MATRIX GEGENBAUER POLYNOMIALS: THE 2×2 FUNDAMENTAL CASES

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Abstract. In this paper, we exhibit explicitly a sequence of 2×2 matrix valued orthogonal polynomials with respect to a weight $W_{p,n}$, for any pair of real numbers p and n such that $0 < p < n$. The entries of these polynomiales are expressed in terms of the Gegenbauer polynomials C_k^λ. Also the corresponding three-term recursion relations are given and we make some studies of the algebra of differential operators associated with the weight $W_{p,n}$.

1. Introduction

The theory of matrix valued orthogonal polynomials, without any consideration of differential equations, goes back to [18] and [19]. In [3], the study of the matrix valued orthogonal polynomials that are eigenfunctions of certain second order symmetric differential operators was started. The first explicit examples of such polynomials were given in [8], [9], [7], [10] and [4]. See also [5], [6], [1], [2], and the references given there.

On the two dimensional sphere $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$, the harmonic analysis with respect to the action of the orthogonal group is contained in the classical theory of the spherical harmonics. In spherical coordinates, the zonal spherical functions on S^2 are the Legendre polynomials. More generally, in the case of the n-dimensional sphere S^n the zonal spherical functions are given in terms of Gegenbauer (or ultraspherical) polynomials of parameter $(n-1)/2$.

This fruitful connection between orthogonal polynomials and representation theory of compact Lie groups is also established in the matrix case: the matrix valued spherical functions of any K-type are closely related to matrix valued orthogonal polynomials. In this way, several examples of matrix orthogonal polynomials which are eigenfunctions of a symmetric differential operator have been obtained by focusing on a group representation approach. See for example [9], [11], [22], [23], [21] and more recently [16] and [24].

The examples of matrix orthogonal polynomials introduced in this paper are motivated by the spherical functions of fundamental K-types associated with the n-dimensional spheres $S^n \simeq G/K$, where $(G, K) = (\mathrm{SO}(n+1), \mathrm{SO}(n))$. These matrix valued spherical functions were studied in detail in [27] and [29]. The "group parameters" of the fundamental K-types are p, $n \in \mathbb{N}$ such that $0 < p < [n/2]$ and they give rise to 2×2 matrix valued orthogonal polynomials.

In this paper we go beyond these group parameters and we extend these parameters continuously. We would like to remark that the group representation theory is a natural source of examples of matrix valued orthogonal polynomials. We keep this in mind in spite of the fact that the results obtained in this paper are self-contained, the proofs are of analytic nature and they do not depend on any previous results on spherical functions.

Given a weight matrix W, it is very natural to study the algebra $\mathcal{D}(W)$, of all differential operators that have a sequence of matrix valued orthogonal polynomials with respect to W as eigenfunctions, see (3). In the classical cases of Hermite, Laguerre and Jacobi weights, the structure of this algebra is well understood: it is a polynomial algebra in a second order differential operator, see [20]. In particular, it is a commutative

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algebra. In the matrix case, the first attempt to go beyond the issue of the existence of one nontrivial element in $\mathcal{D}(W)$ and to study the full algebra is undertaken in [2], with the assistance of symbolic computation, for a few weights W. The first deep study of the algebra $\mathcal{D}(W)$ can be founded in [26], where the author worked out one of the examples introduced in [2]. We refer the reader to [13] for basic definitions and main results concerning the algebra $\mathcal{D}(W)$. The present paper leads to understand completely a second and more promising example of $\mathcal{D}(W)$ in a forthcoming paper, [28]. There are very few examples of non-commutative algebras that arise in a natural setup at the intersection of harmonic analysis and algebras. The study of such examples for the algebra $\mathcal{D}(W)$ considered here is one step in that direction. ++As a consequence of this work, together with F.A. Grünbaum, in [12] we extend to a matrix setup a result that traces its origin and its importance to the work of Claude Shannon in lying the mathematical foundations of information theory, and to a remarkable series of papers by D. Slepian, H. Landau and H. Pollak.

To the best of our knowledge, this is the first example showing in a non-commutative setup that a bispectral property implies that the corresponding global operator of "time and band limiting" admits a commuting local operator. This is a noncommutative analog of the famous prolate spheroidal wave operator.

Now we discuss briefly the content of the paper. In Section 2 we recall the general notions of matrix valued orthogonal polynomials and some results from [13] about the algebra $\mathcal{D}(W)$.

In Section 3, we introduce our sequence $\{P_w\}_{w\in\mathbb{N}_0}$ of 2×2 matrix valued polynomials on $[-1,1]$ whose entries are given in terms of the classical Gegenbauer polynomials, for real parameters p and n such that $0<p<n$, see (4). We prove that these polynomials satisfy $P_wD=\Lambda_wP_w$, where D is a (right-hand side) hypergeometric differential operator and the eigenvalue is a diagonal matrix. This differential operator D is symmetric with respect to the matrix weight W introduced in (12). We use these facts to prove that the polynomials $\{P_w\}_{w\in\mathbb{N}_0}$ are orthogonal with respect to the weight matrix $W=W_{p,n}$ (Theorem 3.6).

We also connect our weight matrix $W_{p,n}$ with the weight considered in [15], where the authors give examples of matrix valued Gegenbauer polynomials, extending for an arbitrary parameter ν the results in [16] for $\nu=1$. See Remark 3.7.

In Section 4 we prove a three-term recursion relation satisfied by $\{P_w\}_{w\in\mathbb{N}_0}$. Section 5 is focused on the study of the algebra $\mathcal{D}(W)$. In our case $\mathcal{D}(W)$ is a noncommutative algebra. We provide a basis $\{D_1,D_2,D_3,D_4,I\}$ of the subspace of the differential operators in $\mathcal{D}(W)$ of order at most two. The differential operators D_1 and D_2 are symmetric operators, while D_3 and D_4 are not. We conjecture that D_1,D_2,D_3,D_4 generates the algebra $\mathcal{D}(W)$.

2. Background on matrix valued orthogonal polynomials

Let $W=W(x)$ be a weight matrix of size N on the real line, that is a complex $N\times N$ matrix valued integrable function on the interval (a,b) such that $W(x)$ is positive definite almost everywhere and with finite moments of all orders. Let $\mathrm{Mat}_N(\mathbb{C})$ be the algebra of all $N\times N$ complex matrices and let $\mathrm{Mat}_N(\mathbb{C})[x]$ be the algebra over $\mathbb{C}$ of all polynomials in the indeterminate x with coefficients in $\mathrm{Mat}_N(\mathbb{C})$. We consider the following Hermitian sesquilinear form in the linear space $\mathrm{Mat}_N(\mathbb{C})[x]$

$$\langle P,Q\rangle=\langle P,Q\rangle_W=\int_a^b P(x)W(x)Q(x)^*\,dx.$$

The following properties are satisfied, for all $P,Q,R\in\mathrm{Mat}_N(\mathbb{C})[x]$, $a,b\in\mathbb{C}$, $T\in\mathrm{Mat}_N(\mathbb{C})$

(1) $\langle aP+bQ,R\rangle=a\langle P,R\rangle+b\langle Q,R\rangle$,
(2) $\langle TP,R\rangle=T\langle P,R\rangle$,
(3) $\langle P,Q\rangle^*=\langle Q,P\rangle$,
(4) $\langle P,P\rangle\geq 0$. Moreover, if $\langle P,P\rangle=0$, then $P=0$.

