Booker, A. R. (2019). A note on Maass forms of icosahedral type. *Mathematische Zeitschrift*, *292*(3-4), 1315-1324. https://doi.org/10.1007/s00209-018-2157-3

Mathematische Zeitschrift
https://doi.org/10.1007/s00209-018-2157-3
Mathematische Zeitschrift

A note on Maass forms of icosahedral type

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Received: 19 December 2017 / Accepted: 13 August 2018

Abstract
Using ideas of Ramakrishnan, we consider the icosahedral analogue of the theorems of Sarnak and Brumley on Hecke–Maass newforms with Fourier coefficients in a quadratic order. Although we are unable to conclude the existence of an associated Galois representation in this case, we show that one can deduce some implications of such an association, including weak automorphy of all symmetric powers and the value distribution of Fourier coefficients predicted by the Chebotarev density theorem.

1 Introduction

In [11], Sarnak showed that a Hecke–Maass newform with integral Fourier coefficients must be associated to a dihedral or tetrahedral Artin representation. Brumley [1] later generalized this to Galois-conjugate pairs of forms with coefficients in the ring of integers of $\mathbb{Q}(\sqrt{d})$ for a fundamental discriminant $d \neq 5$, which are associated to dihedral, tetrahedral or octahedral representations. In this note we consider the remaining case of nondihedral forms with coefficients in $\mathbb{Z}\big[\frac{1+\sqrt{5}}{2}\big]$, which are predicted to correspond to icosahedral representations.

The results of Sarnak and Brumley depend crucially on the existence and cuspidality criteria of the symmetric cube and symmetric fourth power lifts from $\mathrm{GL}(2)$, as established by Kim and Shahidi [3–5]. For the icosahedral case, in order to conclude the existence of an associated Artin representation we would need to know the expected cuspidality criterion for the symmetric sixth power lift (which is not yet known to be automorphic). Appealing to ideas and results of Ramakrishnan [8–10], we show that one can nevertheless derive some of the consequences entailed by the existence of an associated icosahedral representation, including weak automorphy of all symmetric powers and the value distribution of Fourier coefficients predicted by the Chebotarev density theorem. Our precise result is as follows.

Theorem 1.1 *Let $\mathbb{A}$ be the adèle ring of $\mathbb{Q}$, and put $A = \{0, \pm 1, \pm 2, \pm\varphi, \pm\varphi^\tau\}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and τ denotes the nontrivial automorphism of $\mathbb{Q}(\varphi)$. Let $\pi = \bigotimes \pi_v$ and $\pi' = \bigotimes \pi'_v$*

The author was partially supported by EPSRC Grant EP/K034383/1. No data were created in the course of this study.

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be nonisomorphic unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ *with normalized Hecke eigenvalues* $\lambda_\pi(n)$ *and* $\lambda_{\pi'}(n)$*, respectively. Assume that* π *and* π' *are not of dihedral Galois type, and suppose that* $\lambda_\pi(n)$ *and* $\lambda_{\pi'}(n)$ *are elements of* $\mathbb{Z}[\varphi]$ *satisfying* $\lambda_{\pi'}(n) = \lambda_\pi(n)^\tau$ *for every* n*. Then:*

(1) π *corresponds to a Maass form of weight* 0 *and trivial nebentypus character.*
(2) *For any place* v, π_v *is tempered if and only if* π'_v *is tempered.*
(3) *If* S *denotes the set of primes* p *at which* π_p *is not tempered, then*

  (a) $\#\{p \in S : p \le X\} \ll X^{1-\delta}$ *for some* $\delta > 0$*;*
  (b) $\pi_v \cong \pi'_v$ *for all* $v \in S \cup \{\infty\}$*;*
  (c) $\lambda_\pi(p) \in A$ *for every prime* $p \notin S$*;*
  (d) *for each* $k \ge 0$*, there is a unique isobaric automorphic representation* $\Pi_k = \bigotimes \Pi_{k,v}$ *of* $\mathrm{GL}_{k+1}(\mathbb{A})$ *satisfying* $\Pi_{k,p} \cong \mathrm{sym}^k \pi_p$ *for all primes* $p \notin S$ *at which* π_p *is unramified;*
  (e) *if* S *is finite or* $\mathrm{sym}^5 \pi$ *is automorphic then* $S = \emptyset$ *and* π_∞ *is of Galois type (so that* π *corresponds to a Maass form of Laplace eigenvalue* $\frac{1}{4}$*).*

(3) *For each* $\alpha \in A$*, the set of primes* p *such that* $\lambda_\pi(p) = \alpha$ *has a natural density, depending only on the norm* $\alpha\alpha^\tau$*, as follows:*

$\alpha\alpha^\tau$	0	1	4	-1
density	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{120}$	$\frac{1}{10}$

2 Preliminaries

Before embarking on the proof of Theorem 1.1, we first recall some facts about the analytic properties of the standard and Rankin–Selberg L-functions associated to isobaric automorphic representations. We refer to [8, §1] for essential background and terminology.

