Photon spheres in Einstein and Einstein-Gauss-Bonnet theories and circular null geodesics in axially-symmetric spacetimes

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In this article we extend a recent theorem proven by Hod [Upper bound on the radii of black-hole photon spheres, Phys. Lett. B 727, 345 (2013)] to n-dimensional Einstein and Einstein-Gauss-Bonnet theories, which gives an upper bound for the photon sphere radii of spherically symmetric black holes. As applications of these results we give a universal upper bound for the real part of quasinormal modes in the WKB limit and a universal lower bound for the position of the first relativistic image in the strong lensing regime produced by these type of black holes. For the axially-symmetric case, we also make some general comments (independent of the underlying gravitational theory) on the relation between circular null geodesics and the fastest way to circle a black hole.

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I. INTRODUCTION

As is well known, the study of null geodesics at a given spacetime is not only relevant from a theoretical point of view, but also from a practical perspective. Their analysis, in addition to information about the causal structure of the spacetime, can also establish observable consequences of various astrophysical phenomena [1–7].

The role of so-called circular null geodesics is not trivial. In some situations, these geodesics can allow the existence of what is known as a photon surface, which, in the context of four-dimensional spacetimes, is a three-dimensional nonspacelike manifold $\mathcal{P}$, such that every null geodesic whose tangent vector l^a at a given point $p \in \mathcal{P}$ is contained in the tangent space $T_p\mathcal{P}$ of $\mathcal{P}$ at p, always remains in $\mathcal{P}$. In particular, in the case of spherically symmetric spacetimes, a photon sphere can be defined as a $SO(3) \times \mathcal{R}$-invariant photon surface. For rigorous definitions see [8].

The study of photon spheres in the case of spherical symmetry and circular null geodesics in axial symmetric spacetimes is important in astrophysics for several reasons:

(a) They appear explicitly in the study of relativistic images produced by black holes in the strong lensing regime [9,10].
(b) They play an important role in the analysis of quasinormal modes in black hole perturbations [11,12].
(c) They determine the shadow of black holes, or (which is the same), how they look to outside observers [13].
(d) In the case of hairy black holes, they determine a lower bound on the size of the hair [14].
(e) They allow a link between points (a) and (b), or more precisely between gravitational waves and lensing, as recently shown in [15,16].

They have also been analyzed in situations that do not contain black holes, as for example, in Boson stars [17], where the authors showed that in certain configurations photon spheres can occur, or in certain classes of regular metrics with non-negative trace of the energy-momentum tensor, where the existence of at least two photon spheres was shown [18]. Recently, a study of the thermodynamics of a quantum version of photon spheres was presented [19].

Although in axially-symmetric spacetimes there is no notion yet of the analog of photon spheres, there are circular null geodesics on the equatorial plane, and they are also useful for the discussion of the previous points. For all these considerations, the characterization and localization of these particular surfaces or circular null geodesics is relevant.

Recently [20], Hod made some interesting observations about this type of null geodesics. In particular, in the framework of four-dimensional general relativity (GR), he analyzed a general family of spherically symmetric black holes, which satisfy some natural asymptotic and energy conditions, finding an upper bound for the radius of photon spheres in terms of their ADM mass. In the case of axial symmetry, he studied circular null geodesics on the equatorial plane of a Kerr spacetime, finding that they are also the fastest way to circle a Kerr black hole [21] and conjecturing an expression for the minimum orbital time to circle any compact object in GR in terms of its mass. As shown by Pradhan [22], the conclusion that circular null

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geodesics are the fastest way to circle black holes remains valid for the n-dimensional version of the Kerr-Newman metric, namely, for charged Myers-Perry spacetimes.

The reason for and interest in the study of theories of gravity in higher dimensions is motivated by string theory. As a possibility, the Einstein-Gauss-Bonnet (EGB) gravity theory is selected by the low energy limit of the string theory [23,24]. In this theory, corrective terms to Einstein gravity appear, which are quadratic in the curvature of the spacetime. The effect of those Gauss-Bonnet terms is nontrivial for higher dimensions, so the theory of gravity, which includes Gauss-Bonnet terms, is called EGB gravity.

Even without these Gauss-Bonnet corrections, the n-dimensional version of GR is usually studied in astrophysical and theoretical contexts, and in general it is also the case using other alternative gravitational theories. For some works on these topics in EGB or pure GR see [11,25–28].

Because of the evidence garnered of the importance of circular null geodesics in the characterization of spherically and axially-symmetric black holes, we extend and generalize some of the Hod results to higher-dimensional gravitational theories. In particular we study photon spheres in n-dimensional GR and EGB theory and circular null geodesics on the equatorial plane for generic axially-symmetric spacetimes.

The article is organized as follows. In Sec. II we review EGB theory, and prepare the setting for the discussion of photon spheres. In Sec. III, we establish and prove two theorems that state upper bounds on the radii of photon spheres for black holes in GR and EGB theories, in terms of their ADM mass. In Sec. IV, we analyze some of their implications, giving a universal upper bound for the real part of quasinormal frequencies in the WKB limit and a lower bound for the first relativistic image in the strong lensing regime. In Sec. V, we make some general comments on the relation between circular null geodesics in axially-symmetric spacetimes and the fastest way to circle black holes. We also show, by using an explicit counterexample, that a lower bound for the orbital period of circular null geodesics conjectured by Hod in the context of four-dimensional GR [21] cannot be assumed to be valid also in alternative gravitational theories.

II. BACKGROUND AND SETTING

The action that describes Einstein-Gauss-Bonnet gravity coupled with matter fields reads

$$S = \frac{1}{16\pi}\int d^n x\sqrt{-g} \times [R - 2\Lambda + \alpha(R_{abcd}R^{abcd} + R^2 - 4R_{ab}R^{ab})] + S_{\mathrm{matt}},$$

where S_{matt} is the action associated with the matter fields, and α is the Gauss-Bonnet coupling constant associated in the string models, with the tension of these strings. This constant introduces a length scale. In fact, the corrections that this theory produces to GR are noted at short distances, given by the scale $l = \sqrt{4\alpha}$.

