AN INHOMOGENEOUS SINGULAR PERTURBATION PROBLEM FOR THE $p(x)$-LAPLACIAN

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Dedicated to our dear friend and colleague Juan Luis Vázquez on the occasion of his 70th birthday

Abstract. In this paper we study the following singular perturbation problem for the $p_\varepsilon(x)$-Laplacian:

$$(P_\varepsilon(f^\varepsilon, p_\varepsilon)) \qquad \Delta_{p_\varepsilon(x)} u^\varepsilon := \mathrm{div}(|\nabla u^\varepsilon(x)|^{p_\varepsilon(x)-2}\nabla u^\varepsilon) = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon, \quad u^\varepsilon \geq 0,$$

where $\varepsilon > 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, with β a Lipschitz function satisfying $\beta > 0$ in $(0,1)$, $\beta \equiv 0$ outside $(0,1)$ and $\int \beta(s)\, ds = M$. The functions u^ε, f^ε and p_ε are uniformly bounded. We prove uniform Lipschitz regularity, we pass to the limit ($\varepsilon \to 0$) and we show that, under suitable assumptions, limit functions are weak solutions to the free boundary problem: $u \geq 0$ and

$$(P(f,p,\lambda^*)) \qquad \begin{cases} \Delta_{p(x)} u = f & \text{in } \{u > 0\} \\ u = 0, \ |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\} \end{cases}$$

with $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1}\, M\right)^{1/p(x)}$, $p = \lim p_\varepsilon$ and $f = \lim f^\varepsilon$.

In [19] we prove that the free boundary of a weak solution is a $C^{1,\alpha}$ surface near flat free boundary points. This result applies, in particular, to the limit functions studied in this paper.

1. Introduction

Singular perturbation problems of the form

$$Lu^\varepsilon = \beta_\varepsilon(u^\varepsilon) \tag{1.1}$$

with $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, β nonnegative, smooth and supported on $[0,1]$ and L an elliptic second order differential operator have been widely studied due to their appearance in different contexts. One of its main application being to flame propagation. See [3, 4, 7, 29] and also the excellent survey by J. L. Vázquez [26].

A natural generalization is the consideration of inhomogeneous problems

$$Lu^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon \tag{1.2}$$

with f^ε uniformly bounded independently of ε. The inhomogeneous terms may represent sources as well as nonlocal effects, when the family u^ε is uniformly bounded (see [17]).

Problem (1.1) was first studied for a linear uniformly elliptic operator L by Berestycki, Caffarelli and Nirenberg in [3] and then for the heat equation by Caffarelli and Vázquez in [7]. The two phase case for the heat equation was studied by Caffarelli and the authors in [5, 6]. A natural question is

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the identification of the limiting problem as $\varepsilon \to 0$. To this end, estimates uniform in ε are needed. These two questions were the object of the above mentioned articles [3, 7, 5, 6].

For the inhomogeneous problem (1.2) and $L = \Delta$ or $L = \Delta - \partial_t$ these questions were settled in [17, 18].

The homogeneous problem (1.1) in the case of the p-Laplacian was considered in [10] and then, for more general operators with power like growth in [21]. Uniform estimates for the inhomogeneous problem (1.2) and the p-Laplacian were obtained in [22]. Additional results for these type of problems were obtained in [2, 15, 16, 22, 23, 27].

In this paper we study the case where the operator L is the $p_\varepsilon(x)$-Laplacian, defined as

$$\Delta_{p_\varepsilon(x)} u := \operatorname{div}(|\nabla u(x)|^{p_\varepsilon(x)-2}\nabla u),$$

that extends the Laplacian, where $p_\varepsilon(x) \equiv 2$, and the p-Laplacian, where $p_\varepsilon(x) \equiv p$ with $1 < p < \infty$. The $p(x)$-Laplacian has been used in the modeling of electrorheological fluids ([24]) and in image processing ([1], [9]).

We consider the inhomogenous problem (1.2) but we remark that this singular perturbation problem for the $p_\varepsilon(x)$-Laplacian had not been studied even in the homogeneous case (1.1). Moreover, the identification of the limiting problem in the inhomogeneous case had not been done even for $p_\varepsilon(x) \equiv p$.

As stated above, this singular perturbation problem may model flame propagation in a fluid with electromagnetic sensitivity. Hence its interest from a modeling point of view. On the other hand, the presence of a variable exponent $p_\varepsilon(x)$ and a right hand side $f_\varepsilon(x)$ brings new mathematical difficulties, that can be found scattered all along this paper, that were not present in the constant case $p_\varepsilon(x) \equiv p$. An important tool we use is the Harnack Inequality for the inhomogeneous $p(x)$-Laplacian that we recently proved in [28].

More precisely, in this paper we study the following singular perturbation problem for the $p_\varepsilon(x)$-Laplacian:

$$(P_\varepsilon(f^\varepsilon, p_\varepsilon)) \qquad \Delta_{p_\varepsilon(x)} u^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon, \quad u^\varepsilon \geq 0$$

in a domain $\Omega \subset \mathbb{R}^N$. Here $\varepsilon > 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, with β a Lipschitz function satisfying $\beta > 0$ in $(0, 1)$, $\beta \equiv 0$ outside $(0, 1)$ and $\int \beta(s)\, ds = M$.

We assume that u^ε, f^ε are uniformly bounded and that p_ε are uniformly bounded in Lipschitz norm. We prove uniform Lipschitz regularity, we pass to the limit ($\varepsilon \to 0$) and we show that, under suitable assumptions, limit functions are weak solutions to the following free boundary problem: $u \geq 0$ and

$$(P(f, p, \lambda^*)) \qquad \begin{cases} \Delta_{p(x)} u = f & \text{in } \{u > 0\} \\ u = 0, \ |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\} \end{cases}$$

with $\lambda^*(x) = \Big(\frac{p(x)}{p(x)-1}\, M\Big)^{1/p(x)}$, $p = \lim p_\varepsilon$ and $f = \lim f^\varepsilon$.

We remark that, in the inhomogeneous case, there are examples of limit functions that are not solutions to the free boundary problem $P(f, p, \lambda^*)$. These examples were produced with $p_\varepsilon(x) \equiv 2$ in [17]. Hence, some extra assumptions on the limit functions are needed.

In a companion paper [19] we study the regularity of the free boundary for weak solutions of $P(f, p, \lambda^*)$ with $p(x)$ Lipschitz and $\lambda^*(x)$ a Hölder continuous function. In [19] we show that the free boundary is a $C^{1,\alpha}$ surface near flat free boundary points. This regularity result applies in particular to limits of this singular perturbation problem, under the above mentioned assumptions.

These additional assumptions are verified if, for instance, the functions u^ε are local minimizers of an energy functional. We prove this last result in [20]. Moreover, in this special case, we show in [20] that the set of singular points has zero $\mathcal{H}^{N-1}$ measure.

In conclusion, in this first paper of a series on the singular perturbation problem $P_\varepsilon(f^\varepsilon, p_\varepsilon)$ we study the fundamental uniform properties of the solutions and we determine the limiting free boundary problem.

An outline of the paper is as follows: In Section 2 we obtain uniform bounds of the gradients of solutions to the singular perturbation problem $P_\varepsilon(f^\varepsilon, p_\varepsilon)$ (Theorem 2.1). In Section 3 we pass to the limit, in Section 4 we analyze some basic limits and in Section 5 we study the asymptotic behavior of limit functions. Finally, in Section 6 we define the notion of weak solution to the free boundary problem $P(f, p, \lambda^*)$ and we show that, under suitable assumptions, limit functions to the singular perturbation $P_\varepsilon(f^\varepsilon, p_\varepsilon)$ are weak solutions to the free boundary problem $P(f, p, \lambda^*)$ with $\lambda^*(x) = \Big(\frac{p(x)}{p(x)-1}\, M\Big)^{1/p(x)}$ (Theorem 6.1). We also state the result from [19] on the regularity of the interface for weak solutions (Theorem 6.2). We finish the paper with an appendix where we collect some results on variable exponent Sobolev spaces as well as some other results that are used in the paper.

1.1. **Assumptions.** Throughout the paper we let $\Omega \subset \mathbb{R}^N$ a domain.

Assumptions on $p_\varepsilon(x)$ and $p(x)$. We will assume that the functions $p_\varepsilon(x)$ verify

$$1 < p_{\min} \le p_\varepsilon(x) \le p_{\max} < \infty, \qquad x \in \Omega. \tag{1.3}$$

When we are restricted to a ball B_r we use $p^r_{\varepsilon-}$ and $p^r_{\varepsilon+}$ to denote the infimum and the supremum of $p_\varepsilon(x)$ over B_r.

We also assume that $p_\varepsilon(x)$ are continuous up to the boundary and that they have a uniform modulus of continuity $\omega : \mathbb{R} \to \mathbb{R}$, i.e. $|p_\varepsilon(x) - p_\varepsilon(y)| \le \omega(|x-y|)$ if $|x-y|$ is small.

For our main results we need to assume further that $p_\varepsilon(x)$ are uniformly Lipschitz continuous in Ω. In that case, we denote by L the Lipschitz constant of $p_\varepsilon(x)$, namely, $\|\nabla p_\varepsilon\|_{L^\infty(\Omega)} \le L$.

The same assumptions above will be made on the function $p(x)$.

Assumptions on β_ε. We will assume that the functions β_ε are defined by scaling of a single function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying:

i) β is a Lipschitz continuous function,
ii) $\beta > 0$ in $(0,1)$ and $\beta \equiv 0$ otherwise,
iii) $\int_0^1 \beta(s)\, ds = M$.

And then $\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$.

1.2. **Definition of solution to $p(x)$-Laplacian.** Let $p(x)$ be as above and let $g \in L^\infty(\Omega \times \mathbb{R})$. We say that u is a solution to

$$\Delta_{p(x)} u = g(x, u) \quad \text{in} \quad \Omega$$

if $u \in W^{1,p(\cdot)}(\Omega)$ and, for every $\varphi \in W_0^{1,p(\cdot)}(\Omega)$, there holds that

$$\int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = -\int_\Omega \varphi\, g(x,u)\, dx.$$

By the results in [28], it follows that $u \in L^\infty_{\rm loc}(\Omega)$.

1.3. **Notation.**

- N spatial dimension
- $\Omega \cap \partial\{u>0\}$ free boundary
- $|S|$ N-dimensional Lebesgue measure of the set S
- $\mathcal{H}^{N-1}$ $(N-1)$-dimensional Hausdorff measure
- $B_r(x_0)$ open ball of radius r and center x_0
- B_r open ball of radius r and center 0
- $B'_r(x_0)$ open ball of radius r and center x_0 in $\mathbb{R}^{N-1}$
- B'_r open ball of radius r and center 0 in $\mathbb{R}^{N-1}$
- $⨍_{B_r(x_0)} u = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u\,dx$
- $⨍_{\partial B_r(x_0)} u = \frac{1}{\mathcal{H}^{N-1}(\partial B_r(x_0))}\int_{\partial B_r(x_0)} u\,d\mathcal{H}^{N-1}$
- χ_S characteristic function of the set S
- $u^+ = \max(u,0)$, $u^- = \max(-u,0)$
- $\langle\,\cdot\,,\,\cdot\,\rangle$ scalar product in $\mathbb{R}^N$
- $B_\varepsilon(s) = \int_0^s \beta_\varepsilon(\tau)\,d\tau$

2. Uniform bound of the gradient

In this section we consider a family of uniformly bounded solutions to the singular perturbation problem $P_\varepsilon(f^\varepsilon, p_\varepsilon)$ and prove that their gradients are locally uniformly bounded. Our main result in the section is the following theorem

Theorem 2.1. *Assume that $1 < p_{\min} \le p_\varepsilon(x) \le p_{\max} < \infty$ with $p_\varepsilon(x)$ Lipschitz continuous and $\|\nabla p_\varepsilon\|_{L^\infty} \le L$, for some $L > 0$. Let u^ε be a solution of*

$$(P_\varepsilon(f^\varepsilon, p_\varepsilon)) \qquad \Delta_{p_\varepsilon(x)} u^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon, \quad u^\varepsilon \ge 0 \quad \text{in } \Omega,$$

with $\|u^\varepsilon\|_{L^\infty(\Omega)} \le L_1$, $\|f^\varepsilon\|_{L^\infty(\Omega)} \le L_2$. Then, for $\Omega' \subset\subset \Omega$, we have

$$|\nabla u^\varepsilon(x)| \le C \quad \text{in } \Omega',$$

with $C = C(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L, dist(\Omega', \partial\Omega))$, if $\varepsilon \le \varepsilon_0(\Omega, \Omega')$.