Let $\pi = \sigma_1 \boxplus \cdots \boxplus \sigma_n = \bigotimes \pi_v$ be an isobaric automorphic representation of $\mathrm{GL}_d(\mathbb{A})$ for some $d \ge 1$, and assume that the cuspidal summands σ_i have finite-order central characters. Then for any finite set of places $S \supseteq \{\infty\}$, the partial L-function $L^S(s,\pi) = \prod_{v \notin S} L(s,\pi_v)$ converges absolutely for $\Re(s) > 1$ and continues to an entire function, apart from a possible pole at $s = 1$ of order equal to the number of occurrences of the trivial character among the σ_i. Furthermore, $L^S(s,\pi)$ has no zeros in the region $\{s \in \mathbb{C} : \Re(s) \ge 1\}$.

Given two such isobaric representations, π_1 and π_2, we can form the irreducible admissible representation $\pi_1 \boxtimes \pi_2 = \bigotimes(\pi_{1,v} \boxtimes \pi_{2,v})$, where for each place v, $\pi_{1,v} \boxtimes \pi_{2,v}$ is the functorial tensor product defined by the local Langlands correspondence. Then for any set S as above, $L^S(s, \pi_1 \boxtimes \pi_2) = \prod_{v \notin S} L(s, \pi_{1,v} \boxtimes \pi_{2,v})$ agrees with the partial Rankin–Selberg L-function $L^S(s, \pi_1 \times \pi_2)$, which again converges absolutely for $\Re(s) > 1$, continues to an entire function apart from a possible pole at $s = 1$, and does not vanish in $\{s \in \mathbb{C} : \Re(s) \ge 1\}$. The order of the pole is characterized by the facts that (1) it is bilinear with respect to isobaric sum, and (2) if π_1 and π_2 are cuspidal then $L^S(s, \pi_1 \boxtimes \pi_2)$ has a simple pole if $\pi_1 \cong \pi_2^\vee$ and no pole otherwise.

Given an irreducible admissible representation Π of $\mathrm{GL}_d(\mathbb{A})$, let $\mathrm{cond}(\Pi)$ denote its conductor, and let $\{c_n(\Pi)\}_{n=1}^\infty$ be the unique sequence of complex numbers satisfying

$$-\frac{L'}{L}(s,\Pi) = \sum_{n=1}^\infty \frac{\Lambda(n) c_n(\Pi)}{n^s} \quad \text{and} \quad c_n(\Pi) = 0 \text{ whenever } \Lambda(n) = 0.$$

Then $c_n(\Pi)$ is multiplicative in Π, in the sense that if π_1 and π_2 are isobaric representations as above then $c_n(\pi_1 \boxtimes \pi_2) = c_n(\pi_1)c_n(\pi_2)$ for all n coprime to $\gcd(\mathrm{cond}(\pi_1), \mathrm{cond}(\pi_2))$.

Similarly, for any isobaric representation π of $\mathrm{GL}_d(\mathbb{A})$ and any $k \geq 0$, we can form the irreducible admissible representation $\mathrm{sym}^k \pi = \bigotimes \mathrm{sym}^k \pi_v$. If $d = 2$ and π has trivial central character then for all n coprime to $\mathrm{cond}(\pi)$ we have

$$c_n(\mathrm{sym}^k \pi) = P_k(c_n(\pi))$$

for certain polynomials $P_k \in \mathbb{Z}[x]$; in particular,

$$P_0 = 1,\ P_1 = x,\ P_2 = x^2 - 1,\ P_3 = x^3 - 2x,\ P_4 = x^4 - 3x^2 + 1,\ P_5 = x^5 - 4x^3 + 3x.$$

Finally, we recall some standard tools from analytic number theory.

Lemma 2.1 *Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers satisfying $c_n \ll n^{\sigma}$ for some $\sigma \geq 0$, and put*

$$D(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \quad \text{for } \Re(s) > \sigma + 1.$$

Suppose that $(s-1)D(s)$ has analytic continuation to an open set containing $\{s \in \mathbb{C} : \Re(s) \geq 1\}$, and set $r = \mathrm{Res}_{s=1} D(s)$. Then

(1) $\sum_{n \leq X} c_n = rX + o(X)$ *as $X \to \infty$.*
(2) *If $r = 0$ then there exists $\delta > 0$ such that $\sum_{n=1}^{\infty} c_n/n^{1-\delta} < \infty$.*

Proof These are the Wiener–Ikehara theorem [12, Ch. II.7, Thm. 11] and Landau's theorem [12, Ch. II.1, Cor. 6.1], respectively. □

3 Proof of Theorem 1.1

We are now ready for the proof. Our argument is sequential, but to aid the reader we have separated it into six main steps, as follows.