Let us denote $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Given a weight matrix W one can construct sequences of matrix valued orthogonal polynomials, that is sequences $\{P_n\}_{n\in\mathbb{N}_0}$, where P_n is a polynomial of degree n with nonsingular leading coefficient and $\langle P_n, P_m\rangle = 0$ for $n \neq m$. We observe that there exists a unique sequence of monic orthogonal polynomials $\{Q_n\}_{n\in\mathbb{N}_0}$ in $\mathrm{Mat}_N(\mathbb{C})[x]$. By following a standard argument, given for instance in [18] or [19], one shows that the monic orthogonal polynomials $\{Q_n\}_{n\in\mathbb{N}_0}$ satisfy a three-term recursion relation

$$xQ_n(x) = A_nQ_{n-1}(x) + B_nQ_n(x) + Q_{n+1}(x), \qquad n \in \mathbb{N}_0,$$

where $Q_{-1} = 0$ and A_n, B_n are matrices depending on n and not on x.

Two weights W and $\widetilde{W}$ are said to be *similar* if there exists a nonsingular matrix M, which does not depend on x, such that

$$\widetilde{W}(x) = MW(x)M^*, \quad \text{for all } x \in (a, b).$$

Notice that if $\{P_n\}_{n\geq 0}$ is a sequence of orthogonal polynomials with respect to W, and $M \in \mathrm{GL}_N(\mathbb{C})$, then $\{P_nM^{-1}\}_{n\geq 0}$ is orthogonal with respect to $\widetilde{W} = MWM^*$. A weight matrix W reduces to a smaller size if there exists a nonsingular matrix M such that

$$MW(x)M^* = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x)\end{pmatrix}, \qquad \text{for all } x \in (a, b),$$

where W_1 and W_2 are weights of smaller size.

For a given weight matrix and a sequence of orthogonal polynomials, it may be of interest the study of the differential operators having these polynomials as eigenfunctions. Let D be a right-hand side ordinary differential operator with matrix polynomial coefficients $F_i(x)$ of degree less than or equal to i of the form

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \qquad \partial = \frac{d}{dx}, \tag{1}$$

with the action of D on a polynomial function $P(x)$ given by

$$(PD)(x) = \sum_{i=0}^{s} \partial^i(P)(x)F_i(x).$$

We say that the differential operator D is *symmetric* if $\langle PD, Q\rangle = \langle P, QD\rangle$, for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$. It is a matter of careful integration by parts to see that the condition of symmetry for a differential operator of order two is equivalent to a set of three differential equations involving the weight W and the coefficients of the differential operator D.

Proposition 2.1 ([10] or [4]). *Let $W(x)$ be a smooth weight matrix supported on (a, b). Let $D = \partial^2F_2(x) + \partial F_1(x) + F_0$. Then D is symmetric with respect to W if and only if*

$$\begin{cases} F_2W = WF_2^* \\ 2(F_2W)' - F_1W = WF_1^* \\ (F_2W)'' - (F_1W)' + F_0W = WF_0^* \end{cases}$$

with the boundary conditions

$$\lim_{x\to a,b} F_2(x)W(x) = 0, \quad \lim_{x\to a,b} \big(F_1(x)W(x) - WF_1^*(x)\big) = 0.$$

We consider the following subalgebra of the algebra of all right-hand side differential operators with coefficients in $\mathrm{Mat}_N(\mathbb{C})[x]$,

$$\mathcal{D} = \{D = \textstyle\sum_{i=0}^{s} \partial^i F_i \,:\, s \in \mathbb{N}_0,\, F_i \in \mathrm{Mat}_N(\mathbb{C})[x], \deg F_i \leq i\}.$$

Proposition 2.2 ([13], Propositions 2.6 and 2.7). *Let $W = W(x)$ be a weight matrix of size $N \times N$ and let $\{Q_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials in* $\mathrm{Mat}_N(\mathbb{C})[x]$*. If D is a right-hand side ordinary differential operator of order s, as in (1), such that*

$$Q_n D = \Lambda_n Q_n, \qquad \text{for all } n \in \mathbb{N}_0,$$

with $\Lambda_n \in \mathrm{Mat}_N(\mathbb{C})$, then $F_i = F_i(x) = \sum_{j=0}^{i} x^j F_j^i$, $F_j^i \in \mathrm{Mat}_N(\mathbb{C})$, is a polynomial and $\deg(F_i) \leq i$. Moreover D is determined by the sequence $\{\Lambda_n\}_{n\geq 0}$ and

$$\Lambda_n = \sum_{i=0}^{s} [n]_i F_i^i, \qquad \text{for all } n \geq 0, \tag{2}$$

where $[n]_i = n(n-1)\cdots(n-i+1)$, $[n]_0 = 1$.

Given a matrix weight W, the algebra

$$\mathcal{D}(W) = \{D \in \mathcal{D} : P_n D = \Lambda_n(D) P_n, \, \Lambda_n(D) \in \mathrm{Mat}_N(\mathbb{C}), \text{ for all } n \in \mathbb{N}_0\} \tag{3}$$

is introduced in [13], where $\{P_n\}_{n\in\mathbb{N}_0}$ is any sequence of matrix valued orthogonal polynomials with respect to W.

We observe that the definition of $\mathcal{D}(W)$ depends only on the weight matrix W and not on the particular sequence of orthogonal polynomials, since two sequences $\{P_w\}_{w\in\mathbb{N}_0}$ and $\{Q_w\}_{w\in\mathbb{N}_0}$ of matrix orthogonal polynomials with respect to the weight W are related by $P_w = M_w Q_w$, for $w \in \mathbb{N}_0$, with $\{M_w\}_{w\in\mathbb{N}_0}$ invertible matrices (see [13, Corollary 2.5]).

