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Exact partition function in $\mathrm{U}(2) \times \mathrm{U}(2)$ ABJM theory deformed by mass and Fayet-Iliopoulos terms

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Abstract: We exactly compute the partition function for $\mathrm{U}(2)_k \times \mathrm{U}(2)_{-k}$ ABJM theory on $\mathbb{S}^3$ deformed by mass m and Fayet-Iliopoulos parameter ζ. For $k = 1, 2$, the partition function has an infinite number of Lee-Yang zeros. For general k, in the decompactification limit the theory exhibits a quantum (first-order) phase transition at $m = 2\zeta$.

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1 Introduction

The dynamics of two coincident M2 branes on the orbifold $\mathbb{R}^8/\mathbb{Z}_k$ is described by ABJM theory, three-dimensional $\mathrm{U}(2)_k \times \mathrm{U}(2)_{-k}$ supersymmetric Chern-Simons theory with bi-fundamental matter [1]. For this particular gauge group, the ABJM theory has $\mathcal{N} = 8$ superconformal symmetry and is in fact equivalent to Gustavsson-Bagger-Lambert theory [2, 3]. The partition function for the theory on $\mathbb{S}^3$ can be computed by supersymmetric localization [4, 5]. This theory can be deformed, preserving $\mathcal{N} = 4$ supersymmetry, by adding mass and Fayet-Iliopoulos (FI) parameters m, ζ, and the localization technique then reduces the full supersymmetric functional integral to the matrix integral [5]

$$Z = \frac{1}{4} \int \frac{d^2\mu}{(2\pi)^2} \frac{d^2\nu}{(2\pi)^2} \frac{\sinh^2 \frac{\mu_1-\mu_2}{2} \sinh^2 \frac{\nu_1-\nu_2}{2}}{\prod\limits_{i,j} \cosh\left(\frac{\mu_i-\nu_j+m}{2}\right) \cosh\left(\frac{\mu_i-\nu_j-m}{2}\right)} e^{\frac{ik}{4\pi}\sum\limits_i(\mu_i^2-\nu_i^2)-\frac{ik}{2\pi}\zeta\sum\limits_i(\mu_i+\nu_i)} \tag{1.1}$$

where $i, j = 1, 2$. The parameter ζ represents a Fayet-Iliopoulos parameter for the diagonal U(1) subgroup, whereas m corresponds to a mass for the chiral multiplets. The partition function should be understood as a function $Z(2\zeta, m; k)$, but for ease of presentation we will omit its arguments unless needed. For $k = 1$, the theory is mirror dual to $\mathcal{N} = 4$ supersymmetric super Yang-Mills theory with gauge group U(2) coupled to a single fundamental hypermultiplet and a single adjoint hypermultiplet [5].

By shifting the integration variables, $x \equiv \mu - \zeta$, $y \equiv \nu + \zeta$, the partition function becomes

$$Z = \frac{1}{4} \int \frac{d^2x}{(2\pi)^2} \frac{d^2y}{(2\pi)^2} \frac{\sinh^2 \frac{x_1-x_2}{2} \sinh^2 \frac{y_1-y_2}{2}}{\prod\limits_{i,j} \cosh \frac{x_i-y_j+m_1}{2} \cosh \frac{x_i-y_j-m_2}{2}} \mathrm{e}^{\frac{ik}{4\pi}\sum\limits_i(x_i^2-y_i^2)}, \tag{1.2}$$

where m_1, m_2 are

$$m_1 = m + 2\zeta \quad \text{and} \quad m_2 = m - 2\zeta\,. \tag{1.3}$$

Note that ζ has dimension of mass. We are using units where the radius R of the three-sphere is $R = 1$.

The purpose of this note is to explicitly carry out the integration in (1.2). In the $m = \zeta = 0$ case, the integral was computed in [6] (a discussion of the partition function in the more general ABJ case can be found in [7]). On the other hand, the m, ζ-deformed ABJM theory was studied in [8] using the Fermi-gas formulation [9] and at at large N for the $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ gauge group in [10] (with $\zeta = 0$) and in [11] (with general m, $\zeta \neq 0$), where phase transitions in the complex parameter space generated by m_1, m_2 and $g = 2\pi i/k$ were investigated. Our explicit formula will uncover some interesting physical properties of the mass-deformed system with gauge group $\mathrm{U}(2)_k \times \mathrm{U}(2)_{-k}$.

