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Dynamics of a continuous Hénon model

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We study a continuous Hénon system obtained by considering the discrete original model in continuous time. While the dynamics of the continuous model is trivial, we are able to recover the complexity of the discrete model by the introduction of time delays. In particular high period limit cycles and chaotic attractors are observed. We illustrate the results with some numerical simulations. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

In his famous work [9], M. Hénon introduced a discrete system which exhibits chaotic behaviour. The target of the author was to find the simplest model which presents the same essential properties as those encountered for the Lorenz system ([10]).
The system is constructed through the composition of three mappings of the plane into itself, with the purpose to have that the volume is stretched in one direction, and at the same time folded over itself, in the course of one revolution.
The obtained system is the following:

$$\begin{cases} x_{n+1} &= 1 - a x_n^2 + y_n, \\ y_{n+1} &= b x_n. \end{cases} \tag{1.1}$$

where a and b are real parameters.
The system possesses two fixed points if $a > a_0 = \frac{(1-b)^2}{4}$, one of them is always linearly stable, the other is unstable for $a > a_1 = \frac{3(1-b)^2}{4}$. Through numerical simulations they found that for

$$a < a_0, \qquad \text{or} \qquad a > a_3,$$

where a_3 depends on b, the solutions always diverge, while for $a_0 < a < a_3$, depending on the initial values either the trajectories diverge or converge to an attractor. For $a_0 < a < a_1$, the attractor is the stable fixed point, while for $a > a_1$ the attractor consists in periodic set of p points. The value of p increases as a is increased till reaching a critical value a_2. For $a_2 < a < a_3$, the attractor appears to be chaotic and the dynamics is described for the case

$$a = 1.4, b = 0.3.$$

The above numerical results have been rigorously proved in [11]. In particular the author proved the existence of a transversal homoclinic orbit for some parameter values, thus proving the existence of a chaotic attractor.

In the present paper we consider a continuous time version of the Hénon map, and we will show that the dynamics of the continuous system is trivial compared to that of the discrete system. In fact the continuous system presents two fixed point (as in the discrete case) whose stability does not depend on the parameters, that is, one fixed point is always locally stable and the

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other is unstable. Then the solutions either converge to the stable fixed point or diverge to $-\infty$ (see next section). Of course, this behavior is not surprising, since the discrete model describes the intersections of the solutions of a three dimensional differential system with a two dimensional surface and then a more complex behavior is expected. Instead of considering the continuous model in higher dimension, we consider the addition of time delay. This strategy is motivated by the results of Berezowski (see [1], [2] and the applications [5]-[7], [12]) who studied the effects of time delay in continuous physical system. These studies are based on the consideration that most physical systems are characterized by the presence of delays of various types. It can be shown that, even small values of the delays, may lead to qualitative changes in the dynamical behaviour of the system. In particular in [1] they proved the presence of quasi-periodicity and chaos for a system that describes a tubular chemical reactor with superimposed recirculation of mass and/or heat.

We will see that, for the continuous Hénon model, the introduction of time delay may recover the complexity of the discrete model. In fact we will see (through Hopf Bifurcation and stability switches analysis) the appearance of stable limit cycles with increasing period and even the presence of a strange attractor which resembles the famous Hénon Attractor for the values $a = 1.3$ and $b = 0.3$. In the next section we introduce the continuous model and describes its dynamics, while in Section 3 we study the system with one delay parameter. Sections 4 and 5 are devoted to the case in which we have two delay parameters while in Section 6 we present several numerical simulations in order to illustrate the results of the paper. In the last section we present some consideration about the attractor proving the existence of a trapping region (positively invariant set) and of an absorbing set.

2. The continuous model

In this section we introduce the continuous model with two time delay parameters which will be the objective of our paper. First of all, note that if we substract x_n and y_n respectively in the equations of (1.1), we obtain:

$$\begin{cases} x_{n+1} - x_n &= -x_n + \left(1 - ax_n^2 + y_n\right), \\ y_{n+1} - y_n &= -y_n + \left(bx_n\right), \end{cases}$$

which can be considered as the time-discretization of the following continuous (Hénon) model

$$\begin{cases} \sigma_1 \dot{x} &= -x + \left(1 - ax^2 + y\right), \\ \sigma_2 \dot{y} &= -y + \left(bx\right), \end{cases}$$

where $\sigma_1 > 0, \sigma_2 > 0$ are some time-scale parameters.

