

THE OBSTACLE PROBLEM FOR THE INFINITY FRACTIONAL LAPLACIAN

LOURDES MORENO MÉRIDA AND RAÚL EMILIO VIDAL

ABSTRACT. Given g an α -Hölder continuous function defined on the boundary of a bounded domain Ω and given ψ a continuous obstacle defined in $\bar{\Omega}$, in this article, we find u an α -Hölder extension of g in Ω with $u \geq \psi$. This function u minimizes the α -Hölder semi-norm of all possible extensions with these properties and it is a viscosity solution of the associated obstacle problem for the infinity fractional Laplace operator.

1. INTRODUCTION

Let Ω be an open, bounded domain of \mathbb{R}^N and $\alpha \in (0, 1)$. In this paper we will consider the infinite fractional Laplace operator given by

$$Lu(x) = \sup_{y \in \bar{\Omega}, y \neq x} \frac{u(y) - u(x)}{|y - x|^\alpha} + \inf_{y \in \bar{\Omega}, y \neq x} \frac{u(y) - u(x)}{|y - x|^\alpha}, \text{ for } x \in \Omega.$$

Motivated by the results of Chambolle, Lindgren and Monneau (see [6]), we will be interested in solutions of the associated Dirichlet obstacle problem. Concretely, given an α -Hölder function g defined on $\partial\Omega$ and a continuous obstacle ψ defined on $\bar{\Omega}$, we aim to prove the existence and uniqueness of at least a super infinity fractional harmonic function constrained to lie above the obstacle and to take the datum on $\partial\Omega$. More precisely, we consider the following obstacle problem

$$(1) \quad \begin{cases} -Lu(x) = 0, & \text{in } \{x \in \Omega : u(x) > \psi(x)\}, \\ -Lu(x) \geq 0, & \text{in } \{x \in \Omega : u(x) = \psi(x)\}, \\ u(x) \geq \psi(x), & \text{if } x \in \Omega, \\ u(x) = g(x), & \text{if } x \in \partial\Omega, \end{cases}$$

and we will study the existence and uniqueness of a viscosity solution that seems to be natural in this framework.

By a viscosity subsolution (resp. supersolution) of (1) we mean an upper semicontinuous (resp. lower semicontinuous) function u from $\bar{\Omega}$ to \mathbb{R} satisfying that $u \leq g$ (resp. $u \geq g$) on $\partial\Omega$ and the following property: $\forall \varphi \in C^1(\Omega) \cap C(\bar{\Omega})$ such that

$$u \leq \varphi, \text{ in } \bar{\Omega} \text{ (resp. } u \geq \varphi),$$

Key words and phrases. Infinity fractional Laplace operator, viscosity solutions, obstacle problem.

2010 Mathematics Subject Classification: 35D40, 35J60, 35J65.

$$u(x_0) = \varphi(x_0), \text{ for some } x_0 \in \Omega,$$

then

$$\min\{-L\varphi(x_0), \varphi(x_0) - \psi(x_0)\} \leq 0 \text{ (resp. } \geq 0 \text{)}.$$

A viscosity solution is a function which is both a subsolution and a supersolution.

We observe that, if u is a continuous function defined on $\bar{\Omega}$ satisfying $u = g$ on $\partial\Omega$, then we easily deduce, from the above definition, the following characterization of viscosity sub and super solution. Concretely, u is a viscosity subsolution of (1) if for any $\varphi \in C^1(\Omega) \cap C(\bar{\Omega})$ such that

$$u \leq \varphi, \text{ in } \bar{\Omega},$$

$$\psi(x_0) < u(x_0) = \varphi(x_0), \text{ for some } x_0 \in \Omega,$$

then

$$-L\varphi(x_0) \leq 0.$$

On the other hand, u is a viscosity supersolution of (1) if for any $\varphi \in C^1(\Omega) \cap C(\bar{\Omega})$ such that

$$u \geq \varphi, \text{ in } \bar{\Omega},$$

$$u(x_0) = \varphi(x_0), \text{ for some } x_0 \in \Omega,$$

then

$$-L\varphi(x_0) \geq 0 \quad \text{and} \quad u(x_0) \geq \psi(x_0).$$

It is interesting to note that the obstacle function ψ is always a viscosity subsolution of (1). See [11] for more information about viscosity solutions.