The partition function (1.2) manifests the $m_1 \leftrightarrow m_2$ symmetry or, equivalently, $\zeta \to -\zeta$. A less obvious symmetry is $m_2 \to -m_2$, or [8, 11]

$$Z(2\zeta, m; k) = Z(m, 2\zeta; k)\,. \tag{1.4}$$

For the $k = 1$ case, this symmetry already appeared in [5], where it was also explained by the fact that the corresponding brane configuration is self-mirror. The symmetry implies, in particular, that a FI-deformation ζ on the massless theory is equivalent to a mass-deformation $m = 2\zeta$ in the theory with vanishing FI-parameter. The case $m = 2\zeta$ — representing a fixed point of this symmetry — is special, as we shall shortly see. In the dual $\mathcal{N} = 4$ supersymmetric super Yang-Mills theory, $m_2 = 0$ corresponds to coupling the theory to a massless adjoint hypermultiplet.

2 Residue integration

The partition function for the m, ζ-deformed ABJM theory with $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ gauge group can be written in the following form [5, 11]

$$Z(2\zeta, m; k) = \sum_\rho (-1)^\rho \frac{1}{N!} \int d^N\tau \frac{e^{-ikm_2 \sum_i \tau_i}}{\prod_i \cosh(k\pi\tau_i) \cosh\left(\pi(\tau_i - \tau_{\rho(i)}) - \frac{m_1}{2}\right)}\,, \tag{2.1}$$

where the sum goes over permutations. The derivation uses a trigonometric identity, Fourier integrations and only holds for opposite Chern-Simons levels (see section 2 in [11] for details). For $N = 2$, the formula (2.1) then leads to the following expression

$$Z = \frac{1}{2}(Z_1 - Z_2)\,, \tag{2.2}$$

with

$$Z_1 = \int d\tau_1 d\tau_2 \frac{e^{-ikm_2(\tau_1+\tau_2)}}{\cosh(\pi k\tau_1)\cosh(\pi k\tau_2)\cosh^2\left(\frac{m_1}{2}\right)}\,, \tag{2.3}$$

and

$$Z_2 = \int d\tau_1 d\tau_2 \frac{e^{-ikm_2(\tau_1+\tau_2)}}{\cosh(\pi k\tau_1)\cosh(\pi k\tau_2)\cosh\left(\pi(\tau_1-\tau_2) - \frac{m_1}{2}\right)\cosh\left(\pi(\tau_1-\tau_2) + \frac{m_1}{2}\right)}\,, \tag{2.4}$$

Using the identity

$$\frac{1}{\cosh^2\frac{m_1}{2}} - \frac{1}{\cosh\left(\pi\tau - \frac{m_1}{2}\right)\cosh\left(\pi\tau + \frac{m_1}{2}\right)} = \frac{\operatorname{sech}^2\frac{m_1}{2}\,\sinh^2\pi\tau}{\cosh\left(\pi\tau - \frac{m_1}{2}\right)\cosh\left(\pi\tau + \frac{m_1}{2}\right)} \tag{2.5}$$

and the formula for the Fourier transform [11]

$$\int du \frac{e^{-ikm_2 u}}{\cosh\left(\frac{\pi k}{2}(u+v)\right)\cosh\left(\frac{\pi k}{2}(u-v)\right)} = \frac{4\sin(km_2 v)}{k\sinh(\pi k v)\sinh m_2}\,, \tag{2.6}$$

we obtain

$$Z = \frac{1}{k^2\sinh(m_2)\cosh^2\frac{m_1}{2}}\int du \frac{\sin(m_2 u)\sinh^2\frac{\pi u}{k}}{\sinh(\pi u)\cosh\left(\frac{\pi u}{k}-\frac{m_1}{2}\right)\cosh\left(\frac{\pi u}{k}+\frac{m_1}{2}\right)}\,. \tag{2.7}$$

In the limit $m_2 \to 0$, the partition function becomes

$$Z\big|_{m_2=0} = \frac{1}{k^2\cosh^2\frac{m_1}{2}}\int du \frac{u\sinh^2\frac{\pi u}{k}}{\sinh(\pi u)\cosh\left(\frac{\pi u}{k}-\frac{m_1}{2}\right)\cosh\left(\frac{\pi u}{k}+\frac{m_1}{2}\right)}\,. \tag{2.8}$$

In the following, we compute the integrals (2.7), (2.8) by residue integration.