In the present work we will consider a generalization of the previous continuous model including two time delay parameters. Namely, we will analyze the problem

$$\begin{cases} \sigma_1 \dot{x} &= -x + \left(1 - ax_{\tau_1}^2 + y_{\tau_1}\right), \\ \sigma_2 \dot{y} &= -y + \left(bx_{\tau_2}\right), \end{cases}$$

where $a, b \in \mathbb{R}$, $\sigma_1 > 0$ and $\sigma_2 > 0$, are the previous time-scale parameters, $\tau_1 > 0$ and $\tau_2 > 0$ are time delays, and we use the notation $x_\tau = x(t - \tau)$ for short. In order to compare our results with those obtained by Hénon for $a = 1.4$ and $b = 0.3$, we restrict the constants $a, b \in \mathbb{R}$ to be positive. In addition, for simplicity, we will work with $\sigma_1 = \sigma_2 = \sigma$. For results ensuring the global existence and uniqueness of initial value problems associated to the previous system the reader can see, e.g., Hale [8]. Let us now rewrite our system as follows for mathematical convenience

$$\begin{cases} \dot{x} &= -\dfrac{1}{\sigma}x + \dfrac{1}{\sigma} - \dfrac{a}{\sigma}x_{\tau_1}^2 + \dfrac{1}{\sigma}y_{\tau_1}, \\ \dot{y} &= -\dfrac{1}{\sigma}y + \dfrac{b}{\sigma}x_{\tau_2}. \end{cases} \tag{2.1}$$

Like its discrete counterpart, system (2.1) has two steady states $P^+ = (x_*^+, y_*^+)$ and $P^- = (x_*^-, y_*^-)$, where $x_*^\pm$ are the solutions of the equation

$$ax^2 + (1-b)x - 1 = 0,$$

and $y_*^\pm = bx_*^\pm$. In details we have:

$$x_*^\pm = \frac{b - 1 \pm \sqrt{(1-b)^2 + 4a}}{2a}. \tag{2.2}$$

Obviously, they are real since $(1-b)^2 + 4a > 0$. Having assumed $a > 0$, a direct calculation shows that $x_*^+ > 0$ and $x_*^- < 0$.

Let (x_*, y_*) be a steady state of (2.1). In order to determine the stability of (x_*, y_*), we linearize (2.1) at (x_*, y_*), and find that the associated characteristic equation takes the form

$$\lambda^2 + \frac{2}{\sigma}\lambda + \frac{1}{\sigma^2} + \left(\frac{2ax_*}{\sigma^2} + \frac{2ax_*}{\sigma}\lambda\right)e^{-\lambda\tau_1} - \frac{b}{\sigma^2}e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{2.3}$$

Lemma 1 *In absence of delay, the equilibrium point (x_*^+, y_*^+) for model (2.1) is locally asymptotically stable, while the equilibrium point (x_*^-, y_*^-) is unstable.*

Proof. When time delays are not considered, i.e. $\tau_1 = \tau_2 = 0$, the characteristic polynomial for model (1.1) reduces to

$$\lambda^2 + \frac{2(1+ax_*)}{\sigma}\lambda + \frac{1-b+2ax_*}{\sigma^2} = 0. \tag{2.4}$$

It is seen from (2.4) that the determinant of the functional Jacobian matrix is $(1-b+2ax_*)/\sigma^2$ and its trace is $-2(1+ax_*)/\sigma$. As a result, we derive that both eigenvalues of the characteristic polynomial (2.4) are negative when $x_* = x_*^+$ and the eigenvalues have different signs when $x_* = x_*^-$. The statement follows. We recall that P^- is always on the left of P^+, its stable curve $S(t)$ splits the plane into two regions (see figure 1): on the right of $S(t)$ the solutions converge to P^+ while on the left of $S(t)$ the solutions diverges. The latter behaviour is suggested by the preeminence of the term $-\frac{a}{\sigma}x^2$ in the first equations that let $x \to -\infty$. Then, in absence of delay the dynamics is trivial. In figure 1 we represent the vector field of the system for the classical case $a = 1.4$, $b = 0.3$.

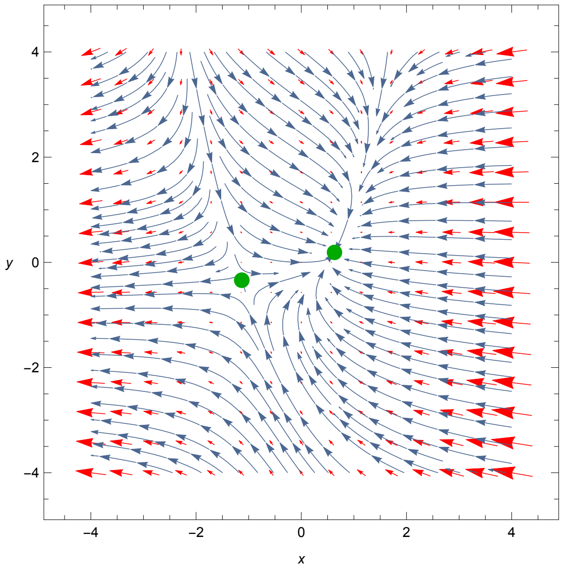


Figure 1. The vector field of the system for $\sigma = 1$, $a = 1.4$, $b = 0.3$.