We also emphasize that in order to have a solution of our obstacle problem (1), it is necessary (due to the boundary conditions) that

$$(2) \quad \psi(x) \leq g(x), \quad \forall x \in \partial\Omega,$$

holds true.

Problem (1) involves a boundary problem in the fractional setting. This kind of problems have been studied extensively, see for instance [7] and [8]. Specifically, Problem (1) involves the infinity fractional Laplace operator. The infinity Laplace operator was considered widely in the literature in the local case, see [1], [2], [5], [12] [14], and [15]; as well as in the nonlocal case (especially fractional), see for instance [4], [6] and [10]. Moreover, in [4], [12] and [15] it is studied the existence of a solution for some obstacle problems. On one hand, in [12], the authors consider the (local) infinity Laplace operator. In particular, they propose a game which involves an obstacle function and they prove that certain limit of some specific values functions is a viscosity solution of the obstacle problem for the infinity Laplacian. On the other side, in [4], the authors consider a nonlocal tug of war game. Motivated by [14] the authors consider here a nonlocal version of the game.

Recently, in [6], given an α -Hölder continuous function g defined on $\partial\Omega$, it is obtained a viscosity solution for the Dirichlet problem

$$\begin{cases} -Lu(x) = 0, & \text{if } x \in \Omega, \\ u(x) = g(x), & \text{if } x \in \partial\Omega, \end{cases}$$

where L is the infinity fractional Laplace operator defined before. Our aim is to extend these results considering the study of the obstacle problem (1).

Specifically, if we will denote the α -Hölder semi-norm of a function u defined on Ω by ¹

$$[u]_\alpha = \sup_{x,y \in \Omega, x \neq y} \frac{|u(y) - u(x)|}{|y - x|^\alpha},$$

our main results are the following theorems.

Theorem 1. *Let Ω be an open, bounded domain of \mathbb{R}^N , $g \in C(\partial\Omega)$ and $\psi \in C(\overline{\Omega})$ such that (2) holds true. Then there exists at most a viscosity solution u of the obstacle problem (1).*

Theorem 2. *Let Ω be an open, bounded and Lipschitz domain of \mathbb{R}^N , $\alpha \in (0, 1)$. If $g \in C^{0,\alpha}(\partial\Omega)$ and $\psi \in C(\overline{\Omega})$ satisfy (2), then there exists a unique viscosity solution u of the obstacle problem (1) which belongs to $C^{0,\alpha}(\overline{\Omega})$. Moreover, the solution u is the best α -Hölder continuous extension of the datum g which lies above the obstacle ψ , in the sense that*

$$[u]_\alpha \leq [z]_\alpha,$$

for any arbitrary α -Hölder extension z of the datum g which satisfies $z \geq \psi$.

Following the arguments of [6], a possible approach to study our problem (1) is to approximate our infinity Laplace operator L with a sequence of approximate operators (see Section 2 below). In this sense, in what follows, given $p > N$, $\frac{N}{p} < \alpha < 1$ and $s := \alpha - \frac{N}{p}$, we consider the fractional Sobolev space $W^{s,p}(\Omega)$ defined by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \left(\frac{|u(y) - u(x)|}{|y - x|^\alpha} \right)^p dy dx < +\infty \right\},$$

and we recall that $(W^{s,p}(\Omega), \|\cdot\|_{s,p})$ is a Banach space, where

$$\|u\|_{s,p} = \left(\int_{\Omega} |u|^p + \int_{\Omega} \int_{\Omega} \left(\frac{|u(y) - u(x)|}{|y - x|^\alpha} \right)^p dy dx \right)^{1/p}, \quad u \in W^{s,p}(\Omega).$$