To compute (2.7) we follow the ideas in [6], where the partition function was computed in the case $m = \zeta = 0$.

Thus we start by writing the integrand as the product of two even functions f, g

$$Z = \frac{1}{k^2\sinh(m_2)\cosh^2\frac{m_1}{2}}\int du f(u)g(u)\,, \tag{2.9}$$

with

$$f(u) = \frac{\sin m_2 u}{\sinh \pi u}\,, \qquad g(u) = \frac{\sinh^2\frac{\pi u}{k}}{\cosh\left(\frac{\pi u}{k}-\frac{m_1}{2}\right)\cosh\left(\frac{\pi u}{k}+\frac{m_1}{2}\right)}\,. \tag{2.10}$$

Under the shift $u \to u + ik$ these functions transform as

$$\begin{aligned} &f(u) \to (-)^k\cosh(m_2 k) f(u) + \text{odd function}\,, \\ &g(u) \to g(u) \end{aligned} \tag{2.11}$$

These properties imply that the integral in (2.9) along the curve $u = x + ik$ with $x \in \mathbb{R}$ will differ from the integration along the real axis by the factor $(-)^k\cosh(m_2 k)$. Therefore, the rectangular contour composed by the real axis, two vertical segments and the displaced real axis $u = x + ik$ becomes appropriate for residue computation in the case $m_2 \neq 0$ (see figure 1).[1]

The residues encircled by the contour comprise the ones arising from the poles of $f(z)$ located at $z = in$ with $n = 1, \ldots, k$ and those of $g(z)$ located at $z_\pm = \pm\frac{m_1 k}{2\pi} + i\frac{k}{2}$. The pole located at $z = ik$ does not contribute due to a double zero in the numerator of $g(z)$. Calling C the closed rectangular contour described above and $\mathcal{F}(z) = f(z)g(z)$ one finds

$$\begin{aligned} \oint_C dz\,\mathcal{F}(z) &= \left(1 - (-)^k\cosh(m_2 k)\right)\int du\,\mathcal{F}(u) \\ &= 2\pi i\left[\sum_{n=1}^{k-1}\mathrm{Res}_{z=in}\,\mathcal{F}(z) + \mathrm{Res}_{z=z_\pm}\,\mathcal{F}(z)\right] \end{aligned}$$

[1] It is easily seen that the vertical contours do not contribute when we push them to infinity.

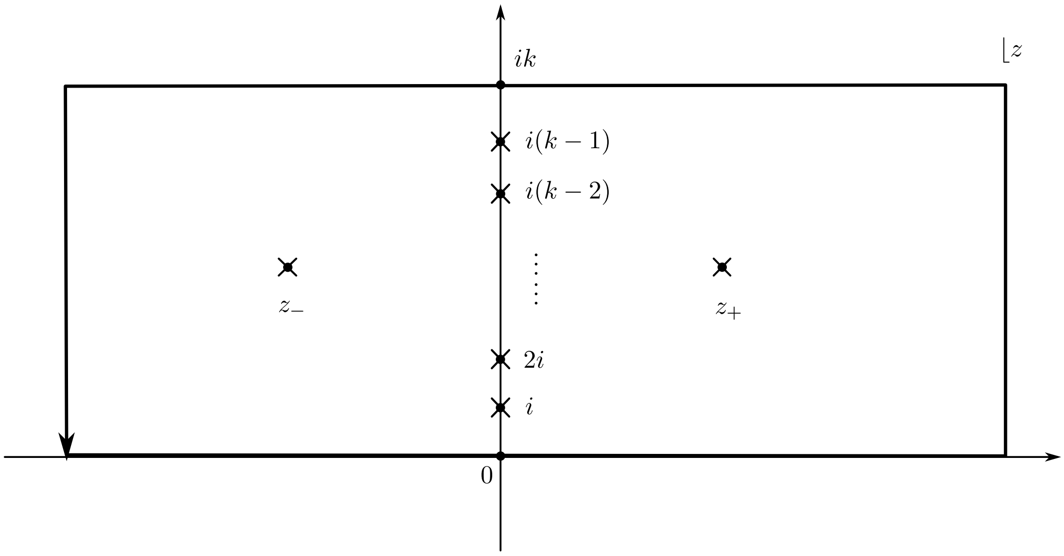

Figure 1. Rectangular contour for residue computation. The poles on the imaginary axis $z = in$ with $n = 1, \ldots, k-1$ arise from the f function, while those at $z_\pm = \pm\frac{m_1 k}{2\pi} + i\frac{k}{2}$ follow from the g function.