An essential tool in the proof will be the following Harnack's Inequality for the inhomogenous $p(x)$-Laplacian equation, proven in [28], Theorem 2.1

Theorem 2.2. *Assume that $p(x)$ is locally log-Hölder continuous in Ω. This is, $p(x)$ has locally a modulus of continuity $\omega(r) = C(\log\frac{1}{r})^{-1}$. Let $x_0 \in \Omega$ and $0 < R \le 1$ such that $\overline{B_{4R}(x_0)} \subset \Omega$. There exists C such that, if $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ is a nonnegative solution of the problem*

$$\Delta_{p(x)} u = f \ \text{ in } \Omega, \tag{2.1}$$

with $f \in L^{q_0}(\Omega)$ for some $\max\{1, \frac{N}{p_-^{4R}}\} < q_0 \le \infty$, then

$$\sup_{B_R(x_0)} u \le C[\inf_{B_R(x_0)} u + R + R\mu]$$

where

$$\mu = [R^{1-\frac{N}{q_0}} ||f||_{L^{q_0}(B_{4R}(x_0))}]^{\frac{1}{p_-^{4R}-1}}.$$

The constant C depends only on N, $p_-^{4R} := \inf_{B_{4R}(x_0)} p$, $p_+^{4R} := \sup_{B_{4R}(x_0)} p$, s, q_0, ω_{4R}, $\mu^{p_+^{4R}-p_-^{4R}}$, $||u||_{L^{sq'}(B_{4R}(x_0))}^{p_+^{4R}-p_-^{4R}}$ and $||u||_{L^{sr_0}(B_{4R}(x_0))}^{p_+^{4R}-p_-^{4R}}$ (for certain $q' = \frac{q}{q-1}$ with $r_0, q \in (1,\infty)$ and $\frac{1}{q_0} + \frac{1}{q} + \frac{1}{r_0} = 1$

depending on N, q_0 and p_-^{4R}). Here $s > p_+^{4R} - p_-^{4R}$ is arbitrary and ω_{4R} is the modulus of log-Hölder continuity of $p(x)$ in $B_{4R}(x_0)$.

We will also use the following result proven in [12], Theorem 1.1,

Theorem 2.3. *Assume that $1 < p_{\min} \le p(x) \le p_{\max} < \infty$, and that $p(x)$ has a modulus of continuity $\omega(r) = C_0 r^{\alpha_0}$ for some $0 < \alpha_0 < 1$. Let $f \in L^\infty(\Omega)$ and let $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be a solution of*

$$\Delta_{p(x)} u = f \text{ in } \Omega. \tag{2.2}$$

Then, $u \in C^{1,\alpha}_{\rm loc}(\Omega)$, where the Hölder exponent α depends on N, $p_{\min}$, $p_{\max}$, $||f||_{L^\infty(\Omega)}$, $||u||_{L^\infty(\Omega)}$, $\omega(r)$ and, for any $\Omega' \subset\subset \Omega$,

$$\|u\|_{C^{1,\alpha}(\bar{\Omega}')} \le C,$$

the constant C depending on N, $p_{\min}$, $p_{\max}$, $||f||_{L^\infty(\Omega)}$, $||u||_{L^\infty(\Omega)}$, $\omega(r)$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

In order to prove Theorem 2.1, we need to prove first some auxiliary results.

Lemma 2.1. *Assume that $1 < p_{\min} \le p_\varepsilon(x) \le p_{\max} < \infty$ with $p_\varepsilon(x)$ Lipschitz continuous and $\|\nabla p_\varepsilon\|_{L^\infty} \le L$, for some $L > 0$. Let u^ε be a solution of $P_\varepsilon(f^\varepsilon, p_\varepsilon)$ in $B_{r_0}(x_0)$ with $\|u^\varepsilon\|_{L^\infty(B_{r_0}(x_0))} \le L_1$, $\|f^\varepsilon\|_{L^\infty(B_{r_0}(x_0))} \le L_2$, such that $u^\varepsilon(x_0) \le 2\varepsilon$. Then, there exists $C > 0$ such that, if $\varepsilon \le 1$,*

$$|\nabla u^\varepsilon(x_0)| \le C,$$

with $C = C(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L, r_0)$.

Proof. Let $v^\varepsilon(x) = \frac{1}{\varepsilon} u^\varepsilon(x_0 + \varepsilon x)$. Then, denoting $\bar{p}_\varepsilon(x) = p_\varepsilon(\varepsilon x + x_0)$ and $\bar{f}^\varepsilon(x) = \varepsilon f^\varepsilon(\varepsilon x + x_0)$, we have, if $\varepsilon \le 1$,

$$\Delta_{\bar{p}_\varepsilon(x)} v^\varepsilon = \beta(v^\varepsilon) + \bar{f}^\varepsilon \text{ in } B_{r_0}. \tag{2.3}$$

We will apply Harnack's Inequality (Theorem 2.2). Let $\bar{r}_0 = \min\{r_0, 4\}$. We first observe that

$$\gamma := (\bar{p}_\varepsilon)_+^{\bar{r}_0} - (\bar{p}_\varepsilon)_-^{\bar{r}_0} = \sup_{B_{\bar{r}_0}} \bar{p}_\varepsilon - \inf_{B_{\bar{r}_0}} \bar{p}_\varepsilon \le L\varepsilon 2\bar{r}_0,$$

so that

$$||v^\varepsilon||^\gamma_{L^\infty(B_{\bar{r}_0})} \le (L_1/\varepsilon)^{L\varepsilon 2\bar{r}_0} \le C_0(L, L_1, r_0).$$

It follows that

$$\sup_{B_{\bar{r}_0/4}} v^\varepsilon \le C_1[v^\varepsilon(0) + \bar{r}_0/4 + \mu\bar{r}_0/4],$$

for $\mu = \left(\frac{\bar{r}_0}{4}\|\beta(v^\varepsilon) + \bar{f}^\varepsilon\|_{L^\infty(B_{\bar{r}_0}(x_0))}\right)^{\frac{1}{(\bar{p}_\varepsilon)_-^{\bar{r}_0} - 1}} \le C_2(L_2, \|\beta\|_{L^\infty}, p_{\min}, r_0)$ and a constant C_1 with $C_1 = C_1(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L, r_0)$.

Now, observing that $v^\varepsilon(0) \le 2$, and using the estimates of Theorem 2.3, we have that

$$|\nabla u^\varepsilon(x_0)| = |\nabla v^\varepsilon(0)| \le C,$$

with $C = C(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L, r_0)$. □

Lemma 2.2. *Assume that* $1 < p_{\min} \le p(x) \le p_{\max} < \infty$ *with* $p(x)$ *Lipschitz continuous and* $\|\nabla p\|_{L^\infty} \le L$*, for some* $L > 0$*. For* $x_0 \in \mathbb{R}^N$*,* $\mu > 0$*,* $\delta > 0$*,* $A > 0$*, consider*

$$\psi(x) = A\left(\frac{e^{-\mu\frac{|x-x_0|^2}{\delta^2}} - e^{-\mu}}{e^{-\mu/16} - e^{-\mu}}\right).$$

Assume moreover that $\delta \le A \le A_0$*. Then, given* $D > 0$*, there exist* $\tilde{\mu} = \tilde{\mu}(N, p_{\min}, p_{\max})$ *and* $\tilde{r} = \tilde{r}(p_{\min}, p_{\max}, L, D, A_0, \mu)$ *such that, if* $\mu \ge \tilde{\mu}$ *and* $\delta \le \tilde{r}$*, there holds that*

$$\Delta_{p(x)}\psi(x) \ge D \quad \text{in } B_\delta(x_0) \setminus \overline{B_{\delta/4}(x_0)}.$$

Proof. For $M > 0$ and $\mu > 0$ let

$$w(x) = M(e^{-\mu|x|^2} - e^{-\mu}). \tag{2.4}$$

The calculations in the proof of Lemma B.4 in [13] show that if $q(x)$ is a Lipschitz continuous function, with $1 < p_{\min} \le q(x) \le p_{\max} < \infty$, there exist $\mu_0 = \mu_0(p_{\max}, p_{\min}, N)$ and $\varepsilon_0 = \varepsilon_0(p_{\min})$ such that, if $\mu \ge \mu_0$ and $\|\nabla q\|_{L^\infty} \le \varepsilon_0$, then

$$e^{\mu|x|^2}(2M\mu)^{-1}|\nabla w|^{2-q(x)}\Delta_{q(x)}w \ge C_1\mu - C_2\|\nabla q\|_{L^\infty}(|\log M| + 1) \quad \text{in } B_1 \setminus B_{1/4},$$

with C_1, C_2 depending only on $p_{\min}$. If, in addition, $\mu \ge \mu_1(p_{\min})$, we get

$$e^{\mu|x|^2}(2M\mu)^{-1}|\nabla w|^{2-q(x)}\Delta_{q(x)}w \ge \frac{C_1}{2}\mu - C_2\|\nabla q\|_{L^\infty}|\log M| \quad \text{in } B_1 \setminus B_{1/4},$$

and therefore,

$$\Delta_{q(x)}w \ge e^{-\mu|x|^2}|\nabla w|^{q(x)-2}2M\mu\left(\frac{C_1}{2}\mu - C_2\|\nabla q\|_{L^\infty}|\log M|\right) \quad \text{in } B_1 \setminus B_{1/4}.$$

So that we have

$$\Delta_{q(x)}w \ge e^{-\mu(p_{\max}-1)}M^{q(x)-1}\mu^{p_{\min}-1}\left(\tilde{C}_1\mu - \tilde{C}_2\|\nabla q\|_{L^\infty}|\log M|\right) \quad \text{in } B_1 \setminus B_{1/4},$$

with $\tilde{C}_1$, $\tilde{C}_2$ depending on $p_{\min}$ and $p_{\max}$ if, in addition, $\mu \ge 1$.

We now observe that, letting in (2.4)

$$M = \frac{A}{\delta(e^{-\mu/16} - e^{-\mu})},$$

we have

$$\psi(x) = A\left(\frac{e^{-\mu\frac{|x-x_0|^2}{\delta^2}} - e^{-\mu}}{e^{-\mu/16} - e^{-\mu}}\right) = \delta M\left(e^{-\mu|\frac{x-x_0}{\delta}|^2} - e^{-\mu}\right) = \delta w\left(\frac{x - x_0}{\delta}\right).$$

We want to show that the constants $\tilde{\mu}$, $\tilde{r}$ in the statement can be chosen in such a way that

$$\Delta_{p(x)}\psi(x) \ge D \quad \text{in } B_\delta(x_0) \setminus \overline{B_{\delta/4}(x_0)}. \tag{2.5}$$

We notice that showing (2.5) is equivalent to showing that

$$\Delta_{\bar{p}(x)}w(x) \ge \delta D \quad \text{in } B_1 \setminus \overline{B_{1/4}}, \tag{2.6}$$

for $\bar{p}(x) = p(x_0 + \delta x)$.

Since $||\nabla\bar{p}||_{L^\infty} = \delta||\nabla p||_{L^\infty} \le \delta L$, the previous calculations give, if μ is as above and $\delta \le r_1 = \frac{\varepsilon_0}{L}$,

$$\Delta_{\bar{p}(x)}w \ge e^{-\mu(p_{\max}-1)}M^{\bar{p}(x)-1}\mu^{p_{\min}-1}\left(\tilde{C}_1\mu - \tilde{C}_2\delta L|\log M|\right) \quad \text{in } B_1 \setminus B_{1/4}.$$

Using that $A \geq \delta$, we have $M \geq e^{\mu/16} \geq 1$, implying that

$$\Delta_{\bar{p}(x)} w \geq e^{-\mu(p_{\max}-1)} \frac{1}{(e^{-\mu/16} - e^{-\mu})^{p_{\min}-1}} \mu^{p_{\min}-1} \left(\tilde{C}_1 \mu - \tilde{C}_2 \delta L \log M \right)$$
$$= C_3(\mu)(\tilde{C}_1 \mu - \tilde{C}_2 \delta L \log M) \qquad \text{in } B_1 \setminus B_{1/4}$$

(here $C_3(\mu)$ is a constant depending on $\mu, p_{\min}, p_{\max}$). Now using that

$$-\delta L \log M \geq -1 - \delta L \mu,$$

if $\delta \leq r_2 = r_2(A_0, L)$ and $\mu \geq \mu_2$, we conclude that

$$\Delta_{\bar{p}(x)} w \geq C_3(\mu) \frac{\tilde{C}_1}{4} \mu \qquad \text{in } B_1 \setminus B_{1/4},$$

if $\mu \geq \mu_3 = \mu_3(p_{\min}, p_{\max})$ and $\delta \leq r_3 = r_3(p_{\min}, p_{\max}, L)$. This is,

$$\Delta_{\bar{p}(x)} w \geq C_5, \qquad \text{in } B_1 \setminus B_{1/4}$$

with $C_5 = C_5(\mu, p_{\min}, p_{\max})$. If we now let $\tilde{\mu} = \max\{\mu_0, \mu_1, \mu_2, \mu_3, 1\}$, fix $\mu \geq \tilde{\mu}$ and take $\delta \leq \tilde{r} = \min\{r_1, r_2, r_3, \frac{C_5}{D}\}$, we conclude that (2.6) holds, thus implying (2.5). □

Lemma 2.3. *Assume that $1 < p_{\min} \leq p_\varepsilon(x) \leq p_{\max} < \infty$ with $p_\varepsilon(x)$ Lipschitz continuous and $\|\nabla p_\varepsilon\|_{L^\infty} \leq L$, for some $L > 0$. Let u^ε be a solution of $P_\varepsilon(f^\varepsilon, p_\varepsilon)$ in B_1 with $\|u^\varepsilon\|_{L^\infty(B_1)} \leq L_1$, $\|f^\varepsilon\|_{L^\infty(B_1)} \leq L_2$ and $0 \in \partial\{u^\varepsilon > \varepsilon\}$. Then, there exists $0 < r_0 < 1$ such that, for $x \in B_{r_0} \cap \{u^\varepsilon > \varepsilon\}$ and $\varepsilon \leq 1$,*

$$u^\varepsilon(x) \leq \varepsilon + C\, dist(x, \{u^\varepsilon \leq \varepsilon\} \cap B_1),$$

with $r_0 = r_0(N, L_1, L_2, p_{\min}, p_{\max}, L)$ and $C = C(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L)$.