Proposition 2.3 ([13], Proposition 2.8). *For each $n \in \mathbb{N}_0$, the mapping $D \mapsto \Lambda_n(D)$ is a representation of $\mathcal{D}(W)$ in $\mathrm{Mat}_N(\mathbb{C})$. Moreover, the sequence of representations $\{\Lambda_n\}_{n\in\mathbb{N}_0}$ separates the elements of $\mathcal{D}(W)$.*

We remark that the result in Proposition 2.3 says that the map

$$D \mapsto (\Lambda_0(D), \Lambda_1(D), \Lambda_2(D), \dots\dots)$$

is an injective morphism of $\mathcal{D}(W)$ into $\mathrm{Mat}_N(\mathbb{C})^{\mathbb{N}_0}$, the direct product of infinite copies, indexed by $\mathbb{N}_0$, of the algebra $\mathrm{Mat}_N(\mathbb{C})$. In particular, if $D_1, D_2 \in \mathcal{D}(W)$ then

$$D_1 = D_2 \quad \text{if and only if} \quad \Lambda_n(D_1) = \Lambda_n(D_2) \text{ for all } n \in \mathbb{N}_0.$$

For any $D \in \mathcal{D}(W)$ there exists a unique differential operator $D^* \in \mathcal{D}(W)$, the adjoint of D in $\mathcal{D}(W)$, such that

$$\langle PD, Q\rangle = \langle P, QD^*\rangle,$$

for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$. See Theorem 4.3 and Corollary 4.5 in [13]. The map $D \mapsto D^*$ is a *-operation in the algebra $\mathcal{D}(W)$. Moreover, it is shown that $\mathcal{S}(W)$, the set of all symmetric operators in $\mathcal{D}(W)$, is a real form of the space $\mathcal{D}(W)$, i.e.

$$\mathcal{D}(W) = \mathcal{S}(W) \oplus i\mathcal{S}(W),$$

as real vector spaces. In particular, the algebra $\mathcal{D}(W)$, together with the involution, is completely determined by $\mathcal{S}(W)$.

Corollary 2.4. *A differential operator $D \in \mathcal{D}(W)$ is a symmetric operator if and only if*

$$\Lambda_n(D)\langle Q_n, Q_n\rangle = \langle Q_n, Q_n\rangle \Lambda_n(D)^*$$

for all $n \in \mathbb{N}_0$.

Also it is worth to recall the following important result from [13].

Proposition 2.5 (Proposition 2.10). *If $D \in \mathcal{D}$ is symmetric then $D \in \mathcal{D}(W)$.*

3. Matrix valued orthogonal polynomials associated with the n-dimensional spheres

Motivated by the results obtained in [27] we introduce the following family of polynomials, for $w \in \mathbb{N}_0$,

$$P_w(x) = P_w^{n,p}(x) = \begin{pmatrix} \frac{1}{n+1} C_w^{\frac{n+1}{2}}(x) + \frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}(x) & \frac{1}{p+w} C_{w-1}^{\frac{n+3}{2}}(x) \\ \frac{1}{n-p+w} C_{w-1}^{\frac{n+3}{2}}(x) & \frac{1}{n+1} C_w^{\frac{n+1}{2}}(x) + \frac{1}{n-p+w} C_{w-2}^{\frac{n+3}{2}}(x) \end{pmatrix}, \tag{4}$$

with parameters $p, n \in \mathbb{R}$ such that $0 < p < n$. Here $C_n^\lambda(x)$ denotes the n-th Gegenbauer polynomial

$$C_w^\lambda(x) = \frac{(2\lambda)_w}{w!} \, {}_2F_1\left(\begin{matrix} -w, \, w+2\lambda \\ \lambda + 1/2 \end{matrix} ; \frac{1-x}{2} \right), \qquad x \in [-1,1],$$

where $(a)_w = a(a+1)\dots(a+w-1)$ denotes the Pochhammer symbol. As usual, we assume $C_w^\lambda(x) = 0$ if $w < 0$. We recall that C_w^λ is a polynomial of degree w, with leading coefficient $\frac{2^w (\lambda)_w}{w!}$.

Let us observe that $\deg(P_w) = w$ and the leading coefficient of P_w is a nonsingular scalar matrix

$$\frac{2^w (\frac{n+1}{2})_w}{(n+1)\, w!} \operatorname{Id} = \frac{1}{w!} 2^{w-1} (\tfrac{n+3}{2})_{w-1} \operatorname{Id}. \tag{5}$$

We start by proving that the polynomials P_w given in (4) are eigenfunctions of the following differential operator D.

Theorem 3.1. *For each $w \in \mathbb{N}_0$, the matrix polynomial P_w is an eigenfunction of the differential operator*

$$D = \partial^2 (1-x^2) - \partial \Big((n+2)x + 2 \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \Big) - \left(\begin{smallmatrix} p & 0 \\ 0 & n-p \end{smallmatrix}\right),$$

with eigenvalue

$$\Lambda_w(D) = \begin{pmatrix} -w(w+n+1) - p & 0 \\ 0 & -w(w+n+1) - n + p \end{pmatrix}.$$

Proof. We need to verify that

$$P_w D = \Lambda_w P_w.$$

We will need to use the following properties of the Gegenbauer polynomials (for the first three see [14] page 40, and for the last one see [25], page 83, equation (4.7.27))

$$(1-x^2) \frac{d^2}{dx^2} C_m^\lambda(x) - (2\lambda+1) x \frac{d}{dx} C_m^\lambda(x) + m(m+2\lambda) C_m^\lambda(x) = 0, \tag{6}$$

$$\frac{d}{dx} C_m^\lambda(x) = 2\lambda \, C_{m-1}^{\lambda+1}(x), \tag{7}$$

$$2(m+\lambda) x \, C_m^\lambda(x) = (m+1) C_{m+1}^\lambda(x) + (m+2\lambda-1) C_{m-1}^\lambda(x), \tag{8}$$

$$\frac{(m+2\lambda-1)}{2(\lambda-1)} C_{m+1}^{\lambda-1}(x) = C_{m+1}^\lambda(x) - x \, C_m^\lambda(x). \tag{9}$$

Also, combining (8) and (9), we have

$$(m+\lambda) C_{m+1}^{\lambda-1}(x) = (\lambda-1) \Big(C_{m+1}^\lambda(x) - C_{m-1}^\lambda(x) \Big). \tag{10}$$

The entry $(1,1)$ of the matrix $P_wD-\Lambda_wP_w$ is

$$\begin{aligned}&(1-x^2)(P_w)''_{11}-(n+2)x(P_w)'_{11}-2(P_w)'_{12}+w(w+n+1)(P_w)_{11}\\&=(1-x^2)\left(\tfrac{1}{n+1}C_w^{\frac{n+1}{2}}+\tfrac{1}{p+w}C_{w-2}^{\frac{n+3}{2}}\right)''-(n+2)x\left(\tfrac{1}{n+1}C_w^{\frac{n+1}{2}}+\tfrac{1}{p+w}C_{w-2}^{\frac{n+3}{2}}\right)'\\&\quad-\tfrac{2}{p+w}\left(C_{w-1}^{\frac{n+3}{2}}\right)'+w(w+n+1)\left(\tfrac{1}{n+1}C_w^{\frac{n+1}{2}}+\tfrac{1}{p+w}C_{w-2}^{\frac{n+3}{2}}\right).\end{aligned}$$

Applying (6) for $\lambda=\frac12(n+1)$, $\lambda=\frac12(n+3)$ and (7) for $\lambda=\frac12(n+3)$, with $m=w$, we have that the entry (1,1) of $P_wD-\Lambda_wP_w$, multiplied by $(p+w)/2$ is

$$-(n+3)C_{w-2}^{\frac{n+5}{2}}+(n+3)\,x\,C_{w-3}^{\frac{n+5}{2}}+(w+n+1)C_{w-2}^{\frac{n+3}{2}}=0,$$

this last identity follows from equation (9) with $\lambda=\frac{n+5}{2}$ and $m=w-3$. Repeating the previous verification, by changing p by $n-p$, it follows that the entry $(2,2)$ of $P_wD-\Lambda_wP_w$ is also zero.