The equations of motion resulting from $\delta S = 0$ are

$$\kappa T_{ab} = \mathcal{G}_{ab} = G^{(0)}_{ab} + G^{(1)}_{ab} + G^{(2)}_{ab},$$

where $\kappa = 8\pi G/c^4$ is the gravitational constant, T_{ab} is the energy-momentum tensor, representing the matter-field distribution resulting from the variation $\delta S_{\mathrm{matt}}/\delta g^{ab}$, and

$$\begin{aligned} G^{(0)}_{ab} &= \Lambda g_{ab} \\ G^{(1)}_{ab} &= R_{ab} - \frac{1}{2}R g_{ab} \\ G^{(2)}_{ab} &= -\alpha\left[\frac{1}{2}g_{ab}(R_{cjek}R^{cjek} - 4R_{cj}R^{cj} + R^2)\right. \\ &\qquad \left. - 2RR_{ab} + 4R_{ac}R^{c}_{b} + 4R_{cj}R^{cj}_{ab} - 2R_{acje}R^{cje}_{b}\right]. \end{aligned}$$

From now on, we use units where $c = 1$ and we adopt α positive since this condition arises from the string theory. We also assume asymptotically flat spacetimes, and therefore $\Lambda = 0$.

Let us consider a spherically symmetric metric given by

$$ds^2 = e^{-2\delta(r)}\mu(r)dt^2 - \mu(r)^{-1}dr^2 - r^2 d\Omega^2_{n-2}, \qquad (1)$$

with $d\Omega^2_{n-2}$ being the metric of the $(n-2)$-sphere

$$d\Omega^2_{n-2} = d\theta_1^2 + \sum_{i=2}^{n-2}\prod_{j=1}^{i-1}\sin^2\theta_j d\theta_i^2 \qquad (2)$$

solution to the Einstein-Gauss-Bonnet equations in higher dimensions. The area of the unit $(n-2)$-sphere is given by

$$S_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}, \qquad (3)$$

with $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, the gamma function.

The Einstein-Gauss-Bonnet equations in terms of the components of the energy-momentum tensor, $T^t_t = -\rho$, $T^r_r = p_r$, and $T^{\theta_i}_{\theta_i} = p_\perp$ read

$$\begin{aligned} \kappa\rho = &-\frac{1}{2r^2}(n-2)[r\mu' + (n-3)(\mu-1)] \\ &+ \frac{\hat{\alpha}}{2r^4}(n-2)(\mu-1)[2r\mu' + (n-5)(\mu-1)], \end{aligned} \qquad (4)$$

$$\begin{aligned} \kappa p_r = -(n-2)\Bigg\{&\frac{1}{2r^2}[2r\mu\delta' - r\mu' - (n-3)(\mu-1)] \\ &- \frac{\hat{\alpha}}{2r^4}(\mu-1)[-2r\mu' + 4r\mu\delta' - (\mu-1)(n-5)]\Bigg\}, \end{aligned} \qquad (5)$$

where $\hat{\alpha} = \alpha(n-3)(n-4)$. We have not written the angular-angular equations because we do not need them. Instead, we use the only nontrivial component of the energy-momentum conservation equation $\nabla_a T^a_r = 0$, which reads

$$p_r' = -\frac{(e^{-2\delta}\mu)'}{2e^{-2\delta}\mu}(\rho + p_r) + \frac{n-2}{r}(p_\perp - p_r). \tag{6}$$

From Eqs. (4)–(5) it follows that

$$\mu[2\alpha(\mu-1) - r^2]\delta' = \kappa\frac{r^3}{(n-2)}(\rho + p_r). \tag{7}$$

We are interested in regular black holes. In particular, we assume that at the horizon r_H, $\mu(r)$, and $\delta(r)$ satisfy

$$\mu(r_H) = 0, \qquad \mu'(r_H) \geq 0, \tag{8}$$

$$\delta(r_H) \leq \infty, \qquad \delta'(r_H) \leq \infty. \tag{9}$$

These conditions, together with (7) imply

$$\rho(r_H) + p_r(r_H) = 0. \tag{10}$$

Because of the asymptotic flatness requirement, we also assume

$$\mu(r \to \infty) = 1, \tag{11}$$

$$\delta(r \to \infty) \to 0. \tag{12}$$

Note that, by requiring a GR limit as $\hat{\alpha} \to 0$, the general solution of (4) can be written as

$$\mu = 1 + \frac{r^2}{2\hat{\alpha}}\left(1 - \sqrt{1 + \frac{8\kappa\hat{\alpha}M(r)}{(n-2)S_{n-2}r^{n-1}}}\right), \tag{13}$$

with $M(r)$ given by

$$M(r) = M_H + S_{n-2}\int_{r_H}^{r} \rho r^{n-2} dr, \tag{14}$$

which can be shown to be the generalized Misner-Sharp mass [29]. In particular, M_H is the horizon mass, and when r goes to infinity, $M(r)$ goes to the ADM mass $\mathcal{M}$. In order to have a finite ADM mass, ρ must satisfy

$$\lim_{r\to\infty} r^{n-1}\rho = 0. \tag{15}$$

In the vacuum case ($T^a_b = 0$), we have

$$\mu = 1 + \frac{r^2}{2\hat{\alpha}}\left(1 - \sqrt{1 + \frac{8\kappa\hat{\alpha}\mathcal{M}}{(n-2)S_{n-2}r^{n-1}}}\right); \tag{16}$$

and the associated vacuum metric is known as the Boulware-Deser-Wheeler (BDW) black hole.

If $\hat{\alpha} \to 0$, Eq. (13) reduces to its GR limit,

$$\mu = 1 - \frac{2\kappa M(r)}{(n-2)S_{n-2}r^{n-3}}. \tag{17}$$

Finally,

$$T = -\rho + p_r + (n-2)p_\perp \tag{18}$$

denotes the trace of the energy-momentum tensor and we assume that matter satisfies the dominant energy condition (DEC) $\rho \geq 0$, $\rho \geq |p_r|, |p_\perp|$. From the DEC, we see then that $M_H \leq M(r) \leq \mathcal{M}$.

III. UPPER BOUND ON THE PHOTON SPHERE RADII

We are now prepared to establish the following theorems:

Theorem 1: Hod's theorem (n-dimensional GR version): Let $(\tilde{M}, g_{ab})$ be a spherically symmetric spacetime, such that (i) it has a regular event horizon, (ii) it is asymptotically flat, (iii) the dominant energy condition is satisfied and the energy-momentum trace is nonpositive, and (iv) it satisfies the n-dimensional Einstein equations. Then this metric admits at least one photon sphere, which in the coordinates given by (1) is characterized by a radius r_γ, which is bounded by the following expression in terms of its total ADM mass $\mathcal{M}$:

$$r_\gamma \leq \left(\frac{\kappa(n-1)}{(n-2)S_{n-2}}\mathcal{M}\right)^{\frac{1}{n-3}}. \tag{19}$$

This theorem can be extended to Einstein-Gauss-Bonnet theory in n dimensions. In particular, in five dimensions we can obtain an upper bound for the photon sphere radius in terms of the ADM mass and the constant $\hat{\alpha}$. For $n > 5$, although we cannot give a general expression for the photon sphere radius bound in terms of the ADM mass (with the exception of $n = 9$), we can state a more general result. These properties are summarized in the following theorem.