We will now explore how the time delays affect the dynamics of (2.1) and will see how to recover the complexity of dynamics observed for the discrete case.
We first observe that it is well known that a root λ of (2.3) depends on time delays continuously. If it will ever leave the left half plane and enter the right half plane on the complex plane as time delays increase, it must cross the purely imaginary axis and a stability switch may occur (through Hopf bifurcation). We need to analyse the boundary of the stability region determined by the equations $\lambda = 0$ and $\lambda = \pm i\omega$. The case $\lambda = 0$ cannot occur since from (2.3) we obtain the absurd $0 = 2ax_* + 1 - b > 0$. Therefore, only the case $\lambda = i\omega$ $(\omega > 0)$ has to be analysed. We will analyze in details this problems in the following sections.

3. Case $\tau_1 = 0$ and $\tau_2 > 0$

In this section we consider the case in which $\tau_1 = 0$ and $\tau_2 > 0$. The characteristic polynomial (2.3) becomes

$$\lambda^2 + A\lambda + B + C\lambda e^{-\lambda\tau_2} = 0, \tag{3.1}$$

where the coefficients are given by the following expression.

$$A = \frac{2(1+ax_*)}{\sigma}, \quad B = \frac{1+2ax_*}{\sigma^2} \quad \text{and} \quad C = -\frac{b}{\sigma^2} < 0. \tag{3.2}$$

For the sake of simplicity we prove the following lemma which analyzes in details the sign of the coefficient of the characteristic polynomial depending on the control parameters.

Lemma 2 *1) Let $x_* = x_*^+ > 0$. Then $A > 0$, $B > 0$ and $C < 0$.*

2) *Let* $x_* = x_*^- < 0$.

i) *If* $b > \frac{1+4a}{2}$*, then* $A > 0$*,* $B > 0$ *and* $C < 0$*.*
ii) *If* $b = \frac{1+4a}{2}$*, then* $A > 0$*,* $B = 0$ *and* $C < 0$*.*
iii) *If* $a < b < \frac{1+4a}{2}$*, then* $A > 0$*,* $B < 0$ *and* $C < 0$*.*
iv) *If* $b = a$*, then* $A = 0$*,* $B < 0$ *and* $C < 0$*.*
v) *If* $b < a$*, then* $A < 0$*,* $B < 0$ *and* $C < 0$*.*

Proof. 1. is immediate; 2. follows from

$$sign(A) = sign(1 + ax_*^-) \quad \text{and} \quad sign(B) = sign(1 + 2ax_*^-), \tag{3.3}$$

noting that from (2.2) one has

$$1 + 2ax_*^- = b - \sqrt{(1-b)^2 + 4a}, \qquad 1 + ax_*^- = \frac{b + 1 - \sqrt{(1-b)^2 + 4a}}{2}. \tag{3.4}$$

Assuming that Eq. (3.1) has a purely imaginary solution of the form $\lambda = i\omega$, $(\omega > 0)$, substituting it into (3.1) and separating the real and imaginary parts lead to

$$-\omega^2 + B = -C\omega \sin \omega\tau_2, \qquad A\omega = -C\omega \cos \omega\tau_2. \tag{3.5}$$

Squaring and adding both equations of (3.5), we obtain the following equation of ω^2

$$\omega^4 + \left(A^2 - C^2 - 2B\right)\omega^2 + B^2 = 0, \tag{3.6}$$

In the next propositions we study the existence of positive roots for equation 3.6 depending on the coefficients (3.2).

Proposition 1 1) *If* $A^2 - C^2 - 2B \geq 0$*, or if* $A^2 - C^2 - 2B < 0$*,* $B < 0$ *and* $A^2 - C^2 > 4B$*, or if* $A^2 - C^2 - 2B < 0$*,* $B > 0$ *and* $A^2 - C^2 \geq 0$*, then Eq. (3.6) has no positive roots.*
2) *If* $A^2 - C^2 - 2B < 0$ *and* $B = 0$*, then Eq. (3.6) has a unique positive root* $\omega_0 = \sqrt{-(A^2 - C^2)}$*.*
3) *If* $A^2 - C^2 - 2B < 0$*,* $B < 0$ *and* $A^2 - C^2 < 4B$*, or if* $A^2 - C^2 - 2B < 0$*,* $B > 0$ *and* $A^2 - C^2 < 0$*, Eq. (3.6) has two positive roots*

$$\omega_{1,2} = \sqrt{\frac{-(A^2 - C^2 - 2B) \pm \sqrt{\left(A^2 - C^2 - 2B\right)^2 - 4B^2}}{2}},$$

where $\omega_1 < \omega_2$*.*

Proof. The statement is straightforward if $A^2 - C^2 - 2B \geq 0$. Let $A^2 - C^2 - 2B < 0$. If $B = 0$, then Eq. (3.6) reduces to $\omega^4 + \left(A^2 - C^2\right)\omega^2 = 0$, which has a unique positive root. Let $B < 0$. Notice that $A^2 - C^2 - 2B < 0$ implies $A^2 - C^2 < 0$. Hence, $\left(A^2 - C^2 - 2B\right)^2 - 4B^2 \neq 0$. We find that $\left(A^2 - C^2 - 2B\right)^2 - 4B^2 > 0$ when $A^2 - C^2 < 4B$, so that there exist two positive solutions for Eq. (3.6). On the other hand, $\left(A^2 - C^2 - 2B\right)^2 - 4B < 0$ when $A^2 - C^2 > 4B$, and so there is no positive solution for Eq. (3.6). Let $B > 0$. If $A^2 - C^2 \geq 0$, then Eq. (3.6) does not have positive roots. If $A^2 - C^2 < 0$, then Eq. (3.6) has two positive roots. This completes the proof. From the previous proposition we are able to study the existence/quantity of positive roots of equation (3.6) depending on the parameters of the system.