We define the functional $E_p : W^{s,p}(\Omega) \rightarrow \mathbb{R}$ by

$$(3) \quad E_p(u) = \int_{\Omega} \int_{\Omega} \left(\frac{|u(y) - u(x)|}{|y - x|^\alpha} \right)^p dy dx,$$

and we study the minimization problem in a specific set. Observe that the operator of the Euler Lagrange equation associated to this functional is

$$(4) \quad L_p u(x) = \int_{\Omega} \left(\frac{|u(y) - u(x)|}{|y - x|^\alpha} \right)^{p-1} \frac{\text{sig}(u(y) - u(x))}{|y - x|^\alpha} dy,$$

where $\text{sig}(x) = \frac{x}{|x|}$ for $x \neq 0$. At least formally, we emphasize that the operator $(L_p(u))^{1/(p-1)}$ should tend to our infinity fractional Laplace operator L when p goes to ∞ . We remark that this formal limit procedure only works when the right hand side is zero (when it is not zero one may expect a different limit equation). This will be the key point in our approach. We want to prove that the unique minimum (belonging to a suitable set) u_p of E_p is a viscosity solution of the obstacle problem associated to the operator L_p . Afterwards, we want to pass to the limit when p tends to infinity. We

¹Recall that $(C^{0,\alpha}(\Omega), \|\cdot\|_\infty + [\cdot]_\alpha)$ is a Banach Space

will prove that the limit of the sequence u_p of approximate solutions is a viscosity solution of (1).

The article is organized as follows: in Section 2, we study the properties of the approximate obstacle problems (associated to the approximate operators L_p) and in Section 3, we prove our main results.

2. APPROXIMATE PROBLEMS

We consider the approximate operators L_p given by (4) and we study in this section the approximate obstacle problems

$$(5) \quad \begin{cases} -L_p u(x) = 0, & \text{in } \{x \in \Omega : u(x) > \psi(x)\}, \\ -L_p u(x) \geq 0, & \text{in } \{x \in \Omega : u(x) = \psi(x)\}, \\ u(x) \geq \psi(x), & \text{if } x \in \Omega, \\ u(x) = g(x), & \text{if } x \in \partial\Omega. \end{cases}$$

In the next lemmas, we prove that the functional E_p given by (3) has a unique minimum (in a specific set) which is a viscosity solution of the approximate obstacle problem (5).

Lemma 1. *Let Ω be an open, bounded and Lipschitz domain of \mathbb{R}^N , $\alpha \in (0, 1)$, $p > \frac{2N}{\alpha}$. If $g \in C^{0,\alpha}(\partial\Omega)$, $\psi \in C(\bar{\Omega})$ and (2) holds, then the functional E_p given by (3) takes a unique minimum u_p in the set*

$$X_{g,\psi} = \{v \in W^{s,p}(\Omega) : v \geq \psi \text{ en } \bar{\Omega}, v = g \text{ en } \partial\Omega\}.$$

Moreover, u_p belongs to $C(\bar{\Omega})$.

Proof. Firstly we observe that, any α -Hölder extension of g which lies above the obstacle ψ belongs to the set $X_{g,\psi}$. Thus, this set is not empty (see [15, Proposition 3.3] for the existence of this extension). In addition, if we take $u \in X_{g,\psi}$ and we fix $k \geq \|\psi\|_\infty + \|g\|_\infty$, then $T_k(u) \in X_{g,\psi}$ and

$$E_p(u) \geq E_p(T_k(u)),$$

where the function $T_k : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} -k, & \text{si } s < -k, \\ s, & \text{si } |s| \leq k, \\ k, & \text{si } s > k. \end{cases}$$

As a consequence, we have that

$$\min_{X_{g,\psi}} E_p(u) = \min_{X_{g,\psi}^k} E_p(u),$$

where

$$X_{g,\psi}^k = \{u \in X_{g,\psi} : \|u\|_\infty \leq k\}.$$

Since the set $X_{g,\psi}^k$ is weakly closed (with the weak topology of $W^{s,p}(\Omega)$), to prove the existence of a minimum in this set, we will study the coercivity and the weak lower semicontinuity of the functional E_p .

On the one hand, we take a sequence $\{u_n\} \subset X_{g,\psi}^k$ such that $\|u_n\|_{s,p} \rightarrow +\infty$. Since $\|u_n\|_\infty \leq k$, this necessarily means that $E_p(u_n) \rightarrow +\infty$. That is, our functional is coercive.