3.1 Initial observations

Let π and π' be as in the statement of the theorem. If π_p and π'_p are both tempered for some prime p, then $\max\{|\lambda_\pi(p)|, |\lambda_\pi(p)^\tau|\} \leq 2$, which holds if and only if $\lambda_\pi(p) \in A$. Thus, conclusion (2) of the theorem implies conclusion (3c). Moreover, the equality $\lambda_{\pi'}(n) = \lambda_\pi(n)^\tau$ for $n \in \{p, p^2\}$ implies that the L-factors $L(s, \pi_p)$ and $L(s, \pi'_p)$ have the same degree, and thus π_p is ramified if and only if π'_p is ramified. We set $N = \gcd(\mathrm{cond}(\pi), \mathrm{cond}(\pi'))$.

Since $\lambda_\pi(n) \in \mathbb{R}$ for every n, π must be self dual. Suppose that π is the automorphic induction of a Hecke character of infinite order. Then the value distribution of $\lambda_\pi(p)$ for primes p is the sum of a point mass of weight $\frac{1}{2}$ at 0 and a continuous distribution. In particular, the set $\{p : \lambda_\pi(p) \in A\}$ has density $\frac{1}{2}$. On the other hand, by [4, Theorem 4.1], the set of p at which π'_p is tempered has lower Dirichlet density at least 34/35, and as observed above, $\lambda_\pi(p) \in A$ for any such p. This is a contradiction, so π cannot be induced from a Hecke character of infinite order. (See [11] for an alternative proof in the Maass form case, based on transcendental number theory.) By hypothesis, π is also not of dihedral Galois type, and it follows that π has trivial central character.

Suppose that π_∞ is a discrete series representation of weight $k \geq 2$. Since π has trivial central character, k must be even. Then π corresponds to a holomorphic newform with Fourier coefficients $\lambda_\pi(n)n^{(k-1)/2}$, which must lie in a fixed number field. Considering primes $n = p$, since $\lambda_\pi(p) \in \mathbb{Z}[\varphi]$ that is only possible if $\lambda_\pi(p) = 0$ for all but finitely many p, contradicting the fact that $L(s, \pi \boxtimes \pi)$ has a pole at $s = 1$. Thus, π_∞ must be a principal or complementary series representation of weight 0, which establishes (1).

Next suppose that π is of tetrahedral or octahedral type. Then $\mathrm{Ad}(\pi) \cong \mathrm{sym}^2 \pi$ corresponds to an irreducible 3-dimensional Artin representation with Frobenius traces $\lambda_\pi(p)^2 - 1$ for all primes $p \nmid N$, and image isomorphic to A_4 or S_4, respectively. In the tetrahedral case, from the character table of A_4 we see that $\lambda_\pi(p)^2 - 1 \in \{3, -1, 0\}$, so that $\lambda_\pi(p) \in \mathbb{Z}$; by strong multiplicity one, that contradicts the hypothesis that $\pi \not\cong \pi'$. In the octahedral case, from the character table of S_4 and the Chebotarev density theorem, $\lambda_\pi(p)^2 - 1 = 1$ for a positive proportion of primes p; that contradicts the hypothesis that $\lambda_\pi(p) \in \mathbb{Z}[\varphi]$.

In summary, we have shown that π corresponds to a Maass form of weight 0 and trivial character, is not in the image of automorphic induction and is not of solvable polyhedral type. By symmetry these conclusions apply to π' as well. Moreover, by Atkin–Lehner theory, if there is a prime $p \mid N$ with $p^2 \nmid N$ then $\lambda_\pi(p)^2 = \lambda_{\pi'}(p)^2 = 1/p$. That contradicts the hypothesis that $\lambda_\pi(n)$ and $\lambda_{\pi'}(n)$ are algebraic integers, so for every $p \mid N$ we must have $p^2 \mid N$ and $\lambda_\pi(p) = \lambda_{\pi'}(p) = 0$.

3.2 Equivalence of $\mathrm{sym}^3 \pi$ and $\mathrm{sym}^3 \pi'$

By the seminal works of Gelbart–Jacquet [2], Ramakrishnan [7], Kim–Shahidi [5] and Kim [3], we know that the representations $\mathrm{sym}^k \pi$ and $\mathrm{sym}^k \pi'$ for $k \leq 4$, $\pi \boxtimes \pi'$, $\pi \boxtimes \mathrm{sym}^2 \pi'$ and $\pi' \boxtimes \mathrm{sym}^2 \pi$ are all automorphic. Moreover, since π and π' are not in the image of automorphic induction and are not of solvable polyhedral type, $\mathrm{sym}^k \pi$ and $\mathrm{sym}^k \pi'$ are cuspidal for $k \leq 4$.

