A little bit of extra functoriality for Ext and the computation of the Gerstenhaber bracket

Mariano Suárez-Álvarez[*]

April 18, 2016

Abstract

We show that the action of the Lie algebra $HH^1(A)$ of outer derivations of an associative algebra A on the Hochschild cohomology $HH^\bullet(A)$ of A given by the Gerstenhaber bracket can be computed in terms of an arbitrary projective resolution of A as an A-bimodule, without having recourse to comparison maps between the resolution and the bar resolution.

In his classic paper *On the cohomology structure of an associative ring* [5], Murray Gerstenhaber introduced a Lie algebra structure on the Hochschild cohomology $HH^\bullet(A)$ of an associative algebra A. This structure played a role in the proof contained in that paper of the commutativity of the cup product of $HH^\bullet(A)$, he himself showed later in [6] that it is related to the deformation theory of A, and it has ever since been regarded as an important piece of the cohomological structure of the algebra. There has been a significant amount of effort expended by many authors in order to study this structure, specially in recent times.

This Lie algebra structure on $HH^\bullet(A)$ is defined in terms of a particular realization of Hochschild cohomology: the algebra A has a canonical bimodule *bar* resolution $B(A)_\bullet$, the Hochschild cohomology $HH^\bullet(A)$ is canonically isomorphic to the cohomology of the complex $\mathsf{hom}_{A^e}(B(A)_\bullet, A)$, and the Lie bracket of $HH^\bullet(A)$ is constructed using certain explicit formulas in terms of cochains in this complex. While this is convenient for many purposes, it is quite inconvenient in one important respect: we never *compute* Hochschild cohomology using the bar resolution. In practice, we pick a projective resolution $P_\bullet$ of A which is better adapted to the task and compute instead the cohomology of the complex $\mathsf{hom}_{A^e}(P_\bullet, A)$, which is —thanks to the yoga of homological algebra— canonically

isomorphic to that of the complex $\mathsf{hom}_{A^e}(B(A)_\bullet, A)$. In principle, we can transport the Lie structure on the complex $\mathsf{hom}_{A^e}(B(A)_\bullet, A)$ to $\mathsf{hom}_{A^e}(P_\bullet, A)$ using those canonical isomorphisms, but actually doing this depends crucially on having explicit comparison morphisms $B(A)_\bullet \rightleftarrows P_\bullet$ between the two resolutions involved. The problem resides in that making such morphisms explicit is notoriously difficult.

In what follows, we present an approach which allows us to compute in this situation *part* of the Gerstenhaber Lie bracket on the cohomology of the complex $\mathsf{hom}_{A^e}(P_\bullet, A)$ *without* having recourse to comparison morphisms. More precisely, it gives a way to compute the restriction of the Lie bracket to $HH^1(A) \times HH^\bullet(A)$ or, in other words, the Lie action of the Lie algebra $HH^1(A)$ of outer derivations of A on $HH^\bullet(A)$. This requires some amount of lifting of maps to resolutions, as it should be expected, but involving only the resolution $P_\bullet$. Carrying this out in concrete examples seems to be quite feasible.

Let us explain the idea and, at the same time, describe the contents of the paper. Suppose that $\delta : A \to A$ is a derivation of the algebra A and that M is a left A-module. We say that a linear map $f : M \to M$ is a *δ-operator* on M if $f(am) = af(m) + \delta(a)m$ for all $a \in A$ and $m \in M$. From such an f we construct in Section 1 a linear map

$$\nabla_f : \mathsf{Ext}_A^\bullet(M, M) \to \mathsf{Ext}_A^\bullet(M, M)$$

as follows: we pick a projective resolution $P_\bullet$ of M, show that there exists a morphism of complexes of *vector spaces* $f_\bullet : P_\bullet \to P_\bullet$ lifting $f : M \to M$ such that each component $f_i : P_i \to P_i$ is a δ-operator, and then define a morphism of complexes $f_\bullet^\sharp : \mathsf{hom}_A(P_\bullet, M) \to \mathsf{hom}_A(P_\bullet, M)$ such that $f_i^\sharp(\phi)(p) = f(\phi(p)) - \phi(f_i(p))$ for each $\phi \in \mathsf{hom}_A(P_i, M)$ and each $p \in P_i$. The map ∇_f is the one induced on homology by $f_\bullet^\sharp$, and the key point here is that it depends only on δ and f and not on the choices made. We view this as exhibiting a little bit of 'extra' functoriality on the Ext functors, now with respect to δ-operators, and find it somewhat surprising.

Next, in section 2 we specialize this to the following situation. We start with a derivation $\delta : A \to A$, we consider the derivation $\delta^e = \delta \otimes 1 + 1 \otimes \delta : A^e \to A^e$ on the enveloping algebra A^e of A, and observe that the map $\delta : A \to A$ is then a δ^e-operator on A viewed as a left A^e-module as usual. Recalling that the Hochschild cohomology $HH^\bullet(A)$ can often be identified with $\mathsf{Ext}_{A^e}^\bullet(A, A)$, our construction then produces a map

$$\nabla_\delta : HH^\bullet(A) \to HH^\bullet(A)$$

which can be computed as described above from any A^e-projective resolution of A endowed with a lifting of δ. In particular, we can use the bar resolution to do this: on it there exists a certain canonical lifting of δ and it turns out that the explicit formulas associated to it for the map ∇_δ are precisely the same ones used by Gerstenhaber to define the map

$$[\delta, -] : HH^\bullet(A) \to HH^\bullet(A).$$

Of course, this means that $\nabla_\delta = [\delta, -]$ and shows that we can compute the restriction of the bracket to $HH^1(A) \times HH^\bullet(A)$ using our favorite resolution, which is what we wanted.

In Section 3 we present this computation of the Gerstenhaber bracket in two "real life" examples: truncated path algebras and crossed products of symmetric algebras $S(V)$ by a finite group G acting linearly. In the two cases —and after a certain amount of work needed to be able to describe explicitly the cohomology itself— we are able to exhibit formulas for the bracket. Finally, in the last section, Section 4, we rapidly explain how a procedure similar to the one sketched above applies to Tor functors and, in particular, to the action of the Lie algebra $HH^1(A)$ on the Hochschild homology $HH_\bullet(A)$.

The very natural problem which we partially solve in this paper, that of finding a way to compute the Gerstenhaber bracket on Hochschild cohomology in term of an arbitrary projective resolution, was posed originally by Gerstenhaber and Samuel Schack in their survey [7] in 1988. It is generally agreed that solving it will require a different perspective on the construction of the bracket. Ten years later, Stefan Schwede gave in [18] a beautiful interpretation of the bracket in terms of actual commutators of paths in the geometric realization of the nerve of a category of Yoneda extensions first considered by Vladimir Retakh in [14] — it does not appear, though, that this interpretation leads to a computational device in practice. The first concrete step forward occurred very recently: in their preprint [13], based on the thesis [12] of the first author, Cris Negron and Sarah Witherspoon describe an alternate approach to the computation of the bracket which, under certain conditions —satisfied, for example, if the algebra is Koszul— allows for the computation of the bracket in terms of a projective resolution. This approach gives a computation of the 'whole' bracket, but has the disadvantage of being very close in practice to the construction of comparison morphisms, which we want to avoid.

In what follows we fix a commutative ring $\Bbbk$ to play the role of ring of scalars. Throughout A will denote a projective $\Bbbk$-algebra, unadorned $\otimes$ and hom will denote $\mathsf{hom}_\Bbbk$ and $\otimes_\Bbbk$, and linear will mean $\Bbbk$-linear. In particular, the Hochschild cohomology $HH^\bullet(A)$ of A as a $\Bbbk$-algebra can and will be identified canonically with the Yoneda algebra $\mathsf{Ext}^\bullet_{A^e}(A, A)$ of A viewed as a left A^e-module, and likewise for homology.

1 A little bit of extra functoriality for Ext

1.1. Let us fix an algebra A and a derivation $\delta : A \to A$. If M is a left A-module, a *δ-operator* on M is a linear map $f : M \to M$ such that for all $a \in A$ and all $m \in M$ we have

$$f(am) = \delta(a)m + af(m).$$

While δ-operators are in general not morphisms of A-modules, we have the following:

Lemma. *If M is a left A-module and f, $f' : M \to M$ are δ-operators on M, then $f - f' : M \to M$ is a morphism of A-modules.*

Proof. This follows at once from the definition. $\qquad\square$

1.2. If M is a left A-module, $f : M \to M$ a δ-operator and $\varepsilon : P_\bullet \to M$ a projective resolution of M,

$$\cdots \longrightarrow P_2 \xrightarrow{\ d_2\ } P_1 \xrightarrow{\ d_1\ } P_0 \xrightarrow{\ \varepsilon\ } M \longrightarrow 0$$

a δ-**lifting** of f to $P_\bullet$ is a sequence $f_\bullet = (f_i)_{i \geq 0}$ of δ-operators $f_i : P_i \to P_i$ such that the diagram

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{\ d_2\ } & P_1 & \xrightarrow{\ d_1\ } & P_0 & \xrightarrow{\ \varepsilon\ } & M & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\
\cdots & \longrightarrow & P_2 & \xrightarrow{\ d_2\ } & P_1 & \xrightarrow{\ d_1\ } & P_0 & \xrightarrow{\ \varepsilon\ } & M & \longrightarrow & 0
\end{array}
$$

is commutative.

1.3. As one can hope, δ-liftings exist and are unique up to reasonable equivalence. The key point to establishing this is the following result:

Lemma. *If $\varepsilon : P \to M$ is a surjective morphism of left A-modules with projective domain and $f : M \to M$ is a δ-operator, then there exists a δ-operator $\tilde{f} : P \to P$ such that the diagram*

$$
\begin{array}{ccc}
P & \xrightarrow{\ \varepsilon\ } & M \\
\downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\
P & \xrightarrow{\ \varepsilon\ } & M
\end{array}
$$

is commutative, $\tilde{f}(\ker \varepsilon) \subseteq \ker \varepsilon$ and the restriction $f|_{\ker \varepsilon} : \ker \varepsilon \to \ker \varepsilon$ is a δ-operator.