which gives

$$\int du\, \mathcal{F}(u) = \frac{2\pi i}{1-(-)^k \cosh(m_2 k)}\left[-\frac{i}{\pi}\sum_{n=1}^{k-1}(-)^n \frac{\sin^2\left(\frac{n\pi}{k}\right)\sinh(m_2 n)}{\cosh\left(\frac{m_1}{2}-\frac{in\pi}{k}\right)\cosh\left(\frac{m_1}{2}+\frac{in\pi}{k}\right)} + \mathsf{R}_k\right] \tag{2.12}$$

where

$$\mathsf{R}_k = \begin{cases} (-)^{\frac{k}{2}}\, \frac{ik}{\pi}\, \frac{\coth\frac{m_1}{2}\sinh\frac{km_2}{2}}{\sinh\frac{km_1}{2}} \cos\frac{km_1 m_2}{2\pi}\,, & k \text{ even} \\ (-)^{\frac{k+1}{2}}\, \frac{ik}{\pi}\, \frac{\coth\frac{m_1}{2}\cosh\frac{km_2}{2}}{\cosh\frac{km_1}{2}} \sin\frac{km_1 m_2}{2\pi}\,, & k \text{ odd} \end{cases} \tag{2.13}$$

Case $m_2 = 0$, k *odd.* It is evident from (2.12) that the $m_2 \to 0$ limit of (2.9) is smooth, the result is

$$Z\big|_{m_2=0} = \frac{1}{k^2\cosh^2 m}\left[\sum_{n=1}^{k-1}(-)^n \frac{n\sin^2\left(\frac{n\pi}{k}\right)}{\cosh\left(m-\frac{in\pi}{k}\right)\cosh\left(m+\frac{in\pi}{k}\right)} - (-)^{\frac{k+1}{2}}\frac{k^2 m\coth m}{\pi\cosh km}\right], \quad k \text{ odd} \tag{2.14}$$

where we have used $m_1 = 2m$.

Case $m_2 = 0$, k *even.* The factor multiplying the bracket in (2.12) prevents taking $m_2 \to 0$ in the even k case. To compute the integral in (2.8) we consider

$$I = \int du \tilde{f}(u) g(u)\,, \tag{2.15}$$

with $g(u)$ as in (2.10) and

$$\tilde{f}(u) = \frac{i}{k}\frac{(u-ik/2)^2}{\sinh \pi u}\,.$$

Upon integration, the odd piece in $\tilde{f}$ vanishes against $g(u)$ and therefore the partition function (2.8) can be written as

$$Z\big|_{m_2=0} = \frac{1}{k^2 \cosh^2 m} I \tag{2.16}$$

The shift $u \to u + ik$ in $\tilde{f}(u)$ gives

$$\tilde{f}(u) \to (-)^{k+1} \tilde{f}(-u)\,.$$

As discussed below (2.11), this property makes the rectangular contour in figure 1 appropriate for computing I by residues.

For the residues analysis we should now consider the pole in $\tilde{f}(z)$ at the origin $z = 0$ but a zero in $g(z)$ eliminates it; along the same lines the residue from $z = ik/2$ is absent since a zero appears for $\tilde{f}$. Calling $\tilde{\mathcal{F}}(z) = \tilde{f}(z)g(z)$ one finds

$$\oint_C dz\, \tilde{\mathcal{F}}(z) = 2I\,,$$

on the other hand

$$\begin{aligned}\oint_C dz\, \tilde{\mathcal{F}}(z) &= 2\pi i \left[\sum_{n=0}^{k-1} \mathrm{Res}_{z=in}\, \tilde{\mathcal{F}}(z) + \mathrm{Res}_{z=z_\pm}\, \tilde{\mathcal{F}}(z) \right] \\ &= 2\pi i \left[\frac{i}{k\pi} \sum_{n=1}^{k-1} (-)^n \left(\frac{k}{2} - n \right)^2 \frac{\sin^2\left(\frac{n\pi}{k}\right)}{\cosh\left(m - \frac{in\pi}{k}\right) \cosh\left(m + \frac{in\pi}{k}\right)} + \tilde{\mathsf{R}}_k \right]. \end{aligned} \tag{2.17}$$

where

$$\tilde{\mathsf{R}}_k = (-)^{\frac{k}{2}} \frac{2i(mk)^2}{\pi^3} \frac{\coth(m) \sinh mk}{\cosh(2mk) - 1}$$