Proof. Let $0 < r_0 < 1/4$ be a constant to be chosen later. For $x_0 \in B_{r_0} \cap \{u^\varepsilon > \varepsilon\}$, take $m_0 = u^\varepsilon(x_0) - \varepsilon$ and $\delta_0 = \text{dist}(x_0, \{u^\varepsilon \leq \varepsilon\} \cap B_1)$. Since $0 \in \partial\{u^\varepsilon > \varepsilon\} \cap B_1$, $\delta_0 \leq r_0$. We want to prove that $m_0 \leq C\delta_0$, with $C = C(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L)$.

Since $B_{\delta_0}(x_0) \subset \{u^\varepsilon > \varepsilon\} \cap B_1$, we have that $u^\varepsilon - \varepsilon > 0$ in $B_{\delta_0}(x_0)$ and $\Delta_{p_\varepsilon(x)}(u^\varepsilon - \varepsilon) = f^\varepsilon$. By Harnack's Inequality (Theorem 2.2)

$$\sup_{B_{\delta_0/4}(x_0)} (u^\varepsilon - \varepsilon) \leq C_1 [\inf_{B_{\delta_0/4}(x_0)} (u^\varepsilon - \varepsilon) + \delta_0/4 + \hat{\mu}\delta_0/4],$$

for $\hat{\mu} = \left(\frac{\delta_0}{4} \|f^\varepsilon\|_{L^\infty(B_{\delta_0}(x_0))} \right)^{\frac{1}{(p_\varepsilon)_-^{\delta_0} - 1}} \leq C_0(L_2, p_{\min})$, with $C_1 = C_1(N, L_1, L_2, p_{\min}, p_{\max}, L)$. It follows that

$$m_0 \leq C_1 \inf_{B_{\delta_0/4}(x_0)} (u^\varepsilon - \varepsilon) + C_2 \delta_0,$$

with $C_2 = C_2(N, L_1, L_2, p_{\min}, p_{\max}, L)$.

If there holds that $m_0 \leq 2C_2\delta_0$, the conclusion follows.

So let us assume that $m_0 > 2C_2\delta_0$. Then, there exists $c_1 = c_1(N, L_1, L_2, p_{\min}, p_{\max}, L)$ such that

$$c_1 m_0 \leq \inf_{B_{\delta_0/4}(x_0)} (u^\varepsilon - \varepsilon).$$

If $c_1 m_0 \leq \delta_0$ there is nothing to prove. So now assume that $c_1 m_0 > \delta_0$.

Let us consider

$$\psi(x) = c_1 m_0 \left(\frac{e^{-\mu \frac{|x - x_0|^2}{\delta_0{}^2}} - e^{-\mu}}{e^{-\mu/16} - e^{-\mu}} \right),$$

with $\mu = \tilde{\mu}(N, p_{\min}, p_{\max})$, the constant in Lemma 2.2.

Then, observing that $c_1 m_0 \le c_1 L_1$, we can apply Lemma 2.2 with $\delta = \delta_0$, $A = c_1 m_0$, $A_0 = c_1 L_1$ and $D = L_2$, if there holds that $\delta_0 \le \tilde{r}$, where $\tilde{r} = \tilde{r}(p_{\min}, p_{\max}, L, D, A_0, \mu)$ is the constant in Lemma 2.2.

If we choose $r_0 = \min\{\tilde{r}, 1/8\}$ above, we have $r_0 = r_0(N, L_1, L_2, p_{\min}, p_{\max}, L)$ and Lemma 2.2 applies, so we get

$$\begin{cases} \Delta_{p_\varepsilon(x)}\psi(x) \ge L_2 \ge f^\varepsilon & \text{in } B_{\delta_0}(x_0) \setminus \overline{B_{\delta_0/4}(x_0)} \\ \psi = 0 & \text{on } \partial B_{\delta_0}(x_0) \\ \psi = c_1 m_0 & \text{on } \partial B_{\delta_0/4}(x_0). \end{cases}$$

By the comparison principle (see the appendix), we have

$$\psi(x) \le u^\varepsilon(x) - \varepsilon \quad \text{in } \overline{B_{\delta_0}(x_0)} \setminus B_{\delta_0/4}(x_0). \tag{2.7}$$

Take $y_0 \in \partial B_{\delta_0}(x_0) \cap \partial\{u^\varepsilon > \varepsilon\}$. Then, $y_0 \in \overline{B_{1/2}}$ and

$$\psi(y_0) = u^\varepsilon(y_0) - \varepsilon = 0. \tag{2.8}$$

Let $v^\varepsilon(x) = \frac{1}{\varepsilon}u^\varepsilon(\varepsilon x + y_0)$, $\bar{p}_\varepsilon(x) = p_\varepsilon(\varepsilon x + y_0)$ and $\bar{f}^\varepsilon(x) = \varepsilon f^\varepsilon(\varepsilon x + y_0)$. Then if $\varepsilon \le 1$ we have that $\Delta_{\bar{p}_\varepsilon(x)} v^\varepsilon = \beta(v^\varepsilon) + \bar{f}^\varepsilon$ in $B_{1/2}$ and $v^\varepsilon(0) = 1$. Therefore, by Harnack's Inequality (Theorem 2.2), using similar arguments as those employed in the proof of Lemma 2.1, we obtain $\max_{\overline{B}_{1/8}} v^\varepsilon \le \widetilde{c} = \widetilde{c}(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L)$.

Now, by Theorem 2.3, we get

$$|\nabla u^\varepsilon(y_0)| = |\nabla v^\varepsilon(0)| \le c_3, \tag{2.9}$$

with $c_3 = c_3(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L)$. Finally, by (2.7), (2.8) and (2.9), we have that $|\nabla\psi(y_0)| \le |\nabla u^\varepsilon(y_0)| \le c_3$. Since $|\nabla\psi(y_0)| = c_1 m_0 \frac{c(\mu)}{\delta_0}$, we obtain

$$m_0 \le \frac{c_3}{c_1 c(\mu)}\delta_0$$

and the result follows.

□

Now, we can prove the following important result

Proposition 2.1. *Assume that $1 < p_{\min} \le p_\varepsilon(x) \le p_{\max} < \infty$ with $p_\varepsilon(x)$ Lipschitz continuous and $\|\nabla p_\varepsilon\|_{L^\infty} \le L$, for some $L > 0$. Let u^ε be a solution of $P_\varepsilon(f^\varepsilon, p_\varepsilon)$ in B_1 with $\|u^\varepsilon\|_{L^\infty(B_1)} \le L_1$ and $\|f^\varepsilon\|_{L^\infty(B_1)} \le L_2$. Assume that $0 \in \partial\{u^\varepsilon > \varepsilon\}$. Then, there exists $0 < r_1 < 1$ such that, for $x \in B_{r_1}$ and $\varepsilon \le 1$,*

$$|\nabla u^\varepsilon(x)| \le C$$

with $r_1 = r_1(N, L_1, L_2, p_{\min}, p_{\max}, L)$ and $C = C(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L)$.

Proof. By Lemma 2.1 we know that if $x_0 \in \{u^\varepsilon \le 2\varepsilon\} \cap B_{3/4}$ then,

$$|\nabla u^\varepsilon(x_0)| \le C_0$$

with $C_0 = C_0(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L)$.

Let $r_0 = r_0(N, L_1, L_2, p_{\min}, p_{\max}, L)$ be as in Lemma 2.3.

Let $x_0 \in B_{r_0/2} \cap \{u^\varepsilon > \varepsilon\}$ and $\delta_0 = \operatorname{dist}(x_0, \{u^\varepsilon \le \varepsilon\})$.

As $0 \in \partial\{u^\varepsilon > \varepsilon\}$ we have that $\delta_0 \le r_0/2$. Therefore, $B_{\delta_0}(x_0) \subset \{u^\varepsilon > \varepsilon\} \cap B_{r_0}$ and then $\Delta_{p_\varepsilon(x)} u^\varepsilon = f^\varepsilon$ in $B_{\delta_0}(x_0)$ and, by Lemma 2.3,

$$u^\varepsilon(x) \le \varepsilon + C_1 \operatorname{dist}(x, \{u^\varepsilon \le \varepsilon\}) \quad \text{in } B_{\delta_0}(x_0), \tag{2.10}$$

with $C_1 = C_1(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L)$.

(1) Suppose that $\varepsilon < \bar{c}\delta_0$ with $\bar{c}$ to be determined. Then, (2.10) gives

$$\sup_{B_{\delta_0}(x_0)} u^\varepsilon \leq \varepsilon + C_1 2\delta_0 \leq (\bar{c} + 2C_1)\delta_0.$$

Now let $v^\varepsilon(x) = \frac{1}{\delta_0} u^\varepsilon(x_0 + \delta_0 x)$ and $p_\varepsilon^{\delta_0}(x) = p_\varepsilon(x_0 + \delta_0 x)$. Then, we have $\Delta_{p_\varepsilon^{\delta_0}(x)} v^\varepsilon = \delta_0 f^\varepsilon(x_0 + \delta_0 x)$ in B_1, with

$$\sup_{B_1} v^\varepsilon = \frac{1}{\delta_0} \sup_{B_{\delta_0}(x_0)} u^\varepsilon \leq (\bar{c} + 2C_1).$$

Therefore, by Theorem 2.3

$$|\nabla u^\varepsilon(x_0)| = |\nabla v^\varepsilon(0)| \leq \widetilde{C},$$

with $\widetilde{C} = \widetilde{C}(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L, \bar{c})$.

(2) Suppose that $\varepsilon \geq \bar{c}\delta_0$. By (2.10) we have

$$u^\varepsilon(x_0) \leq \varepsilon + C_1\delta_0 \leq \Big(1 + \frac{C_1}{\bar{c}}\Big)\varepsilon < 2\varepsilon,$$

if we choose $\bar{c}$ big enough. By Lemma 2.1, we have $|\nabla u^\varepsilon(x_0)| \leq C$, with $C = C(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L)$.

The result follows. □

As a consequence of the previous results we obtain Theorem 2.1. In fact,

Proof of Theorem 2.1. Let $0 < \tau < 1$ be such that $\forall x \in \Omega'$, $\overline{B_{2\tau}(x)} \subset \Omega$, and let $\varepsilon \leq \tau$. Let r_1 be the constant in Proposition 2.1, corresponding to N, $\frac{L_1}{\tau}$, L_2, $p_{\min}, p_{\max}, L$ (i.e., $r_1 = r_1(N, \frac{L_1}{\tau}, L_2, p_{\min}, p_{\max}, L)$).

Let $x_0 \in \Omega'$.

(1) If $\delta_0 = \operatorname{dist}(x_0, \partial\{u^\varepsilon > \varepsilon\}) < \tau r_1$, let $y_0 \in \partial\{u^\varepsilon > \varepsilon\}$ such that $|x_0 - y_0| = \delta_0$. Let $v^\varepsilon(x) = \frac{1}{\tau} u^\varepsilon(y_0 + \tau x)$, $\bar{p}_\varepsilon(x) = p_\varepsilon(y_0 + \tau x)$, $\bar{f}^\varepsilon(x) = \tau f^\varepsilon(y_0 + \tau x)$ and $\bar{x} = \frac{x_0 - y_0}{\tau}$, then $|\bar{x}| < r_1$. There holds that $\|v^\varepsilon\|_{L^\infty(B_1)} \leq \frac{L_1}{\tau}$, $\|\nabla \bar{p}_\varepsilon\|_{L^\infty} \leq L$ and $\|\bar{f}^\varepsilon\|_{L^\infty(B_1)} \leq L_2$.

As $0 \in \partial\{v^\varepsilon > \varepsilon/\tau\}$ and $\Delta_{\bar{p}_\varepsilon(x)} v^\varepsilon = \beta_{\varepsilon/\tau}(v^\varepsilon) + \bar{f}^\varepsilon$ in B_1, we have by Proposition 2.1

$$|\nabla u^\varepsilon(x_0)| = |\nabla v^\varepsilon(\bar{x})| \leq C_1(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L, \tau).$$

(2) If $\delta_0 = \operatorname{dist}(x_0, \partial\{u^\varepsilon > \varepsilon\}) \geq \tau r_1$, there holds that
   (a) $B_{\tau r_1}(x_0) \subset \{u^\varepsilon > \varepsilon\}$, or
   (b) $B_{\tau r_1}(x_0) \subset \{u^\varepsilon \leq \varepsilon\}$.

In the first case, $\Delta_{p_\varepsilon(x)} u^\varepsilon = f^\varepsilon$ in $B_{\tau r_1}(x_0)$. Therefore, by Theorem 2.3

$$|\nabla u^\varepsilon(x_0)| \leq C_2(N, L_1, L_2, p_{\min}, p_{\max}, L, \tau).$$

In the second case, we can apply Lemma 2.1 and we have,

$$|\nabla u^\varepsilon(x_0)| \leq C_3(N, L_1, L_2, \|\beta\|_{L^\infty}, p_{\min}, p_{\max}, L, \tau).$$

The result is proved.