Proof. Let $(p_i, \phi_i)_{i \in I}$ be a projective basis for P, so that $p_i \in P$ and $\phi_i \in \mathsf{hom}_A(P, A)$ for all $i \in I$, and for each $p \in P$ the set $\{i \in I : \phi_i(p) \neq 0\}$ is finite and $p = \sum_{i \in I} \phi_i(p) p_i$, and let $(q_i)_{i \in I}$ be a family of elements of P such that $\varepsilon(q_i) = f(\varepsilon(p_i))$ for all $i \in I$. The function $\tilde{f} : P \to P$ such

$$\tilde{f}(p) = \sum_{i \in I} \big(\phi_i(p) q_i + \delta(\phi_i(p)) p_i \big)$$

for all $p \in P$ is easily seen to satisfy the conditions of the lemma. $\qquad\square$

1.4. We can now deduce in the usual way the existence and uniqueness of δ-liftings:

Lemma. *Let M be a left A-module and let $\varepsilon : P_\bullet \to M$ be a projective resolution.*
  (i) *There exists a δ-lifting $f_\bullet : P_\bullet \to P_\bullet$ of f to $P_\bullet$.*
  (ii) *If $f_\bullet$, $f'_\bullet$ are δ-liftings of a δ-operator $f : M \to M$ to $P_\bullet$, then $f_\bullet$ and $f'_\bullet$ are A-linearly homotopic.*

4

Proof. The first part follows inductively using the result of Lemma **1.3** at each step. On the other hand, in the situation of the second part the diagram

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\ \ } & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\ \varepsilon\ } & M & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle f_2 - f_2'} & & \Big\downarrow{\scriptstyle f_1 - f_1'} & & \Big\downarrow{\scriptstyle f_0 - f_0'} & & \Big\downarrow{\scriptstyle 0} & & \\
\cdots & \xrightarrow{\ \ } & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\ \varepsilon\ } & M & \longrightarrow & 0
\end{array}
$$

is commutative and the vertical arrows are morphisms of left A-modules, so the morphism of complexes $f_\bullet - f_\bullet' : P_\bullet \to P_\bullet$ is homotopic to the zero morphism, through an A-linear homotopy. $\qquad\square$

1.5. Let now M be a left A-module, $f : M \to M$ a δ-operator, $\varepsilon : P_\bullet \to M$ a projective resolution and $f_\bullet : P_\bullet \to P_\bullet$ a δ-lifting of f to $P_\bullet$. If $i \geq 0$ and $\phi \in \mathsf{hom}_A(P_i, M)$, then the map $f_i^\sharp(\phi) : P_i \to M$ given by

$$
f_i^\sharp(\phi)(p) = f(\phi(p)) - \phi(f_i(p))
$$

for all $p \in P_i$ is a morphism of A-modules, so we have a function

$$
f_i^\sharp : \mathsf{hom}_A(P_i, M) \to \mathsf{hom}_A(P_i, M)
$$

which is linear. A computation shows that, in fact, we obtain in this way a morphism of complexes of vector spaces

$$
f_\bullet^\sharp : \mathsf{hom}_A(P_\bullet, M) \to \mathsf{hom}_A(P_\bullet, M).
$$

1.6. To study the dependence of the morphism $f_\bullet^\sharp$ on the data used in its construction we will need the following observation.

Lemma. *Let M be a left A-module, $f : M \to M$ a δ-operator on M, $\varepsilon : P_\bullet \to M$ and $\varepsilon' : P_\bullet' \to M$ projective resolutions and $f_\bullet : P_\bullet \to P_\bullet$ and $f_\bullet' : P_\bullet' \to P_\bullet'$ δ-liftings of f to $P_\bullet$ and to $P_\bullet'$, respectively. If $\alpha_\bullet : P_\bullet' \to P_\bullet$ is a morphism of complexes of A-modules lifting $\mathrm{id}_M : M \to M$, then the diagram*

$$
\begin{array}{ccc}
\mathsf{hom}_A(P_\bullet, M) & \xrightarrow{\ f_\bullet^\sharp\ } & \mathsf{hom}_A(P_\bullet, M) \\
{\scriptstyle \alpha_\bullet^*}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_\bullet^*} \\
\mathsf{hom}_A(P_\bullet', M) & \xrightarrow{\ f_\bullet'^\sharp\ } & \mathsf{hom}_A(P_\bullet', M)
\end{array}
$$

commutes up to homotopy.

Proof. The difference $h_\bullet = \alpha_\bullet f_\bullet' - f_\bullet \alpha_\bullet : P_\bullet' \to P_\bullet$ is, in principle, only a morphism of complexes of vector spaces, but a computation shows that its components are in fact A-linear. As $h_\bullet : P_\bullet' \to P_\bullet$ is then a lifting of the zero map $0 : M \to M$, it is A-linearly homotopic to zero and, therefore, the induced map

$$
h_\bullet^* : \mathsf{hom}_A(P_\bullet, M) \to \mathsf{hom}_A(P_\bullet', M)
$$

is also homotopic to zero. Since $\alpha_\bullet^* \circ f_\bullet^\sharp - f_\bullet'^\sharp \circ \alpha_\bullet^* = h_\bullet^*$, the lemma follows from this. $\quad\square$

1.7. As a first consequence of this lemma, we see that if M is a left A-module, $\delta : M \to M$ a δ-operator on M, $\varepsilon : P_\bullet \to M$ a projective resolution and $f_\bullet : P_\bullet \to P_\bullet$ and $f'_\bullet : P_\bullet \to P_\bullet$ δ-liftings of f to $P_\bullet$, then the maps of complexes $f_\bullet^\sharp$, $f_\bullet'^\sharp : \mathsf{hom}_A(P_\bullet, M) \to \mathsf{hom}_A(P_\bullet, M)$ are homotopic —this is the special case of the lemma in which $P'_\bullet = P_\bullet$, $\varepsilon' = \varepsilon$ and $\alpha_\bullet = \mathrm{id}_{P_\bullet}$— and therefore they induce the same map on cohomology. We may therefore denote this induced map, which depends only on f and not on the choice of the δ-lifting used to compute it, simply by

$$\nabla_{f, P_\bullet}^\bullet : H(\mathsf{hom}_A(P_\bullet, M)) \to H(\mathsf{hom}_A(P_\bullet, M)).$$

Next, if $\varepsilon' : P'_\bullet \to M$ is another projective resolution, $f'_\bullet : P'_\bullet \to P'_\bullet$ a δ-lifting of f to $P'_\bullet$ and $\alpha_\bullet : P'_\bullet \to P_\bullet$ a lifting of $\mathrm{id}_M : M \to M$, the diagram

$$
\begin{array}{ccc}
H(\mathsf{hom}_A(P_\bullet, M)) & \xrightarrow{\nabla_{f, P_\bullet}^\bullet} & H(\mathsf{hom}_A(P_\bullet, M)) \\
{\scriptstyle H(\alpha_\bullet^*)} \downarrow & & \downarrow {\scriptstyle H(\alpha_\bullet^*)} \\
H(\mathsf{hom}_A(P'_\bullet, M)) & \xrightarrow{\nabla_{f, P'_\bullet}^\bullet} & H(\mathsf{hom}_A(P'_\bullet, M))
\end{array}
$$

induced on cohomology by the one in the statement of the lemma *commutes*. Recalling the way the Yoneda functor $\mathsf{Ext}_A^\bullet(M, M)$ is identified with a derived functor, we see that the end result of all we have done is the following:

Theorem A. *If M is a left A-module and $f : M \to M$ is a δ-operator on M, there is a canonical morphism of graded vector spaces*

$$\nabla_f^\bullet : \mathsf{Ext}_A^\bullet(M, M) \to \mathsf{Ext}_A^\bullet(M, M)$$

such that for each projective resolution $\varepsilon : P_\bullet \to M$ and each δ-lifting $f_\bullet : P_\bullet \to P_\bullet$ of f to $P_\bullet$ the diagram

$$
\begin{array}{ccc}
H(\mathsf{hom}_A(P_\bullet, M)) & \xrightarrow{\nabla_{f, P_\bullet}^\bullet} & H(\mathsf{hom}_A(P_\bullet, M)) \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \cong} \\
\mathsf{Ext}_A^\bullet(M, M) & \xrightarrow{\nabla_f^\bullet} & \mathsf{Ext}_A^\bullet(M, M)
\end{array}
$$

in which the vertical arrows are the canonical isomorphisms, commutes. $\qquad\square$

1.8. In keeping with a long standing tradition, the very first example we present is a somewhat trivial one, reserving for the next section the one in which we are really interested.

Lemma. *Suppose that $\delta : A \to A$ is an inner derivation, so that there exists an $r \in A$ with $\delta(a) = [r, a]$ for all $a \in A$. If M is a left A-module and $f : M \to M$ is a δ-operator on M, then there exists a morphism of left A-modules $\bar{f} : M \to M$ such that $f(m) = \bar{f}(m) + rm$ for all $m \in M$, and the map $\nabla_f^\bullet : \mathsf{Ext}_A^\bullet(M, M) \to \mathsf{Ext}_A^\bullet(M, M)$ is such that for all $\phi \in \mathsf{Ext}_A^\bullet(M, M)$ we have $\nabla_f^i(\phi) = \bar{f}_*(\phi) - \bar{f}^*(\phi) = [\bar{f}, \phi]$.*

Here $\bar{f}^*$, $\bar{f}_*$: $\mathsf{Ext}_A^\bullet(M, M) \to \mathsf{Ext}_A^\bullet(M, M)$ are the maps induced by $\bar{f}$ on the first and on the second argument of Ext, respectively, and the commutator in the last formula is the one obtained on $\mathsf{Ext}_A^\bullet(M, M)$ from the Yoneda product.