The $n = \frac{k}{2}$ term in the sum vanishes as expected. The final result is

$$Z\big|_{m_2=0} = -\frac{1}{k \cosh^2 m} \cdot \left[\sum_{n=1}^{k-1} (-)^n \left(\frac{n}{k} - \frac{1}{2} \right)^2 \frac{\sin^2\left(\frac{n\pi}{k}\right)}{\cosh\left(m - \frac{in\pi}{k}\right) \cosh\left(m + \frac{in\pi}{k}\right)} + (-)^{\frac{k}{2}} \frac{2m^2 k}{\pi^2} \frac{\coth(m) \sinh mk}{\cosh(2mk) - 1} \right] \tag{2.18}$$

3 Summary of results and limits

Thus we have obtained

$$Z = \frac{2}{k^2 \sinh(m_2)} \frac{1}{1 - (-1)^k \cosh(m_2 k)} (J_1 - J_2) \tag{3.1}$$

where

$$J_1 = \frac{1}{\cosh^2\left(\frac{m_1}{2}\right)} \sum_{n=1}^{k-1} (-1)^n \frac{\sin^2\left(\frac{n\pi}{k}\right) \sinh(m_2 n)}{\cosh\left(\frac{m_1}{2} - \frac{in\pi}{k}\right) \cosh\left(\frac{m_1}{2} + \frac{in\pi}{k}\right)} \tag{3.2}$$

and

$$J_2 = \begin{cases} (-)^{\frac{k}{2}} \frac{2k \sinh \frac{km_2}{2}}{\sinh(m_1) \sinh \frac{km_1}{2}} \cos \frac{km_1 m_2}{2\pi}\,, & k \text{ even} \\ (-)^{\frac{k+1}{2}} \frac{2k \cosh \frac{km_2}{2}}{\sinh(m_1) \cosh \frac{km_1}{2}} \sin \frac{km_1 m_2}{2\pi}\,, & k \text{ odd} \end{cases} \tag{3.3}$$

Using

$$\frac{2}{1+\cosh\alpha} = \frac{1}{\cosh^2\left(\frac{\alpha}{2}\right)}\,, \qquad \frac{2}{1-\cosh\alpha} = -\frac{1}{\sinh^2\left(\frac{\alpha}{2}\right)}\,, \tag{3.4}$$

we can finally put the partition function in the form

$$Z\big|_{k \text{ even}} = -\frac{1}{k^2 \sinh(m_2) \sinh^2\left(\frac{km_2}{2}\right)} (J_1 - J_2) \tag{3.5}$$

$$Z\big|_{k \text{ odd}} = \frac{1}{k^2 \sinh(m_2) \cosh^2\left(\frac{km_2}{2}\right)} (J_1 - J_2) \tag{3.6}$$

In the formulas (3.5)–(3.6), the symmetry $m_1 \leftrightarrow m_2$ — which is manifest in the integral form (1.2) — is hidden. Interestingly, this symmetry is only recovered upon summation over n. On the other hand, the symmetry $m_2 \to -m_2$ is manifest.

Note that Z is real. While this is expected in a unitary theory, it is not generally the case in Chern-Simons theories (for a discussion, see [12]). In the present case, it is related to the fact the theory is a combination of two Chern-Simons theory with opposite levels.[2]

Consider, as particular examples, the important cases $k = 1, 2$. The partition functions take the form

$$Z\big|_{k=1} = \frac{2}{\sinh(m_1)\sinh(m_2)\cosh\left(\frac{m_1}{2}\right)\cosh\left(\frac{m_2}{2}\right)} \sin\left(\frac{m_1 m_2}{2\pi}\right), \tag{3.7}$$

$$Z\big|_{k=2} = \frac{2}{\sinh^2(m_1)\sinh^2(m_2)} \sin^2\left(\frac{m_1 m_2}{2\pi}\right). \tag{3.8}$$

Now the symmetry $m_1 \leftrightarrow m_2$ has become manifest.