The entry $(1,2)$ of $P_wD-\Lambda_wP_w$ is

$$(1-x^2)(P_w)''_{12}-(n+2)x(P_w)'_{12}-2(P_w)'_{11}+\big(w(w+n+1)-n+2p\big)(P_w)_{12},$$

if we multiply it by $(p+w)$ we get

(11)
$$(1-x^2)\Big(C_{w-1}^{\frac{n+3}{2}}\Big)''-(n+2)x\Big(C_{w-1}^{\frac{n+3}{2}}\Big)'+(w(w+n+1)-n+2p)C_{w-1}^{\frac{n+3}{2}}-2\tfrac{(p+w)}{n+1}\Big(C_w^{\frac{n+1}{2}}\Big)'-2\Big(C_{w-2}^{\frac{n+3}{2}}\Big)'.$$

Applying (6) for $\lambda=(n+3)/2$, $m=w-1$, (7) for $\lambda=(n+1)/2$, $m=w$ and $\lambda=(n+3)/2$, $m=w-1$, one obtain that (11) is

$$2x\Big(C_{w-1}^{\frac{n+3}{2}}\Big)'-2(w-1)C_{w-1}^{\frac{n+3}{2}}-2(n+3)C_{w-3}^{\frac{n+5}{2}}.$$

Now, applying (7) and (9), this expression becomes

$$2(n+3)\Big(C_{w-1}^{\frac{n+5}{2}}-C_{w-3}^{\frac{n+5}{2}}\Big)-2(2w+n+1)C_{w-1}^{\frac{n+3}{2}},$$

which is equal to zero by (10) with $\lambda=\frac{n+5}{2}$ and $m=w-2$. This concludes that the entry $(1,2)$ of $P_wD-\Lambda_wP_w$ is zero. To complete the proof of the theorem we need to verify that the entry $(2,1)$ is also zero. This is obtained making exactly the same computations, by changing p by $n-p$. ☐

We introduce the weight matrix

(12)
$$W(x)=W_{p,n}=(1-x^2)^{\frac n2-1}\begin{pmatrix}p\,x^2+n-p & -nx\\ -nx & (n-p)x^2+p\end{pmatrix},\quad x\in[-1,1].$$

Proposition 3.2. *For $n\neq 2p$, the weight $W(x)$ does not reduce to a smaller size.*

Proof. Assume that there exists a nonsingular matrix $M=\begin{pmatrix}m_{11}&m_{12}\\m_{21}&m_{22}\end{pmatrix}$ such that

$$MW(x)M^*=\begin{pmatrix}w_1(x)&0\\0&w_2(x)\end{pmatrix}.$$

The entry $(1,2)$ of $MW(x)M^*$ is

$$x^2\big(p\,m_{11}\overline{m}_{21}+(n-p)m_{12}\overline{m}_{22}\big)-\big(m_{11}\overline{m}_{22}+m_{12}\overline{m}_{21}\big)n\,x+(n-p)m_{11}\overline{m}_{21}+p\,m_{12}\overline{m}_{22},$$

from here we see that

$$m_{11}\overline{m}_{22} + m_{12}\overline{m}_{21} = 0,$$

$$p\, m_{11}\overline{m}_{21} + (n-p)m_{12}\overline{m}_{22} = 0, \tag{13}$$

$$(n-p)m_{11}\overline{m}_{21} + p\, m_{12}\overline{m}_{22} = 0. \tag{14}$$

By combining equations (13) and (14) we have that $(n-2p)m_{11}\overline{m}_{21} = 0$. The assumption $n \neq 2p$, together with (9), implies $\det(M) = 0$, which is a contradiction. □

Remark 3.3. For $n = 2p$, the weight matrix W reduces to two scalar weights. The corresponding scalar polynomials are Jacobi polynomials $P_w^{\alpha,\beta}$ with $(\alpha, \beta) = (n/2+1, n/2-1)$ and $(\alpha, \beta) = (n/2-1, n/2+1)$, respectively. In fact, by taking $M = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ we have that

$$MW(x)M^* = 2p\,(1-x^2)^{\frac{n}{2}-1} \begin{pmatrix} (1-x)^2 & 0 \\ 0 & (1+x)^2 \end{pmatrix}.$$

Remark 3.4. We have that the weight matrices $W_{p,n}$ and $W_{n-p,n}$ are similar. In fact, by taking $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we get

$$MW_{p,n}M^* = W_{n-p,n}.$$

From Proposition 2.1 and following straightforward computations, one can prove the following result.

Proposition 3.5. *The differential operator*

$$D = \partial^2\,(1-x^2) - \partial\left((n+2)x + 2\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right) - \left(\begin{smallmatrix} p & 0 \\ 0 & n-p \end{smallmatrix}\right)$$

is symmetric with respect to the weight function $W(x)$.

In the scalar case, if D is a symmetric differential operator with respect to W and $\{P_w\}_{w\in\mathbb{N}_0}$ is a family of eigenfunctions of D with different eigenvalues, then the sequence $\{P_w\}_{w\in\mathbb{N}_0}$ is automatically orthogonal with respect to W. In the matrix case this is not always true since

$$\Lambda_w\langle P_w, P_{w'}\rangle = \langle P_w D, P_{w'}\rangle = \langle P_w, P_{w'}D\rangle = \langle P_w, P_{w'}\rangle\Lambda_{w'} \tag{15}$$

does not imply that $\langle P_w, P_{w'}\rangle = 0$, for $w \neq w'$. Therefore, we prove the orthogonality in the next theorem.