On the other hand, we take a sequence $\{u_n\} \subset X_{g,\psi}^k$ such that u_n weakly converges to a function u in $W^{s,p}(\Omega)$. Since $W^{s,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$ (see [9, Corollary 1.2]) and the norm $\|\cdot\|_{s,p}$ is a w.l.s.c. function, then

$$\liminf E_p(u_n) \geq E_p(u)$$

as we desired.

Consequently, the functional E_p has a minimum u_p in the set $X_{g,\psi}$ and again by [9, Theorem 8.2], $u_p \in C(\bar{\Omega})$. Moreover, since the functional is convex, this minimum is unique. \square

Lemma 2. *Let Ω be an open, bounded and Lipschitz domain of \mathbb{R}^N , $g \in C^{0,\alpha}(\partial\Omega)$ and $\psi \in C(\bar{\Omega})$ satisfying (2). If $p > 2N/\alpha$, then the minimum u_p (given by Lemma 1) is a viscosity solution of (5).*

Proof. Firstly, recall that the minimum u_p (given by Lemma 1) belongs to $C(\bar{\Omega})$ and satisfies $u_p = g$ on $\partial\Omega$ and $u_p \geq \psi$ in $\bar{\Omega}$. We will prove that u_p is a viscosity sub and super solution. On the one hand, we claim that u_p is a viscosity subsolution of (5). Indeed, we take $\varphi \in C^1(\Omega) \cap C(\bar{\Omega})$ such that

$$u_p \leq \varphi, \quad \text{in } \bar{\Omega},$$

$$\psi(x_0) < u_p(x_0) = \varphi(x_0), \quad \text{for some } x_0 \in \Omega,$$

and we prove $-L\varphi(x_0) \leq 0$. Without loss of generality, we suppose that φ touches u_p only at the point x_0 ; otherwise it is sufficient to replace φ by $\varphi(x) + \delta|x - x_0|^2$ with δ small enough. We define the functions

$$\varphi^\epsilon = \max(u_p, \varphi - \epsilon),$$

and

$$\varphi_\epsilon = \min(u_p, \varphi - \epsilon).$$

Since, we suppose that $\varphi(x_0) = u_p(x_0) > \psi(x_0)$, for ϵ small enough, $\varphi_\epsilon \geq \psi$ in Ω and moreover, $\varphi_\epsilon = u_p$ on $\partial\Omega$. Hence, φ_ϵ belongs to $X_{g,\psi}$ and using that u_p is a minimum of E_p in this set, we have

$$E_p(\varphi_\epsilon) \geq E_p(u_p).$$

From this inequality and using the following convexity inequality (see [6, Lemma 6.2])

$$|\max(a, c) - \max(b, d)|^p + |\min(a, c) - \min(b, d)|^p \leq |a - b|^p + |c - d|^p,$$

for all $p \geq 1$, we deduce that

$$E_p(\varphi_\epsilon) + E_p(\varphi^\epsilon) \leq E_p(u_p) + E_p(\varphi) \leq E_p(\varphi_\epsilon) + E_p(\varphi),$$

that is

$$E_p(\varphi^\epsilon) \leq E_p(\varphi).$$

The convexity of E_p implies

$$E_p((1-t)\varphi + t\varphi^\epsilon) \leq (1-t)E_p(\varphi) + tE_p(\varphi^\epsilon) \leq E_p(\varphi),$$

and then we have

$$\frac{E_p((1-t)\varphi + t\varphi^\epsilon) - E_p(\varphi)}{t} \leq 0.$$

Let call

$$f(t) = E_p((1-t)\varphi + t\varphi^\epsilon).$$

From the above inequality and using the convexity of the function f , we have

$$f'(0) \leq \frac{f(t) - f(0)}{t} \leq 0,$$

and then

$$p \int_{\Omega} \int_{\Omega} H(x, y) dy dx \leq 0,$$

where

$$H(x, y) = \left| \frac{\varphi(y) - \varphi(x)}{|y - x|^\alpha} \right|^{p-1} \frac{\operatorname{sgn}(\varphi(y) - \varphi(x))}{|y - x|^\alpha} (\varphi^\epsilon(y) - \varphi(y) + \epsilon - \varphi^\epsilon(x) + \varphi(x) - \epsilon).$$