Theorem 2: n-dimensional EGB version: Let $(\tilde{M}, g_{ab})$ be a spherically symmetric spacetime such that (a) it satisfies conditions (i), (ii), and (iii) of theorem 1, and (b) it satisfies the n-dimensional EGB equations. Then this metric admits at least one photon sphere, whose radius r_γ in the coordinates given by (1) is always bounded by the photon sphere radius r_γ^* of a vacuum spherically symmetric BDW black hole with the same total ADM mass $\mathcal{M}$, i.e., $r_\gamma \leq r_\gamma^*$. In particular in five dimensions the photon sphere radius is bounded by the following expression in terms of its total ADM mass $\mathcal{M}$:

$$r_\gamma \leq \frac{\sqrt{6}[\kappa\mathcal{M}(\kappa\mathcal{M} - 3\hat{\alpha}\pi^2)]^{1/4}}{3\pi}. \tag{20}$$

Note that the inequality (20) is saturated for a five-dimensional BDW black hole with mass $\mathcal{M}$.

Proof: Common part. Since the Einstein equations can be recovered from (4)–(5) with $\hat{\alpha} = 0$, in the proof there is a common part for both theorems. Basically, we follow the same steps as Hod's proof [20], although some changes are needed to incorporate the n-dimensional case. Because of the spherical symmetry, null geodesics may without loss of generality be taken to be on the equatorial plane. From the metric (1) it is easy to show that the equation for circular null geodesics $\dot{r}_\gamma = (\dot{r}_\gamma)' = 0$ (where a dot means derivative with respect to an affine parameter and $'$ means derivative with respect to r) reads [11,30]

$$\tilde{N}(r_\gamma) = 0, \tag{21}$$

with

$$\tilde{N}(r) = 2e^{-2\delta(r)}\mu(r) - r[e^{-2\delta(r)}\mu(r)]'. \tag{22}$$

By defining the function

$$N(r) = e^{2\delta(r)}\tilde{N}(r), \tag{23}$$

we similarly obtain

$$N(r_\gamma) = 2[1 + r_\gamma \delta'(r_\gamma)]\mu(r_\gamma) - r_\gamma \mu'(r_\gamma) = 0. \tag{24}$$

From the trace of the energy-momentum tensor and (23), we can write (6) as

$$p_r' = \frac{1}{2\mu r}(N(\rho + p_r) + 2\mu T - 2n\mu p_r). \tag{25}$$

Now if we define

$$P = r^n p_r, \tag{26}$$

we arrive at

$$P'(r) = \frac{r^{n-1}}{2\mu}(N(\rho + p_r) + 2\mu T). \tag{27}$$

In terms of the energy-momentum components, N reads

$$N = \frac{1}{2\hat{\alpha}(\mu - 1) - r^2}\Big\{(n-1)\hat{\alpha}\mu^2 - [(n-1)r^2 + 2(n-3)\hat{\alpha}]\mu + (n-5)\hat{\alpha} + (n-3)r^2 + \frac{2}{n-2}\kappa r^4 p_r\Big\}, \tag{28}$$

which in the n-dimensional GR case reduces to

$$N_{GR} = (n-1)\mu - (n-3) - \frac{2\kappa}{(n-2)}r^2 p_r. \tag{29}$$

Let us observe that (24) admits at least one solution as follows from the fact that the conditions (8) and (23) imply $N(r_H) \le 0$, and taking into account that $\lim_{r\to\infty} r^2 p_r = 0$ [which follows from the dominant energy condition and (15)], we see that $N(r \to \infty) \to 2$; therefore, there is a r_γ where $N(r_\gamma) = 0$. What is more, because we are interested in the innermost null circular orbit, $N(r)$ must satisfy

$$N(r_H \le r < r_\gamma) < 0. \tag{30}$$

Then, by using the dominant energy condition, we obtain

$$p_r(r_H) = -\rho(r_H) \le 0, \qquad \Rightarrow p_r(r_H) \le 0 \Leftrightarrow P(r_H) \le 0. \tag{31}$$

Similarly from (27), (30), and the condition of nonpositive trace $T \le 0$, we see that $P'(r_H \le r < r_\gamma) \le 0$. From this last condition and (31) we have

$$p_r(r_\gamma) \le 0. \tag{32}$$

Let us analyze the implications of this relation in the Einstein and Einstein-Gauss-Bonnet cases separately.

Completion of the proof of theorem 1. In the Einstein case, from (32) and (29) we conclude that

$$\mu(r_\gamma) \le \frac{n-3}{n-1}, \tag{33}$$

which by using (17) implies

$$r_\gamma \le \left(\frac{\kappa(n-1)}{(n-2)S_{n-2}}M(r_\gamma)\right)^{\frac{1}{n-3}}, \tag{34}$$

or in terms of the total ADM mass $\mathcal{M}$,

$$r_\gamma \le \left(\frac{\kappa(n-1)}{(n-2)S_{n-2}}\mathcal{M}\right)^{\frac{1}{n-3}}. \tag{35}$$

Completion of the proof of theorem 2. Let us start by noting that in (28) the following expression appears in the denominator:

$$2\hat{\alpha}(\mu - 1) - r^2 = -\frac{r^2}{2\hat{\alpha}}\sqrt{1 + \frac{8\kappa\hat{\alpha}M(r)}{(n-2)S_{n-2}r^{n-1}}} < 0. \tag{36}$$

Therefore, from (28) and (32) we deduce that at r_γ,

$$N^* \equiv \{(n-1)\hat{\alpha}\mu^2 - [(n-1)r^2 + 2(n-3)\hat{\alpha}]\mu + (n-5)\hat{\alpha} + (n-3)r^2\} = -\frac{2}{n-2}\kappa r^4 p_r \ge 0. \tag{37}$$

The implications of this equation for the allowed r_γ will be discussed first in five dimensions because this case is exactly solvable, and after that, we extend the analysis to higher dimensions.