□

3. Passage to the limit

Since we have that $|\nabla u^\varepsilon|$ is locally bounded by a constant independent of ε, we have that there exists a function $u \in Lip_{\rm loc}(\Omega)$ such that, for a subsequence $\varepsilon_j \to 0$, $u^{\varepsilon_j} \to u$. In this section we will prove some properties of the function u.

Lemma 3.1. *Let u^ε be a family of solutions to*

$$(P_\varepsilon(f^\varepsilon, p_\varepsilon)) \qquad \Delta_{p_\varepsilon(x)} u^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon, \quad u^\varepsilon \geq 0$$

in a domain $\Omega \subset \mathbb{R}^N$. Let us assume that $||u^\varepsilon||_{L^\infty(\Omega)} \leq L_1$ and $||f^\varepsilon||_{L^\infty(\Omega)} \leq L_2$ for some $L_1 > 0$, $L_2 > 0$. Assume moreover that $1 < p_{\min} \leq p_\varepsilon(x) \leq p_{\max} < \infty$ and that $p_\varepsilon(x)$ are Lipschitz continuous with $\|\nabla p_\varepsilon\|_{L^\infty} \leq L$, for some $L > 0$.

Then, for any sequence $\varepsilon_j \to 0$ there exist a subsequence $\varepsilon'_j \to 0$ and functions $u \in Lip_{\rm loc}(\Omega)$, $f \in L^\infty(\Omega)$ and $p \in Lip(\Omega)$, with $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ and $\|\nabla p\|_{L^\infty} \leq L$, such that

1. *$u^{\varepsilon'_j} \to u$ uniformly on compact subsets of Ω,*
2. *$f^{\varepsilon'_j} \rightharpoonup f$ $*-$weakly in $L^\infty(\Omega)$,*
3. *$p_{\varepsilon'_j} \to p$ uniformly on compact subsets of Ω,*
4. *$\Delta_{p(x)} u \geq f$ in the distributional sense in Ω,*
5. *$\Delta_{p(x)} u = f$ in $\{u > 0\}$.*
6. *There exists a nonnegative Radon measure μ such that $\beta_{\varepsilon'_j}(u^{\varepsilon'_j}) \rightharpoonup \mu$ as measures in Ω', for every $\Omega' \subset\subset \Omega$.*
7. *There holds*
$$-\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\mu + \int_\Omega f \varphi \, dx$$
for every $\varphi \in C_0^\infty(\Omega)$.
8. *$\nabla u^{\varepsilon'_j} \rightharpoonup \nabla u$ weakly in $L^{p(\cdot)}_{\rm loc}(\Omega)$.*
9. *If $p(x) \equiv p_0$, with p_0 a constant, then $\nabla u^{\varepsilon'_j} \to \nabla u$ in $L^{p_0}_{\rm loc}(\Omega)$.*

Proof. (1) and (8) follow by Theorem 2.1. (2) and (3) are immediate.

In order to prove (5), take $E \subset\subset E' \subset\subset \{u > 0\}$. Then, $u \geq c > 0$ in E'. Therefore, $u^{\varepsilon'_j} > c/2$ in E' for ε'_j small. If we take $\varepsilon'_j < c/2$ –as $\Delta_{p_{\varepsilon'_j}(x)} u^{\varepsilon'_j} = f^{\varepsilon'_j}$ in $\{u^{\varepsilon'_j} > \varepsilon'_j\}$– we have that $\Delta_{p_{\varepsilon'_j}(x)} u^{\varepsilon'_j} = f^{\varepsilon'_j}$ in E'. Therefore, by Theorem 2.3, $\|u^{\varepsilon'_j}\|_{C^{1,\alpha}(\bar{E})} \leq C$.

Thus, for a subsequence, we have

$$\nabla u^{\varepsilon'_j} \to \nabla u \quad \text{uniformly in } E.$$

Therefore, $\Delta_{p(x)} u = f$ in E.

In order to prove (6), let us take $\Omega' \subset\subset \Omega$, and $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, with $\varphi = 1$ in Ω' as a test function in $P_{\varepsilon_j}(f^{\varepsilon_j}, p_{\varepsilon_j})$. Since $\|\nabla u^{\varepsilon'_j}\| \leq C$ in Ω', there holds that

$$C(\varphi) \geq \int_\Omega \beta_{\varepsilon'_j}(u^{\varepsilon'_j}) \varphi \, dx \geq \int_{\Omega'} \beta_{\varepsilon'_j}(u^{\varepsilon'_j}) \, dx. \tag{3.1}$$

Therefore, $\beta_{\varepsilon'_j}(u^{\varepsilon'_j})$ is bounded in $L^1_{\rm loc}(\Omega)$, so that, there exists a locally finite measure μ such that

$$\beta_{\varepsilon'_j}(u^{\varepsilon'_j}) \rightharpoonup \mu \quad \text{as measures.}$$

That is, for every $\varphi \in C_0(\Omega)$,

$$\int_\Omega \beta_{\varepsilon'_j}(u^{\varepsilon'_j})\varphi\, dx \to \int_\Omega \varphi\, d\mu.$$

We will divide the reminder of the proof into several steps.

Let $\Omega' \subset\subset \Omega$. We will show that for every $v \in C_0^\infty(\Omega')$ there holds that

$$\int_{\Omega'} |\nabla u^{\varepsilon'_j}|^{p_{\varepsilon'_j}(x)-2}\nabla u^{\varepsilon'_j}\cdot \nabla v\, dx \to \int_{\Omega'} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v\, dx. \tag{3.2}$$

Let us denote, for $\eta \in \mathbb{R}^N$, $A^{\varepsilon_j}(x,\eta) = |\eta|^{p_{\varepsilon_j}(x)-2}\eta$ and $A(x,\eta) = |\eta|^{p(x)-2}\eta$.

By Theorem 2.1, we have $|\nabla u^{\varepsilon_j}| \le C$ in Ω'. Therefore for a subsequence ε'_j we have that there exists $\xi \in (L^\infty(\Omega'))^N$ such that,

$$\begin{aligned} &\nabla u^{\varepsilon'_j} \rightharpoonup \nabla u && * - \text{weakly in } (L^\infty(\Omega'))^N\\ &A^{\varepsilon_j}{}'(x,\nabla u^{\varepsilon'_j}) \rightharpoonup \xi && * - \text{weakly in } (L^\infty(\Omega'))^N\\ &u^{\varepsilon'_j} \to u && \text{uniformly in } \Omega'. \end{aligned} \tag{3.3}$$

For simplicity we call $\varepsilon'_j = \varepsilon$, $A^{\varepsilon_j}(x,\eta) = A^\varepsilon(\eta)$ and $A(x,\eta) = A(\eta)$.

Step 1. Let us prove that for any $v \in C(\Omega') \cap W^{1,\infty}(\Omega')$ there holds that

$$\int_{\Omega'} (\xi - A(\nabla u))\nabla v\, dx = 0. \tag{3.4}$$

In fact, as A^ε is monotone (i.e $\big(A^\varepsilon(\eta) - A^\varepsilon(\zeta)\big)\cdot(\eta-\zeta) \ge 0\ \forall \eta,\zeta \in \mathbb{R}^N$) we have that, for any $w \in W^{1,\infty}(\Omega')$,

$$I = \int_{\Omega'} \big(A^\varepsilon(\nabla u^\varepsilon) - A^\varepsilon(\nabla w)\big)(\nabla u^\varepsilon - \nabla w)\, dx \ge 0. \tag{3.5}$$

Therefore, if $\psi \in C_0^\infty(\Omega')$,

$$\begin{aligned} &-\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u^\varepsilon\, dx - \int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)\nabla w\, dx - \int_{\Omega'} A^\varepsilon(\nabla w)(\nabla u^\varepsilon - \nabla w)\, dx\\ &= -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u^\varepsilon\, dx - \int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon\, dx + I\\ &= -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u\, dx - \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon - u)\psi\, dx - \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon-u)(1-\psi)\, dx\\ &\quad - \int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon\, dx + I\\ &\ge -\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u\, dx + \int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)\nabla(u^\varepsilon - u)\psi\, dx + \int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)(u^\varepsilon - u)\nabla\psi\, dx\\ &\quad - \int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon - u)(1-\psi)\, dx - \int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon\, dx + \int_{\Omega'} f^\varepsilon(u^\varepsilon - u)\psi\, dx, \end{aligned} \tag{3.6}$$

where in the last inequality we are using (3.5) and equation $P_\varepsilon(f^\varepsilon, p_\varepsilon)$.

Now, take $\psi = \psi_j \to \chi_{\Omega'}$ a.e., with $0 \le \psi_j \le 1$. If Ω' is smooth we can choose the functions so that $\int |\nabla\psi_j|\, dx \le C\text{Per}\ \Omega'$. Therefore,

$$\Big|\int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)(u^\varepsilon - u)\nabla\psi_j\, dx\Big| \le C\|u^\varepsilon - u\|_{L^\infty(\Omega')}\int_{\Omega'} |\nabla\psi_j|\, dx \le C\|u^\varepsilon - u\|_{L^\infty(\Omega')}.$$

Also

$$\Big|\int_{\Omega'} f^\varepsilon(u^\varepsilon-u)\psi_j\,dx\Big|\le C\|u^\varepsilon-u\|_{L^\infty(\Omega')},$$

and

$$\Big|\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)(u^\varepsilon-u)\,dx\Big|\le C\|u^\varepsilon-u\|_{L^\infty(\Omega')}.$$

So that, with this choice of $\psi=\psi_j$ in (3.6), we obtain

$$\begin{aligned}
&-\int_{\Omega'}\beta_\varepsilon(u^\varepsilon)u^\varepsilon\,dx-\int_{\Omega'}A^\varepsilon(\nabla u^\varepsilon)\nabla w\,dx-\int_{\Omega'}A^\varepsilon(\nabla w)(\nabla u^\varepsilon-\nabla w)\,dx\\
&\ge-\int_{\Omega'}\beta_\varepsilon(u^\varepsilon)u\,dx+\int_{\Omega'}A^\varepsilon(\nabla u^\varepsilon)\nabla(u^\varepsilon-u)\,dx-C\|u^\varepsilon-u\|_{L^\infty(\Omega')}-\int_{\Omega'}A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon\,dx\\
&=-\int_{\Omega'}\beta_\varepsilon(u^\varepsilon)u\,dx-\int_{\Omega'}A^\varepsilon(\nabla u^\varepsilon)\nabla u\,dx-C\|u^\varepsilon-u\|_{L^\infty(\Omega')}\\
&\ge-\int_{\Omega'}\beta_\varepsilon(u^\varepsilon)u^\varepsilon\,dx-\int_{\Omega'}A^\varepsilon(\nabla u^\varepsilon)\nabla u\,dx-C\|u^\varepsilon-u\|_{L^\infty(\Omega')}.
\end{aligned}$$

Therefore, canceling $\int_{\Omega'}\beta_\varepsilon(u^\varepsilon)u^\varepsilon\,dx$ first, and then, letting $\varepsilon\to0$ we get by using (3.3) and (3) that

$$-\int_{\Omega'}\xi\nabla w\,dx-\int_{\Omega'}A(\nabla w)(\nabla u-\nabla w)\,dx\ge-\int_{\Omega'}\xi\nabla u\,dx$$

and then,

$$\int_{\Omega'}(\xi-A(\nabla w))(\nabla u-\nabla w)\,dx\ge0. \tag{3.7}$$

Take now $w=u-\lambda v$ with $v\in C(\Omega')\cap W^{1,\infty}(\Omega')$ and $\lambda>0$. Dividing by λ and taking $\lambda\to0^+$ in (3.7), we obtain

$$\int_{\Omega'}(\xi-A(\nabla u))\nabla v\,dx\ge0.$$

Replacing v by $-v$ we obtain (3.4). Then, (3.2) holds which implies (7) and (4).

In order to prove (9) let us now assume that $p(x)\equiv p_0$, with p_0 a constant. Then we now have $A(x,\eta)=A(\eta)=|\eta|^{p_0-2}\eta$.