Note that the partition functions for $k = 1, 2$ have zeros. Restoring the R dependence, the zeros are located at

$$m_1 m_2 R^2 = 2\pi^2 n\,, \qquad n = \pm 1, \pm 2, \dots \tag{3.9}$$

They represent Lee-Yang zeros (see, for example, [13]). In the infinite volume, $R \to \infty$, the zeros condense in a certain line, and a phase transition should emerge. The fact that the partition function has zeros seems to be related to the fact that the coupling, $g = 2\pi i/k$, is imaginary for real k. Indeed, from the general expressions (3.2)–(3.3) we see that the arguments of the sine and cosine functions in (3.7), (3.8) contain a factor π/k. If the coupling g is (unphysically) continued to the real line by taking $k \to ik$, the partition function zeros would then lie on the imaginary g-axis, in accordance with the Lee-Yang theorem (see [11] for a related discussion).

For the undeformed ABJM theory, the $k = 1$ case is of special interest, since it is conjectured to describe the dynamics of two M2 branes in eleven-dimensional Minkowski

[2]We thank Miguel Tierz for comments on this point.

spacetime. An interesting question is what is the origin of these Lee-Yang singularities in the brane realization.

The partition function $Z(2\zeta, m; k)$ does not have any zeros for $k > 2$. For higher values of k, the partition function becomes more involved, below we quote explicitly the $k = 3$ and $k = 4$ cases

$$Z\big|_{k=3} = \frac{2}{3}\,\frac{2 - \sin\left(\frac{3m_1m_2}{2\pi}\right)\operatorname{csch}\left(\frac{m_1}{2}\right)\operatorname{csch}\left(\frac{m_2}{2}\right)}{(\cosh m_1 + \cosh 2m_1)(\cosh m_2 + \cosh 2m_2)} \tag{3.10}$$

$$Z\big|_{k=4} = \frac{1 - \operatorname{sech}(m_1) - \operatorname{sech}(m_2) + \cos\left(\frac{2m_1m_2}{\pi}\right)\operatorname{sech}(m_2)\operatorname{sech}(m_1)}{8\sinh^2 m_1 \sinh^2 m_2} \tag{3.11}$$

Note that the symmetry under the exchange $m_1 \leftrightarrow m_2$ is manifest.

Asymptotic formulas. Let us consider the limit of a large sphere, $mR \gg 1$, at fixed k. Assuming $m_1 > 0$, $m_2 > 0$ and restoring the R dependence, we find

$$Z\big|_{k=1} \sim 32\, e^{-\frac{3}{2}(m_1+m_2)R} \sin\left(\frac{m_1m_2R^2}{2\pi}\right), \tag{3.12}$$

$$Z\big|_{k=2} \sim 32\, e^{-2(m_1+m_2)R} \sin^2\left(\frac{m_1m_2R^2}{2\pi}\right), \tag{3.13}$$

$$Z\big|_{k>2} \sim \frac{64}{k^2}\, e^{-2(m_1+m_2)R} \sin^2\left(\frac{\pi}{k}\right). \tag{3.14}$$

The general asymptotic formula with arbitrary sign for m_2 and $m_2 \neq 0$, is obtained by replacing m_2 by $|m_2|$.

The absolute value implies a discontinuity in the first derivative of $F = -\ln Z$. This indicates a first-order phase transition in the parameter m_2 at $m_2 = 0$, i.e., when the two mass scales $m, 2\zeta$ cross. Explicitly, at large R, we have

$$F = 2\big(|m_1| + |m_2|\big)R + O(1)\,, \qquad k > 1\,. \tag{3.15}$$

Hence

$$\frac{d\Delta F}{dm_2}\bigg|_{m_2=0} = 4R\,, \qquad \Delta F \equiv F_{m_2>0} - F_{m_2<0}\,. \tag{3.16}$$

For $k = 1$ the discontinuity in the first derivative of ΔF is equal to $3R$, as can be seen from (3.12).