Therefore, a change of variable implies

$$\int_{\Omega} (\varphi^\epsilon - \varphi + \epsilon)(x) (-L_p \varphi(x)) dx \leq 0.$$

Now we argue by contradiction. Suppose that $-L_p \varphi(x_0) > 0$. By continuity, which holds under our assumptions, there is a small ball $B_r(x_0)$ such that $-L_p \varphi > 0$ in $B_r(x_0)$. Since $\varphi^\epsilon = \max(u_p, \varphi - \epsilon)$, for ϵ small enough, we have $\operatorname{supp}(\varphi^\epsilon - \varphi + \epsilon) \subset B_r(x_0)$. We also observe that $\varphi^\epsilon - \varphi + \epsilon \geq 0$. Consequently, we deduce

$$0 < \int_{B_r(x_0)} (\varphi^\epsilon - \varphi + \epsilon)(x) (-L_p \varphi(x)) dx \leq 0,$$

which is a contradiction.

In the same way, one can prove that u is a viscosity supersolution. \square

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Suppose that u and v are two viscosity solutions of the obstacle problem (1) and define the set

$$W = \{x \in \Omega : u(x) > v(x)\}.$$

We claim that W is an empty set. Indeed, arguing by contradiction, we suppose that W is not empty. Since $v \geq \psi$ in Ω , we have $v \geq \psi$ and $u > \psi$ in W . Consequently, the functions u and v satisfy

$$\begin{cases} -Lu = 0 & \text{in } W, \\ u = u & \text{on } \partial W, \end{cases} \quad \begin{cases} -Lv \geq 0 & \text{in } W, \\ v = u & \text{on } \partial W, \end{cases}$$

which implies, using the comparison principle [6, Proposition 11.2], that $v \geq u$ in W . This is a contradiction and the claim is proved. Reversing the role of u and v gives that $u = v$ which conclude the proof. \square

To prove Theorem 2 we need the following technical result.

Lemma 3. [6, Lemma 6.5] *For $\varphi \in C^1(\Omega)$, $p \geq 1$ and $\alpha \in (0, 1)$, we define*

$$f_p(y) = \frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^\alpha} \quad \text{and} \quad f(y) = \frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha},$$

where $x_p \rightarrow x_0 \in \Omega$ as $p \rightarrow \infty$. Then,

$$\lim_{p \rightarrow \infty} \left\| \frac{f_p^+(y)}{|y - x_p|^{\alpha/p}} \right\|_{L^p(\Omega)} = \|f^+\|_{L^\infty(\Omega)},$$

with $f^\pm(x) = \max(\pm f(x), 0)$. The same also holds for f_p^- .

Proof of Theorem 2. Let $\{u_p\}$ be a sequence of viscosity solutions u_p of (5) given by Lemma 1. Our aim is to pass to the limit when p goes to infinite. Firstly, given $\alpha \in (0, 1)$, we prove that the sequence $\{u_p\}$ is bounded in $W^{s,q}(\Omega)$ for any $q > 2N/\alpha$. Indeed, by construction, there is a positive constant k such that

$$\|u_p\|_\infty \leq k, \quad \forall p.$$

Now, we take any $p > 2N/\alpha$ and we fix a number q such that $2N/\alpha < q < p$. Let z be a Hölder extension of g such that $z \geq \psi$ (see [15, Proposition 3.3]). Since the functional E_p takes a unique minimum u_p in the set $X_{g,\psi}$, then

$$E_p(u_p) \leq E_p(z) \leq |\Omega|^2 [z]_\alpha^p,$$

and by Hölder inequality

$$(6) \quad E_q(u_p) \leq E_p(u_p)^{\frac{q}{p}} |\Omega|^{\frac{2(p-q)}{p}} \leq |\Omega|^2 [z]_\alpha^q,$$