Theorem 3.6. *When $n \neq 2p$ the matrix polynomials $\{P_w\}_{w\in\mathbb{N}_0}$ are orthogonal polynomials with respect to the matrix valued inner product*

$$\langle P, Q\rangle = \int_{-1}^{1} P(x)W(x)Q(x)^*\,dx.$$

Proof. We know that P_w is a polynomial of degree w and its leading coefficient is a nonsingular diagonal matrix (see (5)). We only have to verify that for $w \neq w'$, $\langle P_w, P_{w'}\rangle_W = 0$. Since P_w is an eigenfunction of the differential operator D, which is symmetric with respect to W, we have that (15) holds with

$$\Lambda_w = \begin{pmatrix} \lambda_{w,1} & 0 \\ 0 & \lambda_{w,2} \end{pmatrix} = \begin{pmatrix} -w(w+n+1)-p & 0 \\ 0 & -w(w+n+1)-n+p \end{pmatrix},$$

see Theorem 3.1. Therefore, for $i, j = 1, 2$ we have $\lambda_{w,i}\langle P_{w,i}, P_{w',j}\rangle = \lambda_{w',j}\langle P_{w,i}, P_{w',j}\rangle$, where $P_{w,i}$ is the i-th row of the polynomial P_w, and

$$\langle P_{w,i}, P_{w',j}\rangle = \int_{-1}^{1} P_{w,i}(x)\,W(x)P^*_{w',j}(x)\,dx \in \mathbb{C}.$$

It is not difficult to verify that $\lambda_{w,i} \neq \lambda_{w',j}$, for $w \neq w'$ or $i \neq j$. Then we have

$$\langle P_{w,i}, P_{w',j}\rangle = 0, \qquad \text{for } w \neq w' \text{ or } i \neq j. \tag{16}$$

Therefore $\langle P_w, P_{w'}\rangle = 0$, for $w \neq w'$, which concludes the proof of the theorem. □

Remark 3.7. Recently, in [15] the authors study some families on matrix valued polynomials, depending on one real parameter $\nu > 0$, of arbitrary size $(2\ell+1)\times(2\ell+1)$ with $\ell \in \frac{1}{2}\mathbb{N}$. These weights are not irreducible. For $\ell = 1, \frac{3}{2}, 2$ appears some irreducible 2×2 blocks $W_+^{(\nu)}$ and $W_-^{(\nu)}$. See Remark 2.8 (ii) there.

The case $\ell = 3/2$ does not match with the examples considered in this paper. The cases $\ell = 1$ and $\ell = 2$ are particular cases of our weight matrices $W_{p,n}$ by choosing our parameters $(p,n) = (\nu, 2\nu+1)$ and $(p,n) = (\nu, 2\nu+3)$, for $\ell = 1$ and $\ell = 2$ respectively. In fact, with $L = \begin{pmatrix} 0 & \sqrt{2} \\ -1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ we get

$$\begin{aligned} W_+^{(\nu)} &= \frac{(\nu+2)}{(2\nu+1)} L\, W_{\nu,2\nu+1} L^* && \text{for } \ell = 1, \\ W_-^{(\nu)} &= \frac{(\nu+4)(\nu+2)}{(2\nu+1)(2\nu+3)} D\, W_{\nu,2\nu+3} D^* && \text{for } \ell = 2. \end{aligned}$$

The case $\nu = 1$ was previously studied in [16] and [17].

4. Three-term recursion relation

The main result of this section is a three-term recursion relation satisfied by the sequence of orthogonal polynomials studied in this paper. We give a proof by using some properties of the Gegenbauer polynomials.

Theorem 4.1. *The orthogonal polynomials $\{P_w\}_{w\in\mathbb{N}_0}$ satisfy the three-term recursion relation*

$$x\, P_w(x) = A_w P_{w-1}(x) + B_w P_w(x) + C_w P_{w+1}(x),$$

where

$$A_w = \begin{pmatrix} \frac{(n+w)(p+w-1)(n-p+w+1)}{(p+w)(n-p+w)(2w+n+1)} & 0 \\ 0 & \frac{(n+w)(p+w+1)(n-p+w-1)}{(p+w)(n-p+w)(2w+n+1)} \end{pmatrix},$$

$$B_w = \begin{pmatrix} 0 & \frac{-p}{(p+w)(p+w+1)} \\ \frac{-(n-p)}{(n-p+w)(n-p+w+1)} & 0 \end{pmatrix}, \qquad C_w = \tfrac{w+1}{2w+n+1} I.$$

Proof. To verify the $(1,1)$-entry of the equation in the statement of the theorem we need to prove that

$$\begin{aligned} x\Big(\tfrac{1}{n+1} C_w^{\frac{n-1}{2}-1}(x) + \tfrac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}(x)\Big) &= \tfrac{(n+w)(p+w-1)(n-p+w+1)}{(2w+n+1)(p+w)(n-p+w)} \Big(\tfrac{1}{n+1} C_{w-1}^{\frac{n-1}{2}-1}(x) + \tfrac{1}{p+w} C_{w-3}^{\frac{n+3}{2}}(x)\Big) \\ &\quad - \tfrac{p}{(p+w)(p+w+1)(n-p+w)} C_{w-1}^{\frac{n+3}{2}}(x) + \tfrac{w+1}{2w+n+1} \Big(\tfrac{1}{n+1} C_{w+1}^{\frac{n-1}{2}-1}(x) + \tfrac{1}{p+w-1} C_{w-1}^{\frac{n+3}{2}}(x)\Big). \end{aligned} \tag{17}$$

By replacing the identities given by (8) for $\lambda = \frac{n+1}{2}$, $m = w$ and $\lambda = \frac{n+3}{2}$, $m = w-2$, one obtain that (17) is equivalent to

$$\begin{aligned} &\tfrac{(w+n)}{(n+1)(2w+n+1)} \Big(-1 + \tfrac{(p+w-1)(n-p+w-1)}{(p+w)(n-p+w)}\Big) C_{w-1}^{\frac{n+1}{2}}(x) \\ &+ \Big(-\tfrac{p}{(p+w)(p+w+1)(n-p+w)} + \tfrac{w+1}{(2w+n+1)(p+w+1)} - \tfrac{w-1}{(p+w)(2w+n-1)}\Big) C_{w-1}^{\frac{n+3}{2}}(x) \\ &+ \tfrac{(n+w)}{p+w} \Big((\tfrac{n-p+w-1}{(2w+n+1)(n-p+w)} - \tfrac{1}{2w+n-1}\Big) C_{w-3}^{\frac{n+3}{2}}(x) = 0. \end{aligned} \tag{18}$$

Thus, by using the relation (10) for $\lambda = \frac{n+3}{2}$ and $m = w-2$, the identity in (18) follows after some straightforward computations.

Now we verify that the equation for the $(1,2)$-entry in the statement of the theorem holds. We need to verify that the following identity holds

$$\begin{aligned}\frac{1}{p+w}\, x\, C_{w-1}^{\frac{n+3}{2}}(x) &= \frac{(n+w)(n-p+w+1)}{(p+w)(2w+n+1)(n-p+w)} C_{w-2}^{\frac{n+3}{2}}(x) \\ &- \frac{p}{(p+w)(p+w+1)}\left(\frac{1}{n+1} C_{w}^{\frac{n+1}{2}}(x) + \frac{1}{n-p+w} C_{w-2}^{\frac{n+3}{2}}(x)\right) + \frac{w+1}{(2w+n+1)(p+w+1)} C_{w}^{\frac{n+3}{2}}(x).\end{aligned} \tag{19}$$

From (10) for $\lambda = \frac{n+3}{2}$ and $m = w-1$ we have that the right-hand side of (19) is

$$\frac{n+w+1}{(p+w)(2w+n+1)} C_{w-2}^{\frac{n+3}{2}}(x) + \frac{w}{(p+w)(2w+n+1)} C_{w}^{\frac{n+3}{2}}(x).$$

Therefore, (19) is proved, since it is equivalent to (8) with $\lambda = \frac{n+3}{2}$ and $m = w-1$.