For brevity of notation, we set

$$a_n = c_n(\pi) \quad \text{and} \quad b_n = a_n^\tau = c_n(\pi').$$

Note that $a_p = \lambda_\pi(p)$ for all primes p, and $a_n = 0$ whenever $(n, N) > 1$. For $f \in \mathbb{R}[x, y]$, let

$$D_f(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n) f(a_n, b_n)}{n^s} = \sum_{(n,N)=1} \frac{\Lambda(n) f(a_n, b_n)}{n^s} + f(0,0) \sum_{p \mid N} \frac{\log p}{p^s - 1}.$$

For any f such that $(s-1)D_f(s)$ has an analytic continuation to an open set containing $\{s \in \mathbb{C} : \Re(s) \geq 1\}$, we define

$$r(f) = \mathrm{Res}_{s=1} D_f(s).$$

In particular, by the properties of Rankin–Selberg L-functions described in §2, $r(P_i(x)P_j(y))$ is defined for $i, j \leq 4$. Note also that $r(f)$ is linear in f.

Consider

$$\begin{aligned} F &= (x-y)^2((x-y)^2 - 5) \\ &= P_4(x) - 4P_3(x)y + 6P_2(x)P_2(y) - 4xP_3(y) + P_4(y) + 4P_2(x) - 6xy + 4P_2(y) \end{aligned} \tag{3.1}$$

Note that for $u, v \in \mathbb{Z}$ with $u \equiv v \pmod 2$, we have

$$F\left(\frac{u+v\sqrt{5}}{2}, \frac{u-v\sqrt{5}}{2}\right) = 25v^2(v^2-1) \geq 0.$$

Since $\operatorname{sym}^k \pi$ and $\operatorname{sym}^k \pi'$ are cuspidal for $k \leq 4$ and $\pi \not\cong \pi'$, we have $r(F) = 6r(P_2(x)P_2(y))$.

Suppose that $\operatorname{sym}^2 \pi \cong \operatorname{sym}^2 \pi'$. Then $a_n = \pm b_n$ for all n; writing $a_n = \frac{u_n+v_n\sqrt{5}}{2}$ as above, it follows that $2 \mid v_n$, so that

$$F(a_n, b_n) = 25v_n^2(v_n^2-1) \geq 75v_n^2 = 15(a_n-b_n)^2.$$

This implies

$$6 = r(F) \geq 15r((x-y)^2) = 30,$$

which is absurd. Hence, $\operatorname{sym}^2 \pi \not\cong \operatorname{sym}^2 \pi'$ and $r(F) = 0$.

By [13, Theorem B], this in turn implies that $\pi \boxtimes \operatorname{sym}^2 \pi'$ and $\pi' \boxtimes \operatorname{sym}^2 \pi$ are cuspidal. Also, in view of the identity

$$(xy)^2 = (P_2(x)+1)(P_2(y)+1) = P_2(x)P_2(y) + P_2(x) + P_2(y) + 1,$$

we have $r((xy)^2) = 1$, so that $\pi \boxtimes \pi'$ is cuspidal.

Since $F(a_n, b_n)$ is nonnegative, by Lemma 2.1(2) there exists $\varepsilon > 0$ such that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)F(a_n, b_n)}{n^{1-\varepsilon}} < \infty.$$

Applying Cauchy–Schwarz and the inequality

$$(x-y)^4 F(x,y) = (x-y)^6((x-y)^2-5) \leq (x-y)^8 \leq 128(x^8+y^8),$$

we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-\varepsilon/3}}(a_n-b_n)^2 F(a_n,b_n)\right)^2 &\leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-\varepsilon}} F(a_n,b_n) \cdot \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\varepsilon/3}}(a_n-b_n)^4 F(a_n,b_n) \\ &\leq 128 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-\varepsilon}} F(a_n,b_n) \cdot \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\varepsilon/3}}(a_n^8+b_n^8).\end{aligned}$$

Noting that $x^8 = (P_4(x)+3P_2(x)+2)^2$, the final sum on the right-hand side converges, by Rankin–Selberg. Thus, the series defining $D_{(x-y)^2F}(s)$ converges absolutely for $\Re(s) \geq 1-\varepsilon/3$, so that $r((x-y)^2F) = 0$.