Proof. Let $\lambda : M \to M$ be the map given by multiplication by r. A computation shows that $\bar{f} = f - \lambda : M \to M$ is a morphism of left A-modules. Let now $\varepsilon : P_\bullet \to M$ be a projective resolution, let $\bar{f}_\bullet : P_\bullet \to P_\bullet$ be a lifting of $\bar{f}$ to $P_\bullet$ and let $\lambda_\bullet : P_\bullet \to P_\bullet$ be the map given by multiplication by r. Then $f_\bullet = \bar{f}_\bullet + \lambda_\bullet : P_\bullet \to P_\bullet$ is a δ-lifting of f to $P_\bullet$ and if $i \geq 0$ and $\phi \in \mathsf{hom}_A(P_i, M)$, we have $f_i^\sharp(\phi) = \bar{f}_*(\phi) - \bar{f}_i^*(\phi)$. This proves the lemma. $\qquad\square$

2 The Gerstenhaber bracket on Hochschild cohomology

2.1. As we did in the previous section, we fix an algebra A and a derivation $\delta : A \to A$. If M is a left A-module, there is a standard projective resolution $\varepsilon : B(M)_\bullet \to A$, called the ***bar resolution***, with $B(M)_i = A^{\otimes(i+1)} \otimes M$ for each $i \geq 0$, differentials given by

$$d_i(a_0 \otimes \cdots \otimes a_i \otimes m) = \sum_{i=0}^{i-1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_i \otimes m$$
$$+ (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i m$$

for all $i \geq 1$ and augmentation map $\varepsilon : B(M)_0 \to M$ such that $\varepsilon(a_0 \otimes m) = a_0 m$. If $f : M \to M$ is a δ-operator on M, there is a canonical δ-lifting $f_\bullet : B(M)_\bullet \to B(M)_\bullet$ of f to $B(M)_\bullet$ given by

$$f_i(a_0 \otimes \cdots \otimes a_i \otimes m) = \sum_{i=0}^{i-1} a_0 \otimes \cdots \otimes a_{i-1} \otimes \delta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_i \otimes m$$
$$+ a_0 \otimes \cdots \otimes a_i \otimes f(m),$$

as a computation will show. From this we obtain an explicit description of the morphism $f_\bullet^\sharp : \mathsf{hom}_A(B(M)_\bullet, M) \to \mathsf{hom}_A(B(M)_\bullet, M)$: if $i \geq 0$ and $\phi : B(M)_i \to M$ is A-linear, then

$$f_i^\sharp(\phi)(a_0 \otimes \cdots \otimes a_i \otimes m) = f(\phi(a_0 \otimes \cdots \otimes a_i \otimes m))$$
$$- \sum_{i=0}^{i-1} \phi(a_0 \otimes \cdots \otimes a_{i-1} \otimes \delta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_i \otimes m)$$
$$- \phi(a_0 \otimes \cdots \otimes a_i \otimes f(m))$$

2.2. Let now $A^e = A \otimes A^{\mathsf{op}}$ be the enveloping algebra of A —so that we may identify A-bimodules with left A^e-modules— and consider A as a left A^e-module as usual. From the derivation $\delta : A \to A$ we can construct a new derivation $\delta^e = \delta \otimes \mathrm{id}_A + \mathrm{id}_A \otimes \delta : A^e \to A^e$,

and it turns out that the map $\delta : A \to A$ is then a δ^e-operator on A. Recalling that in our context we may identify $\mathsf{Ext}^\bullet_{A^e}(A, A)$ with the Hochschild cohomology $HH^\bullet(A)$, our general construction from the previous section gives us a map

$$\nabla_\delta : HH^\bullet(A) \to HH^\bullet(A). \tag{1}$$

We want to see what this map is.

If for each $i \geq 0$ we turn the left A-module $B(A)_i$ into an A-bimodule with right action given by

$$a_0 \otimes \cdots \otimes a_i \otimes a \cdot b = a_0 \otimes \cdots \otimes a_i \otimes ab,$$

then the projective resolution $\varepsilon : B(A)_\bullet \to A$ of A as a left A-module constructed in **2.1** becomes a projective resolution of A as an A^e-module. Moreover, the δ-lifting $\delta_\bullet : B(A)_\bullet \to B(A)_\bullet$ of the δ-operator $\delta : A \to A$ constructed there is a δ^e-lifting, as one can easily check, so that we may use it to compute the map (1) up to the canonical identification of $HH^\bullet(A)$ with $H(\mathsf{hom}_{A^e}(B(A)_\bullet, A))$.

Let now $C^\bullet(A)$ be the standard complex which computes Hochschild cohomology, which has $C^i(A) = \mathsf{hom}(A^{\otimes i}, A)$ for each $i \geq 0$ and differentials $d^i : C^i(A) \to C^{i+1}(A)$ given by

$$d^i(\phi)(a_1 \otimes \cdots \otimes a_{i+1}) = a_1 \phi(a_2 \otimes \cdots \otimes a_{i+1})$$

$$+ \sum_{j=1}^{i} (-1)^{j+1} \phi(a_1 a \otimes \cdots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \cdots \otimes a_{i+1})$$

$$+ (-1)^{i+1} \phi(a_1 \otimes \cdots \otimes a_i) a_{i+1}.$$

Of course, there is an isomorphism of complexes $\tau_\bullet : \mathsf{hom}_{A^e}(B(A)_\bullet, A) \to C^\bullet(A)$ such that $\tau_i(\phi)(a_1 \otimes \cdots \otimes a_i \otimes 1) = \phi(1 \otimes a_1 \otimes \cdots \otimes a_i)$ for all $i \geq 0$ and all $\phi \in \mathsf{hom}_{A^e}(B(A)_i, A)$. If now we let $[-, -]$ be the Gerstenhaber bracket on $C^\bullet(A)$, as constructed in [5], then the diagram

$$
\begin{array}{ccc}
\mathsf{hom}_{A^e}(B(A)_\bullet, A) & \xrightarrow{\tau_\bullet} & C^\bullet(A) \\
{\scriptstyle \delta^\sharp_\bullet} \downarrow & & \downarrow {\scriptstyle [\delta, -]} \\
\mathsf{hom}_{A^e}(B(A)_\bullet, A) & \xrightarrow{\tau_\bullet} & C^\bullet(A)
\end{array}
$$

commutes. This means that the map ∇_δ of (1) is in fact simply $[\delta, -]$. The point of all this is that Theorem **A** from the previous section tells us that we can compute ∇_δ using *any* projective resolution of A as an A-bimodule, provided we are able to construct a δ^e-lifting of δ.

2.3. If in the situation of the previous paragraph the derivation δ with which we start is inner, so that there is an $r \in A$ with $\delta = [r, -]$, then δ^e is also inner, as $\delta^e = [r^e, -]$ with $r^e = r \otimes 1 - 1 \otimes r$. Moreover, if $\varepsilon : P_\bullet \to A$ is any resolution of A as an A-bimodule, then there is a δ^e-lifting $\delta_\bullet : P_\bullet \to P_\bullet$ of $\delta : A \to A$ to $P_\bullet$ such that for all $i \geq 0$ and all $p \in P_i$ we have $\delta_i(p) = rp - pr$. The associated map $\delta_\bullet^\sharp : \mathsf{hom}_{A^e}(P_\bullet, A) \to \mathsf{hom}_{A^e}(P_\bullet, A)$ is identically zero, so we have that $\nabla_\delta : HH^\bullet(A) \to HH^\bullet(A)$ itself is zero, as it should.

3 Examples

Monomial algebras

3.1. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver and let $\Bbbk Q$ be the corresponding path algebra; if $v \in Q_0$ is a vertex, we write e_v the corresponding idempotent. Let R be a set of paths in Q of length at least 2 such that no element of R divides another and consider the monomial algebra $A = \Bbbk Q/(R)$. We write E the subalgebra of A spanned by the vertices; whenever Z is a set of paths in Q, the vector space $\Bbbk Z$ which has Z as a basis has a natural structure of E-bimodule.

We let $\varepsilon : \mathsf{Br}_\bullet \to A$ be the projective resolution of A as an A-bimodule constructed by Michael Bardzell in [1]; a useful companion to Bardzell's paper is the work [19] of Emil Sköldberg, where a contracting homotopy on $\mathsf{Br}_\bullet$ is exhibited. There is a sequence $(R_i)_{i \geq 0}$ of sets of paths in Q such that $R_0 = Q_0$ is the set of vertices, $R_1 = Q_1$ is the set of arrows, $R_2 = R$ and $\mathsf{Br}_i = A \otimes_E \Bbbk R_i \otimes_E A$ for all $i \geq 0$. The augmentation $\varepsilon : \mathsf{Br}_0 = A \otimes_E A \to A$ is the map induced by the multiplication of A, and for each $i \geq 1$ the differential $d : \mathsf{Br}_i \to \mathsf{Br}_{i-1}$ has the property that whenever u and w are paths in Q which are not in the ideal (R) and $v \in R_i$, and we can form the concatenation uvw, then $d_i(u \otimes v \otimes w)$ is a $\Bbbk$-linear combination of elementary tensors $u' \otimes v' \otimes w'$ of $\mathsf{Br}_{i-1} = A \otimes_E R_{i-1} \otimes_E A$ with u' and w' paths in Q not in (R) and a path v' in R_{i-1} such that the concatenation $u'v'w'$ exists and coincides with the path uvw.