Step 2. Let us prove that

$$\int_{\Omega'}A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon\to\int_{\Omega'}A(\nabla u)\nabla u. \tag{3.8}$$

By passing to the limit in the equation

$$0=\int_{\Omega'}A^\varepsilon(\nabla u^\varepsilon)\nabla\phi+\int_{\Omega'}\beta_\varepsilon(u^\varepsilon)\phi+\int_{\Omega'}f^\varepsilon\phi\,dx, \tag{3.9}$$

we have, by Step 1, that for every $\phi\in C_0(\Omega')\cap W^{1,\infty}(\Omega')$,

$$0=\int_{\Omega'}A(\nabla u)\nabla\phi+\int_{\Omega'}\phi\,d\mu+\int_{\Omega'}f\phi\,dx. \tag{3.10}$$

On the other hand, taking $\phi=u^\varepsilon\psi$ in (3.9) with $\psi\in C_0^\infty(\Omega')$ we have that

$$0=\int_{\Omega'}A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon\psi\,dx+\int_{\Omega'}A^\varepsilon(\nabla u^\varepsilon)u^\varepsilon\nabla\psi\,dx+\int_{\Omega'}\beta_\varepsilon(u^\varepsilon)u^\varepsilon\psi\,dx+\int_{\Omega'}f^\varepsilon u^\varepsilon\psi\,dx.$$

Using that $A^\varepsilon(\nabla u^\varepsilon)u^\varepsilon\nabla\psi \to A(\nabla u)u\nabla\psi$ a.e. in Ω', with $|A^\varepsilon(\nabla u^\varepsilon)u^\varepsilon\nabla\psi| \le C$ in Ω', we get

$$\int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)u^\varepsilon\nabla\psi\,dx \to \int_{\Omega'} A(\nabla u)u\nabla\psi\,dx$$
$$\int_{\Omega'} \beta_\varepsilon(u^\varepsilon)u^\varepsilon\psi\,dx \to \int_{\Omega'} u\psi d\mu.$$

Then we obtain

$$0 = \lim_{\varepsilon\to 0}\Big(\int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon\psi\,dx\Big) + \int_{\Omega'} A(\nabla u)u\nabla\psi\,dx + \int_{\Omega'} u\psi d\mu + \int_{\Omega'} fu\psi\,dx.$$

Now taking, $\phi = u\psi$ in (3.10) we have

$$0 = \int_{\Omega'} A(\nabla u)\nabla u\psi\,dx + \int_{\Omega'} A(\nabla u)u\nabla\psi\,dx + \int_{\Omega'} u\psi\,d\mu + \int_{\Omega'} fu\psi\,dx.$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_{\Omega'} A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon\psi\,dx = \int_{\Omega'} A(\nabla u)\nabla u\psi\,dx.$$

Then,

$$\begin{aligned}
&\Big|\int_{\Omega'}(A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon - A(\nabla u)\nabla u)\,dx\Big| \\
&\le \Big|\int_{\Omega'}(A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon - A(\nabla u)\nabla u)\psi\,dx\Big| + \Big|\int_{\Omega'}(A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon)(1-\psi)\,dx\Big| \\
&\quad + \Big|\int_{\Omega'} A(\nabla u)\nabla u(1-\psi)\,dx\Big| \\
&\le \Big|\int_{\Omega'}(A^\varepsilon(\nabla u^\varepsilon)\nabla u^\varepsilon - A(\nabla u)\nabla u)\psi\,dx\Big| + C\int_{\Omega'}|1-\psi|\,dx
\end{aligned}$$

so that taking $\varepsilon \to 0$ and then $\psi \to 1$ a.e. with $0 \le \psi \le 1$ we obtain (3.8). This is,

$$\int_{\Omega'} |\nabla u^\varepsilon|^{p_\varepsilon(x)}dx \to \int_{\Omega'} |\nabla u|^{p_0}dx. \tag{3.11}$$

Step 3. Let us prove that

$$\int_{\Omega'} |\nabla u^\varepsilon|^{p_0}\,dx \to \int_{\Omega'} |\nabla u|^{p_0}\,dx. \tag{3.12}$$

We first observe that

$$\Big|\int_{\Omega'} |\nabla u^\varepsilon|^{p_\varepsilon(x)}\,dx - \int_{\Omega'} |\nabla u^\varepsilon|^{p_0}\,dx\Big| \le \int_{\Omega'} \big||\nabla u^\varepsilon|^{p_\varepsilon(x)} - |\nabla u^\varepsilon|^{p_0}\big|\,dx \to 0. \tag{3.13}$$

Here we have used that $\big||\nabla u^\varepsilon|^{p_\varepsilon(x)} - |\nabla u^\varepsilon|^{p_0}\big| \to 0$ a.e. in Ω' with $\big||\nabla u^\varepsilon|^{p_\varepsilon(x)} - |\nabla u^\varepsilon|^{p_0}\big| \le C$ in Ω'.

Thus, (3.12) follows from (3.11) and (3.13).

Step 4. End of the proof of (9).

Since $u^\varepsilon \rightharpoonup u$ weakly in $W^{1,p_0}_{\rm loc}(\Omega)$ and $||u^\varepsilon||_{W^{1,p_0}(\Omega')} \to ||u||_{W^{1,p_0}(\Omega')}$, for every $\Omega' \subset\subset \Omega$, it follows that $u^\varepsilon \to u$ in $W^{1,p_0}_{\rm loc}(\Omega)$. In particular, $\nabla u^\varepsilon \to \nabla u$ in $L^{p_0}_{\rm loc}(\Omega)$. This completes the proof of the lemma. □

Lemma 3.2. *Let v be a continuous nonnegative function in a domain $\Omega \subset \mathbb{R}^N$, $v \in W^{1,p(\cdot)}(\Omega)$, such that $\Delta_{p(x)} v = g$ in $\{v > 0\}$ with $g \in L^\infty(\Omega)$. Then $\lambda_v := \Delta_{p(x)} v - g\chi_{\{v>0\}}$ is a nonnegative Radon measure with support on $\Omega \cap \partial\{v > 0\}$.*

Proof. The proof follows as in the case $p(x) \equiv 2$, that was done in [18], Lemma 2.1. □

Corollary 3.1. *Let u^{ε_j} be a family of solutions to $P_{\varepsilon_j}(f^{\varepsilon_j}, p_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $1 < p_{\min} \leq p_{\varepsilon_j}(x) \leq p_{\max} < \infty$ and $p_{\varepsilon_j}(x)$ Lipschitz continuous with $\|\nabla p_{\varepsilon_j}\|_{L^\infty} \leq L$, for some $L > 0$. Assume that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \rightharpoonup f$ $*-$weakly in $L^\infty(\Omega)$, $p_{\varepsilon_j} \to p$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$. Then,*

$$\Delta_{p(x)} u - f\chi_{\{u>0\}} = \lambda_u \qquad \text{in } \Omega,$$

with λ_u a nonnegative Radon measure supported on the free boundary $\Gamma = \Omega \cap \partial\{u > 0\}$.

Proof. It is an immediate consequence of Lemma 3.1 and Lemma 3.2. □

Lemma 3.3. *Let u^{ε_j} be a family of solutions to $P_{\varepsilon_j}(f^{\varepsilon_j}, p_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $1 < p_{\min} \leq p_{\varepsilon_j}(x) \leq p_{\max} < \infty$ and $p_{\varepsilon_j}(x)$ Lipschitz continuous with $\|\nabla p_{\varepsilon_j}\|_{L^\infty} \leq L$, for some $L > 0$. Assume that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \rightharpoonup f$ $*-$weakly in $L^\infty(\Omega)$, $p_{\varepsilon_j} \to p$ uniformly on compact subsets of Ω and $\varepsilon_j \to 0$.*

Let $x_0 \in \Omega$ and $x_n \in \Omega$ be such that $u(x_0) = 0$, $u(x_n) = 0$ and $x_n \to x_0$ as $n \to \infty$. Let $\lambda_n \to 0$, $u_{\lambda_n}(x) = \frac{1}{\lambda_n} u(x_n + \lambda_n x)$ and $(u^{\varepsilon_j})_{\lambda_n}(x) = \frac{1}{\lambda_n} u^{\varepsilon_j}(x_n + \lambda_n x)$. Assume that $u_{\lambda_n} \to U$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}^N$. Then, there exists $j(n) \to +\infty$ such that for every $j_n \geq j(n)$ there holds that $\frac{\varepsilon_{j_n}}{\lambda_n} \to 0$ and

1) $(u^{\varepsilon_{j_n}})_{\lambda_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$,
2) $\nabla(u^{\varepsilon_{j_n}})_{\lambda_n} \to \nabla U$ in $L^{p_0}_{\text{loc}}(\mathbb{R}^N)$ with $p_0 = p(x_0)$.

Proof. The result follows from Lemma 3.1 exactly as Lemma 3.2 in [5]. □

4. Basic Limits

In this section we analyze some limits that are crucial in the understanding of general limits.

We start with the following lemma

Lemma 4.1. *Let u^{ε_j}, f^{ε_j}, p_{ε_j}, ε_j, u, f and p be as in Lemma 3.3.*

Then there exists $\chi \in L^1_{\text{loc}}(\Omega)$ such that, for a subsequence, $B_{\varepsilon_j}(u^{\varepsilon_j}) \to \chi$ in $L^1_{\text{loc}}(\Omega)$, with $\chi \equiv M$ in $\{u > 0\}$ and $\chi(x) \in \{0, M\}$ a.e. in Ω. If, in addition, $f^{\varepsilon_j} \rightharpoonup 0$ in $\{u \equiv 0\}^\circ$, there holds that $\chi \equiv M$ or $\chi \equiv 0$ on every connected component of $\{u \equiv 0\}^\circ$.

Proof. We first observe that, for every $K \subset\subset \Omega$, there holds

$$\int_K |\nabla B_{\varepsilon_j}(u^{\varepsilon_j})| = \int_K \beta_{\varepsilon_j}(u^{\varepsilon_j})|\nabla u^{\varepsilon_j}| \leq C_K \int_K \beta_{\varepsilon_j}(u^{\varepsilon_j}), \tag{4.1}$$

where the last term is bounded by a constant C'_K due to estimate (3.1).

Since $0 \leq B_{\varepsilon_j}(u^{\varepsilon_j}) \leq M$, then, there exists $\chi \in L^1_{\text{loc}}(\Omega)$ such that, for a subsequence, $B_{\varepsilon_j}(u^{\varepsilon_j}) \to \chi$ in $L^1_{\text{loc}}(\Omega)$.

Proceeding as in the case $p(x) \equiv 2$ (see [18], Lemma 3.1) we deduce that $\chi \equiv M$ in $\{u > 0\}$ and $\chi(x) \in \{0, M\}$ a.e. in Ω.

Finally, if $f^{\varepsilon_j} \rightharpoonup 0$ in $\{u\equiv 0\}^\circ$, we take $K \subset\subset \{u \equiv 0\}^\circ$ in (4.1) and we observe that the last term there goes to zero since, by (6) and (7) in Lemma 3.1, $\beta_{\varepsilon_j}(u^{\varepsilon_j}) \rightharpoonup \mu$ locally as measures, with $\mu = 0$ in K. Thus the result follows. □

Proposition 4.1. *Let u^{ε_j} be solutions to $P_{\varepsilon_j}(f^{\varepsilon_j}, p_{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ with $1 < p_{\min} \leq p_{\varepsilon_j}(x) \leq p_{\max} < \infty$ and $p_{\varepsilon_j}(x)$ Lipschitz continuous with $\|\nabla p_{\varepsilon_j}\|_{L^\infty} \to 0$. Let $x_0 \in \Omega$ and suppose u^{ε_j} converge to $u_0 = \alpha(x-x_0)_1^+$ uniformly on compact subsets of Ω, with $\alpha \in \mathbb{R}$, $f^{\varepsilon_j} \rightharpoonup 0$ $*-$weakly in $L^\infty(\Omega)$, $p_{\varepsilon_j} \to p_0$ uniformly on compact subsets of Ω, with $p_0 \in \mathbb{R}$, and $\varepsilon_j \to 0$. Then*

$$\alpha = 0 \quad or \quad \alpha = \Big(\frac{p_0}{p_0-1}\, M\Big)^{1/p_0},$$

with $\int \beta(s)\, ds = M$.

Proof. Assume, for simplicity, that $x_0 = 0$. Since $u^{\varepsilon_j} \geq 0$, we have that $\alpha \geq 0$. If $\alpha = 0$ there is nothing to prove. So let us assume that $\alpha > 0$.

Let $\psi \in C_0^\infty(\Omega)$. We claim that there holds that

$$\begin{aligned}(4.2)\qquad &-\int_\Omega \frac{|\nabla u^{\varepsilon_j}|^{p_{\varepsilon_j}}}{p_{\varepsilon_j}}\psi_{x_1}\, dx + \int_\Omega |\nabla u^{\varepsilon_j}|^{p_{\varepsilon_j}-2}\nabla u^{\varepsilon_j}\cdot\nabla\psi\, u^{\varepsilon_j}_{x_1}\, dx + \int_\Omega f^{\varepsilon_j} u^{\varepsilon_j}_{x_1}\psi\, dx = \\ &\int_\Omega \frac{|\nabla u^{\varepsilon_j}|^{p_{\varepsilon_j}}}{p_{\varepsilon_j}} \log|\nabla u^{\varepsilon_j}|(p_{\varepsilon_j})_{x_1}\psi\, dx - \int_\Omega \frac{|\nabla u^{\varepsilon_j}|^{p_{\varepsilon_j}}}{p^2_{\varepsilon_j}}(p_{\varepsilon_j})_{x_1}\psi\, dx + \int_\Omega B_{\varepsilon_j}(u^{\varepsilon_j})\psi_{x_1}\, dx.\end{aligned}$$

In fact, let $\Omega' \subset\subset \Omega$ be smooth and let v_n be such that

$$(4.3)\qquad \begin{cases}\operatorname{div}\big((\frac{1}{n} + |\nabla v_n|^2)^{\frac{p_\varepsilon(x)-2}{2}}\nabla v_n\big) = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon = g^\varepsilon & \text{in } \Omega' \\ v_n = u^\varepsilon & \text{on } \partial\Omega',\end{cases}$$

were for simplicity we have denoted $\varepsilon_j = \varepsilon$. By the results in [12] and [8], $v_n \in C^{1,\alpha}(\overline{\Omega}') \cap W^{2,2}_{\text{loc}}(\Omega')$, with $||v_n||_{C^{1,\alpha}(\overline{\Omega}')} \leq C$, with C independent of n, and therefore, there exists v_0 such that, for a subsequence,

$$\begin{aligned} v_n &\to v_0 \quad \text{uniformly in } \Omega' \\ \nabla v_n &\to \nabla v_0 \quad \text{uniformly in } \Omega'.\end{aligned}$$

We get $\Delta_{p_\varepsilon(x)} v_0 = \Delta_{p_\varepsilon(x)} u^\varepsilon = g^\varepsilon$ in Ω', with $v_0 = u^\varepsilon$ in $\partial\Omega'$ and therefore, $v_0 = u^\varepsilon$.