Next, we compute that

$$\begin{aligned}20P_3(x)P_3(y) = 20 &- (x-y)^2F(x,y) + P_2(x)P_4(x) - 6P_3(x) \cdot yP_2(x) + 15P_4(x)P_2(y) \\ &+ 15P_2(x)P_4(y) - 6xP_2(y) \cdot P_3(y) + P_2(y)P_4(y) + 14P_4(x) - 38P_3(x)y \\ &+ 60P_2(x)P_2(y) - 38xP_3(y) + 14P_4(y) + 38P_2(x) - 48xy + 38P_2(y).\end{aligned}$$

Evaluating r of both sides and using that $r((x-y)^2F) = 0$, we see that $r(P_3(x)P_3(y)) = 1$, whence $\operatorname{sym}^3 \pi \cong \operatorname{sym}^3 \pi'$. Similarly, we have

$$xP_2(y) \cdot yP_2(x) = P_3(x)P_3(y) + xP_3(y) + yP_3(x) + xy,$$

from which it follows that $\pi \boxtimes \mathrm{sym}^2 \pi' \cong \pi' \boxtimes \mathrm{sym}^2 \pi$. Also, from

$$P_4(x) - P_4(y) = (x + y)(P_3(x) - P_3(y) + xP_2(y) - yP_2(x)),$$

we get $P_4(a_n) = P_4(b_n)$, so that $\mathrm{sym}^4 \pi_p \cong \mathrm{sym}^4 \pi'_p$ for all $p \nmid N$. By strong multiplicity one, $\mathrm{sym}^4 \pi \cong \mathrm{sym}^4 \pi'$.

3.3 Nontempered and archimedean places

In view of the identity

$$x^2(P_3(x)-P_3(y)) + (x^2+xy-1)(xP_2(y)-yP_2(x)) = (x - y)(x^2 - x - 1)(x^2 + x - 1),$$

for every n we have either $a_n = b_n \in \mathbb{Z}$ or $a_n \in \{\pm\varphi, \pm\varphi^\tau\}$. For any prime $p \nmid N$, it follows that if either of π_p, π'_p is nontempered then $\pi_p \cong \pi'_p$.

Next we show that this conclusion holds for ramified and archimedean places as well. If π_p is nontempered then, as explained in [6, Remark 1], π_p is a twist of an unramified complementary series representation, i.e. $\pi_p \cong (|\cdot|_p^s \boxplus |\cdot|_p^{-s}) \otimes \chi$ for some $s > 0$ and unitary character χ of $\mathbb{Q}_p^\times$. Since $\mathrm{sym}^3 \pi_p \cong \mathrm{sym}^3 \pi'_p$, π'_p must also be nontempered, so we similarly have $\pi'_p \cong (|\cdot|_p^{s'} \boxplus |\cdot|_p^{-s'}) \otimes \chi'$ for some $s' > 0$ and unitary character χ'. Thus,

$$\mathrm{sym}^3(|\cdot|_p^s \boxplus |\cdot|_p^{-s}) \otimes \chi^3 \cong \mathrm{sym}^3(|\cdot|_p^{s'} \boxplus |\cdot|_p^{-s'}) \otimes (\chi')^3,$$

from which it follows that $s = s'$ and $\chi^3 = (\chi')^3$. Comparing central characters, we deduce that $\chi = \chi'$, whence $\pi_p \cong \pi'_p$. Running through this argument again with the roles of π and π' reversed, we obtain conclusions (2) and (3b) of the theorem for finite places.

Similarly, we have $\pi_\infty = (|\cdot|_\mathbb{R}^s \boxplus |\cdot|_\mathbb{R}^{-s}) \otimes \mathrm{sgn}^\epsilon$ for some $\epsilon \in \{0, 1\}$ and $s \in i\mathbb{R} \cup (0, \frac{1}{2})$, and comparing the parameters of $\mathrm{sym}^3 \pi_\infty$ and $\mathrm{sym}^3 \pi'_\infty$, we conclude that $\pi_\infty \cong \pi'_\infty$.

3.4 Value distribution of $\lambda_\pi(p)$

Making use of the isomorphism $\pi \boxtimes \mathrm{sym}^2 \pi' \cong \pi' \boxtimes \mathrm{sym}^2 \pi$, if $0 < j < i \le 8 - j$ then

$$\begin{aligned} r(x^i y^j) &= r(x^{i-2}y^{j-1}(yP_2(x) + y)) = r(x^{i-2}y^{j-1}(xP_2(y) + y)) \\ &= r(x^{i-1}y^{j+1}) + r(x^{i-2}y^j) - r(x^{i-1}y^{j-1}), \end{aligned}$$

and similarly with the roles of x and y reversed. By systematic application of this rule and linearity, we reduce the computation of $r(x^i y^j)$ for $i + j \le 8$ to that of $r(P_i(x)P_j(y))$, $r(P_i(x)P_j(x))$ and $r(P_i(y)P_j(y))$ for $i, j \le 4$, all of which are determined from the conclusions obtained in §3.2. After some computation we arrive at the following table of values for $r(x^i y^j)$:

$i \setminus j$	0	1	2	3	4	5	6	7	8
0	1	0	1	0	2	0	5	0	14
1	0	0	0	0	0	0	0	1	
2	1	0	1	0	2	0	6		
3	0	0	0	1	0	4			
4	2	0	2	0	5				
5	0	0	0	4					
6	5	0	6						
7	0	1							
8	14								

This enables us to compute $r(f)$ for any f of total degree at most 8 without having to work out the full expansion as in (3.1).