For the entries $(2,2)$ and $(2,1)$ we proceed in a similar way, by observing that we need to do the same computations as in the cases $(1,1)$ and $(1,2)$ respectively, by changing p by $n-p$. This concludes the proof of the theorem. □

The sequence of monic orthogonal polynomials is given by

$$Q_w = \frac{w!(n+1)}{2^w \left(\frac{n+1}{2}\right)_w} P_w, \qquad w \in \mathbb{N}_0. \tag{20}$$

The first polynomials of the sequence $\{Q_w\}_{w\in\mathbb{N}_0}$ are

$$Q_0 = \mathrm{Id}, \qquad Q_1 = \begin{pmatrix} x & \frac{1}{p+1} \\ \frac{1}{n-p+1} & x \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} x^2 - \frac{p}{(n+3)(p+2)} & \frac{2}{p+2}x \\ \frac{2}{n-p+2}x & x^2 - \frac{n-p}{(n+3)(n-p+2)} \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} x^3 - \frac{3(p+1)}{(n+5)(p+3)}x & \frac{3}{p+3}x^2 - \frac{3}{(n+5)(p+3)} \\ \frac{3}{n-p+3}x^2 - \frac{3}{(n+5)(n-p+3)} & x^3 - \frac{3(n-p+1)}{(n+5)(n-p+3)}x \end{pmatrix}.$$

Remark 4.2. Observe that from (16) and (20) we have that $\langle Q_w, Q_w\rangle$ is always a diagonal matrix. Moreover one can verify that

$$\langle Q_w, Q_w\rangle = \|Q_w\|^2 = \frac{\pi 2^{[w/2]}\Gamma(n/2+1+[w/2])}{w!(n+2w+1)\Gamma((n+3)/2)} \prod_{k=1}^{[(w-1)/2]}(n+2k+1)\begin{pmatrix} \frac{p\,(n-p+w+1)}{p+w} & 0 \\ 0 & \frac{(n-p)(p+w+1)}{n-p+w}\end{pmatrix}.$$

5. The algebra $\mathcal{D}(W)$

In this section we discuss some properties of the structure of the algebra $\mathcal{D}(W)$, defined in (3), for our weight matrix $W(x)$ introduced in (12). We are not interested in the cases when $p = n-p$, since the weight reduces to classical scalar weights, see Remark 3.3. We observe that in our example, the polynomials $\{P_w\}_{w\in\mathbb{N}_0}$, given in (4), and the monic orthogonal polynomials $\{Q_w\}_{w\in\mathbb{N}_0}$ have the same sequence of eigenvalues, since they are related by a scalar multiple, see (20).

First of all we observe that the space of differential operators of *order zero* in $\mathcal{D}(W)$ consists of scalar multiplies of the identity operator. In fact, a differential operator of order zero, having the sequence of monic orthogonal polynomials $\{Q_w\}_w$ as eigenfunctions, is a constant matrix L such that

$$Q_w L = \Lambda_w\, Q_w, \qquad \text{for all } w \in \mathbb{N}_0.$$

From (2) we have that $\Lambda_w = L$ for every w. When $w = 1$, we obtain that the entries of L satisfy $L_{11} = L_{22}$ and $(p+1)L_{12} = (n-p+1)L_{21}$. Thus, looking at the case $w = 2$ we get $(n-2p)L_{12} = 0$. Therefore we obtain that any operator of order zero L in $\mathcal{D}(W)$ is a multiple of the identity matrix.

Now we study differential operators of order at most two in the algebra $\mathcal{D}(W)$. Let $\{Q_w\}_{w\in\mathbb{N}_0}$ the sequence of monic orthogonal polynomials with respect to W and D a differential operator of order at most two in $\mathcal{D}(W)$. From Proposition 2.2 we have

$$D = \partial^2(A_2x^2 + A_1x + A_0) + \partial(B_1x + B_0) + C \in \mathcal{D}(W)$$

if and only if

$$Q_wD = \big(w(w-1)A_2 + wB_1 + C\big)Q_w, \qquad \text{for all } w \in \mathbb{N}_0.$$

Here $A_2, A_1, A_0, B_1, B_0, C$ are 2×2 complex matrices. Let us denote $Q_{w,j}$ the coefficients of the polynomial Q_w, i.e., $Q_w = \sum_{j=0}^{w} Q_{w,j}\, x^j$, with $Q_{w,w} = I$. Therefore $D \in \mathcal{D}(W)$ if and only if

$$\begin{aligned} j(j-1)Q_{w,j}A_2 + j(j+1)Q_{w,j+1}A_1 + (j+1)(j+2)Q_{w,j+2}A_0 + jQ_{w,j}B_1 \\ + (j+1)Q_{w,j+1}B_0 + Q_{w,j}C - \big(w(w-1)A_2 + wB_1 + C\big)Q_{w,j} = 0 \end{aligned}$$

for all $w \in \mathbb{N}_0$ and $j = 0,\dots,w$. For $j = w-1$ and $j = 0$ we respectively obtain

$$\begin{aligned} (w-1)(w-2)Q_{w,w-1}A_2 + w(w-1)A_1 + (w-1)Q_{w,w-1}B_1 + wB_0 + Q_{w,w-1}C \\ - \big(w(w-1)A_2 + wB_1 + C\big)Q_{w,w-1} = 0 \end{aligned} \tag{21}$$

and

$$2Q_{w,2}A_0 + Q_{w,1}B_0 + Q_{w,0}\,C - \big(w(w-1)A_2 + wB_1 + C\big)Q_{w,0} = 0. \tag{22}$$

Now from (21) considering $w = 1$ and $w = 2$, and from (22) considering $w = 2$, we respectively obtain

$$\begin{aligned} &B_0 = (B_1 + C)Q_{1,0} - Q_{1,0}C, \qquad 2A_1 = (2A_2 + 2B_1 + C)Q_{2,1} - Q_{2,1}B_1 - 2B_0 - Q_{2,1}C, \\ &2A_0 = (2A_2 + 2B_1 + C)Q_{2,0} - Q_{2,1}B_0 - Q_{2,0}C. \end{aligned}$$

From the expression of Q_1 and Q_2, given at the end of Section 4, we know that

$$Q_{1,0} = \begin{pmatrix} 0 & \frac{1}{p+1} \\ \frac{1}{n-p+1} & \end{pmatrix}, \qquad Q_{2,1} = \begin{pmatrix} 0 & \frac{2}{p+2} \\ \frac{2}{n-p+2} & 0 \end{pmatrix}, \qquad Q_{2,0} = \tfrac{-p}{(n+3)} \begin{pmatrix} \frac{1}{(p+2)} & 0 \\ 0 & \frac{1}{(n-p+2)} \end{pmatrix}.$$