The differentials $d : \mathsf{Br}_1 \to \mathsf{Br}_0$ and $d : \mathsf{Br}_2 \to \mathsf{Br}_1$, in particular, are such that $d_1(1 \otimes \alpha \otimes 1) = \alpha \otimes 1 - 1 \otimes \alpha$ whenever α is an arrow, and

$$d_2(1 \otimes r \otimes 1) = \sum_{i=1}^{n} \alpha_1 \cdots \alpha_{i-1} \otimes \alpha_i \otimes \alpha_{i+1} \cdots \alpha_n$$

whenever $r = \alpha_1 \cdots \alpha_n$ is an element of R of length n. It follows from this that we can identify $\mathsf{hom}_{A^e}(\mathsf{Br}_1, A)$ with the vector space $\mathsf{hom}_{E^e}(\Bbbk Q_1, A)$, whose elements are the linear functions $\delta : \Bbbk Q_1 \to A$ which map each arrow α to a linear combination of paths in Q which are parallel to α. Such a map is in the kernel of the differential $d_2^* : \mathsf{hom}_{A^e}(\mathsf{Br}_1, A) \to \mathsf{hom}_{A^e}(\mathsf{Br}_2, A)$ iff for each element $r = \alpha_1 \cdots \alpha_n$ of R we have

$$\sum_{i=1}^{n} \alpha_1 \cdots \alpha_{i-1} \delta(\alpha_i) \alpha_{i+1} \cdots \alpha_n = 0$$

in the algebra A, and indeed this condition is satisfied iff the map δ can be extended to a E-linear derivation $A \to A$.

3.2. If $c : Q_1 \to \Bbbk$ is an arbitrary function defined on the set of arrows of Q, we may consider the extension $c : Q_* \to \Bbbk$ such that $c(w) = \sum_{i=1}^n c(\alpha_i)$ for each path $w = \alpha_1 \cdots \alpha_n$ in Q, and then the E^e-linear map $\delta_c : A \to A$ such that $\delta_c(w) = c(w)w$ for all paths in Q. One sees immediately that this is an E^e-linear derivation. We say that derivations of the form δ_c for some $c : Q_1 \to \Bbbk$ are ***diagonal***.

We consider the derivation $\delta_c^e = \delta_c \otimes 1 + 1 \otimes \delta_c : A^e \to A^e$ on the enveloping algebra A^e, as in **2.2**, and view $\delta_c : A \to A$ as a δ_c^e-operator on the left A^e-module A. There is a δ_c^e-lifting $(\delta_c)_\bullet : \mathsf{Br}_\bullet \to \mathsf{Br}_\bullet$ of δ_c to the resolution $\mathsf{Br}_\bullet$ such that for each $i \geq 0$, each $u, w \in Q_*$ and each $v \in R_i$ such that the concatenation uvw exists, we have

$$(\delta_c)_i(u \otimes v \otimes w) = c(uvw) \cdot u \otimes v \otimes w.$$

We may use this δ_c^e-lifting to compute the Gerstenhaber bracket $[\delta_c, -]$ on $HH^\bullet(A)$, if we identify it with the cohomology of the complex $\mathsf{hom}_{A^e}(\mathsf{Br}_\bullet, A)$. Indeed, the space $\mathsf{hom}_{A^e}(\mathsf{Br}_i, A)$ of i-cochains in this complex can be identified with $\mathsf{hom}_{E^e}(\Bbbk R_i, A)$, and clearly has as a basis the set of all E^e-linear maps $\phi_{r,u} : \Bbbk R_i \to A$ with $r \in R_i$ and u a path in Q parallel to r and which is non-zero in A, given by $\phi_{r,u}(s) = 0$ for all $s \in R_i \setminus \{r\}$ and $\phi_{r,u}(r) = u$, so it is sufficient to compute $[\delta_c, \phi_{r,u}]$, and this is, according to what we have done so far,

$$[\delta_c, \phi_{r,u}] = \nabla_{\delta_c}(\phi_{r,u}) = (\delta_c)_i^\sharp(\phi_{r,u}) = (c(u) - c(r))\phi_{r,u}. \tag{2}$$

If the set R satisfies the condition that

whenever $\alpha : i \to j$ is an arrow of Q we have $\mathsf{dim}\, e_i A e_j = 1$, so that there are no non-zero paths in A parallel to an arrow apart from the arrow itself,

then it is easy to see that *all* elements of $HH^1(A)$ are represented by diagonal derivations and in this situation Lucrecia Román has obtained the formula (2) in her thesis [15] after fearlessly computing comparison morphisms $\mathsf{Br}_\bullet \rightleftarrows B(A)_\bullet$ and then using the usual formula for the Gerstenhaber bracket on the standard complex $C^\bullet(A)$.

3.3. Let us suppose now that we have an integer $N \geq 2$ and that R is the set Q_N of *all* paths of length N in Q; the algebra A is then what is usually called a ***truncated algebra***. In this case the Bardzell resolution $\mathsf{Br}_\bullet$ admits a very simple description, which we now recall. Let $\zeta : \mathbb{N}_0 \to \mathbb{N}_0$ be the function such that $\zeta(2k) = Nk$ and $\zeta(2k + 1) = Nk + 1$ for all $k \in \mathbb{N}_0$. Then for all $i \geq 0$ we have $R_i = Q_{\zeta(i)}$, the set of all paths of length $\zeta(i)$ in the quiver Q, and the differential on Br_i is such that for each $w \in Q_{\zeta(i)}$ we have

$$d(1 \otimes w \otimes 1) = \sum_{\substack{aub=w \\ u \in Q_{N(k-1)+1}}} a \otimes u \otimes b \qquad \text{if } i = 2k \text{ is even}$$

10

and

$$d(1 \otimes w \otimes 1) = a \otimes r \otimes 1 - 1 \otimes l \otimes b \qquad\qquad \text{if } i = 2k+1 \text{ is odd,}$$

where in this last case $a, b \in Q_1$ and $r, l \in Q_{Nk}$ are such that $w = ar = lb$.

Lemma. *An E^e-linear map $\delta : \Bbbk Q_1 \to A$ is a 1-cocycle in the complex $\mathsf{hom}_{A^e}(\mathsf{Br}_\bullet, A)$ if it takes values in $\mathsf{rad}\, A$, and if the quiver Q has more than one vertex and is connected then this condition is also necessary.*

Proof. The sufficiency is clear, so we deal only with the second part. Let $\delta : \Bbbk Q_1 \to A$ be a 1-cocycle and let $d_2^* : \mathsf{hom}_{A^e}(\mathsf{Br}_1, A) \to \mathsf{hom}_{A^e}(\mathsf{Br}_2, A)$ be the differential. There are E^e-linear maps $\delta_0 : \Bbbk Q_1 \to \Bbbk E$ and $\delta_+ : \Bbbk Q_1 \to \mathsf{rad}\, A$ such that $\delta = \delta_0 + \delta_+$. If $r = \alpha_1 \cdots \alpha_N \in R$, then $d_2^*(\delta_+)(r) = \sum_{i=1}^{N} \alpha_1 \cdots \delta_+(\alpha_i) \cdots \alpha_N$ is a linear combination of paths of length at least N, so that in fact $d_2^*(\delta_+) = 0$ and, in particular, we have $d_2^*(\delta_0) = 0$. On the other hand, if we write ΩQ the set of loops in Q, there is a function $\lambda : \Omega Q \to \Bbbk$ such that for all $\alpha \in Q_1$ we have

$$\delta_0(\alpha) = \begin{cases} \lambda(\alpha)s(\alpha), & \text{if } \alpha \in \Omega Q; \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha \in \Omega Q$, then $\alpha^N \in R$ and $d_2^*(\delta_0)(\alpha^N) = N\lambda(\alpha)\alpha^{N-1} = 0$, so that $N\lambda(\alpha) = 0$. As Q has more than one vertex and is connected, there is an arrow $\beta \in Q_1 \setminus \Omega Q$ such that one of $\alpha^{N-1}\beta$ or $\beta\alpha^{N-1}$ is in R, and then either $d_2^*(\delta_0)(\alpha^{N-1}\beta) = (N-1)\lambda(\alpha)\alpha^{N-2}\beta = 0$ or $d_2^*(\delta_0)(\beta\alpha^{N-1}) = (N-1)\lambda(\alpha)\beta\alpha^{N-2} = 0$, and therefore $(N-1)\lambda(\alpha) = 0$. It follows that $\lambda(\alpha) = 0$ and we see that in fact $\delta_0 = 0$. $\qquad\square$

We now fix an E^e-linear map $\delta : \Bbbk Q_1 \to \mathsf{rad}\, A$ and assume moreover that δ is ***homogeneous***, so that there is an $l \in \{1, \ldots, N-1\}$ such that the image of δ is in $\Bbbk Q_l$; the ***degree*** of δ is then the number $l - 1$. We will write the E^e-linear derivation $A \to A$ which extends the 1-cocycle δ by the same letter. If $n, m \geq 0$, there is a unique E^e-linear map $\Delta_n^m : \Bbbk Q \to A \otimes_E \Bbbk Q_m \otimes_E A$ such that for each path $w \in Q_*$ we have $\Delta_n^m(w) = 0$ if $|w| < n + m$ and

$$\Delta_n^m(w) = a \otimes u \otimes b,$$

with aub the unique factorization of w with $|a| = n$ and $|u| = m$. A verification shows that there is a δ^e-lifting $\delta_\bullet : \mathsf{Br}_\bullet \to \mathsf{Br}_\bullet$ of δ to the complex $\mathsf{Br}_\bullet$ such that for each $i \geq 0$ and each $w = \alpha_1 \cdots \alpha_{\zeta(i)} \in Q_{\zeta(i)}$ we have

$$\delta_i(1 \otimes w \otimes 1) = \begin{cases} \Delta_{l-1}^{\zeta(i)}(\delta(w)), & \text{if } i \text{ is even;} \\ \Delta_{l-1}^{\zeta(i)}(\delta(w)) + \displaystyle\sum_{j=0}^{l-2} \Delta_j^{\zeta(i)}(\alpha_1 \cdots \alpha_{\zeta(i)-1}\delta(\alpha_{\zeta(i)})), & \text{if } i \text{ is odd.} \end{cases}$$

To exemplify how we can use this, we propose to describe the Lie action of $HH^1(A)$ on the cohomology $HH^\bullet(A)$. To do this we need some information on this cohomology, of

11

course. In low degrees, it was computed by Claude Cibils in [3] and this was extended to all degrees by Ana Locateli in [10] when the ground field has characteristic zero and by Yunge Xu, Yang Han and Wenfeng Jiang for the general case in [20]. What these authors do, though, is to find the dimensions of the cohomology groups and for our purpose this is not enough, as we need the actual cocycles. To keep things simple, we will content ourselves with a special case and with obtaining information only on the even part of cohomology.