By writing N^* as

$$N^* = (n-1)\hat{\alpha}(\mu - B_{(n)-})(\mu - B_{(n)+}), \tag{38}$$

we see that in five dimensions, (37) implies that at r_γ, some of these conditions hold:

$$\mu \leq \frac{1}{2\hat{\alpha}}\left(\hat{\alpha} + r^2 - \sqrt{\hat{\alpha}^2 + r^4}\right) \equiv B_{(5)-}, \tag{39}$$

or

$$\mu \geq \frac{1}{2\hat{\alpha}}\left(\hat{\alpha} + r^2 + \sqrt{\hat{\alpha}^2 + r^4}\right) \equiv B_{(5)+}, \tag{40}$$

where in order to write these inequalities we use the fact that $B_{(5)-}$ and $B_{(5)+}$ are both positive real functions and that $B_{(5)-} < B_{(5)+}$.

We also note [using $M_H \leq M(r) \leq \mathcal{M}$] that independently of the dimension, μ is bounded from below by

$$\mu \geq 1 + \frac{r^2}{2\hat{\alpha}}\left(1 - \sqrt{1 + \frac{8\kappa\hat{\alpha}\mathcal{M}}{(n-2)S_{n-2}r^{n-1}}}\right) \equiv \mu_{(n)\mathcal{M}}, \tag{41}$$

and by

$$\mu \leq 1 + \frac{r^2}{2\hat{\alpha}}\left(1 - \sqrt{1 + \frac{8\kappa\hat{\alpha}M_H}{(n-2)S_{n-2}r^{n-1}}}\right) \equiv \mu_{(n)M_H} \tag{42}$$

from above. Consequently, the validity of condition (39) or (40) implies

$$\mu_{(5)\mathcal{M}}(r_\gamma) \leq B_{(5)-}(r_\gamma), \tag{43}$$

or

$$\mu_{(5)M_H}(r_\gamma) \geq B_{(5)+}(r_\gamma), \tag{44}$$

respectively. Moreover, the four functions $\mu_{\mathcal{M}}(r)$, $\mu_{(5)M_H}$, $B_{(5)+}(r)$, and $B_{(5)-}(r)$ are all monotonically increasing functions. From this fact, and observing that $B_{(5)+}(r) \geq 1$, and $\mu_{(5)M_H}(r \to \infty) = 1$, it follows that

$$\mu(r) \leq \mu_{(5)M_H}(r) < B_{(5)+}(r) \quad \forall r; \tag{45}$$

thus, it is not possible for any real r_γ to satisfy the inequality (40).

Let us study inequality (43). Observing that for $r \geq r_H$,

$$0 < B_{(5)-}(r) < B_{(5)-}(r \to \infty) = \frac{1}{2}, \tag{46}$$

and

$$\mu_{(5)\mathcal{M}}(r = r_H) \leq 0, \qquad \mu_{(5)\mathcal{M}}(r \to \infty) = 1, \tag{47}$$

we see that (43) will be valid for all r such that $r_H \leq r \leq r_\gamma^*$, with r_γ^* being the solution of

$$\mu_{(5)\mathcal{M}}(r_\gamma^*) = B_{(5)-}(r_\gamma^*). \tag{48}$$

The only positive real solution of this equation is

$$r_\gamma^* = \frac{\sqrt{6}[\kappa\mathcal{M}(\kappa\mathcal{M} - 3\hat{\alpha}\pi^2)]^{1/4}}{3\pi}. \tag{49}$$

On the other hand, as $\mu(r_H) = 0$, $B_{(5)-}(r_H) > 0$, and $\mu(r) \geq \mu_{(5)\mathcal{M}}(r)$, we conclude that $\mu(r) > B_{(5)-}(r)$ for all $r > r_\gamma^*$. Consequently, the radius r_γ that satisfies (39) must necessarily be bounded from above by r_γ^*, that is

$$r_\gamma \leq \frac{\sqrt{6}[\kappa\mathcal{M}(\kappa\mathcal{M} - 3\hat{\alpha}\pi^2)]^{1/4}}{3\pi}. \tag{50}$$

In the n-dimensional case, as was mentioned above, we cannot find an explicit upper bound in terms of the ADM mass; however, we can show that even in these situations, the photon sphere radius will always be bounded by the photon sphere radius of a BDW black hole with the same mass. It can be shown as follows.

First, in the general case, following a similar analysis as in Eqs. (38)–(40), we see that (37) is satisfied at r_γ if

$$\mu \leq \frac{1}{2\hat{\alpha}}\left(\frac{2(n-3)}{n-1}\hat{\alpha} + r^2 - \sqrt{\frac{16}{(n-1)^2}\hat{\alpha}^2 + r^4}\right) \equiv B_{(n)-}, \tag{51}$$

or if

$$\mu \geq \frac{1}{2\hat{\alpha}}\left(\frac{2(n-3)}{n-1}\hat{\alpha} + r^2 + \sqrt{\frac{16}{(n-1)^2}\hat{\alpha}^2 + r^4}\right) \equiv B_{(n)+}, \tag{52}$$

which using (41)–(42) implies

$$\mu_{(n)\mathcal{M}}(r_\gamma) \leq B_{(n)-}(r_\gamma), \tag{53}$$

or

$$\mu_{(n)M_H}(r_\gamma) \geq B_{(n)+}(r_\gamma). \tag{54}$$

For the same reasons as in the five-dimensional case [$B_{(n)+}(r) \geq 1$, $\mu_{(n)M_H}(r \to \infty) = 1$, and both being monotonically increasing functions], we conclude that there is no real r_γ so that (54) [and hence (52)] can be satisfied. On the other hand, from the fact that $B_{(n)-}$ and $\mu_{(n)\mathcal{M}}$ are monotonically increasing functions, and taking into account that

$$0 \leq B_{(n)-}(r) < B_{(n)-}(r \to \infty) = \frac{n-3}{n-1} < 1 \quad \forall r \geq r_H, \tag{55}$$

and $\mu_{(n)\mathcal{M}}(r_H) \leq 0$, $\mu_{(n)\mathcal{M}}(r \to \infty) = 1$, we see that (53) holds for all $r_H \leq r < r_\gamma^*$ with r_γ^* satisfying

$$\mu_{(n)\mathcal{M}}(r_\gamma^*) = B_{(n)-}(r_\gamma^*). \tag{56}$$

As $\mu(r_H) = 0, B_{(n)-}(r_H) > 0$ and $\mu(r) \geq \mu_{(n)\mathcal{M}}(r)$, and following the same arguments as in the five-dimensional situation, we conclude that the radius r_γ where (39) holds will also satisfy $r_\gamma \leq r_\gamma^*$. Furthermore, from (28), (30), and (38), and observing that

$$N^* = [2\hat{\alpha}(\mu - 1) - r^2]N - \frac{2}{n-2}\kappa r^4 p_r, \tag{57}$$

we see that (56) is equivalent to requiring $N(r_\gamma^*) = 0$ for a BDW black hole with ADM mass $\mathcal{M}$ ($\rho = p_r = p_\perp = 0$). In other words, r_γ^* can be interpreted as the radius of the photon sphere for a BDW black hole with mass $\mathcal{M}$. As a result, the bound is saturated in these cases.