By using (20) and (4) it is easy to see that

$$Q_{w,w-1} = \begin{pmatrix} 0 & \frac{w}{p+w} \\ \frac{w}{n-p+w} & \end{pmatrix}, \qquad \text{for all } w \in \mathbb{N}.$$

To determine the matrices $A_2 = (a_{ij})$, $B_1 = (b_{ij})$ and $C = (c_{ij})$, we first combine the entries in the diagonal of the matrix (21) to obtain

$$\begin{aligned} 2(n+2)a_{21} &= \frac{\big((n+p+2)b_{21} - 2c_{21}\big)}{p+1} + \frac{(p+2)(p+w)(2c_{12} - (n-p)b_{12})}{(n-p+1)(n-p+2)(n-p+w)}, \\ 2(n+2)a_{12} &= \frac{\big((2n-p+2)b_{12} - 2c_{12}\big)}{n-p+1} + \frac{(n-p+2)(n-p+w)(2c_{21} - p\,b_{21})}{(p+1)(p+2)(p+w)}. \end{aligned}$$

Since these identities are valid for any integer $w \geq 3$ we conclude that, if $n \neq 2p$ then $2c_{12} = (n-p)b_{12}$ and $2c_{21} = p\,b_{21}$. Therefore $b_{21} = 2(p+1)a_{21}$ and $b_{12} = 2(n-p+1)a_{12}$.

By combining the nondiagonal entries of (21) we have

$$(n-2p+1)\big((n+2)a_{11} - b_{11}\big) = (n-2p-1)\big((n+2)a_{22} - b_{22}\big)$$

and

$$c_{11} - c_{22} = (p+1)(p+2)a_{22} - p(p+1)a_{11} + p\,b_{11} - (p+1)b_{22}.$$

Equation (22) with $w = 3$ says that

$$2Q_{3,2}A_0 + Q_{3,1}B_0 + Q_{3,0}\,C - \big(6A_2 + 3B_1 + C\big)Q_{3,0} = 0.$$

Now, by using the expression of $Q_3 = x^3 + Q_{3,2}x^2 + Q_{3,1}x + Q_{3,0}$ given at the end of Section 4, it is not difficult to see that $b_{11} = (n+2)a_{11}$. Thus $b_{22} = (n+2)a_{22}$, and $c_{11} - c_{22} = p(n-p+1)a_{11} - (p+1)(n-p)a_{22}$.

Therefore, the matrices $A_2, A_1, A_0, B_1, B_0, C$ are given in terms of the entries of A_2 and c_{11}, as we state in the following theorem.

Theorem 5.1. *The differential operators of order at most two in $\mathcal{D}(W)$ are of the form*

$$D = \partial^2 F_2(x) + \partial F_1(x) + F_0,$$

where

$$\begin{aligned} F_2(x) =& x^2 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + x \begin{pmatrix} a_{12} - a_{21} & a_{11} - a_{22} \\ a_{22} - a_{11} & a_{21} - a_{12} \end{pmatrix} + \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}, \\ F_1(x) =& x \begin{pmatrix} (n+2)a_{11} & 2(n-p+1)a_{12} \\ 2(p+1)a_{21} & (n+2)a_{22} \end{pmatrix} + \begin{pmatrix} -pa_{21} + (n-p+2)a_{12} & (n-p+2)a_{11} - (n-p)a_{22} \\ -pa_{11} + (p+2)a_{22} & (p+2)a_{21} - (n-p)a_{12} \end{pmatrix}, \\ F_0 =& \begin{pmatrix} p\,(n-p+1)a_{11} + c & (n-p)(n-p+1)a_{12} \\ p\,(p+1)a_{21} & (p+1)(n-p)a_{22} + c \end{pmatrix}. \end{aligned} \tag{23}$$

with a_{11}, a_{12}, a_{21}, a_{22}, c arbitrary complex numbers.

Proof. We have already proved that any differential operator of order at most two in $\mathcal{D}(W)$ is of this form for some constant $a_{11}, a_{12}, a_{21}, a_{22}, c \in \mathbb{C}$. Let $\mathcal{D}_2$ be the complex vector space of the differential operators in $\mathcal{D}(W)$ of order at most two. Then we have that $\dim \mathcal{D}_2 \leq 5$.

From Proposition 2.1 it is not difficult to see that a differential operator D of order two, with coefficients given by (23), is a symmetric operator if and only if

$$a_{11}, a_{22}, c \in \mathbb{R} \qquad \text{and} \qquad p\,a_{21} = (n-p)\,\overline{a}_{12}.$$

From Proposition 2.5 any symmetric operator $D \in \mathcal{D}$ belongs to the algebra $\mathcal{D}(W)$. Thus there exists (at least) five $\mathbb{R}$-linearly independent symmetric operators in $\mathcal{D}_2$. Therefore $\dim \mathcal{D}_2 = 5$ and this concludes the proof of the theorem. □

Corollary 5.2. *There are no operators of order one in the algebra $\mathcal{D}(W)$.*

The elements of the sequence $\{Q_w\}_w$ are eigenfunctions of the operators $D \in \mathcal{D}(W)$ and they satisfy $Q_w D = \Lambda_w(D) Q_w$,for $w \in \mathbb{N}_0$. We explicitly state the eigenvalues Λ_w using formula (2): for a differential operator $D = \partial^2 F_2 + \partial F_1 + F_0$ we have

$$\Lambda_w(D) = w(w-1)F_2^2 + wF_1^1 + F_0^0,$$

with F_i^i (i=1,2,3) the leading coefficient of the polynomial coefficient F_i of the differential operator D. Therefore we get

Corollary 5.3. *Let $D \in \mathcal{D}(W)$, defined as in Theorem 5.1. The monic orthogonal polynomials $\{Q_w\}_w$ satisfy*

$$Q_w D = \Lambda_w(D) Q_w, \qquad \textit{for } w \in \mathbb{N}_0,$$

where the eigenvalue $\Lambda_w(D)$ is given by

$$\Lambda_w(D) = \begin{pmatrix} (w+p)(w+n-p+1)a_{11} + c & (w+n-p)(w+n-p+1)a_{12} \\ (w+p)(w+p+1)a_{21} & (w+n-p)(w+p+1)a_{22} + c \end{pmatrix}.$$

Now we introduce a useful basis for the differential operators of order at most two in the algebra $\mathcal{D}(W)$: the identity I and

$$D_1 = \partial^2 \begin{pmatrix} x^2 & x \\ -x & -1 \end{pmatrix} + \partial \begin{pmatrix} (n+2)x & n-p+2 \\ -p & 0 \end{pmatrix} + \begin{pmatrix} p\,(n-p+1) & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_2 = \partial^2 \begin{pmatrix} -1 & -x \\ x & x^2 \end{pmatrix} + \partial \begin{pmatrix} 0 & p-n \\ p+2 & (n+2)x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (p+1)(n-p) \end{pmatrix},$$
$$D_3 = \partial^2 \begin{pmatrix} -x & -1 \\ x^2 & x \end{pmatrix} + \partial \begin{pmatrix} -p & 0 \\ 2(p+1)x & p+2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ p(p+1) & 0 \end{pmatrix},$$
$$D_4 = \partial^2 \begin{pmatrix} x & x^2 \\ -1 & -x \end{pmatrix} + \partial \begin{pmatrix} n-p+2 & 2(n-p+1)x \\ 0 & p-n \end{pmatrix} + \begin{pmatrix} 0 & (n-p)(n-p+1) \\ 0 & 0 \end{pmatrix}.$$