Proposition 2 1) *Let* $x_* = x_*^+$*.*

i) *If* $\sigma \geq \frac{b}{\left[2\left(1+ax_*^+\right)\right]}$*, then Eq. (3.6) has no positive roots.*
ii) *If* $\sigma < \frac{b}{\left[2\left(1+ax_*^+\right)\right]}$*, then Eq. (3.6) has two different positive roots.*

2) *Let* $x_* = x_*^-$*.*

i) *If* $\sigma \geq \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$ *or if* $\sigma < \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$*, with* $b > \frac{1+4a}{2}$ *and* $\sigma \geq \frac{b}{\left[2\left(1+ax_*^-\right)\right]}$*, or* $a < b < \frac{1+4a}{2}$ *and* $\sigma \geq \frac{b}{\left[2\left(1+ax_*^-\right)\right]}$*, or* $b = a$*, or* $b < a$ *and* $\sigma > \frac{b}{\left(-2ax_*^-\right)}$ *or* $a < b < \frac{1+4a}{2}$ *and* $\sigma > \frac{b}{\left(-2ax_*^-\right)}$*, then Eq. (3.6) has no positive roots.*
ii) *If* $\sigma < \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$ *with* $b > \frac{1+4a}{2}$ *and* $\sigma < \frac{b}{\left[2\left(1+ax_*^-\right)\right]}$*, or* $a < b < \frac{1+4a}{2}$ *and* $\sigma < \frac{b}{\left(-2ax_*^-\right)}$*, or* $b < a$ *and* $\sigma < \frac{b}{-2ax_*^-}$*, then Eq. (3.6) has two different positive roots* $\omega_{1,2}$*.*
iii) *If* $\sigma < \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$*, with* $b = \frac{1+4a}{2}$*,* $\sigma < \frac{b}{\left[2\left(1+ax_*^-\right)\right]}$ *i.e. with* $\sigma < \frac{1+4a}{4}$*, then Eq. (3.6) has a unique positive root* ω_0*.*

Proof. A direct computation shows

$$A^2 - C^2 - 2B = \frac{\left(1 + 2ax_*\right)^2 + 1}{\sigma^2} - \frac{b^2}{\sigma^4}.$$

Consequently,

$$A^2 - C^2 - 2B \geq 0$$

if

$$\sigma \geq \frac{b}{\sqrt{(1+2ax_*)^2+1}},$$

and $A^2 - C^2 - 2B < 0$ if

$$\sigma < \frac{b}{\sqrt{(1+2ax_*)^2+1}}.$$

The statement follows after a long and tedious analysis using the previous Lemma 2, Proposition 1, (3.3) and (3.4). The critical values $\tau_{2,j}^r$ $(r = 0, 1, 2; j = 0, 1, 2, \ldots)$ of the delay τ_2 corresponding to ω_r are obtained solving equations in (3.5). This yields

$$\sin(\omega_r \tau_{2,j}^r) = \frac{\omega_r^2 - B}{C\omega_r}, \qquad \cos(\omega_r \tau_{2,j}^r) = -\frac{A}{C}.$$

Thus,

$$\tau_{2,j}^r = \begin{cases} \dfrac{1}{\omega_r}\left\{\cos^{-1}\left(-\dfrac{A}{C}\right) + 2j\pi\right\}, & \text{if } \omega_r^2 - B \geq 0, \\[2ex] \dfrac{1}{\omega_r}\left\{\left[2\pi - \cos^{-1}\left(-\dfrac{A}{C}\right)\right] + 2j\pi\right\}, & \text{if } \omega_r^2 - B < 0. \end{cases} \tag{3.7}$$

The next proposition analyzes the direction of the bifurcation.