Lemma. *Suppose that Q has no sources and no sinks and that it is not an oriented cycle. If $k \geq 1$, then the subspace of $2k$-cocycles in the complex $\hom_{A^e}(\mathsf{Br}_\bullet, A)$ is $\hom_{E^e}(\Bbbk Q_{Nk}, \Bbbk Q_{N-1})$.*

Here the simplifying assumption is that there are no sources or sinks, while the exclusion of the case of an oriented cycle is due to the fact that this is really an exceptional case. Using this lemma we can easily obtain the promised result:

Corollary. *A homogeneous derivation of nonzero degree acts by zero on the even part $HH^{\mathsf{even}}(A)$ of the Hochschild cohomology.*

Proof. It is enough to check that if $l \in \{2, \ldots, N-\}$ and $\delta : \Bbbk Q_1 \to \Bbbk Q_l$ is a homogeneous cocycle of degree $l - 1$ and $\delta_\bullet$ is the lifting of δ to the complex $\mathsf{Br}_\bullet$ described above, then $\delta_{2k}^\sharp(\phi) = 0$ for all $k \geq 1$ and all $\phi \in \hom_{E^e}(\Bbbk Q_{Nk}, \Bbbk Q_{N-1})$, since the lemma tells us that these are all the $2k$-cocycles. This is a trivial computation. $\qquad\square$

Proof of the lemma. If $\phi : \Bbbk Q_{Nk} \to A$ is a $2k$-cochain in that complex, there are E^e-linear maps $\phi_i : \Bbbk Q_{Nk} \to \Bbbk Q_l$ for each $i \in \{0, \ldots, N-1\}$ such that $\phi = \sum_{i=0}^{N-1} \phi_i$, and it is clear, since the algebra is monomial, that ϕ is a cocycle if and only if all the ϕ_i are. Now, an E^e-linear map $\Bbbk Q_{Nk} \to \Bbbk Q_{N-1}$ is automatically a cocycle, so the lemma will be proved if we show that

> if $0 \leq l < N - 1$ and $\phi : \Bbbk Q_{Nk} \to \Bbbk Q_l$ is an E^e-linear map which is a
> $2k$-cocycle in the complex $\hom_{A^e}(\mathsf{Br}_\bullet, A)$, then $\phi = 0$.

To do that, let us fix an integer l such that $0 \leq l < N - 1$ and an E^e-linear map $\phi : \Bbbk Q_{Nk} \to \Bbbk Q_l$ which is a $2k$-cocycle with values in $\Bbbk Q_l$. This means, precisely, that for each path $\alpha_1 \ldots \alpha_{Nk+1} \in Q_{Nk+1}$ we have

$$\phi(\alpha_1 \cdots \alpha_{Nk})\alpha_{Nk+1} = \alpha_1\phi(\alpha_2 \cdots \alpha_{Nk+1}). \tag{3}$$

Let r be an integer such that $0 \leq 2r \leq l$ and suppose that

> *for each $w = \alpha_1 \cdots \alpha_{Nk} \in Q_{Nk}$ there exists a $\bar\phi(w) \in \Bbbk Q_{l-2r}$ which
> is a linear combination of paths from $s(\alpha_{r+1})$ to $t(\alpha_{Nk-r})$ such that* $\tag{4}$
> $\phi(w) = \alpha_1 \cdots \alpha_r \bar\phi(w)\alpha_{Nk-r+1} \cdots \alpha_{Nk}.$

Notice that this holds when $r = 0$. Let $w = \alpha_1 \cdots \alpha_{Nk} \in Q_{Nk}$. As there are no sinks in Q, there is an arrow $\alpha_{Nk+1} \in Q_1$ such that $w\alpha_{Nk+1}$ is a path. If we put $w' = \alpha_2 \cdots \alpha_{Nk+1}$,

then we have from (3) and the hypothesis (4) that

$$\alpha_1 \cdots \alpha_r \bar{\phi}(w) \alpha_{Nk-r+1} \cdots \alpha_{Nk+1} = \alpha_1 \cdots \alpha_{r+1} \bar{\phi}(w') \alpha_{Nk-r+2} \cdots \alpha_{Nk+1},$$

so that in fact

$$\bar{\phi}(w) \alpha_{Nk-r+1} = \alpha_{r+1} \bar{\phi}(w'). \tag{5}$$

If $l - 2r \geq 2$, this implies that all the paths appearing in $\bar{\phi}(w)$ start with α_{r+1} and, by symmetry, they also end in α_{Nk-r}. In other words, there exists a $\bar{\bar{\phi}}(w) \in \Bbbk Q_{l-2r-2}$ which is a sum of paths from $s(\alpha_{r+2})$ to $t(\alpha_{Nk-r-1})$ such that $\bar{\phi}(w) = \alpha_{r+1} \bar{\bar{\phi}}(w) \alpha_{Nk-r}$. In this case we have that the condition (4) holds with r replaced with $r+1$, and we may therefore proceed inductively.

If instead $l - 2r = 1$, then the equation (5) tells us there exists a scalar $\lambda(w) \in \Bbbk$ which is zero if $\alpha_{r+1} \neq \alpha_{Nk-r}$ such that $\bar{\phi}(w) = \lambda(w) \alpha_{r+1}$, so in this case we have $\phi(w) = \lambda(w) \alpha_1 \cdots \alpha_r \alpha_{r+1} \alpha_{Nk-r+1} \cdots \alpha_{Nk}$. Finally, if $l - 2r = 0$, then the equation (5) implies that there is a scalar $\lambda(w) \in \Bbbk$ which is zero if $\alpha_{r+1} \neq \alpha_{Nk-r+1}$ and such that $\bar{\phi}(w) = \lambda(w) e_{s(\alpha_{r+1})}$, and therefore $\phi(w) = \lambda(w) \alpha_1 \cdots \alpha_r \alpha_{Nk-r+1} \cdots \alpha_{Nk}$.

In this way we conclude that there is a function $\lambda : Q_{Nk} \to \Bbbk$ such that for each $w = \alpha_1 \cdots \alpha_{Nk} \in Q_{Nk}$ we have

$$\phi(w) = \begin{cases} \lambda(w) \alpha_1 \cdots \alpha_r \alpha_{Nk-r+1} \cdots \alpha_{Nk}, & \text{if } l = 2r \text{ is even;} \\ \lambda(w) \alpha_1 \cdots \alpha_r \alpha_{r+1} \alpha_{Nk-r+1} \cdots \alpha_{Nk}, & \text{if } l = 2r+1 \text{ is odd;} \end{cases}$$

with

$$\lambda(w) = 0 \text{ if } \alpha_{r+1} \neq \alpha_{Nk-r+1} \text{ and } l = 2r \text{ is even, or if } \alpha_{r+1} \neq \alpha_{Nk-r} \\ \text{and } l = 2r+1 \text{ is odd.} \tag{6}$$

We define a relation $\sim$ on the set Q_{Nk} so that for each $w, w' \in Q_{Nk}$ we have $w \sim w'$ iff there exist arrows $\alpha, \beta \in Q_1$ such that $w\alpha = \beta w'$, and let $\approx$ be the least equivalence relation on Q_{Nk} coarser that $\sim$. Now, if $w, w' \in Q_{Nk}$ are such that $w \sim w'$, there is a path $\alpha_1 \cdots \alpha_{Nk+1} \in Q_{Nk+1}$ such that $w = \alpha_1 \cdots \alpha_{Nk}$ and $w' = \alpha_2 \cdots \alpha_{Nk+1}$. From equation (3) we have

$$\lambda(w) \alpha_1 \cdots \alpha_r \alpha_{Nk-r+1} \cdots \alpha_{Nk+1} = \lambda(w') \alpha_1 \cdots \alpha_{r+1} \alpha_{Nk-r+2} \cdots \alpha_{Nk+1}$$

if $l = 2r$ is even, and

$$\lambda(w) \alpha_1 \cdots \alpha_r \alpha_{r+1} \alpha_{Nk-r+1} \cdots \alpha_{Nk+1} = \lambda(w') \alpha_1 \cdots \alpha_{r+1} \alpha_{r+2} \alpha_{Nk-r+2} \cdots \alpha_{Nk+1}$$

if $l = 2r+1$ is odd. In any of the two cases we find that $\lambda(w) = \lambda(w')$, and it follows from this that λ is constant on the equivalence classes of the relation $\approx$.