In order to get (4.2) we take as test function in the weak formulation of (4.3) the function $\psi v_{n x_1}$, with $\psi \in C_0^\infty(\Omega')$. It follows that

$$\begin{aligned}(4.4)\qquad &-\int_\Omega \Big(\frac{1}{n} + |\nabla v_n|^2\Big)^{\frac{p_\varepsilon-2}{2}} \nabla v_n\cdot\nabla v_{n x_1}\,\psi\, dx = \\ &\qquad \int_\Omega \Big(\frac{1}{n} + |\nabla v_n|^2\Big)^{\frac{p_\varepsilon-2}{2}} \nabla v_n\cdot\nabla\psi\, v_{n x_1}\, dx + \int_\Omega g^\varepsilon v_{n x_1}\psi\, dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}(4.5)\qquad &-\int_\Omega \frac{(\frac{1}{n} + |\nabla v_n|^2)^{\frac{p_\varepsilon}{2}}}{p_\varepsilon}\psi_{x_1}\, dx = \int_\Omega \frac{(\frac{1}{n} + |\nabla v_n|^2)^{\frac{p_\varepsilon}{2}}}{p_\varepsilon}\,\frac{1}{2}\log\Big(\frac{1}{n} + |\nabla v_n|^2\Big)(p_\varepsilon)_{x_1}\psi\, dx \\ &\quad - \int_\Omega \frac{(\frac{1}{n} + |\nabla v_n|^2)^{\frac{p_\varepsilon}{2}}}{p_\varepsilon^2}(p_\varepsilon)_{x_1}\psi\, dx + \int_\Omega \Big(\frac{1}{n} + |\nabla v_n|^2\Big)^{\frac{p_\varepsilon-2}{2}}\nabla v_n\cdot\nabla v_{n x_1}\,\psi\, dx.\end{aligned}$$

Then, recalling that $g^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon$, we obtain from (4.4) and (4.5)

$$-\int_\Omega \frac{(\frac{1}{n}+|\nabla v_n|^2)^{\frac{p_\varepsilon}{2}}}{p_\varepsilon}\psi_{x_1}\,dx + \int_\Omega \Big(\frac{1}{n}+|\nabla v_n|^2\Big)^{\frac{p_\varepsilon-2}{2}}\nabla v_n\cdot\nabla\psi\, v_{n x_1}\,dx + \int_\Omega f^\varepsilon v_{nx_1}\psi\,dx =$$
$$\int_\Omega \frac{(\frac{1}{n}+|\nabla v_n|^2)^{\frac{p_\varepsilon}{2}}}{p_\varepsilon}\log\Big(\frac{1}{n}+|\nabla v_n|^2\Big)^{\frac12} p_{\varepsilon x_1}\psi\,dx - \int_\Omega \frac{(\frac{1}{n}+|\nabla v_n|^2)^{\frac{p_\varepsilon}{2}}}{p_\varepsilon^2} p_{\varepsilon x_1}\psi\,dx - \int_\Omega \beta_\varepsilon(u^\varepsilon)v_{nx_1}\psi\,dx.$$

Passing to the limit as $n\to\infty$ and integrating by parts in the last term, we get (4.2).

Now, by Lemma 4.1, we have that there exists $\chi\in L^1_{\rm loc}(\Omega)$ such that, for a subsequence, $B_{\varepsilon_j}(u^{\varepsilon_j})\to\chi$ in $L^1_{\rm loc}(\Omega)$. This, together with the strong convergence result in Lemma 3.1 and the fact that $\|\nabla p_{\varepsilon_j}\|_{L^\infty}\to 0$ gives, when passing to the limit in (4.2),

$$-\int_\Omega \frac{|\nabla u_0|^{p_0}}{p_0}\psi_{x_1}\,dx + \int_\Omega |\nabla u_0|^{p_0-2}\nabla u_0\cdot\nabla\psi\,(u_0)_{x_1}\,dx = \int_\Omega \chi\psi_{x_1}\,dx. \tag{4.6}$$

Now let $\overline{B_s}(0)\subset\Omega$. Using that, by Lemma 4.1, $\chi\equiv M$ in $B_s(0)\cap\{x_1>0\}$ and $\chi\equiv\overline{M}$ in $B_s(0)\cap\{x_1<0\}$, for a constant $\overline{M}$, with $\overline{M}=0$ or $\overline{M}=M$, and the fact that $\nabla u_0=\alpha\chi_{\{x_1>0\}}e_1$, we obtain for $\psi\in C_0^\infty(B_s(0))$

$$-\int_{\{x_1>0\}}\frac{\alpha^{p_0}}{p_0}\psi_{x_1}\,dx + \int_{\{x_1>0\}}\alpha^{p_0}\psi_{x_1}\,dx = M\int_{\{x_1>0\}}\psi_{x_1} + \overline{M}\int_{\{x_1<0\}}\psi_{x_1}.$$

Then, integrating by parts, we get

$$\Big(-\frac{\alpha^{p_0}}{p_0}+\alpha^{p_0}\Big)\int_{\{x_1=0\}}\psi\,dx' = M\int_{\{x_1=0\}}\psi\,dx' - \overline{M}\int_{\{x_1=0\}}\psi\,dx'.$$

Thus, $(-\frac{\alpha^{p_0}}{p_0}+\alpha^{p_0}) = M-\overline{M}$. Since we have assumed that $\alpha>0$, it follows that $\overline{M}=0$ and therefore, $\alpha=\Big(\frac{p_0}{p_0-1}M\Big)^{1/p_0}$.

□

5. Asymptotic behavior of limit functions

In this section we analyze the behavior of limit functions near the free boundary.

For the next result we will need the following definition

Definition 5.1. Let u be a continuous nonnegative function in a domain $\Omega\subset\mathbb{R}^N$. Let $x_0\in\Omega\cap\partial\{u>0\}$. We say that x_0 is a regular point from the positive side if there is a ball $B\subset\{u>0\}$ with $x_0\in\partial B$.

Theorem 5.1. *Let u^{ε_j}, f^{ε_j}, p_{ε_j}, ε_j, u, f and p be as in Lemma 3.3.*
Let $x_0\in\Omega\cap\partial\{u>0\}$. Assume one of the following conditions holds:

- (D) *There exist $\gamma>0$ and $0<c<1$ such that, for every $x\in B_\gamma(x_0)\cap\partial\{u>0\}$ which is regular from the positive side and $r\le\gamma$, there holds that $|\{u=0\}\cap B_r(x)|\ge c|B_r(x)|$.*
- (L) *There exist $\gamma>0$, $\theta>0$ and $s_0>0$ such that for every point $y\in B_\gamma(x_0)\cap\partial\{u>0\}$ which is regular from the positive side, and for every ball $B_r(z)\subset\{u>0\}$ with $y\in\partial B_r(z)$ and $r\le\gamma$, there exists a unit vector $\tilde e_y$, with $\langle\tilde e_y, z-y\rangle>\theta||z-y||$, such that $u(y-s\tilde e_y)=0$ for $0<s<s_0$.*

Then,

$$\limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)| = 0 \qquad \textit{or} \qquad \limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)| = \lambda^*(x_0),$$

where $\lambda^*(x) = \Big(\frac{p(x)}{p(x)-1}\,M\Big)^{1/p(x)}$ *and* $\int \beta(s)\,ds = M$.

Remark 5.1. In [20] we prove that if u^{ε_j}, f^{ε_j}, p_{ε_j}, ε_j, u, f and p are as in Theorem 5.1, with u^{ε_j} local minimizers of an energy functional then, u satisfies condition (D) in Theorem 5.1 at every point in $\Omega\cap\partial\{u>0\}$.

Proof of Theorem 5.1. Let

$$\alpha := \limsup_{\substack{x\to x_0\\ u(x)>0}} |\nabla u(x)|.$$

Since $u\in Lip_{\rm loc}(\Omega)$, $\alpha<\infty$. If, $\alpha=0$ there is nothing to prove. So, suppose that $\alpha>0$. By the definition of α there exists a sequence $z_k\to x_0$ such that

$$u(z_k)>0, \qquad |\nabla u(z_k)|\to\alpha.$$

Let y_k be the nearest point from z_k to $\Omega\cap\partial\{u>0\}$ and let $d_k=|z_k-y_k|$.

Consider the blow up sequence u_{d_k} with respect to $B_{d_k}(y_k)$. This is, $u_{d_k}(x)=\frac{1}{d_k}u(y_k+d_kx)$. Since u is locally Lipschitz, and $u_{d_k}(0)=0$ for every k, there exists $u_0\in Lip(\mathbb{R}^N)$, such that (for a subsequence) $u_{d_k}\to u_0$ uniformly on compact sets of $\mathbb{R}^N$.

Since $\Delta_{p(x)}u=f$ in $\{u>0\}$, by interior Hölder gradient estimates (see, for instance, [12]), we have that $\Delta_{p_0}u_0=0$ in $\{u_0>0\}$ with $p_0=p(x_0)$.

Now, set $\bar z_k=(z_k-y_k)/d_k\in\partial B_1$. We may assume that $\bar z_k\to\bar z\in\partial B_1$. Take

$$\nu_k := \frac{\nabla u_{d_k}(\bar z_k)}{|\nabla u_{d_k}(\bar z_k)|} = \frac{\nabla u(z_k)}{|\nabla u(z_k)|}.$$

For a subsequence, and after a rotation, we can assume that $\nu_k\to e_1$. Observe that $B_{2/3}(\bar z)\subset B_1(\bar z_k)$ for k large, and therefore $\Delta_{p_0}u_0=0$ there. By interior Hölder gradient estimates, we have $\nabla u_{d_k}\to\nabla u_0$ uniformly in $B_{1/3}(\bar z)$, and therefore $\nabla u(z_k)\to\nabla u_0(\bar z)$. Thus, $\nabla u_0(\bar z)=\alpha\, e_1$ and, in particular, $\partial_{x_1}u_0(\bar z)=\alpha$.

Next, we claim that $|\nabla u_0|\le\alpha$ in $\mathbb{R}^N$. In fact, let $R>1$ and $\delta>0$. Then, there exists $\tau_0>0$ such that $|\nabla u(x)|\le\alpha+\delta$ for any $x\in B_{\tau_0R}(x_0)$. For $|z_k-x_0|<\tau_0R/2$ and $d_k<\tau_0/2$ we have $B_{d_kR}(z_k)\subset B_{\tau_0R}(x_0)$ and therefore, $|\nabla u_{d_k}(x)|\le\alpha+\delta$ in B_R for k large. Passing to the limit, we obtain $|\nabla u_0|\le\alpha+\delta$ in B_R, and since δ and R were arbitrary, the claim holds.

Since ∇u_0 is Hölder continuous in $B_{1/3}(\bar z)$, there holds that $\nabla u_0\neq 0$ in a neighborhood of $\bar z$. Thus, $u_0\in W^{2,2}$ in a ball $B_r(\bar z)$ for some $r>0$ (see, for instance, [25] or [8]) and, since

$$\int |\nabla u_0|^{p_0-2}\nabla u_0\cdot\nabla\varphi\,dx=0 \quad \text{for every } \varphi\in C_0^\infty(B_r(\bar z)),$$

taking $\psi\in C_0^\infty(B_r(\bar z))$ and $\varphi=\psi_{x_1}$, and integrating by parts we see that, for $w=\frac{\partial u_0}{\partial x_1}$ and suitable coefficients $a_{ij}(\nabla u_0)$,

$$\sum_{i,j=1}^N \int_{B_r(\bar z)} a_{ij}\big(\nabla u_0(x)\big)w_{x_j}\psi_{x_i}\,dx=0.$$

This is, w is a solution to a uniformly elliptic equation

$$\mathcal{T}w := \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}\big(\nabla u_0(x)\big)w_{x_j}\Big) = 0.$$

Now, since $w \le \alpha$ in $B_r(\bar z)$, $w(\bar z) = \alpha$ and $\mathcal{T}w = 0$ in $B_r(\bar z)$, by the strong maximum principle we conclude that $w \equiv \alpha$ in $B_r(\bar z)$.

Now, since we can repeat this argument around any point where $w = \alpha$, by a continuation argument, we have that $w = \alpha$ in $B_1(\bar z)$.

Therefore, $\nabla u_0 = \alpha\, e_1$ in $B_1(\bar z)$ and we have, for some $y \in \mathbb{R}^N$, $u_0(x) = \alpha(x_1 - y_1)$ in $B_1(\bar z)$. Since $u_0(0) = 0$, there holds that $y_1 = 0$ and $u_0(x) = \alpha x_1$ in $B_1(\bar z)$. Finally, since $\Delta_{p_0} u_0 = 0$ in $\{u_0 > 0\}$ by a continuation argument we have that $u_0(x) = \alpha x_1$ in $\{x_1 \ge 0\}$.