Proposition 3 *$\lambda = i\omega_r$ $(r = 0, 1, 2)$ is a simple root of the characteristic equation (3.1) satisfying*

$$\left[\frac{dRe(\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2,j}^0,\omega=\omega_0} > 0, \; \left[\frac{dRe(\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2,j}^1,\omega=\omega_1} < 0, \; \left[\frac{dRe(\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2,j}^2,\omega=\omega_2} > 0.$$

Proof. Let $\lambda(\tau_2)$ be the root of (3.1) near $\tau_2 = \tau_{2,j}^r$ such that $Re(\lambda(\tau_{2,j}^r)) = 0$ and $Im(\lambda(\tau_{2,j}^r)) = \omega_r$. By differentiating (3.1) with respect to τ_2, we obtain

$$\left[2\lambda + A + (C - C\tau_2\lambda)\, e^{-\lambda\tau_2}\right] \frac{d\lambda}{d\tau_2} = C\lambda^2 e^{-\lambda\tau_2}. \tag{3.8}$$

Therefore,

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{2\lambda + A + (C - C\tau_2\lambda)\, e^{-\lambda\tau_2}}{C\lambda^2 e^{-\lambda\tau_2}} = \frac{(2\lambda + A)\, e^{\lambda\tau_2}}{C\lambda^2} + \frac{1}{\lambda^2} - \frac{\tau_2}{\lambda}.$$

Recalling (3.1), $e^{\lambda\tau_2} = -(C\lambda)/(\lambda^2 + A\lambda + B)$, and so we have

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = -\frac{2\lambda + A}{\lambda(\lambda^2 + A\lambda + B)} + \frac{1}{\lambda^2} - \frac{\tau_2}{\lambda}.$$

A calculation yields

$$Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\tau_2=\tau_{2,j}^r,\omega=\omega_r} = \frac{2\omega_r^2 + A^2 - 2B}{(B - \omega_r^2)^2 + A^2\omega_r^2} - \frac{1}{\omega_r^2}.$$

Notice that (3.6) implies $C^2\omega_r^2 = (B - \omega_r^2)^2 + A^2\omega_r^2$. As a result,

$$Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\tau_2=\tau_{2,j}^r,\omega=\omega_r} = \frac{2\omega_r^2 + A^2 - C^2 - 2B}{C^2\omega_r^2},$$

which, using (3.7), gives

$$\begin{aligned} sign\left\{\left.\frac{d\,(Re\lambda)}{d\tau_2}\right|_{\tau_2=\tau_{2,j}^r,\omega=\omega_r}\right\} &= sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau_2=\tau_{2,j}^r,\omega=\omega_r}\right\} \\ &= sign\left\{2\omega_r^2 + A^2 - C^2 - 2B\right\}. \end{aligned}$$

If $\omega_r = \omega_0$, then $\omega_0^2 + A^2 - C^2 - 2B = 0$. Thus,

$$sign\left\{2\omega_r^2 + A^2 - C^2 - 2B\right\} = sign\left\{\omega_0^2\right\}.$$

If $\omega_r = \omega_{1,2}$, then
$$2\omega_r^2 + A^2 - C^2 - 2B = \pm\sqrt{\left(A^2 - C^2 - 2B\right)^2 - 4B^2}.$$
Hence, it is positive when $\omega_r = \omega_1$ and negative when $\omega_r = \omega_2$. It remains to prove that $\lambda = i\omega_r$ is a simple root of (3.1). If we suppose by contradiction that $\lambda = i\omega_r$ is a multiple root of (3.1), then from (3.8) we obtain $C\omega_r^2 e^{-i\omega_r \tau_{2,j}^r} = 0$, leading to an absurd. This completes the proof. Consider the fixed point P^+ for model (2.1) that is locally asymptotically stable for $\tau_2 = 0$. From the previous discussion we have that when Eq. (3.6) has no positive real root, all eigenvalues of the characteristic equation (3.1) have negative real parts for arbitrary $\tau_2 > 0$. The equilibrium point remains stable for all $\tau_2 > 0$. When Eq. (3.6) has only the positive root ω_0, then $\lambda = i\omega_0$ satisfies $[dRe(\lambda)/d\tau_2]_{\omega=\omega_0} > 0$, and therefore there exists a critical delayed value $\tau_{2,0}^0$ such that all the eigenvalues of the characteristic equation (3.1) have negative real parts for $\tau_2 \in (0, \tau_{2,0}^0)$ and at least one root with a positive real part for $\tau_2 > \tau_{2,0}^0$. The equilibrium point remains stable in $(0, \tau_{2,0}^0)$ and loses its stability via a Hopf bifurcation at $\tau_2 = \tau_{2,0}^0$. When Eq. (3.6) has two positive roots $\omega_1 < \omega_2$, then $[dRe(\lambda)/d\tau_2]_{\omega=\omega_1} < 0$ and $[dRe(\lambda)/d\tau_2]_{\omega=\omega_2} > 0$, and there exists a finite number of τ_2 intervals in which all eigenvalues of the characteristic equation (3.1) have negative real parts. If time delay is fixed into these intervals, the equilibrium point is locally asymptotically stable, while it is unstable for values outside these delayed ranges. Therefore, the system dynamics switches from stable to unstable, and then back to stable when delay increases and crosses the critical delayed values.

Consider now the fixed point P^- of the model (2.1) that is unstable for $\tau_2 = 0$. If Eq. (3.6) has no positive solutions or the unique solution ω_0, then no stability switches exist. If Eq. (3.6) has two positive solutions ω_1 and ω_2, then many stability switches may exist.