Let us make some remarks before continuing. In order to see how the bound (19) varies in terms of the dimension, we plot the quotient $r_\gamma/(G\mathcal{M})^{\frac{1}{n-3}}$ in Fig. 1, assuming a continuous n. The physical values must be taken only for natural n in the graphic. We see that in eight dimensions this quotient reaches its minimum value. However, as for $n > 4$, the dependence in the mass is not linear; the dimension that minimizes the value of the upper bound for the radius r_γ at a given fixed $\mathcal{M}$ will depend on the value of this mass.

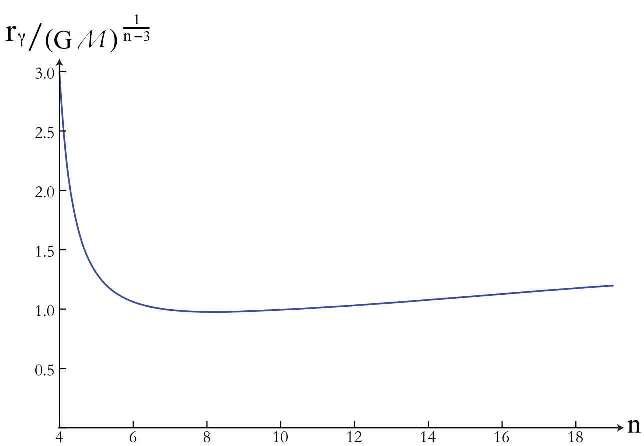

FIG. 1 (color online). Upper bound for the quotient $r_\gamma/(G\mathcal{M})^{\frac{1}{n-3}}$ in terms of the dimension n of the spacetime.

Let us also observe that if we take $\alpha \to 0$ in (50), we recover the upper bound that we found in the GR case.

As a final comment, we mention that (56) can also be solved in nine dimensions, obtaining the result

$$r_\gamma \leq 98(14)^{1/3}\left[\frac{h^{2/3} - 686(14)^{2/3}\pi^2\kappa\hat{\alpha}\mathcal{M}}{\pi^3 h^{1/3}}\right]^{1/4}, \tag{58}$$

with

$$h = \pi\kappa\mathcal{M}\left[18\kappa\mathcal{M} + \sqrt{2\kappa\mathcal{M}(162\kappa\mathcal{M} + 7\pi^4\hat{\alpha}^3)}\right]. \tag{59}$$

IV. APPLICATIONS: QUASINORMAL MODES AND STRONG LENSING

Recently [11], Cardoso *et al.* showed a correspondence between the quasinormal modes associated to a black hole, in the eikonal limit, and some properties of photon spheres. In particular, in this limit, the quasinormal frequencies ω_{QNM} can be computed from analytical WKB approximation methods [31], obtaining

$$\omega_{QNM} = l\Omega_\infty - i|\lambda|(m + 1/2), \tag{60}$$

where

$$\Omega_\infty = \frac{[e^{-2\delta(r_\gamma)}\mu(r_\gamma)]^{1/2}}{r_\gamma} \tag{61}$$

is the angular velocity of circular null geodesics at r_γ as measured by asymptotic observers, λ is the Lyapunov exponent associated with this kind of geodesics, l is the angular momentum of the perturbation (and it is assumed that $l \gg 1$), and m is the overtone number. Based on this fact, Hod in [20] presented a simple and universal bound for the real part of the quasinormal frequencies in the WKB limit, expressed in terms of the event horizon radius of the associated black hole. Now we generalize this expression to n-dimensional GR and EGB theory, and we also discuss other simple bounds for other observables.

From (60), we see that the real part of these frequencies is given by

$$\omega_l \equiv \Re[\omega_{QNM}] = l\Omega_\infty. \tag{62}$$

By using (51), which can be rewritten as

$$\mu(r_\gamma) \leq 2\frac{\frac{n-5}{n-1}\hat{\alpha} + \frac{n-3}{n-1}r_\gamma^2}{\frac{n-3}{n-1}\alpha + r_\gamma^2 + \sqrt{\frac{16\hat{\alpha}^2}{(n-1)^2} + r_\gamma^4}}, \tag{63}$$

and using the fact that $e^{-2\delta} \leq 1$ [which follows from (7) and the asymptotic condition for $\delta(r)$], we see that (61) implies

$$\Omega_\infty \leq 2\left[\frac{\frac{n-5}{(n-1)r_\gamma^2}\hat{\alpha} + \frac{n-3}{n-1}}{\frac{n-3}{n-1}\alpha + r_\gamma^2 + \sqrt{\frac{16\hat{\alpha}^2}{(n-1)^2} + r_\gamma^4}}\right]^{1/2}. \tag{64}$$

Hence, from relation (62) we obtain

$$\omega_l \leq 2l\left[\frac{\frac{n-5}{(n-1)r_\gamma^2}\hat{\alpha} + \frac{n-3}{n-1}}{\frac{n-3}{n-1}\alpha + r_\gamma^2 + \sqrt{\frac{16\hat{\alpha}^2}{(n-1)^2} + r_\gamma^4}}\right]^{1/2}. \tag{65}$$