For the general theory with gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$, large N phase transitions in the complex parameter $Ng = 2\pi i N/k$ were studied in [10, 11]. These phase transitions require taking infinite volume and, at the same time, a strong coupling limit with fixed kR — a limit that already appeared in the context of supersymmetric $\mathrm{U}(N)$ Chern-Simons theory with massive fundamental matter in [14, 15]. It should be noted that such decompactification limit is different from the present (more physical) limit of large R at fixed k.

Another interesting aspect of (3.14) is that it is in a form suitable for a weak coupling expansion in powers of $1/k$:

$$Z\big|_{k>2} \sim -\frac{32}{k^2}\, e^{-2(m_1+m_2)R} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{2\pi}{k}\right)^{2n}. \tag{3.17}$$

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The perturbative expansion has an infinite radius of convergence. However, the original theory on the three-sphere of *finite* radius R has an asymptotic perturbative expansion, with $2n!$ asymptotic behavior for the $1/k^{2n}$ term. This can be seen by using the integral form (2.7) and generalizing the study of [16, 17] on the resurgence properties of the perturbation series of ABJM theory. Now, expanding the integrand in (2.7), one finds a series with finite radius of convergence determined by the poles of $\text{sech}(\pi u/k \pm m_1/2)$ in the complex u-plane. The integral over u then adds an extra $(2n)!$, leading to an asymptotic (but Borel summable) perturbation series.

4 The special case $m_2 = 0$

The $m_2 = 0$ case is special and must be considered separately. In particular, it represents the critical point in the phase transitions that arise in the decompactification limit. In section 2 we have obtained the following formulas:

Odd k:

$$Z\big|_{m_2=0} = \frac{1}{k^2\cosh^2 m}\sum_{n=1}^{k-1}(-)^n \frac{n\sin^2\frac{\pi n}{k}}{\cosh\left(m+\frac{i\pi n}{k}\right)\cosh\left(m-\frac{i\pi n}{k}\right)} + \frac{(-)^{\frac{k-1}{2}}2m}{\pi\cosh(km)\sinh(2m)}\,. \tag{4.1}$$

Even k:

$$\begin{aligned}Z\big|_{m_2=0} = &\frac{1}{k\cosh^2 m}\sum_{n=1}^{k-1}(-)^{n+1}\left(\frac{n}{k}-\frac{1}{2}\right)^2\frac{\sin^2\left(\frac{n\pi}{k}\right)}{\cosh\left(m-\frac{in\pi}{k}\right)\cosh\left(m+\frac{in\pi}{k}\right)}\\ &+(-)^{\frac{k}{2}+1}\frac{4m^2}{\pi^2}\frac{\sinh mk}{\sinh(2m)\big(\cosh(2mk)-1\big)}\end{aligned} \tag{4.2}$$

In particular,

$$\begin{aligned}Z\big|_{k=1} &= \frac{2m}{\pi\cosh(m)\sinh(2m)}\,,\\ Z\big|_{k=2} &= \frac{2m^2}{\pi^2\sinh^2(2m)}\,.\end{aligned} \tag{4.3}$$

Note that the partition function does not have zeros in this case.

Asymptotic formulas $m_2 = 0$. Consider again the limit of a large sphere, $mR \gg 1$, at fixed k, but now with $m_2 = 0$. We find

$$Z\big|_{k=1} \sim \frac{8mR}{\pi}\,e^{-3mR}, \tag{4.4}$$

$$Z\big|_{k=2} \sim \frac{8}{\pi^2}\,m^2R^2\,e^{-4mR}, \tag{4.5}$$

$$Z\big|_{k>2} \sim \frac{4}{k^2}\,e^{-4mR}\tan^2\frac{\pi}{k}\,. \tag{4.6}$$

Note that these formulas differ from the asymptotic formulas (3.12)–(3.14) given above for $Z(m_1, m_2)$ at $m_2 = 0$. This is expected, since the latter were obtained by assuming $|m_1 R|$, $|m_2 R| \to \infty$.

Unlike the $m_2 \neq 0$ case, the perturbation series for this flat-theory limit has now finite radius of convergence $|\pi/k| < \pi/2$, therefore perturbation series is convergent for all $k > 2$, where the formula applies. On the other hand, just like the general $m_2 \neq 0$ case, the theory on a finite-radius $\mathbb{S}^3$ has an asymptotic perturbation series with $2n!$ asymptotic behavior.

Finally, it would be interesting to study supersymmetric Wilson loops in the present mass/FI deformed theory, along the lines of [18].

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