On the other hand, as $u_0 \ge 0$, $\Delta_{p_0} u_0 = 0$ in $\{u_0 > 0\}$ and $u_0 = 0$ in $\{x_1 = 0\}$ we have, by Lemma A.1, that

$$u_0(x) = -\bar\alpha x_1 + o(|x|) \quad \text{ in } \{x_1 < 0\}$$

for some $\bar\alpha \ge 0$.

Now, define for $\lambda > 0$, $(u_0)_\lambda(x) = \frac{1}{\lambda} u_0(\lambda x)$. There exist a sequence $\lambda_n \to 0$ and $u_{00} \in Lip(\mathbb{R}^N)$ such that $(u_0)_{\lambda_n} \to u_{00}$ uniformly on compact sets of $\mathbb{R}^N$. We have $u_{00}(x) = \alpha x_1^+ + \bar\alpha x_1^-$.

We will show that $\bar\alpha = 0$.

In fact, first assume condition (D) holds. We observe that, for any R, there holds for large k, that

$$|\{u = 0\} \cap B_{d_k R}(y_k)| \ge c|B_{d_k R}(y_k)|,$$

implying that

$$|\{u_{d_k} = 0\} \cap B_R(0)| \ge c|B_R(0)|,$$

and therefore

$$|\{u_0 = 0\} \cap B_R(0)| \ge c|B_R(0)|, \qquad \text{and} \qquad |\{u_{00} = 0\} \cap B_1(0)| \ge c|B_1(0)|.$$

This shows that $\bar\alpha = 0$.

Now assume condition (L) holds. Then, for every k there exists a unit vector $\tilde e_k$ such that

$$\langle \tilde e_k, \frac{z_k - y_k}{d_k}\rangle > \theta \quad \text{and} \quad u(y_k - s d_k \tilde e_k) = 0 \quad \text{ for } \quad 0 < s < s_0.$$

So that

$$u_{d_k}(-s\tilde e_k) = 0 \quad \text{ for } \quad 0 < s < s_0.$$

For a subsequence we have $\tilde e_k \to \tilde e$, and $\frac{z_k - y_k}{d_k} \to \bar z$, with $\langle \tilde e, \bar z\rangle \ge \theta$, implying that $u_0(-s\tilde e) = 0$ for $0 < s < s_0$ and thus, $u_{00}(-\tilde e) = 0$.

We now observe that, since we have seen that $B_1(\bar z) \subset \{u_0(x) = \alpha x_1\} = \{x_1 > 0\}$ and $0 \in \partial B_1(\bar z)$, it follows that $\bar z = e_1$. Therefore $0 = u_{00}(-\tilde e) = \bar\alpha\langle \tilde e, e_1\rangle \ge \bar\alpha\theta$.

So that $\bar\alpha = 0$ under condition (L) as well.

Now, by Lemma 3.3 we see that there exists a sequence $\delta_n \to 0$ and solutions u^{δ_n} to $P_{\delta_n}(f^{\delta_n}, p_{\delta_n})$ such that $u^{\delta_n} \to u_0$ uniformly on compact sets of $\mathbb{R}^N$, with $f^{\delta_n} \rightharpoonup 0$ $*-$weakly in L^∞ on compact sets of $\mathbb{R}^N$, $p_{\delta_n} \to p(x_0)$ uniformly on compact sets of $\mathbb{R}^N$ and $\|\nabla p_{\delta_n}\|_{L^\infty} \to 0$ on compact sets of $\mathbb{R}^N$.

Applying a second time Lemma 3.3 we find a sequence $\tilde\delta_n \to 0$ and solutions $u^{\tilde\delta_n}$ to $P_{\tilde\delta_n}(f^{\tilde\delta_n}, p_{\tilde\delta_n})$ such that $u^{\tilde\delta_n} \to u_{00}$ uniformly on compact sets of $\mathbb{R}^N$, with $f^{\tilde\delta_n} \rightharpoonup 0$ $*-$weakly in L^∞ on compact sets of $\mathbb{R}^N$, $p_{\tilde\delta_n} \to p(x_0)$ uniformly on compact sets of $\mathbb{R}^N$ and $\|\nabla p_{\tilde\delta_n}\|_{L^\infty} \to 0$ on compact sets of $\mathbb{R}^N$. Now we can apply Proposition 4.1 and we conclude that $\alpha = \lambda^*(x_0)$. □

Definition 5.2. Let v be a continuous nonnegative function in a domain $\Omega \subset \mathbb{R}^N$. We say that v is nondegenerate at a point $x_0 \in \Omega \cap \{v = 0\}$ if there exist $c > 0$, $r_0 > 0$ such that one of the following conditions holds:

$$\fint_{B_r(x_0)} v\,dx \geq cr \quad \text{for } 0 < r \leq r_0, \tag{5.1}$$

$$\fint_{\partial B_r(x_0)} v\,dx \geq cr \quad \text{for } 0 < r \leq r_0, \tag{5.2}$$

$$\sup_{B_r(x_0)} v \geq cr \quad \text{for } 0 < r \leq r_0. \tag{5.3}$$

We say that v is uniformly nondegenerate on a set $\Gamma \subset \Omega \cap \{v = 0\}$ in the sense of (5.1) (resp. (5.2), (5.3)) if the constants c and r_0 in (5.1) (resp. (5.2), (5.3)) can be taken independent of the point $x_0 \in \Gamma$.

Remark 5.2. Assume $v \geq 0$ is locally Lipschitz continuous in a domain $\Omega \subset \mathbb{R}^N$, $v \in W^{1,p(\cdot)}(\Omega)$ with $\Delta_{p(x)} v \geq f\chi_{\{v>0\}}$, where $f \in L^\infty(\Omega)$, $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ and $p(x)$ is Lipschitz continuous. Then the three concepts of nondegeneracy in Definition 5.2 are equivalent (for the idea of the proof, see Remark 3.1 in [16], where the case $p(x) \equiv 2$ and $f \equiv 0$ is treated).

Remark 5.3. In [20] we prove that if u^{ε_j}, f^{ε_j}, p_{ε_j}, ε_j, u, f and p are as in Lemma 3.3, with u^{ε_j} local minimizers of an energy functional then, u is locally uniformly nondegenerate on $\Omega\cap\partial\{u > 0\}$.

Theorem 5.2. *Let u^{ε_j}, f^{ε_j}, p_{ε_j}, ε_j, u, f and p be as in Lemma 3.3.*

Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and suppose that u is uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ in a neighborhood of x_0. Assume there is a ball B contained in $\{u = 0\}$ touching x_0, then

$$\limsup_{\substack{x\to x_0\\ u(x)>0}} \frac{u(x)}{dist(x, B)} = \lambda^*(x_0), \tag{5.4}$$

where $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1} M\right)^{1/p(x)}$ and $\int \beta(s)\,ds = M$.

Proof. Let ℓ be the finite limit on the left hand side of (5.4) and let $y_k \to x_0$ with $u(y_k) > 0$ be such that

$$\frac{u(y_k)}{d_k} \to \ell, \quad d_k = \operatorname{dist}(y_k, B).$$

Consider the blow up sequence u_k with respect to $B_{d_k}(x_k)$, where $x_k \in \partial B$ are points with $|x_k-y_k| = d_k$, this is, $u_k(x) = \frac{u(x_k+d_kx)}{d_k}$. Choose a subsequence with blow up limit u_0, such that there exists

$$e := \lim_{k\to\infty} \frac{y_k - x_k}{d_k}.$$

As in Theorem 5.1, we see that $\Delta_{p_0} u_0 = 0$ in $\{u_0 > 0\}$ with $p_0 = p(x_0)$.

By construction, $u_0(e) = \ell = \ell\langle e, e\rangle$, $u_0(x) \leq \ell\langle x, e\rangle$ for $\langle x, e\rangle \geq 0$, $u_0(x) = 0$ for $\langle x, e\rangle \leq 0$.

Let us see that $\ell > 0$. In fact, if $\ell = 0$, then $u_0 \equiv 0$. Since $u(y_k) > 0$ and $u(x_k) = 0$, there exists $z_k \in \partial\{u > 0\}$ in the segment between y_k and x_k. By the nondegeneracy assumption,

$$\sup_{B_r(z_k)} u \geq cr \quad \text{for } 0 < r \leq r_0,\ c > 0$$

and, in particular,

$$\sup_{B_{d_k}(z_k)} u \geq cd_k \quad \text{for } k \geq k_0.$$

Then, there exists a_k such that $|a_k - z_k| \leq d_k$ and $u(a_k) \geq cd_k$. Then, letting $\bar{x}_k = \frac{a_k - x_k}{d_k}$, we get that $u_k(\bar{x}_k) \geq c$, with $|\bar{x}_k| \leq 2$. It follows that there exists $\bar{x}$ with $|\bar{x}| \leq 2$ such that $u_0(\bar{x}) \geq c > 0$, which is a contradiction.

We now observe that $\nabla u_0(e) = \ell\, e$, and thus, $|\nabla u_0(e)| = \ell > 0$. Using that ∇u_0 is continuous in $\{u_0 > 0\}$ we deduce, from the fact that $\Delta_{p_0} u_0 = 0$ in $\{u_0 > 0\}$, that $u_0 \in W^{2,2}_{\text{loc}}$ in $\{u_0 > 0\} \cap \{|\nabla u_0| > 0\}$. Then, u_0 is a solution of $Lv = 0$ in $\{u_0 > 0\} \cap \{|\nabla u_0| > 0\}$ where

$$Lv := \sum_{i,j=1}^{N} b_{ij}(\nabla u_0) v_{x_i x_j}$$

is the uniformly elliptic operator given by

$$b_{ij}(z) = \delta_{ij} + \frac{(p_0 - 2)}{|z|^2} z_i z_j.$$

Since $w(x) = \ell\langle x, e\rangle$ also satisfies $Lw = 0$ we have, from the strong maximum principle, that u_0 and w must coincide in a neighborhood of the point e.

By continuation we have that $u_0(x) = \ell\langle x, e\rangle^+$. Thus, applying Lemma 3.3 as we did in Theorem 5.1 and using Proposition 4.1, we get that $\ell = \lambda^*(x_0)$. □

Definition 5.3. We say that ν is the inward unit normal to the free boundary $\partial\{u > 0\}$ at a point $x_0 \in \partial\{u > 0\}$ in the measure theoretic sense, if $\nu \in \mathbb{R}^N$, $|\nu| = 1$ and

$$\lim_{r \to 0} \frac{1}{r^N} \int_{B_r(x_0)} |\chi_{\{u>0\}} - \chi_{\{x\,/\,\langle x - x_0, \nu\rangle > 0\}}|\, dx = 0. \tag{5.5}$$

Theorem 5.3. *Let u^{ε_j}, f^{ε_j}, p_{ε_j}, ε_j, u, f and p be as in Lemma 3.3.*

Let $x_0 \in \Omega \cap \partial\{u > 0\}$ be such that $\partial\{u > 0\}$ has at x_0 an inward unit normal ν in the measure theoretic sense and suppose that u is nondegenerate at x_0. Assume, in addition, that either condition (D) or condition (L) in Theorem 5.1 holds at x_0. Then,

$$u(x) = \lambda^*(x_0)\langle x - x_0, \nu\rangle^+ + o(|x - x_0|),$$

where $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1} M\right)^{1/p(x)}$ and $\int \beta(s)\, ds = M$.

Proof. Assume that $x_0 = 0$, and $\nu = e_1$. Take $u_\lambda(x) = \frac{1}{\lambda} u(\lambda x)$. Let $\rho > 0$ such that $B_\rho \subset\subset \Omega$. Since $u_\lambda \in Lip(B_{\rho/\lambda})$ uniformly in λ, $u_\lambda(0) = 0$, there exist $\lambda_j \to 0$ and $U \in Lip(\mathbb{R}^N)$ such that $u_{\lambda_j} \to U$ uniformly on compact sets of $\mathbb{R}^N$. From Lemma 3.1, $\Delta_{p(\lambda x)} u_\lambda = \lambda f(\lambda x)$ in $\{u_\lambda > 0\}$. Using the fact that e_1 is the inward normal in the measure theoretic sense, we have, for fixed k,

$$|\{u_\lambda > 0\} \cap \{x_1 < 0\} \cap B_k| \to 0 \quad \text{as } \lambda \to 0.$$

Hence, $U = 0$ in $\{x_1 < 0\}$. Moreover, U is nonnegative in $\{x_1 > 0\}$, $\Delta_{p_0} U = 0$ in $\{U > 0\}$ with $p_0 = p(x_0)$ and U vanishes in $\{x_1 \leq 0\}$. Then, by Lemma A.1 we have that there exists $\alpha \geq 0$ such that

$$U(x) = \alpha x_1^+ + o(|x|).$$

By Lemma 3.3 we see that there exist a sequence $\delta_n \to 0$ and solutions u^{δ_n} to $P_{\delta_n}(f^{\delta_n}, p_{\delta_n})$ such that $u^{\delta_n} \to U$ uniformly on compact sets of $\mathbb{R}^N$, with $f^{\delta_n} \rightharpoonup 0$ $*-$weakly in L^∞ on compact sets of $\mathbb{R}^N$, $p_{\delta_n} \to p(x_0)$ uniformly on compact sets of $\mathbb{R}^N$ and $\|\nabla p_{\delta_n}\|_{L^\infty} \to 0$ on compact sets of $\mathbb{R}^N$.