The corresponding eigenvalues are

$$\Lambda_w(D_1) = \begin{pmatrix} (w+p)(w+n-p+1) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Lambda_w(D_2) = \begin{pmatrix} 0 & 0 \\ 0 & (w+p+1)(w+n-p) \end{pmatrix},$$
$$\Lambda_w(D_3) = \begin{pmatrix} 0 & 0 \\ (w+p)(w+p+1) & 0 \end{pmatrix}, \qquad \Lambda_w(D_4) = \begin{pmatrix} 0 & (w+n-p)(w+n-p+1) \\ 0 & 0 \end{pmatrix}.$$

Remark 5.4. The differential operator D appearing in Theorem 3.1 is $D = -D_1 - D_2 + p(n-p)I$.

We observe here that, for example,

$$\Lambda_w(D_1)\Lambda_w(D_3) \neq \Lambda_w(D_3)\Lambda_w(D_1), \quad \text{for all } w \in \mathbb{N}_0.$$

By using Proposition 2.3 we obtain that $D_1D_3 \neq D_3D_1$, which in turn implies the following result.

Corollary 5.5. *The algebra $\mathcal{D}(W)$ is not commutative.*

By following the same argument, through the sequence of eigenvalues, we obtain the following relations among the differential operators D_1, D_2, D_3, D_4.

$$D_1D_2 = 0, \quad D_2D_1 = 0, \quad D_1D_3 = 0, \quad D_4D_1 = 0, \quad D_2D_4 = 0, \quad D_3D_2 = 0, \quad D_3^2 = 0, \quad D_4^2 = 0,$$
$$D_3D_1 = D_2D_3 - (n-2p)D_3, \quad D_1D_4 = D_4D_2 - (n-2p)D_4, \quad D_3D_4 = D_2^2 - (n-2p)D_2,$$
$$D_4D_3 = D_1^2 + (n-2p)D_1.$$

Conjecture 5.6.

(1) *There are no operators of odd order in $\mathcal{D}(W)$.*
(2) *The second order differential operators in $\mathcal{D}(W)$ generate the algebra $\mathcal{D}(W)$.*

For a differential operator of order two $D = \partial^2 F_2 + \partial F_1 + F_0 \in \mathcal{D}(W)$, the explicit expression of the adjoint operator D^* is

$$D^* = \partial^2 G_2 + \partial G_1 + G_0,$$

where the polynomials G_i, $i = 0, 1, 2$, are defined by

$$G_0 = \langle Q_0, Q_0\rangle \Lambda_0(D)^* \langle Q_0, Q_0\rangle^{-1}, \qquad G_1 = \langle Q_1, Q_1\rangle \Lambda_1(D)^* \langle Q_1, Q_1\rangle^{-1} Q_1(x) - Q_1(x)G_0,$$
$$G_2 = \langle Q_2, Q_2\rangle \Lambda_2(D)^* \langle Q_2, Q_2\rangle^{-1} Q_2(x) - \partial(Q_2)G_1(x) - Q_2(x)G_0,$$

see Theorem 4.3 in [13].

Also from Corollary 4.5 in [13], we obtain the expression for the corresponding eigenvalues for the adjoint operator D^*, in terms of the eigenvalues of the differential operator D and the norm of the polynomials Q_w,

$$\Lambda_w(D^*) = \langle Q_w, Q_w\rangle \Lambda_w(D)^* \langle Q_w, Q_w\rangle^{-1}, \quad \text{for all } w.$$

By using the expressions of $\langle Q_i, Q_i\rangle$, given at the end of Section 4, and making straightforward computations, we can verify that

$$D_1^* = D_1, \quad D_2^* = D_2, \quad \text{and} \quad D_3^* = \tfrac{p}{n-p} D_4.$$

Therefore

$$E_3 = (n-p)D_3 + pD_4 \qquad \text{and} \qquad E_4 = i\big((n-p)D_3 - pD_4\big)$$

are also symmetric operators, because for any $D \in \mathcal{D}(W)$ the operators $D + D^*$ and $i(D - D^*)$ are symmetric operators. Explicitly,

$$E_3 = (n-p)D_3 + pD_4 = \partial^2 \begin{pmatrix} -x(n-2p) & x^2p - n + p \\ x^2(n-p) - p & x(n-2p) \end{pmatrix} + \partial \begin{pmatrix} 2p & 2p(n-p+1)x \\ 2(p+1)(n-p)x & 2(n-p) \end{pmatrix} + \begin{pmatrix} 0 & p(n-p)(n-p+1) \\ p(p+1)(n-p) & 0 \end{pmatrix},$$

$$-iE_4 = (n-p)D_3 - pD_4 = \partial^2 \begin{pmatrix} -nx & -x^2p - n + p \\ x^2(n-p) + p & nx \end{pmatrix} + \partial \begin{pmatrix} -2p(n-p+1) & -2p(n-p+1)x \\ 2(p+1)(n-p)x & 2(n-p)(p+1) \end{pmatrix} + \begin{pmatrix} 0 & -p(n-p)(n-p+1) \\ p(p+1)(n-p) & 0 \end{pmatrix}.$$

The corresponding eigenvalues are

$$\Lambda_w(E_3) = \begin{pmatrix} 0 & p(n-p+w)(n-p+w+1) \\ (n-p)(p+w)(p+w+1) & 0 \end{pmatrix},$$

$$\Lambda_w(-iE_4) = \begin{pmatrix} 0 & -p(n-p+w)(n-p+w+1) \\ (n-p)(p+w)(p+w+1) & 0 \end{pmatrix}.$$

Remark 5.7. In [16] the authors study matrix valued orthogonal polynomials related to spherical functions on the group $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$. The weight matrix is $W_+^{(\nu)}$, with $\nu = 1$ in the notation of Remark 3.7. Let us denote $\widetilde{D}_1$, $\widetilde{D}_2$ and $\widetilde{D}_3$ the differential operators D_1,D_2 and D_3 appearing in Theorem 8.1 in [16]. Then we have the following relations with our operators D_1, D_2, D_3 and D_4 for the case $n = 3$ and $p = 1$

$$\widetilde{D}_1 = L(D_1 + D_2 - 3)L^{-1}, \quad \widetilde{D}_2 = LD_2L^{-1}, \quad \widetilde{D}_3 = -\sqrt{2}\, L(2D_3 + D_4)L^{-1}.$$

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