In order to compare this with the Hod bound, we give a weaker inequality, but one that is simpler and more universal than (65) in the sense that it does not depend on the particular properties of the black hole. Let us note that, in our study cases, $\hat{\alpha} \geq 0$ and $r_\gamma \geq r_H$; therefore,

$$\begin{aligned}\Omega_\infty &\leq 2\left[\frac{\frac{n-5}{(n-1)r_\gamma^2}\hat{\alpha} + \frac{n-3}{n-1}}{\frac{n-3}{n-1}\hat{\alpha} + r_\gamma^2 + \sqrt{\frac{16\hat{\alpha}^2}{(n-1)^2} + r_\gamma^4}}\right]^{1/2} \\ &\leq 2\left[\frac{\frac{n-5}{(n-1)r_H^2}\hat{\alpha} + \frac{n-3}{n-1}}{\frac{n-3}{n-1}\hat{\alpha} + r_H^2 + \sqrt{\frac{16\hat{\alpha}^2}{(n-1)^2} + r_H^4}}\right]^{1/2} \\ &\leq \left[\frac{\frac{n-5}{(n-1)r_H^2}\hat{\alpha} + \frac{n-3}{n-1}}{r_H^2}\right]^{1/2}.\end{aligned} \tag{66}$$

Combining this inequality with (62), we finally obtain

$$\omega_l \leq l\left[\frac{\frac{n-5}{(n-1)r_H^2}\hat{\alpha} + \frac{n-3}{n-1}}{r_H^2}\right]^{1/2}, \tag{67}$$

which implies

$$\omega_l r_H \leq l\left[\frac{n-5}{(n-1)r_H^2}\hat{\alpha} + \frac{n-3}{n-1}\right]^{1/2}. \tag{68}$$

This expression generalizes to n-dimensional EGB theory, the universal bound found by Hod in the framework of four-dimensional GR [20]. In particular, in the GR case it reduces to

$$\omega_l r_H \leq l\left[\frac{n-3}{n-1}\right]^{1/2}, \tag{69}$$

which, when $n=4$, agrees with the expression given in [20]. Note also that in five dimensions the first factor of (68) does not survive, and we get the simple bound

$$\omega_l r_H \leq \frac{1}{\sqrt{2}}l. \tag{70}$$

At this point, we can ask if there is a similar universal bound for the Lyapunov exponent; however, the answer is negative. The Lyapunov exponent for circular null geodesics is defined by [11]

$$\lambda = \sqrt{\frac{r_\gamma^2[e^{-2\delta(r_\gamma)}\mu(r_\gamma)]}{2L^2}V''(r_\gamma)}, \tag{71}$$

with

$$V''(r_\gamma) = \frac{L^2}{r_\gamma^4 e^{-2\delta(r_\gamma)}}\{e^{-2\delta(r_\gamma)}\mu(r_\gamma) - r_\gamma^2[e^{-2\delta(r_\gamma)}\mu(r_\gamma)]''\}, \tag{72}$$

and L the orbital angular momentum of the circular null geodesics. After some simple computations it can be shown that in n-dimensional GR,

$$V''(r_\gamma) = \frac{L^2 N'(r_\gamma)}{r_\gamma^3}. \tag{73}$$

This makes it possible to write (71) as

$$\lambda = \sqrt{[e^{-2\delta(r_\gamma)}\mu(r_\gamma)]\frac{N'(r_\gamma)}{2r_\gamma}}. \tag{74}$$

From (29) we have

$$\begin{aligned}N'(r_\gamma) &= (n-1)\mu'(r_\gamma) \\ &\quad - \frac{2\kappa}{(n-2)}[2r_\gamma p_r(r_\gamma) + r_\gamma^2 p_r'(r_\gamma)] \\ &\geq (n-1)\mu'(r_\gamma) + 2\kappa r_\gamma p_r(r_\gamma),\end{aligned} \tag{75}$$

where in order to establish the last inequality, we have taken into account that at r_γ, p_r and P' are both nonpositive numbers. However, from the first factor in (74), which is the factor $e^{-2\delta(r_\gamma)}\mu(r_\gamma)$, we can only ensure that $e^{-2\delta(r_\gamma)}\mu(r_\gamma) \leq \mu(r_\gamma)$, and therefore this inequality and (75) impose bounds in opposite directions, thereby preventing a universal bound (independent of the matter content) from being written for the Lyapunov exponent.

Coming back to the inequality (66), it can also be used to impose a universal bound over the location of the first relativistic image in the strong lensing regime. In general, if we assume that the observer is at a distance D_{ol} from the lens, then the first relativistic image subtends an angle θ_∞ [10,15], which can be expressed in terms of the circular orbital frequency Ω_∞ as

$$\theta_\infty = \frac{1}{D_{ol}\Omega_\infty}. \tag{76}$$

From this relation and using (66), we obtain the universal and simple lower bound

$$\theta_\infty \geq \frac{1}{D_{ol}}\left[\frac{r_H^2}{\frac{n-5}{(n-1)r_H^2}\hat{\alpha} + \frac{n-3}{n-1}}\right]^{1/2}. \tag{77}$$

In the GR limit it reduces to

$$\theta_\infty \geq \frac{r_H}{D_{ol}}\left[\frac{n-1}{n-3}\right]^{1/2}, \tag{78}$$

which, for the four-dimensional case gives

$$\theta_\infty \geq \sqrt{3}\frac{r_H}{D_{ol}}. \tag{79}$$

V. GENERAL COMMENT ON THE FASTEST WAY TO CIRCLE AXIALLY-SYMMETRIC SPACETIMES

It would be very interesting to find a generalization of some of the previous results for the case of axial symmetry. Even in this case, the circular null geodesics on the equatorial plane share some of the properties found in the spherical symmetric case. For example, Wei and Liu [16] found universal relations between the first relativistic image and the quasinormal frequencies by analyzing equatorial circular null geodesics in arbitrary axially-symmetric black holes, thus extending some of the results of spherical symmetry.

Recently, Hod [21] also showed that the fastest way to circle a black hole in general relativity is through circular null geodesics. In particular, he demonstrated that for any spherically symmetric spacetime in four dimensions, the minimum orbital time as measured by an asymptotic observer is realized by circular null geodesics, independently of the underlying gravitational theory. Moreover, he showed that a similar result can be obtained in GR for the case of a Kerr black hole by noting that the equatorial circular orbit with minimum traveling time coincides with the solution obtained by solving the equation governing circular null geodesics. More recently, in [22], Pradhan, in his study of charged Myers-Perry black holes in higher dimensions, made the observation that Hod's conclusion regarding the fastest way to circle a black hole remains valid in this more general family of metrics.

As mentioned, the fact that circular null geodesics minimize the orbital period in the case of spherically symmetric spacetimes is always valid. However, in [21], when Hod analyzed the Kerr metric, he obtained two different equations (one for the fastest circular orbit, and another for the circular null geodesics; see Eqs. (21) and (31) in his paper), and he showed that they admit the same kind of solutions; therefore, it was not clear from his discussion how general these results were. In other words, one can ask whether the agreement between equatorial circular null geodesics and the fastest way to circle black holes is a property unique to GR or a geometrical property general to any axially-symmetric metric, independently of any gravity theory.