which implies that the sequence $\{u_p\}$ is bounded in $W^{s,q}(\Omega)$. By the Sobolev embedding (see [9, Theorem 8.2]) we deduce that, up to a subsequence, u_p strongly converges to a function u in $C(\bar{\Omega})$. Moreover, since $u_p = g$ on $\partial\Omega$, and $u_p \geq \psi$ in Ω , then we also have that the function u satisfies

$$u = g, \text{ on } \partial\Omega, \quad u \geq \psi, \text{ in } \Omega.$$

Now, we will prove that u is a viscosity sub and super solution of problem (1). On the one hand, we claim that u is a viscosity subsolution. Indeed, we take $\varphi \in C^1(\Omega) \cap C(\bar{\Omega})$ such that

$$u \leq \varphi, \text{ in } \bar{\Omega},$$

$$\psi(x_0) < u(x_0) = \varphi(x_0), \text{ for some } x_0 \in \Omega,$$

and, without loss of generality, we suppose that φ touches u only at the point x_0 (x_0 is a strict maximum of $u - \varphi$). Hence,

$$M_p := \sup_{\bar{\Omega}} (u_p - \varphi) = (u_p - \varphi)(x_p),$$

where

$$x_p \mapsto x_0, \quad M_p \mapsto 0.$$

Moreover, since $\varphi(x_0) > \psi(x_0)$, we can suppose that $\varphi(x_p) > \psi(x_p)$ for p large enough. This shows that

$$\begin{cases} u_p \leq \varphi_p := \varphi + M_p, \\ \psi(x_p) < u_p(x_p) = \varphi(x_p). \end{cases}$$

The fact that u_p is a viscosity solution implies

$$0 \geq -L_p \varphi_p(x_p) = -L_p \varphi(x_p),$$

that is,

$$0 \geq - \int_{\Omega} \left| \frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^\alpha} \right|^{p-1} \frac{\text{sgn}(\varphi(y) - \varphi(x_p))}{|y - x_p|^\alpha} dy,$$

or equivalently,

$$\left\| \left(\frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^{\alpha + \frac{\alpha}{p-1}}} \right)^+ \right\|_{L^{p-1}(\Omega)} \geq \left\| \left(\frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^{\alpha + \frac{\alpha}{p-1}}} \right)^- \right\|_{L^{p-1}(\Omega)}.$$

Thanks to Lemma 3, we can pass to the limit in this inequality to obtain

$$\sup_{y \in \Omega} \left(\max \left(\frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha}, 0 \right) \right) + \inf_{y \in \Omega} \left(\min \left(\frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha}, 0 \right) \right) \geq 0.$$

Since φ is C^1 at x_0 , it is clear that $L^+\varphi(x_0) \geq 0$ and $L^-\varphi(x_0) \leq 0$, where

$$L^+\varphi(x_0) = \sup_{y \in \Omega, y \neq x_0} \frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha}, \quad L^-\varphi(x_0) = \inf_{y \in \Omega, y \neq x_0} \frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha}.$$

Summing up, we deduce

$$-L\varphi(x_0) \leq 0,$$

as we desired.

Finally, in the same way, one can prove that u is a viscosity super solution. Therefore, u is a viscosity solution of the obstacle problem (1).

To conclude, we characterize the function limit u . In order to do it, let z be any Hölder extension of g such that $z \geq \psi$. By (6), we have

$$E_q(u_p) \leq |\Omega|^{2/q} [z]_\alpha^q,$$

which implies, passing to the limit as p goes to ∞ ,

$$(E_q(u))^{1/q} \leq |\Omega|^{2/q} [z]_\alpha.$$

As a consequence, when q tends to ∞ , we obtain

$$\left\| \frac{u(y) - u(x)}{|y - x|^\alpha} \right\|_{L^\infty(\Omega \times \Omega)} \leq [z]_\alpha,$$

i.e., we have proved that $u \in C^{0,\alpha}(\overline{\Omega})$ and moreover

$$[u]_\alpha \leq [z]_\alpha,$$

for any arbitrary Hölder extension z of the datum g which satisfies $z \geq \psi$, as we desired. \square

Acknowledgements. The first author was supported by MINECO-FEDER grant MTM2015-68210-P, Junta de Andalucía FQM-116 and Ministerio de Educación, Cultura y Deporte (Spain) FPU grant FPU12/02395. The second author was partially supported by IEMath-Granada, CONICET and Secyt-UNC.