We claim that in fact

$$\textit{there is only one equivalence class for the relation } \approx. \tag{7}$$

13

There is a preorder $\preceq$ on the set of vertices Q_0 such that whenever $i, j \in Q_0$ we have $i \preceq j$ iff there is a path in Q from j to i, and associated to $\preceq$ there is an equivalence relation $\asymp$ on Q_0 such that $i \asymp j$ iff $i \preceq j$ and $j \preceq i$. The equivalence classes of $\asymp$ are the ***strongly connected components*** of the quiver and the preorder $\preceq$ induces an actual order on the quotient $Q_0/\asymp$; in particular, we may speak of maximal and minimal strongly connected components. Our claim (7) now follows easily from the following two facts:

- If $w \in Q_{Nk}$, there is a path $w' \in Q_{Nk}$ which is totally contained in a maximal strongly connected component of Q and such that $w \approx w'$, and the same is true replacing 'maximal' by 'minimal'.
- If w and w' are elements of Q_{Nk} totally contained in possibly different maximal strongly connected components of Q, then $w \approx w'$.

Let us prove the first one, leaving the other for the reader. Let $w \in Q_{Nk}$. Let $i \in Q_0$ be a vertex in a strongly connected component C of Q_0 which is maximal among those elements in $Q_0/\asymp$ greater than the one containing $s(w)$. As i is not a source, there exists an arrow α with $t(\alpha) = i$; since the component C is maximal, there is a path u from i to $s(\alpha)$ which never leaves C and, since αu is a closed path starting at i, we see that there exists a path $w' \in Q_{Nk}$ contained in C and ending in i. On the other hand, the choice of i implies that there exists a path w_1 in Q going from i to $s(w)$. Considering all the factors of length Nk of the path $w'w_1 w$, we see at once that $w' \approx w$, as we wanted.

It follows now from (7) that the function $\lambda : Q_{Nk} \to \Bbbk$ is constant. As Q is not an oriented cycle, there exists a vertex $i \in Q_0$ and two different arrows α and α' such that either $s(\alpha) = s(\alpha') = i$ or $t(\alpha) = t(\alpha') = i$. Suppose, for example, that we are in the first case; the other can be handled in the same way. Since there are no sources and sinks in Q, if $l = 2r$ is even, there are paths $u \in Q_{Nk-r}$, and $v, v' \in Q_{r-1}$, and if $l = 2r + 1$ is odd, there are paths $u \in Q_{Nk-r-1}$ and $v, v' \in Q_r$ such that, in either case, $w = u\alpha v$ and $w' = u\alpha' v'$ are paths of length Nk. In view of our observation (6), at least one of the scalars $\lambda(w)$ and $\lambda(w')$ is zero. With this we can therefore conclude that $\phi = 0$. □

3.4. In the presence of sinks and sources, the homogeneous derivations of positive degree of a truncated algebra may well act non-trivially, as the following simple example shows. Fix $N \geq 3$ and $k \geq 4$, consider the quiver Q
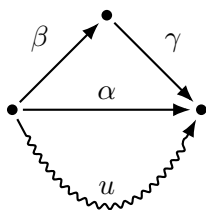
in which α, β and γ are arrows and u is a path of length $\zeta(k)$, and let A be the quotient of the path algebra $\Bbbk Q$ by the ideal generated by Q_N. Identifying $HH^\bullet(A)$ with the cohomology of the complex $\mathsf{hom}_{A^e}(\mathsf{Br}_\bullet, A)$, we see at once that $HH^0(A) \cong \Bbbk$, that

$HH^1(A)$ is the vector space spanned by two linearly independent diagonal derivations and the E^e-linear derivation $\delta : A \to A$, homogeneous of degree 1, such that $\delta(\alpha) = \beta\gamma$ and $\delta(\omega) = 0$ for all arrows ω different from α, that $HH^k(A)$ is 2-dimensional, spanned by the E^e-linear maps ϕ_1, $\phi_2 : \Bbbk Q_{\zeta(k)} \to A$ such that $\phi_1(u) = \alpha$ and $\phi_2(u) = \beta\gamma$, and that all other cohomology groups are zero. Moreover, computing the Lie action of $HH^1(A)$ of $HH^k(A)$ using the liftings constructed above for truncated algebras shows immediately that $[\delta_2, \phi_0] = -\phi_1$. In particular, the derivation δ acts non-trivially on $HH^k(A)$.

Crossed products

3.5. Let A be an algebra and let G be a finite group acting on A; we will suppose throughout that our ground field ring is a field in which the order of G is invertible and which splits G. We may construct the cross product algebra $A \rtimes G$ which as a vector space is $A \otimes \Bbbk G$, with $\Bbbk G$ the group algebra of G, and where multiplication is such that $a \otimes g \cdot b \otimes g = ag(b) \otimes gh$.

The group acts diagonally on the enveloping algebra A^e, so we can consider also the crossed product $A^e \rtimes G$. An $A^e \rtimes G$-module structure on a vector space M may be described as an A^e-module structure on which G acts in a compatible way, in the sense that $g(amb) = g(a)g(m)g(b)$ for all $g \in G$, a, $b \in A$ and $m \in M$. If M is such an $A^e \rtimes G$-module, we denote $M \rtimes G$ the $(A \rtimes G)^e$-module with underlying vector space $M \otimes \Bbbk G$ and left and right actions by $A \rtimes G$ given by

$$a \otimes g \cdot m \otimes h = ag(m) \otimes gh, \qquad\qquad m \otimes h \cdot a \otimes g = mh(a) \otimes hg,$$

respectively, whenever $a \otimes g \in A \rtimes G$ and $m \otimes h \in M \rtimes G$. On the other hand, if M is an $(A \rtimes G)^e$-module, we denote M^{ad} the $A^e \rtimes G$-module which coincides with M as an A^e-module and on which G acts so that

$$g \cdot m = gmg^{-1}$$

for all $g \in G$ and all $m \in M$. These two constructions are related in the following way:

Lemma. *If M is an $A^e \rtimes G$-module and N is an $(A \rtimes G)^e$-module, there are isomorphisms*

$$\mathsf{hom}_{(A \rtimes G)^e}(M \rtimes G, N) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{hom}_{A^e}(M, N^{\mathsf{ad}})^G$$

natural in M and N.

On the right we are taking invariants with respect to the action of G on $\mathsf{hom}_{A^e}(M, N^{\mathsf{ad}})$ given by $(g \cdot f)(m) = g \cdot f(g^{-1} \cdot m)$ for each $g \in G$ and each A^e-linear map $f : M \to N^{\mathsf{ad}}$.

Proof. We may put $\Phi(f)(m) = f(m \otimes 1)$ for all $f \in \mathsf{hom}_{(A \rtimes G)^e}(M \rtimes G, N)$ and all $m \in M$, and $\Psi(f)(m \otimes g) = f(m)g$ for all $f \in \mathsf{hom}_{A^e}(M, N^{\mathsf{ad}})$ and all $m \otimes g \in M \rtimes G$. $\qquad\square$

The following result is an immediate consequence of the lemma, since the functor $(-)^G$ which computes invariants is exact.

Corollary. *If P is an $A^e \rtimes G$-module which is projective as an A^e-module, then $P \rtimes G$ is a projective $(A \rtimes G)^e$-module.* $\qquad\square$

We view A as an $A^e \rtimes G$-module in the obvious way and let $P_\bullet \to A$ be a resolution of A as an $A^e \rtimes G$-module by modules which are projective as A^e-modules; such a resolution exists: for example, as $A^e \rtimes G$ is projective as a left A^e-module, it suffices to take $P_\bullet$ to be an $A^e \rtimes G$-projective resolution of A, but one can often be much more economical. The corollary implies then that $P_\bullet \rtimes G \to A \rtimes G$ is a projective resolution of $A \rtimes G$ as an $(A \rtimes G)^e$-module. In particular the cohomology of the complex $\mathsf{hom}_{(A \rtimes G)^e}(P_\bullet \rtimes G, A \rtimes G)$ can be identified to the Hochschild cohomology $HH^\bullet(A \rtimes G)$ of $A \rtimes G$. As this complex is, according to the lemma, naturally isomorphic to $\mathsf{hom}_{A^e}(P_\bullet, A \rtimes G)^G$ and since G acts semisimply, taking homology in this second complex we see that $HH^\bullet(A \rtimes G)$ is isomorphic to $H^\bullet(A, A \rtimes G)^G$; this result is usually obtained using the spectral sequence constructed by Dragoş Ştefan in [17], but for our purposes we need the isomorphism to come out of an actual resolution.

Let $\delta : A \rtimes G \to A \rtimes G$ be a derivation of $A \rtimes G$. The restriction $\delta|_{\Bbbk G} : \Bbbk G \to A \rtimes G$ is a derivation of the group algebra $\Bbbk G$ with values in the $\Bbbk G$-bimodule $A \rtimes G$ and therefore, since $\Bbbk G$ is a separable algebra, this restriction is inner: there exists an element $u \in A \rtimes G$ such that $\delta(g) = [u, g]$ for all $g \in G$. It follows that the derivation $\delta - [u, -] : A \rtimes G \to A \rtimes G$, which is cohomologous to δ, is ***normalized***, that is, it vanishes on G and we conclude that, up to inner derivations, we can assume that derivations of $A \rtimes G$ are normalized.

3.6. We specialize now the discussion to the following situation. Let G be a finite group, let $\rho : G \to \mathsf{GL}(V)$ be a representation of G on a finite dimensional vector space V, and consider the corresponding action of G on the symmetric algebra $S(V)$ and the associated crossed product $S(V) \rtimes G$. We then have available the bimodule Koszul resolution $K_\bullet = S(V) \otimes \Lambda^\bullet V \otimes S(V) \to S(V)$ of $S(V)$ as an $S(V)^e$-module, and it turns out, when we endow $K_\bullet$ with its natural action of G, that the resolution is a complex of $S(V)^e \rtimes G$-modules.