Define $U_\lambda(x) = \frac{1}{\lambda} U(\lambda x)$, then $U_\lambda \to \alpha x_1^+$ uniformly on compact sets of $\mathbb{R}^N$. Applying a second time Lemma 3.3 we find a sequence $\tilde{\delta}_n \to 0$ and solutions $u^{\tilde{\delta}_n}$ to $P_{\tilde{\delta}_n}(f^{\tilde{\delta}_n}, p_{\tilde{\delta}_n})$ such that $u^{\tilde{\delta}_n} \to \alpha x_1^+$ uniformly on compact sets of $\mathbb{R}^N$, with $f^{\tilde{\delta}_n} \rightharpoonup 0$ $*-$weakly in L^∞ on compact sets of $\mathbb{R}^N$, $p_{\tilde{\delta}_n} \to p(x_0)$ uniformly on compact sets of $\mathbb{R}^N$ and $\|\nabla p_{\tilde{\delta}_n}\|_{L^\infty} \to 0$ on compact sets of $\mathbb{R}^N$.

By the nondegeneracy assumption on u, we have

$$\frac{1}{r^N} \int_{B_r} u_{\lambda_j}\, dx \geq cr$$

and then

$$\frac{1}{r^N} \int_{B_r} U_{\lambda_j}\, dx \geq cr.$$

Therefore $\alpha > 0$. Now, by Proposition 4.1, $\alpha = \lambda^*(x_0)$.

We have shown that

$$U(x) = \begin{cases} \lambda^*(x_0) x_1 + o(|x|) & x_1 > 0 \\ 0 & x_1 \leq 0. \end{cases}$$

Then, using that $\Delta_{p(\lambda x)} u_\lambda = \lambda f(\lambda x)$ in $\{u_\lambda > 0\}$, by interior Hölder gradient estimates we have $\nabla u_{\lambda_j} \to \nabla U$ uniformly on compact subsets of $\{U > 0\}$. Then, by Theorem 5.1, $|\nabla U| \leq \lambda^*(x_0)$ in $\mathbb{R}^N$. As $U = 0$ on $\{x_1 = 0\}$ we have, $U \leq \lambda^*(x_0) x_1$ in $\{x_1 > 0\}$.

We claim that either $U \equiv \lambda^*(x_0) x_1$ in $\{x_1 > 0\}$ or else $U < \lambda^*(x_0) x_1$ in $\{x_1 > 0\}$.

In fact, if there exists $\bar{x}$ with $\bar{x}_1 > 0$ such that the equality holds at $\bar{x}$, then we proceed exactly as we did in the proof of Theorem 5.2 and deduce, from the strong maximum principle, that equality holds in a neighborhood of $\bar{x}$. Then, by continuation, we get $U \equiv \lambda^*(x_0) x_1$ in $\{x_1 > 0\}$.

So let us now assume that $U < \lambda^*(x_0) x_1$ in $\{x_1 > 0\}$. Let $\delta > 0$ be such that $U(\delta e_1) > 0$. Let w be such that

$$\begin{cases} \Delta_{p_0} w = 0 & \text{in } B_\delta^+ \\ w = 0 & \text{on } \{x_1 = 0\} \\ w = U & \text{on } \partial B_\delta \cap \{x_1 > 0\}. \end{cases}$$

Since $\Delta_{p_0} U \geq 0$ (this follows, for instance, from the application of Lemma 3.2 with $g = 0$ and $p(x) = p_0$), we have that $w \geq U$ in B_δ^+. Therefore $w \geq \lambda^*(x_0) x_1 + o(|x|)$ in B_δ^+.

We also have $w \leq \lambda^*(x_0) x_1$ in B_δ^+. Moreover, $w < \lambda^*(x_0) x_1$ in B_δ^+, because this holds on $\partial B_\delta \cap \{x_1 > 0\}$, and with the same argument employed above we can see that, if equality holds at a point in B_δ^+, then it must hold everywhere on B_δ^+.

On the other hand, we know that $w \in C^{1,\alpha}(\overline{B_\sigma^+})$ for any $\sigma < \delta$, and since $w \geq \lambda^*(x_0) x_1 + o(|x|)$ in B_δ^+, then $|\nabla w(0)| > 0$, implying that $|\nabla w| > 0$ in $\overline{B_\gamma^+}$ for some $\gamma > 0$.

Since, in B_γ^+, both w and $\lambda^*(x_0) x_1$ are solutions to $Lv = 0$, with L a linear uniformly elliptic operator in nondivergence form, with $w < \lambda^*(x_0) x_1$ in B_γ^+, from the Hopf's boundary principle we get that $w \leq (\lambda^*(x_0) - \rho)x_1 + o(|x|)$ for some $\rho > 0$ in B_γ^+. This is in contradiction with the fact that $w \geq \lambda^*(x_0)x_1 + o(|x|)$ in B_δ^+.

This shows that $U \equiv \lambda^*(x_0)x_1$ in $\{x_1 > 0\}$. The proof is complete. □

6. Weak solutions to the free boundary problem $P(f, p, \lambda^*)$

In this section we give a notion of weak solution to the free boundary problem $P(f, p, \lambda^*)$ and we show that, under suitable assumptions, limit functions to problems $P_\varepsilon(f^\varepsilon, p_\varepsilon)$ are weak solutions, in this sense, to the free boundary problem with $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1}\, M\right)^{1/p(x)}$, $p = \lim p_\varepsilon$ and $f = \lim f^\varepsilon$.

As a consequence, we are able to apply to limit functions the result on the regularity of the free boundary we prove in [19] (see Theorem 6.2 below).

Definition 6.1. Let $\Omega \subset \mathbb{R}^N$ be a domain. Let p be a measurable function in Ω with $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$, λ^* continuous in Ω with $0 < \lambda_{\min} \leq \lambda^*(x) \leq \lambda_{\max} < \infty$ and $f \in L^\infty(\Omega)$. We call u a weak solution of $P(f, p, \lambda^*)$ in Ω if

(1) u is continuous and nonnegative in Ω, $u \in W^{1,p(\cdot)}(\Omega)$ and $\Delta_{p(x)} u = f$ in $\Omega \cap \{u > 0\}$.

(2) For $D \subset\subset \Omega$ there are constants $0 < c_{\min} \leq C_{\max}$ and $r_0 > 0$ such that for balls $B_r(x) \subset D$ with $x \in \partial\{u > 0\}$ and $0 < r \leq r_0$

$$c_{\min} \leq \frac{1}{r} \sup_{B_r(x)} u \leq C_{\max}.$$

(3) For $\mathcal{H}^{N-1}$ a.e. $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$ (this is, for $\mathcal{H}^{N-1}$-almost every point x_0 such that $\partial\{u > 0\}$ has an exterior unit normal $\nu(x_0)$ in the measure theoretic sense) u has the asymptotic development

$$u(x) = \lambda^*(x_0)\langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|). \tag{6.1}$$

(4) For every $x_0 \in \Omega \cap \partial\{u > 0\}$,

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} |\nabla u(x)| \leq \lambda^*(x_0).$$

If there is a ball $B \subset \{u = 0\}$ touching $\Omega \cap \partial\{u > 0\}$ at x_0, then

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} \frac{u(x)}{\mathrm{dist}(x, B)} \geq \lambda^*(x_0).$$

From the definition of weak solution above, and the results in the previous sections we obtain:

Theorem 6.1. *Let u^{ε_j}, f^{ε_j}, p_{ε_j}, ε_j, u, f and p be as in Lemma 3.3.*

Assume that u is locally uniformly nondegenerate on $\Omega \cap \partial\{u > 0\}$ and that at every point $x_0 \in \Omega \cap \partial\{u > 0\}$ either condition (D) or condition (L) in Theorem 5.1 holds. Then, u is a weak solution to the free boundary problem: $u \geq 0$ and

$$\begin{cases} \Delta_{p(x)} u = f & \text{in } \{u > 0\} \\ u = 0,\ |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\} \end{cases} \tag{$P(f,p,\lambda^*)$}$$

with $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1}\, M\right)^{1/p(x)}$ and $M = \int \beta(s)\, ds$.

Proof. The result follows from Theorem 2.1, Lemma 3.1, Remark 5.2 and Theorems 5.1, 5.2 and 5.3. □

Remark 6.1. In [20] we prove that if u^{ε_j}, f^{ε_j}, p_{ε_j}, ε_j, u, f and p are as in Lemma 3.3, with u^{ε_j} local minimizers of an energy functional, u is under the assumptions of Theorem 6.1.

In [19] we prove the following result for weak solutions that applies, in particular, to limit functions u as those in Theorem 6.1, at every point in $\Omega \cap \partial_{\text{red}}\{u > 0\}$.

Theorem 6.2. *Let $p \in Lip(\Omega)$ and λ^* Hölder continuous in Ω. Let u be a weak solution of $P(f, p, \lambda^*)$ in Ω. Let $x_0 \in \Omega \cap \partial_{\text{red}}\{u > 0\}$ be such that u has the asymptotic development (6.1). There exists $r_0 > 0$ such that $B_{r_0}(x_0) \cap \partial\{u > 0\}$ is a $C^{1,\alpha}$ surface for some $0 < \alpha < 1$. It follows that, in $B_{r_0}(x_0)$, u is C^1 up to $\partial\{u > 0\}$ and the free boundary condition is satisfied in the classical sense. In addition, there is a neighborhood $\mathcal{U}$ of x_0 such that $\nabla u \neq 0$ in $\mathcal{U} \cap \{u > 0\}$, $u \in W^{2,2}_{\text{loc}}(\mathcal{U} \cap \{u > 0\})$ and the equation is satisfied in a pointwise sense in $\mathcal{U} \cap \{u > 0\}$. If moreover ∇p and f are Hölder continuous in Ω, then $u \in C^2(\mathcal{U} \cap \{u > 0\})$ and the equation is satisfied in the classical sense in $\mathcal{U} \cap \{u > 0\}$.*

Appendix A

In this appendix we collect some result on Lebesgue and Sobolev spaces with variable exponent as well as some other results that are used in the paper.

Let $p : \Omega \to [1, \infty)$ be a measurable bounded function, called a variable exponent on Ω and denote $p_{\max} = \operatorname{esssup} p(x)$ and $p_{\min} = \operatorname{essinf} p(x)$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular $\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx$ is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leq 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

One central property of these spaces (since p is bounded) is that $\varrho_{p(\cdot)}(u_i) \to 0$ if and only if $\|u_i\|_{p(\cdot)} \to 0$, so that the norm and modular topologies coincide. In fact, we have

Proposition A.1. *There holds*

$$\min\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}}, \Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\} \leq \|u\|_{L^{p(\cdot)}(\Omega)}$$
$$\leq \max\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}}, \Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\}.$$

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}$ a Banach space.

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

In some occasions, it is necessary to assume extra hypotheses on the regularity of $p(x)$. We say that p is log-Hölder continuous if there exists a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{|\log|x - y||}$$

if $|x-y|<1/2$.

If one assumes that p is log-Hölder continuous then, there holds that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$.

Some important results for these spaces are

Theorem A.1. *Let $p'(x)$ such that*

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1.$$

Then $L^{p'(\cdot)}(\Omega)$ is the dual of $L^{p(\cdot)}(\Omega)$. Moreover, if $p_{\min}>1$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive.

Theorem A.2. *Let $q(x)\le p(x)$, then $L^{p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ continuously.*

We also have the following Hölder's inequality

Theorem A.3. *Let $p'(x)$ be as in Theorem A.1. Then there holds*

$$\int_\Omega |f||g|\,dx\le 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

for all $f\in L^{p(\cdot)}(\Omega)$ and $g\in L^{p'(\cdot)}(\Omega)$.

The following version of Poincare's inequality holds

Theorem A.4. *Let Ω be bounded. Assume that $p(x)$ is log-Hölder continuous in Ω. For every $u\in W_0^{1,p(\cdot)}(\Omega)$, the inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)}\le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

holds with a constant C depending on N, $\operatorname{diam}(\Omega)$ *and the log-Hölder modulus of continuity of $p(x)$.*

For the proof of these results and more about these spaces, see [11, 14] and the references therein.

Remark A.1. For any $x\in\Omega$, $\xi,\eta\in\mathbb{R}^N$ fixed we have the following inequalities

$$|\eta-\xi|^{p(x)}\le C(|\eta|^{p(x)-2}\eta-|\xi|^{p(x)-2}\xi)(\eta-\xi) \qquad \text{if } p(x)\ge 2,$$

$$|\eta-\xi|^2\Big(|\eta|+|\xi|\Big)^{p(x)-2}\le C(|\eta|^{p(x)-2}\eta-|\xi|^{p(x)-2}\xi)(\eta-\xi) \qquad \text{if } p(x)<2.$$

These inequalities imply that the function $A(x,\xi)=|\xi|^{p(x)-2}\xi$ is strictly monotone. Then, the comparison principle for the $p(x)$-Laplacian holds since it follows from the monotonicity of $A(x,\xi)$.

We will also need

Lemma A.1. *Let $1<p_0<+\infty$. Let u be Lipschitz continuous in $\overline{B_1^+}$, $u\ge 0$ in B_1^+, $\Delta_{p_0}u=0$ in $\{u>0\}$ and $u=0$ on $\{x_N=0\}$. Then, in B_1^+ u has the asymptotic development*

$$u(x)=\alpha x_N+o(|x|),$$

with $\alpha\ge 0$.

Proof. See [5] for $p_0=2$, [10] for $1<p_0<+\infty$ and [21] for a more general operator. □

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