Consider

$$H = (xy + 1)x^2(x^2 - 1)(x^2 - 4).$$

Then $H(a_n, b_n) \geq 0$ for all n, and for $p \nmid N$, π_p and π'_p are tempered if and only if $H(a_p, b_p) = 0$. We verify by the above that $r(H) = 0$, so by Lemma 2.1(2) there exists $\delta \in (0, 1]$ such that $\sum_{n=1}^{\infty} \Lambda(n)H(a_n, b_n)/n^{1-\delta} < \infty$. Thus, with S as in the statement of the theorem, we have

$$\sum_{\substack{p\in S\\ p\leq X}} (\log p)a_p^8 \leq \sum_{\substack{n\leq X\\ a_n\notin A}} \Lambda(n)a_n^8 \ll \sum_{n\leq X} \Lambda(n)H(a_n, b_n) \leq X^{1-\delta} \sum_{n=1}^{\infty} \frac{\Lambda(n)H(a_n, b_n)}{n^{1-\delta}} \ll X^{1-\delta}. \tag{3.2}$$

Including the possible contribution from ramified primes, which are finite in number, we see that π_p and π'_p are tempered for all but $O(X^{1-\delta})$ primes $p \leq X$, which proves (3a).

Next, for each $\alpha \in A$ we define a polynomial $f_\alpha \in \mathbb{R}[x, y]$ of total degree at most 6, as follows:

α	f_α	$r(f_\alpha)/f_\alpha(\alpha, \alpha^\tau)$
0	$(xy + 1)(x^2 - 1)(x^2 - 4)$	$\frac{1}{4}$
± 1	$(x^2 + y^2 - 3)x(x + \alpha)(x^2 - 4)$	$\frac{1}{6}$
± 2	$(xy + 1)x(x^2 - 1)(x + \alpha)$	$\frac{1}{120}$
$\pm\varphi, \pm\varphi^\tau$	$(x - y)(1 \pm x \pm y)(x - \alpha^\tau)$	$\frac{1}{10}$

In each case, we have $f_\alpha(\beta, \beta^\tau) \geq 0$ for all $\beta \in \mathbb{Z} \cup A$, with $f_\alpha(\beta, \beta^\tau) = 0$ for $\beta \in A \setminus \{\alpha\}$ and $f_\alpha(\alpha, \alpha^\tau) > 0$. Also, since $\deg f_\alpha \leq 6$, we have $f_\alpha(a_n, b_n) \ll a_n^6 \leq a_n^8$ whenever $a_n \notin A$. Thus, by (3.2) and Lemma 2.1(1),

$$\begin{aligned}\sum_{\substack{p\leq X\\ \lambda_\pi(p)=\alpha}} \log p &= O(X^{1/2}) + \sum_{\substack{n\leq X\\ a_n=\alpha}} \Lambda(n) = O(X^{1/2} + X^{1-\delta}) + \frac{1}{f_\alpha(\alpha, \alpha^\tau)} \sum_{n\leq X} \Lambda(n) f_\alpha(a_n, b_n) \\ &= \frac{r(f_\alpha)}{f_\alpha(\alpha, \alpha^\tau)} X + o(X) \quad \text{as } X \to \infty.\end{aligned}$$

By partial summation, it follows that $\{p : \lambda_\pi(p) = \alpha\}$ has natural density $r(f_\alpha)/f_\alpha(\alpha, \alpha^\tau)$, whose values are shown in the table. This proves (4).

3.5 Weak automorphy of symmetric powers

Let $G = \mathrm{SL}_2(\mathbb{F}_5)$, which is the smallest group supporting a 2-dimensional icosahedral representation [13, §2]. Then G has nine irreducible representations, with dimensions 1, 2, 2, 3, 3, 4, 4, 5 and 6. Their characters all take values in $\mathbb{Z}[\varphi]$ and can be written as

$$\chi_0 = 1, \quad \chi_1 = \chi, \quad \chi_2 = \chi^\tau, \quad \chi_3 = P_2(\chi), \quad \chi_4 = P_2(\chi^\tau),$$
$$\chi_5 = \chi\chi^\tau, \quad \chi_6 = P_3(\chi), \quad \chi_7 = P_4(\chi), \quad \chi_8 = \chi P_2(\chi^\tau),$$

where χ is one of the characters of dimension 2 and χ^τ is its Galois conjugate. (Our numbering scheme is more or less arbitrary, and was made for notational convenience below.) The character table is as follows:

	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}4&0\\0&4\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&2\\4&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}2&2\\4&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}2&0\\0&3\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}4&1\\0&4\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}4&2\\0&4\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$
χ_0	1	1	1	1	1	1	1	1	1
χ_1	2	-2	1	-1	0	φ	φ^τ	$-\varphi$	$-\varphi^\tau$
χ_2	2	-2	1	-1	0	φ^τ	φ	$-\varphi^\tau$	$-\varphi$
χ_3	3	3	0	0	-1	φ	φ^τ	φ	φ^τ
χ_4	3	3	0	0	-1	φ^τ	φ	φ^τ	φ
χ_5	4	4	1	1	0	-1	-1	-1	-1
χ_6	4	-4	-1	1	0	1	1	-1	-1
χ_7	5	5	-1	-1	1	0	0	0	0
χ_8	6	-6	0	0	0	-1	-1	1	1

Let $\langle\ ,\ \rangle$ denote the inner product on $L^2(G)$. For each $k \geq 0$ and $i \in \{0, \dots, 8\}$, let

$$m_{k,i} = \langle P_k(\chi), \chi_i \rangle$$

be the multiplicity of χ_i in the kth symmetric power of χ, so that

$$P_k(\chi) = \sum_{i=0}^{8} m_{k,i}\chi_i.$$

In view of the character table, for any $\alpha \in A$ we may choose $g \in G$ with $\chi(g) = \alpha$ and evaluate both sides of the above at g to get

$$P_k(\alpha) = \sum_{i=0}^{8} m_{k,i} h_i(\alpha, \alpha^\tau),$$

where we write

$$h_0 = 1, \quad h_1 = x, \quad h_2 = y, \quad h_3 = P_2(x), \quad h_4 = P_2(y),$$
$$h_5 = xy, \quad h_6 = P_3(x), \quad h_7 = P_4(x), \quad h_8 = xP_2(y).$$

On the automorphic side, we make the parallel definitions

$$\sigma_0 = 1, \quad \sigma_1 = \pi, \quad \sigma_2 = \pi', \quad \sigma_3 = \mathrm{sym}^2\pi, \quad \sigma_4 = \mathrm{sym}^2\pi',$$
$$\sigma_5 = \pi \boxtimes \pi', \quad \sigma_6 = \mathrm{sym}^3\pi, \quad \sigma_7 = \mathrm{sym}^4\pi, \quad \sigma_8 = \pi \boxtimes \mathrm{sym}^2\pi',$$

and we set

$$\Pi_k = \boxplus_{i=0}^{8} (\underbrace{\sigma_i \boxplus \cdots \boxplus \sigma_i}_{m_{k,i}\text{ times}}) \quad \text{for } k \geq 0.$$

By construction, if $p \notin S$ and π_p is unramified, then for every power $n = p^j$ we have

$$c_n(\mathrm{sym}^k \pi) = P_k(a_n) = \sum_{i=0}^{8} m_{k,i} h_i(a_n, b_n) = \sum_{i=0}^{8} m_{k,i} c_n(\sigma_i) = c_n(\Pi_k).$$

Thus, $\mathrm{sym}^k \pi_p \cong \Pi_{k,p}$, as claimed.

It remains to prove the uniqueness of Π_k. Suppose that Π'_k is another such isobaric representation. Then for any $i \in \{0, \dots, 8\}$,

$$\sum_{(n,N)=1} \frac{\Lambda(n) c_n(\Pi'_k \boxtimes \sigma_i)}{n^s} = \sum_{(n,N)=1} \frac{\Lambda(n) c_n(\Pi_k \boxtimes \sigma_i)}{n^s} + \sum_{(n,N)=1} \frac{\Lambda(n)(c_n(\Pi'_k) - c_n(\Pi_k)) c_n(\sigma_i)}{n^s}.$$

Since $c_n(\Pi'_k) = c_n(\mathrm{sym}^k \pi) = c_n(\Pi_k)$ whenever $(n, N) = 1$ and $a_n \in A$, by Cauchy–Schwarz we have

$$\left(\sum_{(n,N)=1} \frac{\Lambda(n) |(c_n(\Pi'_k) - c_n(\Pi_k)) c_n(\sigma_i)|}{n^{1-\delta/3}} \right)^2 \le \sum_{(n,N)=1} \frac{\Lambda(n) |c_n(\Pi'_k) - c_n(\Pi_k)|^2}{n^{1+\delta/3}} \cdot \sum_{\substack{(n,N)=1 \\ a_n \notin A}} \frac{\Lambda(n) c_n(\sigma_i)^2}{n^{1-\delta}}.$$