As explained above, the complex $\mathsf{hom}_{(S(V) \rtimes G)^e}(K_\bullet \rtimes G, S(V) \rtimes G)$, which computes the Hochschild cohomology $HH^\bullet(S(V) \rtimes G)$, is isomorphic to $\mathsf{hom}_{S(V)^e}(K_\bullet, S(V) \rtimes G)^G$ which, in turn, can be identified with the complex $\mathsf{hom}(\Lambda^\bullet(V), S(V) \rtimes G)^G$. There is, moreover, a decomposition

$$\mathsf{hom}(\Lambda^\bullet(V), S(V) \rtimes G) = \bigoplus_{g \in G} \mathsf{hom}(\Lambda^\bullet(V), S(V) \rtimes g) \tag{8}$$

If $g \in G$, we let V^g be the fixed subspace of g in V, V_g the subspace of V spanned by eigenvectors of g corresponding to eigenvalues different from 1, and $d(g) = \mathsf{dim}\, V_g$. As $V = V^g \oplus V_g$ and this decomposition is preserved by g, we have $S(V) = S(V^g) \otimes S(V_g)$, $\Lambda^\bullet(V) = \Lambda^\bullet(V^g) \otimes \Lambda^\bullet(V_g)$, and there is an obvious map of complexes,

$$\vee : \mathsf{hom}(\Lambda^\bullet(V^g), S(V^g)) \otimes \mathsf{hom}(\Lambda^\bullet(V_g), S(V_g) \rtimes g) \to \mathsf{hom}(\Lambda^\bullet(V), S(V) \rtimes g),$$

which can be seen to be a quasi-isomorphism; this is part of the content of Theorem XI.3.1 in [2], for example. The complex $\mathsf{hom}(\Lambda^\bullet(V^g), S(V^g))$ has zero differential; on the other hand, the cohomology of the complex $\mathsf{hom}(\Lambda^\bullet(V_g), S(V_g) \rtimes g)$ is zero except in degree $d(g)$, where it is one dimensional and spanned by the cohomology class of any non-zero linear map $\omega_g : \Lambda^{d(g)}(V_g) \to \Bbbk g$, as shown in Lemma 3.4 of [4]; we fix one such map once and for all. All this means that the inclusion of complexes

$$\mathsf{hom}(\Lambda^\bullet(V^g), S(V^g)) \vee \omega_g \hookrightarrow \mathsf{hom}(\Lambda^\bullet(V), S(V) \rtimes g)$$

is a quasi-isomorphism, with the subcomplex having zero differential. Since everything in sight is G-equivariant, and taking into account the decomposition (8), the same is true of the inclusion

$$\left(\bigoplus_{g \in G} \mathsf{hom}(\Lambda^\bullet(V^g), S(V^g)) \vee \omega_g \right)^G \hookrightarrow \mathsf{hom}(\Lambda^\bullet(V), S(V) \rtimes G)^G.$$

If g, h are in G, then $h \cdot \omega_g$ and $\omega_{hgh^{-1}}$ are two non-zero elements of the 1-dimensional vector space $\mathsf{hom}(\Lambda^{d(hgh^{-1})}(V_{hgh^{-1}}), \Bbbk hgh^{-1})$, so there exists a scalar $\lambda(h, g) \in \Bbbk^\times$ such that $h \cdot \omega = \lambda(h, g)\omega_{hgh^{-1}}$. In this way we find a function $\lambda : G \times G \to \Bbbk^\times$ and the associativity of the action of G implies that $\lambda(gh, k) = \lambda(h, k)\lambda(g, hkh^{-1})$ for all g, h, $k \in G$.

If $g \in G$, we let G_g be the centralizer of g in G. The map $\chi_g : h \in G_g \mapsto \lambda(h, g) \in \mathbb{C}^\times$ is a group morphism. The group G acts on $\mathsf{hom}(\Lambda^\bullet(V^g), S(V^g))$, so we may restrict that action to G_c and consider the subspace of *semi-invariants* $\mathsf{hom}(\Lambda^\bullet(V^g), S(V^g))^{G_g}_{\chi_g}$, that is, of the elements $f \in \mathsf{hom}(\Lambda^\bullet(V^g), S(V^g))$ such that $h \cdot f = \chi_g(h)f$ for all $h \in C_g$. Let now $\langle G \rangle$ be the set of conjugacy classes of G and for each $c \in \langle G \rangle$ let g_c be a fixed element of c. Then we have an isomorphism Φ_c

$$\Phi_c : \mathsf{hom}(\Lambda^\bullet(V^{g_c}), S(V^{g_c}))^{G_{g_c}}_{\chi_{g_c}}[-d(g)] \to \left(\bigoplus_{g \in c} \mathsf{hom}(\Lambda^\bullet(V^g), S(V^g)) \vee \omega_g \right)^G$$

such that for each $f \in \mathsf{hom}(\Lambda^\bullet(V^{g_c}), S(V^{g_c}))^{G_{g_c}}_{\chi_{g_c}}$ we have

$$\Phi_c(f) = \sum_{g \in G} \lambda(g, g_c)g(f) \vee \omega_{gg_c g^{-1}}.$$

It follows now that we have an isomorphism

$$\bigoplus_{c \in \langle G \rangle} \Phi_c : \bigoplus_{c \in \langle G \rangle} \mathsf{hom}(\Lambda^\bullet(V^{g_c}), S(V^{g_c}))^{G_{g_c}}_{\chi_{g_c}}[-d(g)] \to \left(\bigoplus_{g \in G} \mathsf{hom}(\Lambda^\bullet(V^g), S(V^g)) \vee \omega_g \right)^G,$$

and in this way we arrive at a well-known and very explicit description of the Hochschild cohomology of the crossed product algebra $S(V) \rtimes G$,

$$HH^\bullet(S(V) \rtimes G) = \bigoplus_{c \in \langle G \rangle} \mathsf{hom}(\Lambda^\bullet(V^{g_c}), S(V^{g_c}))^{G_{g_c}}_{\chi_{g_c}}[-d(g_c)]$$

17

originally obtained by Marco Farinati in [4] and Victor Ginzburg and Dmitry Kaledin in [8]; the paper [16] is a good reference for this, too. In particular, if we let $\langle G \rangle_1$ be the set of conjugacy classes $c \in \langle G \rangle$ such that $d(g_c) = 1$, we have

$$HH^1(S(V) \rtimes G) = \mathsf{hom}(V, S(V))^G \oplus \bigoplus_{c \in \langle G \rangle_1} S(V^{g_c})^{G_{g_c}}_{\chi_{g_c}}$$

and something nice happens: if $c \in \langle G \rangle_1$, then $\chi_{g_c}(g_c) = \mathsf{det}\,\rho(g_c) \neq 1$ and therefore $S(V^{g_c})^{G_{g_c}}_{\chi_{g_c}} = 0$, since $g_c \in G_{g_c}$. We thus see that, in fact,

$$HH^1(S(V) \rtimes G) = \mathsf{hom}(V, S(V))^G.$$

Tracing back the isomorphisms involved, we can describe this identification explicitly as follows. If $r : V \to S(V)$ is a G-equivariant linear map, then one of the universal properties of the symmetric algebra $S(V)$ implies that there is a unique derivation $\bar{r} : S(V) \to S(V)$ which extends r and it turns out to be G-equivariant. There is then a normalized derivation $\delta_r : S(V) \rtimes G \to S(V) \rtimes G$ such that $\delta_r(fg) = \bar{r}(f)g$ for all $f \in S(V)$ and $g \in G$. The class of this δ_r is the element of $HH^1(S(V) \rtimes G)$ corresponding to the map r.

3.7. We are finally in position to describe the Lie module structure of $HH^\bullet(S(V) \rtimes G)$ over the Lie algebra $HH^1(S(V) \rtimes G)$ using our results from Section 2. We fix a G-equivariant map $r : V \to S(V)$ and let $\delta = \delta_r : S(V) \rtimes G \to S(V) \rtimes G$ be the corresponding derivation described above. Let $T(V)$ be the tensor algebra on V and denote $\pi : T(V) \to S(V)$ the natural surjection. Since π is G-equivariant, it admits a G-equivariant section $\sigma : S(V) \to T(V)$. On the other hand, and using now a universal property of the tensor algebra, there exists a unique linear derivation $D : T(V) \to S(V) \otimes V \otimes S(V)$ of the algebra $T(V)$ with values in $S(V) \otimes V \otimes S(V)$ endowed with it obvious $T(V)$-bimodule structure such that $D(v) = 1 \otimes v \otimes 1$ for all $v \in V$, and it is G-equivariant. If $v \in V$, we will write the element $D(\sigma(r(v)))$ of $S(V) \otimes V \otimes S(V)$ in the form $v_{(1)} \otimes v_{(2)} \otimes v_{(3)}$ with an implicit sum, à la Sweedler. There is a δ^e-lifting $\delta_\bullet : K_\bullet \rtimes G \to K_\bullet \rtimes G$ of δ to the resolution $K_\bullet \rtimes G$ of $S(V) \rtimes G$ such that

$$\delta_p(1 \otimes v_1 \wedge \cdots \wedge v_p \otimes 1 \rtimes 1) = \sum_{i=1}^{p} v_{i(1)} \otimes v_1 \wedge \cdots \wedge v_{i(2)} \wedge \cdots \wedge v_p \otimes v_{i(3)} \rtimes 1,$$

as one can check by direct computation. From this lifting we can construct the map of complexes

$$\delta^\sharp_\bullet : \mathsf{hom}_{(S(V) \rtimes G)^e}(K_\bullet \rtimes G, S(V) \rtimes G) \to \mathsf{hom}_{(S(V) \rtimes G)^e}(K_\bullet \rtimes G, S(V) \rtimes G)$$

which up to natural isomorphisms in the lemma is identified with the map

$$\delta^\sharp_\bullet : \mathsf{hom}(\Lambda^\bullet(V), S(V) \rtimes G)^G \to \mathsf{hom}(\Lambda^\bullet(V), S(V) \rtimes G)^G$$

18

given in each degree $p \geq 0$ by

$$(\delta_p^\sharp \phi)(v_1 \wedge \cdots \wedge v_p) = \delta(\phi(v_1 \wedge \cdots \wedge v_p)) - \sum_{i=1}^{p} v_{i(1)} \phi(v_1 \wedge \cdots \wedge v_{i(2)} \wedge \cdots \wedge v_p) v_{i(3)}$$

for all $\phi \in \mathsf{hom}(\Lambda^p(V), S(V) \rtimes G)^G$. The right hand side in this equation therefore a representative for the Gerstenhaber bracket $[\delta, \phi]$.