Based on the above analysis, we have the following results on the stability of the equilibria (x_*^+, y_*^+) and (x_*^-, y_*^-).

Theorem 1 *Let $\tau_{2,0}^0$ be defined as in (3.7).*

1) *The equilibrium (x_*^+, y_*^+) of system (2.1) is locally asymptotically stable for $\tau_2 \geq 0$ if $\sigma \geq \frac{b}{\left[2\left(1+ax_*^+\right)\right]}$, and it has stability switches for $\tau_2 > 0$ if $\sigma < \frac{b}{\left[2\left(1+ax_*^+\right)\right]}$. A Hopf bifurcation occurs at each stability switch.*
2) *The equilibrium (x_*^-, y_*^-) of system (2.1) is unstable for $\tau_2 \geq 0$ if one of the following conditions is fulfilled:*
   - $\diamond$ $\sigma \geq \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$,
   - $\diamond$ $\sigma < \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$, $b > \frac{(1+4a)}{2}$ *and* $\sigma \geq \frac{b}{\left[2\left(1+ax_*^-\right)\right]}$,
   - $\diamond$ $\sigma < \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$, $a < b < \frac{(1+4a)}{2}$ *and* $\sigma \geq \frac{b}{\left[2\left(1+ax_*^-\right)\right]}$,
   - $\diamond$ $\sigma > \frac{b}{\left(-2ax_*^-\right)}$,
   - $\diamond$ $\sigma < \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$, $b < a$ *and* $\sigma > \frac{b}{\left(-2ax_*^-\right)}$,
   - $\diamond$ $\sigma < \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$, *with* $b = \frac{(1+4a)}{2}$ *and* $\sigma < \frac{b}{\left[2\left(1+ax_*^-\right)\right]}$ *i.e. with* $\sigma < \frac{(1+4a)}{4}$.
3) *The equilibrium (x_*^-, y_*^-) has stability switches for $\tau_2 > 0$ if one of the following conditions is fulfilled*
   - $\diamond$ $\sigma < \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$, $b > \frac{1+4a}{2}$ *and* $\sigma < \frac{b}{\left[2\left(1+ax_*^-\right)\right]}$,
   - $\diamond$ $\sigma < \frac{b}{\sqrt{\left(1+2ax_*^-\right)^2+1}}$, $b < \frac{(1+4a)}{2}$ *and* $b \neq a$.

   A Hopf bifurcation occurs at each stability switch.

4. Case $\tau_1 > 0$, τ_2 in its stable intervals

In this case we consider both delay parameters different from zero. In order to simplify the analysis we consider the characteristic equation (2.3) with τ_2 in its stable intervals and regard τ_1 as a parameter. For convenience, we rewrite (2.3) as
$$\lambda^2 + M\lambda + N + (P + Q\lambda)\, e^{-\lambda\tau_1} + Re^{-\lambda(\tau_1+\tau_2)} = 0, \tag{4.1}$$
where
$$M = \frac{2}{\sigma}, \qquad N = \frac{1}{\sigma^2}, \qquad P = \frac{2ax_*}{\sigma^2}, \qquad Q = \frac{2ax_*}{\sigma}, \qquad R = -\frac{b}{\sigma^2}.$$
Let $\lambda = i\omega$ $(\omega > 0)$ be a root of (4.1). Then, we obtain
$$-\omega^2 + N = (R\sin\omega\tau_2 - Q\omega)\sin\omega\tau_1 - (R\cos\omega\tau_2 + P)\cos\omega\tau_1 \tag{4.2}$$
and
$$M\omega = (R\sin\omega\tau_2 - Q\omega)\cos\omega\tau_1 + (R\cos\omega\tau_2 + P)\sin\omega\tau_1. \tag{4.3}$$

Squaring and adding Eqs. (4.3) and (4.4) lead to

$$g(\omega) = \omega^4 + \left(M^2 - Q^2 - 2N\right)\omega^2 + (2QR\sin\omega\tau_2)\,\omega$$
$$+N^2 - P^2 - R^2 - 2PR\cos\omega\tau_2 = 0. \tag{4.4}$$

It is easy to check that $g(+\infty) = +\infty$. In case

$$g(0) = N^2 - (P+R)^2 < 0,$$

there is at least a positive ω satisfying Eq. (4.4). Notice that $g(0) < 0$ for $|N| < |P+R|$, i.e. when

$$\left|-1 \pm \sqrt{(1-b)^2 + 4a}\right| > 1.$$

We find that $g(0) < 0$ if the equilibrium point (x_*, y_*) of system (2.1) is (x_*^-, y_*^-), while $g(0)$ is not necessarily negative if $(x_*, y_*) = (x_*^+, y_*^+)$. Henceforth, we assume that Eq. (4.4) has finitely many positive roots $\omega_1, \omega_2, ..., \omega_N$. For every fixed ω_l, $l = 1, 2, ..., N$, there exists a sequence $\tau_{1,l}^j > 0$ $(j = 1, 2, ...)$ such that (4.4) holds. Let

$$\tau_1^c = \min\left\{\tau_{1,l}^j, l = 1, 2, ..., N, j = 1, 2, ...\right\}. \tag{4.5}$$