Additionally, Hod made the interesting conjecture that there should be a lower bound for the traveling time, given in terms of the ADM mass of the black hole [21]. More precisely, he conjectured a lower bound on the orbital periods T_∞ of circular null geodesics around compact objects with mass $\mathcal{M}$ (as seen from far away observers). In mathematical terms he states that (in units with $G = c = 1$)

$$T_\infty \geq 4\pi\mathcal{M}. \tag{80}$$

In particular, the equality is satisfied by an extreme Kerr black hole. This conjecture is physically motivated by taking into account the rotational dragging that is maximal in the case of an extremal black hole. If this conjecture were correct, it could also be used to establish another universal bound for the possible location of the first relativistic image of an object lensed by a black hole. In fact, if (80) were valid we should obtain $\Omega_\infty = 2\pi/T_\infty \leq \frac{1}{2\mathcal{M}}$, which would imply

$$\theta_\infty \geq \frac{2\mathcal{M}}{D_{ol}}. \tag{81}$$

In other words, the expression (81) would also be valid for a generic rotating black hole, at least in the GR regime.

Even given the reasonableness of conjecture (80), it has not yet been proven. However, it leads one to wonder whether this conjecture can be kept as potentially valid in alternative gravitational theories. It is our purpose in this section to answer these questions. In particular, we make the observation that the orbital period for circular orbits *always* coincides with the circular null geodesics for any axially-symmetric spacetime, independently of the gravitational theory. We also answer the second question negatively on the validity of (80) for alternative black hole candidates by giving an explicit counterexample where the bound assumed by Hod is not satisfied. What is more, we show that the conjecture can be violated even in the case of a nonrotating black hole. In order to do that, we discuss a special family of Kaluza-Klein black holes.

Let us start with the first question. We begin, with the more general conformally stationary and axially-symmetric metric in n dimensions, without making reference to any gravitational theory.

These kinds of metrics are given by

$$ds^2 = e^{2\Phi(r,\theta,t)}(g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + r^2\cos(\theta)^2 d\Omega_{n-3}^2), \quad (82)$$

with

$$d\Omega_{n-3}^2 = d\phi_2^2 + \sin^2\phi_2[d\phi_3^2 + \sin^2\phi_3(\cdots d\phi_{n-4}^2)], \quad (83)$$

and with the components of the metric not depending on ϕ. As this metric is axially symmetric, there are equatorial orbits, and in particular we can compute from this metric the orbital time for circular null curves, i.e., curves such that $ds^2 = 0, \theta = \pi/2$ and $r = r_\gamma$. The orbital period for these curves is

$$T_\infty = 2\pi\frac{-g_{t\phi} \pm \sqrt{\Delta}}{g_{tt}}, \quad (84)$$

where

$$\Delta = g_{t\phi}^2 - g_{tt}g_{\phi\phi}, \quad (85)$$

and the $+/-$ signs correspond to counter-rotating/corotating orbits, respectively. If we compute their extremes we obtain

$$\frac{dT}{dr} = \frac{\pi}{\sqrt{\Delta}}G = 0, \quad (86)$$

with

$$G_\pm = \frac{1}{g_{tt}^2}\left[(\mp 2g_{t\phi}\sqrt{\Delta} + \Delta + g_{t\phi}^2)\frac{dg_{tt}}{dr} + (-2g_{tt}g_{t\phi} \pm 2g_{tt}\sqrt{\Delta})\frac{dg_{t\phi}}{dr} + g_{tt}^2\frac{dg_{\phi\phi}}{dr}\right]. \quad (87)$$

So, if we assume that Δ is a regular function and we only consider the region exterior to all horizons (where $\Delta = 0$), the minimum orbital period is obtained for those curves that satisfy $G_\pm = 0$.

On the other hand, we can study the condition for the existence of circular null geodesics on the equatorial plane. We start from the conservation equations

$$E = -(g_{tt}\dot{t} + g_{t\phi}\dot{\phi}), \quad (88)$$

$$L = g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi}, \quad (89)$$

$$p_r = g_{rr}\dot{r}, \quad (90)$$

which represent the energy of the null particle as measured by an asymptotic observer, the orbital angular momentum, and the radial component of the linear momentum, respectively. These equations can be solved for $\dot{t}$ and $\dot{\phi}$ and replaced in the Hamiltonian H,

$$H = -E\dot{t} + L\dot{\phi} + p_r\dot{r} = 0, \quad (91)$$

obtaining the following equation for $\dot{r}$:

$$\dot{r}^2 = V_r, \quad (92)$$

with

$$V_r = -\frac{E^2}{g_{rr}\Delta}(g_{\phi\phi} + 2g_{t\phi}b + g_{tt}b^2), \quad (93)$$

and $b = L/E$, the impact parameter. Circular null geodesics must satisfy $\dot{r}_\gamma = \dot{r}'_\gamma = 0$, or equivalently $V_r = V'_r = 0$. From $V_r = 0$, we obtain

$$b = \frac{\pm\sqrt{\Delta} - g_{t\phi}}{g_{tt}}, \quad (94)$$

and from $V'_r = 0$ it follows that

$$\frac{d}{dr}g_{\phi\phi} + 2b\frac{d}{dr}g_{t\phi} + b^2\frac{d}{dr}g_{tt} = 0. \quad (95)$$

Finally, by replacing the expression for b given by (94) in the last equation, we conclude that the circular null geodesics, if they exist, must satisfy

$$\frac{1}{g_{tt}^2}\left[(\mp 2g_{t\phi}\sqrt{\Delta} + \Delta + g_{t\phi}^2)\frac{dg_{tt}}{dr} + (-2g_{tt}g_{t\phi} \pm 2g_{tt}\sqrt{\Delta})\frac{dg_{t\phi}}{dr} + g_{tt}^2\frac{dg_{\phi\phi}}{dr}\right] = 0, \quad (96)$$

which agrees with the expression for $G_\pm$ given previously. As a result, its critical points always coincide with the values of the circular geodesics, generalizing in this way the results given by Hod for spherically symmetric spacetimes, and the particular cases of Kerr [21] and Myers Perry [22]. Let us remark that there is not inconsistency between this result and the fact that in [21] Hod obtained two different algebraic equations [Eqs. (21) and (31) in [21] for the minimum orbital period and geodesic motion, respectively]. In fact, the circular null geodesics are obtained by solving $\dot{r}_\gamma = 0$ and $\dot{r}'_\gamma = 0$, both being functions of b. If we first solve $\dot{r}'_\gamma = 0$ for b and replace its value in $\dot{r}_\gamma = 0$ (as the path followed by Hod), we obtain Eq. (31) of [21]. Alternatively, if we first solve for b from $\dot{r}_\gamma = 0$ and replace it in $\dot{r}'_\gamma = 0$, we arrive to his Eq. (21).

