kappa_2}$. Without loss of generality, we assume that $a > 0$. If $a < 1, P(s)$ has no real solution and if $a = 1$, the unique solution is $s = 1$. Finally, if $a > 1$, the two distinct roots of $P(s)$ are $a \pm \sqrt{a^2-1}$, one of which is less than one and the other greater than one. Hence, we get $\tau := (\cot)^{-1}\left(a + \sqrt{a^2-1}\right)$, so we get the required formula. Observe that if $|\kappa_1| > 1$ and $|\kappa_2| > 1$, we have $\alpha + \gamma - 1 = q = 1$, i.e., the left hand side inequality above is sharp.

VI. EXAMPLES

Here, we briefly discuss how some of the examples of Ref. 9 can be recovered from our construction.

The two families of marginally trapped surfaces found by Chen and Van der Veken in $\mathbb{R}^4_1$ are

$$L_1(x, y) := (x, y, f(x), f(x)),$$

where f is an arbitrary differentiable function with $f''(x)$ being nowhere zero, and

$$\begin{aligned} L_2(x, y) := \Bigg(& y\cos x - \int_0^x r(x)\sin x dx, y\sin x + \int_0^x r(x)\cos x dx,\\ & q(x)y + \int_0^x r(x)q'(x)dx, q(x)y + \int_0^x r(x)q'(x)dx\Bigg), \end{aligned}$$

where q and r are defined in an open interval $I \ni 0$ satisfying $q''(x) + q(x) \neq 0$ for each $x \in I$.

Thus, $L_1(x, y) = (\iota(x, y), 0) + f(x)(\nu_0, 1)$, where $\nu_0 = (0, 0, 1)$, so we are in the first case (null second fundamental form) of Theorem 1. Moreover, since

$$T(x, y) := \left(y\cos x - \int_0^x r(x)\sin x\, dx, y\sin x + \int_0^x r(x)\cos x\, dx\right)$$

is simply a reparametrization of an open subset of the plane, setting

$$\tau(x, y) := q(x)y + \int_0^x r(x)q'(x)dx,$$

the second family takes the form $L_2(x, y) = (\iota \circ T(x, y) + \tau(x, y)\nu_0, \tau(x, y))$, so we are again in the case of null second fundamental form.

Next, consider the immersion in $d\mathbb{S}^4$ given by

$$L_3(x, y) := \left(\sin x\cos y, \sin y, \cos x\cos y, f(x)\cos y, f(x)\cos y\right),$$

where f is an arbitrary differentiable function defined on an open interval I satisfying $f'' + f \neq 0$ at each point in I. It takes the form

$$L_3(x, y) = (\sin x \cos y, \sin y, \cos x \cos y, 0, 0) + f(x) \cos y (0, 0, 0, 1, 1)$$
$$= (\iota(x, y), 0) + \tau(x, y)(\nu, 1),$$

where $\tau(x, y) := f(x) \cos y$, $\nu := (0, 0, 0, 1)$ and $\iota(x, y) : \mathbb{S}^2 \to \mathbb{S}^3$ is the totally geodesic embedding given in coordinates by $\iota(x, y) = (\sin x \cos y, \sin y, \cos x \cos y, 0)$. Hence, L_3 has null second fundamental form (first case of Theorem 2).

Finally, consider the immersion in $Ad\mathbb{S}^4$ given by

$$L_4(x, y) := \left(e^y - 2\sinh y, xe^y, x^2 e^y - \frac{1}{2}e^y, \frac{3}{2}e^y - 2\sinh y, x^2 e^y\right).$$

A straightforward calculation shows that a normalized, null normal vector along L_4 is $\bar{\nu} = (-1, 0, 1, -1, 1) = (\nu, 1)$. Since $\bar{\nu}$ is constant, $\langle \bar{h}(.,.), \bar{\nu}\rangle_2$ vanishes, so is in particular the second fundamental form $\bar{h}(.,.)$ is null and L_4 is marginally trapped. Observe moreover that $\varphi := \psi - \tau\nu = \psi - x^2 e^y(-1, 0, 1, -1)$ is an immersion whose normal unit vector $\nu = (-1, 0, 1, -1)$ is constant, therefore φ is totally geodesic.

We leave to the reader the easy task to check that all other examples of Ref. 9 have null second fundamental form.

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