When $\tau_1 = \tau_1^c$, then the characteristic equation (4.1) has a pair of purely imaginary roots $\pm i\omega_c$. Let $\lambda(\tau_1)$ be the root of (4.2) near $\tau_1 = \tau_1^c$ satisfying $Re(\lambda(\tau_1^c)) = 0$ and $Im(\lambda(\tau_1^c)) = \omega_c$. Substituting $\lambda(\tau_1)$ into the left side of (4.1) and taking the derivative with respect to τ_1 gives

$$\left[2\lambda + M + Qe^{-\lambda\tau_1} - (P + Q\lambda)\tau_1 e^{-\lambda\tau_1} - R(\tau_1+\tau_2)e^{-\lambda(\tau_1+\tau_2)}\right]\frac{d\lambda}{d\tau_1} = \left[(P+Q\lambda)\lambda + R\lambda e^{-\lambda\tau_2}\right]e^{-\lambda\tau_1}. \tag{4.6}$$

Proceeding similarly to the case $\tau_1 = 0$ and $\tau_2 > 0$, we can verify the transversality condition of Hopf bifurcation to be given by

$$sign\left[\frac{dRe(\lambda)}{d\tau_1}\right]_{\tau_1=\tau_1^c} = sign\left[Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\right]_{\tau_1=\tau_1^c} = sign\,(a_1b_1 - a_2b_2)\,,$$

where

$$a_1 = M - Q\omega_c\tau_1^c\sin\omega_c\tau_1^c + (Q - P\tau_1^c)\cos\omega_c\tau_1^c + R(\tau_1^c+\tau_2)\sin\omega_c\tau_1^c\sin\omega_c\tau_2 - R(\tau_1^c+\tau_2)\sin\omega_c\tau_1^c\cos\omega_c\tau_2, \tag{4.7}$$

$$a_2 = -\omega_c\,(2 - Q\tau_1^c\cos\omega_c\tau_1^c) + (Q - P\tau_1^c)\sin\omega_c\tau_1^c - R(\tau_1^c+\tau_2)\sin\omega_c\tau_1^c\cos\omega_c\tau_2 - R(\tau_1^c+\tau_2)\cos\omega_c\tau_1^c\sin\omega_c\tau_2, \tag{4.8}$$

and

$$b_1 = \omega_c\left[(R\sin\omega_c\tau_2 - Q\omega_c)\cos\omega_c\tau_1^c + (P + R\cos\omega_c\tau_2)\sin\omega_c\tau_1^c\right], \tag{4.9}$$

$$b_2 = -\omega_c\left[(R\sin\omega_c\tau_2 - Q\omega_c)\sin\omega_c\tau_1^c - (P + R\cos\omega_c\tau_2)\cos\omega_c\tau_1^c\right]. \tag{4.10}$$

Then, if

$$sign\,(a_1b_1 - a_2b_2) > 0 \qquad (\text{resp. } sign\,(a_1b_1 - a_2b_2) < 0),$$

each crossing of the real part of characteristic roots at τ_1^c is from left (resp. right) to right (resp. left). Finally, notice that $\lambda = i\omega_c$ is a simple root of (4.2). Otherwise, from (4.6) we have

$$(P + Qi\omega_c)i\omega_c + Ri\omega_c e^{-i\omega_c\tau_2},$$

which means

$$R\cos\omega\tau_2 + P = 0$$

and

$$R\sin\omega\tau_2 - Q\omega = 0.$$

From (4.2) and (4.3), one would get the absurd $-\omega^2 + N = 0$ and $M\omega = 0$.

According to the general Hopf bifurcation theorem for Functional Differential Equations in [8], we derive the following results.

Theorem 2 *Let $g(\omega)$, τ_1^c and a_1, b_1, a_2, b_2 be defined as in (4.4), (4.5), (4.7)-(4.10), respectively, and τ_2 in its stable intervals.*

1) *If $g(\omega)$ has no positive root, then the equilibrium (x_*, y_*) of system (2.1) is locally asymptotically stable or unstable for any given time delay τ_1, depending on whether or not the system free of time delay is stable.*

2) *Suppose that the fixed point (x_*, y_*) for the system without delay is locally asymptotically stable (resp. unstable). Moreover, suppose that the system with delay satisfies the following conditions:*
   - ⋄ *$g(\omega)$ has only one positive root*
   - ⋄ *$sign\,(a_1b_1 - a_2b_2) > 0$ (resp. $sign\,(a_1b_1 - a_2b_2) < 0$),*

   then there exists one critical time delay $\tau_1^c > 0$ such that the equilibrium (x_*, y_*) of system (2.1) is locally asymptotically stable (resp. unstable) for $\tau_1 \in [0, \tau_1^c)$ and becomes unstable (resp. locally asymptotically stable) when $\tau_1 > \tau_1^c$. System (2.1) undergoes a Hopf bifurcation at the equilibrium (x_*, y_*) for $\tau_1 = \tau_1^c$.
   Suppose that the fixed point (x_*, y_*) for the system without delay is locally asymptotically stable (resp. unstable) and $sign\,(a_1b_1 - a_2b_2) < 0$ (resp. $sign\,(a_1b_1 - a_2b_2) > 0$), then the equilibrium (x_*, y_*) of system (2.1) remains locally asymptotically stable (resp.unstable) as τ_1 increases.