REFERENCES

- [1] Aronsson, G. *On certain singular solutions of the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$* . Manuscripta mathematica, vol. 47, no 1-3, (1984), p. 133-151.
- [2] Aronsson, G., Crandall, M., and Juutinen, P. *A tour of the theory of absolutely minimizing functions*. Bulletin of the American mathematical society, vol. 41, no 4, (2004), p. 439-505.
- [3] Barles, G.; Chasseigne, E. and Imbert, C. *On the Dirichlet problem for second-order elliptic integro-differential equations*. Indiana University Mathematics Journal, vol. 57, no 1, (2008), p. 213-146.
- [4] Bjorland C., Caffarelli L. and Figalli A. *Nonlocal TugofWar and the Infinity Fractional Laplacian*. Communications on Pure and Applied Mathematics, vol. 65, no 3, (2012), p. 337-380.
- [5] Bhattacharya, T.; Dibenedetto, E. and Manfredi, J. *Limits as $p \rightarrow \infty$ of $\Delta_p u_p = f$ and related extremal problems*. Rend. Sem. Mat. Univ. Pol. Torino, Fascicolo Speciale Nonlinear PDEs, (1989), p. 15-68.

- [6] Chambolle, A., Lindgren, E. and Monneau, R. *A Hölder infinity Laplacian*. Optimisation and Calculus of Variations, vol. 18, no 03, (2012), p. 799-835.
- [7] Caffarelli, L., Salsa, S., and Silvestre, L. *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*. Inventiones mathematicae, vol 171, no 2, (2008), p. 425-461.
- [8] Dipierro, S., Savin, O., and Valdinoci, E. *A nonlocal free boundary problem*. SIAM Journal on Mathematical Analysis, vol. 47, no 6, (2015), p. 4559-4605.
- [9] Di Nezza, E.; Palatucci, G. and Valdinoci, E. *Hitchhikers guide to the fractional Sobolev spaces*. Bulletin des Sciences Mathématiques. vol. 136, no 5, (2012), p. 521-573.
- [10] Ferreira, R. and Pérez-Llanos, M. *Limit problems for a fractional p -Laplacian as $p \rightarrow \infty$* . Nonlinear Differ. Equ. Appl. no 2, Art. 14, 28pp., (2016). doi:10.1007/s00030-016-0368-z
- [11] Lions Crandall, M.; Ishit, H. and Lions, P. *Users guide to viscosity solutions of second order partial differential equations*. Bulletin of the American Mathematical Society, vol. 27, no 1, (1992), p. 1-67.
- [12] Manfredi, J.; Rossi, J. and Somersille, S. *An obstacle problem for Tug-of-War games*. Communications on Pure and Applied Analysis. Vol. 14(1), (2015), p. 217-228.
- [13] Mcshne, E. *Extension of range of functions*. Bulletin of the American Mathematical Society, vol. 40, no 12, (1934), p. 837-842.
- [14] Peres, Y., Schramm, O., Sheffield, S. and Wilson, D. *Tug-of-war and the infinity Laplacian*. Journal of the American Mathematical Society, vol. 22, no 1, (2009), p. 167-210.
- [15] Rossi, J.; Teixeira, E. and Urbano J. *Optimal regularity at the free boundary for the infinity obstacle problem*. Interfaces and Free Boundaries. Vol. 17(3), (2015), p. 381-398.

(L. Moreno-Mérida)

UNIVERSIDAD DE GRANADA, GRANADA, ESPAÑA.

EMAIL: lumore@ugr.es

(R. E. Vidal)

FAMAF, UNIVERSIDAD NACIONAL DE CORDOBA, (5000), CORDOBA, ARGENTINA.

EMAIL: vidal@mate.uncor.edu