As mentioned above, in [21], by comparing the total orbital period for rotating Kerr black holes with those of Schwarzschild ones, a lower bound for the orbital periods T_∞ (as measured from far away observers) of circular null geodesics around compact objects with mass $\mathcal{M}$, Eq. (80), was conjectured.

Now, by giving an explicit counterexample, we show that, in contrast to the geometrical status of the fastest way to circle black holes, this conjecture cannot be generally valid in other black hole candidates coming from alternative gravitational theories.

The counterexample is based on the Kaluza-Klein black hole [32]. Shadows of these kinds of black holes were recently analyzed in [33]. Let us consider the following metric, which satisfies the vacuum Einstein equations in five dimensions.

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) + dy^2. \tag{97}$$

Then, by doing a compactification of the extra dimension and a boost transformation with velocity v in the y-direction, and by projecting in the 4-manifold, a new metric is obtained representing a charged spherically symmetric black hole together with a dilaton field [32]. The four-dimensional metric reads

$$ds^2 = -\frac{1-\frac{2m}{r}}{B}dt^2 + \frac{B}{1-\frac{2m}{r}}dr^2 + Br^2(d\theta^2 + \sin(\theta)^2 d\phi^2), \tag{98}$$

with

$$B = \sqrt{1 + \frac{2mv^2}{r(1-v^2)}}. \tag{99}$$

This metric together with the scalar field

$$\Phi = -\frac{\sqrt{3}}{2}\ln B, \tag{100}$$

and the electromagnetic potential $\mathcal{A}$

$$\mathcal{A} = \frac{v}{2(1-v^2)}\frac{1-\frac{2m}{r}}{B^2}dt, \tag{101}$$

satisfies the set of Einstein-Maxwell-dilaton field equations that follow from the action

$$S = \int d^4x\sqrt{-g}\,[R + 2(\nabla\Phi)^2 - e^{2\sqrt{2}\Phi}F^2]. \tag{102}$$

This metric represents a black hole with ADM mass $\mathcal{M}$ and charge Q given in terms of the parameter m and the velocity v by

$$\mathcal{M} = m\left[1 + \frac{v^2}{2(1-v^2)}\right], \tag{103}$$

$$Q = \frac{mv}{1-v^2}. \tag{104}$$

One of the characteristics of this type of metric is that the ADM mass, the dilation field, and the charge depend on the boost parameter. Note that the event horizon is located at $r_H = 2m$, which for a fixed mass $\mathcal{M}$, shrinks to zero when v goes to 1. Let us compute now the minimum orbital time that a circular null geodesic takes to orbit this black hole. In order to do this, we must solve for the corotating orbits, that is $G_- = 0$, which, using the expression (87) in this case gives

$$(v^2-2)^2r^2 - 2\mathcal{M}(v^2-2)(4v^2-3)r - 16\mathcal{M}^2v^2(1-v^2) = 0. \tag{105}$$

The physical solution of this equation is

$$r_\gamma = \frac{3-4v^2+\sqrt{9-8v^2}}{2-v^2}\mathcal{M}. \tag{106}$$

By replacing this radius in (84), we obtain

$$T_\infty = 2\pi\frac{3-4v^2+\sqrt{-8v^2+9}}{2-v^2}\sqrt{\frac{3+\sqrt{-8v^2+9}}{-1+\sqrt{-8v^2+9}}}\mathcal{M}. \tag{107}$$

A plot of this function is shown in Fig. 2. At $v = 0$ (Schwarzschild case) it takes the value $T_\infty = 6\sqrt{3}\pi\mathcal{M}$, and it goes to zero when $v \to 1$.

Consequently, the orbital period around a black hole with mass $\mathcal{M}$ can be made arbitrarily small. Therefore, there is a value of v from which the Hod conjecture cannot be satisfied for this type of black hole. Numerically, we found

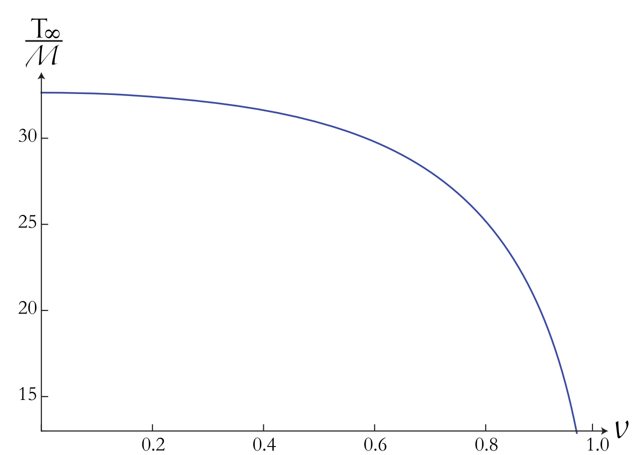

FIG. 2 (color online). Plot of the quotient $T_\infty/\mathcal{M}$ as a function of v. We assume $G = 1$.

that for $v > 0.96591$ all the orbital periods T_γ are smaller than $4\pi\mathcal{M}$.

VI. SUMMARY

In this work, we have found upper bounds for the radius of photon spheres in n-dimensional GR and EGB theories, thereby generalizing the previous work of Hod in [20]. In the general situation of a black hole dressed with matter fields, we have seen that the photon sphere radius is always smaller than that corresponding to a vacuum black hole with the same mass. It would be interesting to know how generic these results are, in the sense of how much they depend on gravitational field equations. In particular, it would be natural to study whether these results hold for the more general Lovelock gravitational theory or if they can be extended to other theories like $f(R)$ ones.

With respect to the study of circular null geodesics in axial symmetry, we have observed that equatorial circular null geodesics always minimize the orbital time around a black hole. It is a geometrical result. Given the astrophysical importance of this type of geodesics, it would also be interesting to attempt a proof of the Hod conjecture (80). We wish to deal with these and other associated problems in the near future.

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