4   Tensor products, Tor and Hochschild homology

4.1. Let A be an algebra and $\delta : A \to A$ a derivation, as before. Let M and N be a right and a left A-module, respectively, and let $f : M \to M$ and $g : N \to N$ be δ-operators on M and N. There is a linear map $f \boxtimes g : M \otimes_A N \to M \otimes_A N$ such that $(f \boxtimes g)(m \otimes n) = f(m) \otimes n + m \otimes g(n)$ for all $m \in M$ and all $n \in N$, and this map depends naturally on the data used to construct it in the obvious sense.

If $\eta : Q_\bullet \to N$ is a projective resolution of N as a left A-module and $g_\bullet : Q_\bullet \to Q_\bullet$ is a δ-lifting of g to $Q_\bullet$, then we may consider the complex $M \otimes_A Q_\bullet$ and the morphism $f \boxtimes g_\bullet : M \otimes_A Q_\bullet \to M \otimes_A Q_\bullet$ which in each homological degree $i \geq 0$ is given by $f \boxtimes g_i$ and which induces upon passing to homology a map

$$H(f \boxtimes g_\bullet) : H(M \otimes_A Q_\bullet) \to H(M \otimes_A Q_\bullet).$$

There is an analogue of Lemma **1.6** and therefore proceeding as we did to prove Theorem **A** we obtain:

Theorem B. *If M and N are a right and a left A-module, respectively, and $f : M \to M$ and $g : N \to N$ are δ-operators, then there is a canonical morphism of graded vector spaces*

$$\nabla_\bullet^{f,g} : \mathsf{Tor}_\bullet^A(M, N) \to \mathsf{Tor}_\bullet^A(M, N)$$

such that for each projective resolution $\eta : Q_\bullet \to N$ and each δ-lifting $g_\bullet : Q_\bullet \to Q_\bullet$ of g to $Q_\bullet$ the diagram

$$
\begin{array}{ccc}
H(M \otimes_A Q_\bullet) & \xrightarrow{\;\nabla_\bullet^{f,g,Q_\bullet}\;} & H(M \otimes_A Q_\bullet) \\
\cong \downarrow & & \downarrow \cong \\
\mathsf{Tor}_\bullet^A(M, N) & \xrightarrow{\;\nabla_\bullet^{f,g}\;} & \mathsf{Tor}_\bullet^A(M, N)
\end{array}
$$

commutes. $\qquad\qquad\square$

If in the situation of the theorem we also have a projective resolution $\varepsilon : P_\bullet \to M$ of M and a δ-lifting $f_\bullet : P_\bullet \to P_\bullet$ of f to $P_\bullet$, we may consider the (total complex of

the) tensor product $P_\bullet \otimes_A Q_\bullet$ and on it the linear map $f_\bullet \boxtimes g_\bullet : P_\bullet \otimes_A Q_\bullet \to P_\bullet \otimes_A Q_\bullet$ constructed in the obvious way. As the diagram

$$
\begin{array}{ccc}
P_\bullet \otimes_A Q_\bullet & \xrightarrow{\ f_\bullet \boxtimes g_\bullet\ } & P_\bullet \otimes_A Q_\bullet \\
{\scriptstyle \varepsilon \otimes \mathrm{id}_{Q_\bullet}}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon \otimes \mathrm{id}_{Q_\bullet}} \\
M \otimes_A Q_\bullet & \xrightarrow{\ f \boxtimes g_\bullet\ } & M \otimes_A Q_\bullet
\end{array}
$$

commutes, taking homology and observing that the morphism $\varepsilon \otimes \mathrm{id}_{Q_\bullet}$ induces the canonical isomorphism $H(P_\bullet \otimes_A Q_\bullet) \cong H(M \otimes_A Q_\bullet)$, we find that the diagram

$$
\begin{array}{ccc}
H(P_\bullet \otimes_A Q_\bullet) & \xrightarrow{\ H(f_\bullet \boxtimes g_\bullet)\ } & H(P_\bullet \otimes_A Q_\bullet) \\
{\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\
\mathsf{Tor}^A_\bullet(M,N) & \xrightarrow{\ \nabla^{f,g}_\bullet\ } & \mathsf{Tor}^A_\bullet(M,N)
\end{array}
$$

with vertical arrows the canonical isomorphisms, commutes and, proceeding symmetrically, that the same happens with

$$
\begin{array}{ccc}
H(P_\bullet \otimes_A N) & \xrightarrow{\ H(f_\bullet \boxtimes g)\ } & H(P_\bullet \otimes_A N) \\
{\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\
\mathsf{Tor}^A_\bullet(M,N) & \xrightarrow{\ \nabla^{f,g}_\bullet\ } & \mathsf{Tor}^A_\bullet(M,N)
\end{array}
$$

This means that the computation of the map $\nabla^{f,g}_\bullet$ is as "balanced" as that of Tor itself.

4.2. If $\delta^e = \delta \otimes 1 + 1 \otimes \delta : A^e \to A^e$ is the derivation of the enveloping algebra induced by δ and if we view $\delta : A \to A$ as a δ^e-operator both on the left A^e-module A, as in **2.2**, and on the right A^e-module A, and recalling that in our situation we can identify the Hochschild homology $HH_\bullet(A)$ with $\mathsf{Tor}^{A^e}_\bullet(A,A)$, the construction of Theorem **B** gives us a map

$$
\nabla^{\delta,\delta}_\bullet : HH_\bullet(A) \to HH_\bullet(A)
$$

which we can compute in terms of any projective resolution of A as an A^e-bimodule, provided we can δ^e-lift δ to it. If we write what this amounts to when we use the A^e-projective resolution $\varepsilon : B(A)_\bullet \to A$ and the δ^e-lifting described in **2.2**, we find immediately that the map $\nabla^{\delta,\delta}_\bullet$ induced on Hochschild homology by δ coincides with the one considered by Tom Goodwillie in [9] or Jean-Louis Loday in Section 4.1 of [11].

References

[1] M. J. Bardzell, *The alternating syzygy behavior of monomial algebras*, J. Algebra **188** (1997), no. 1, 69–89. MR1432347

[2] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956. MR0077480

[3] C. Cibils, *Rigidity of truncated quiver algebras*, Adv. Math. **79** (1990), no. 1, 18–42. MR1031825

[4] M. Farinati, *Hochschild duality, localization, and smash products*, J. Algebra **284** (2005), no. 1, 415–434. MR2115022

[5] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288. MR0161898

[6] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. (2) **79** (1964), 59–103. MR0171807

[7] M. Gerstenhaber and S. D. Schack, *Algebraic cohomology and deformation theory*, Deformation theory of algebras and structures and applications (Il Ciocco, 1986), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 247, Kluwer Acad. Publ., Dordrecht, 1988, pp. 11–264. MR981619

[8] V. Ginzburg and D. Kaledin, *Poisson deformations of symplectic quotient singularities*, Adv. Math. **186** (2004), no. 1, 1–57. MR2065506

[9] T. G. Goodwillie, *Cyclic homology, derivations, and the free loopspace*, Topology **24** (1985), no. 2, 187–215. MR793184

[10] A. C. Locateli, *Hochschild cohomology of truncated quiver algebras*, Comm. Algebra **27** (1999), no. 2, 645–664. MR1671958

[11] J.-L. Loday, *Cyclic homology*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, 1998. MR1600246

[12] C. Negron, *Alternate Approaches to the Cup Product and Gerstenhaber Bracket on Hochschild Cohomology*, Ph. D. thesis, University of Washington, Seattle, WA, United States, 2015. MR3438933

[13] C. Negron and S. Witherspoon, *An alternate approach to the Lie bracket on Hochschild cohomology* (2015), available at http://arxiv.org/abs/1406.0036v3.

[14] V. S. Retakh, *Homotopy properties of categories of extensions*, Uspekhi Mat. Nauk **41** (1986), no. 6(252), 179–180 (Russian). MR890505

[15] L. Román, *La cohomología de Hochschild de álgebras de cuerdas y su estructura de álgebra de Gerstenhaber*, Ph. D. thesis, Universidad Nacional del Sur, Bahía Blanca, Argentina, 2016.

[16] A. V. Shepler and S. Witherspoon, *Hochschild cohomology and graded Hecke algebras*, Trans. Amer. Math. Soc. **360** (2008), no. 8, 3975–4005. MR2395161

[17] D. Ştefan, *Hochschild cohomology on Hopf Galois extensions*, J. Pure Appl. Algebra **103** (1995), no. 2, 221–233. MR1358765

[18] S. Schwede, *An exact sequence interpretation of the Lie bracket in Hochschild cohomology*, J. Reine Angew. Math. **498** (1998), 153–172. MR1629858

[19] E. Sköldberg, *A contracting homotopy for Bardzell's resolution*, Math. Proc. R. Ir. Acad. **108** (2008), no. 2, 111–117. MR2475805

[20] Y.-g. Xu, Y. Han, and W.-f. Jiang, *Hochschild cohomology of truncated quiver algebras*, Sci. China Ser. A **50** (2007), no. 5, 727–736. MR2355359

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires. Ciudad Universitaria, Pabellón I (1428) Ciudad de Buenos Aires - Argentina
E-mail address: mariano@dm.uba.ar