Since $\Pi'_{k,p} \cong \mathrm{sym}^k \pi_p$ is tempered for all unramified $p \notin S$, the cuspidal summands of Π'_k must be unitary, so the first sum on the right-hand side converges by Rankin–Selberg. As for the second, for $a_n \notin A$ we have $c_n(\sigma_i)^2 \ll H(a_n, b_n)$, so it converges as well. Therefore,

$$\mathrm{Res}_{s=1} \sum_{(n,N)=1} \frac{\Lambda(n) c_n(\Pi'_k \boxtimes \sigma_i)}{n^s} = \mathrm{Res}_{s=1} \sum_{(n,N)=1} \frac{\Lambda(n) c_n(\Pi_k \boxtimes \sigma_i)}{n^s} = m_{k,i},$$

so σ_i occurs as a summand of Π'_k with multiplicity $m_{k,i}$. Since Π_k and Π'_k are both representations of $\mathrm{GL}_{k+1}(\mathbb{A})$, we have $\Pi'_k \cong \Pi_k$, as desired. This establishes (3d).

3.6 Temperedness and Galois type at ∞

Suppose that $\mathrm{sym}^5 \pi$ is automorphic. Then it agrees with an isobaric representation at all unramified finite places. By the uniqueness of Π_5, we must have $\mathrm{sym}^5 \pi_p \cong \Pi_{5,p} = \pi_p \boxtimes \mathrm{sym}^2 \pi'_p$ for all $p \nmid N$. When $\lambda_\pi(p) = \lambda_{\pi'}(p)$, this implies the relation

$$\lambda_\pi(p)^5 - 4\lambda_\pi(p)^3 + 3\lambda_\pi(p) = \lambda_\pi(p)(\lambda_\pi(p)^2 - 1),$$

so that $\lambda_\pi(p) \in \{0, \pm 1, \pm 2\}$. Thus, π_p is tempered for all $p \nmid N$. In particular, S is finite.

Finally, suppose that S is finite. Then, by what we have already shown, π is *s-icosahedral* in the sense of Ramakrishnan [9]. Appealing to [9, Theorem A], we conclude that π is tempered and π_∞ is of Galois type. This establishes (3e) and concludes the proof.

References

1. Brumley, Farrell: Maass cusp forms with quadratic integer coefficients. Int. Math. Res. Not. **2003**(18), 983–997 (2003)
2. Gelbart, Stephen, Jacquet, Hervé: A relation between automorphic representations of GL(2) and GL(3). Ann. Sci. École Norm. Sup. (4) **11**(4), 471–542 (1978)
3. Kim, Henry H.: Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. J. Amer. Math. Soc. **16**(1), 139–183 (2003). (**With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak**)
4. Kim, Henry H., Shahidi, Freydoon: Cuspidality of symmetric powers with applications. Duke Math. J. **112**(1), 177–197 (2002)
5. Kim, Henry H., Shahidi, Freydoon: Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the symmetric cube for GL_2. Ann. of Math. (2) **155**(3), 837–893 (2002). (**With an appendix by Colin J. Bushnell and Guy Henniart**)
6. Martin, K., Ramakrishnan, D.: A comparison of automorphic and Artin L-series of GL(2)-type agreeing at degree one primes, Advances in the theory of automorphic forms and their L-functions, Contemp. Math., vol. 664, pp. 339–350. Amer. Math. Soc., Providence (2016)
7. Ramakrishnan, Dinakar: Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2). Ann. of Math. (2) **152**(1), 45–111 (2000)
8. Ramakrishnan, Dinakar: Remarks on the symmetric powers of cusp forms on GL(2), Automorphic forms and L-functions I. Global aspects, Contemp. Math., vol. 488, Amer. Math. Soc., Providence, RI, pp. 237–256 (2009)
9. Ramakrishnan, D.: Icosahedral fibres of the symmetric cube and algebraicity, On certain L-functions, Clay Math. Proc., vol. 13, pp. 483–499. Amer. Math. Soc., Providence (2011)
10. Ramakrishnan, D.: Recovering cusp forms on GL(2) from symmetric cubes, SCHOLAR—a scientific celebration highlighting open lines of arithmetic research, Contemp. Math., vol. 655, pp. 181–189. Amer. Math. Soc., Providence (2015)
11. Sarnak, P.: Maass cusp forms with integer coefficients, A panorama of number theory or the view from Baker's garden (Zürich, 1999, pp. 121–127. Cambridge Univ. Press, Cambridge (2002)
12. Tenenbaum, Gérald: Introduction to analytic and probabilistic number theory, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015, Translated from the 2008 French edition by Patrick D. F. Ion
13. Wang, Song: On the symmetric powers of cusp forms on GL(2) of icosahedral type. Int. Math. Res. Not. **2003**(44), 2373–2390 (2003)