3) *If $g(\omega)$ has at least two positive roots, then a finite number of stability switches may occur as the time delay τ_1 increases from zero to the positive infinity, and the system becomes unstable at last. A Hopf bifurcation occurs at each stability switch.*

5. Case $\tau_1 = \tau_2 = \tau$

In this section we consider the case in which $\tau_1 = \tau_2$. The characteristic equation (2.3) becomes

$$\lambda^2 + A\lambda + B + (C\lambda + D)\, e^{-\lambda\tau} + E e^{-2\lambda\tau} = 0, \tag{5.1}$$

where its coefficients are given by

$$A = \frac{2}{\sigma}, \qquad B = \frac{1}{\sigma^2}, \qquad C = \frac{2ax_*}{\sigma}, \qquad D = \frac{2ax_*}{\sigma^2}, \qquad E = -\frac{b}{\sigma^2}.$$

In order to analyze the distribution of characteristic roots of (5.1) we will use the method proposed in [4].
If $\lambda = i\omega$, $(\omega > 0)$ is a root of (5.1), then

$$-\omega^2 + Ai\omega + B + (Ci\omega + D)\, e^{-i\omega\tau} + E e^{-2i\omega\tau} = 0. \tag{5.2}$$

Let us consider the following two cases:

$$(\omega\tau)/2 \neq (\pi/2) + j\pi, \qquad j \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}\,,$$

and

$$(\omega\tau)/2 = (\pi/2) + j\pi, \qquad j \in \mathbb{N}^0.$$

Suppose the first condition is verified. Letting $\theta = \tan\left[(\omega\tau)/2\right]$, we have

$$e^{-i\omega\tau} = (1 - i\theta)/(1 + i\theta).$$

Separating the real and imaginary parts in (5.2), we see that θ satisfies

$$\left\{\begin{array}{rcl} \left(\omega^2 - B + D - E\right)\theta^2 - 2A\omega\theta & = & \omega^2 - B - D - E, \\ (C - A)\,\omega\theta^2 + \left(-2\omega^2 + 2B - 2E\right)\theta & = & -(A + C)\,\omega. \end{array}\right. \tag{5.3}$$

Denote

$$\mathcal{D}(\omega) = \begin{vmatrix} \omega^2 - B + D - E & -2A\omega \\ (C - A)\,\omega & -2\omega^2 + 2B - 2E \end{vmatrix}, \tag{5.4}$$

$$\mathcal{E}(\omega) = \begin{vmatrix} \omega^2 - B - D - H & -2A\omega \\ -(A + C)\,\omega & -2\omega^2 + 2B - 2E \end{vmatrix}, \tag{5.5}$$

$$\mathcal{F}(\omega) = \begin{vmatrix} \omega^2 - B + D - E & \omega^2 - B - D - E \\ (C - A)\,\omega & -(A + C)\,\omega \end{vmatrix}. \tag{5.6}$$

If $\mathcal{D}(\omega) \neq 0$, one can solve from Eqs. (5.3) that

$$\theta^2 = \frac{\mathcal{E}(\omega)}{\mathcal{D}(\omega)}, \qquad \theta = \frac{\mathcal{F}(\omega)}{\mathcal{D}(\omega)}.$$

Furthermore, from (5.4)-(5.6), we obtain that ω satisfies

$$\mathcal{D}(\omega)\mathcal{E}(\omega) = [\mathcal{F}(\omega)]^2, \tag{5.7}$$

which is a polynomial equation for $z = \omega^2$ with degree 4,

$$z^4 + s_1 z^3 + s_2 z^2 + s_3 z + s_4 = 0, \tag{5.8}$$

with

$$\begin{aligned}
s_1 &= 2A^2 - 4B - C^2, \\
s_2 &= 6B - 2E^2 - 4A^2B - D^2 + A^4 - A^2C^2 + 2BC^2 + 2C^2E, \\
s_3 &= 2BD^2 - A^2D^2 - 4B^3 + 2A^2B^2 - B^2C^2 - 2BC^2E + 4ACDE - 2D^2E + 4BE^2 - 2A^2E^2 - C^2E^2, \\
s_4 &= (B-E)^2[-D^2 + (B+E)^2].
\end{aligned}$$

Notice that if $\mathcal{D}(\omega) = 0$, in order to make sure the solvability of Eqs. (5.3) for θ, we have $\mathcal{E}(\omega) = \mathcal{F}(\omega) = 0$. Then, ω is still a solution of Eq. (5.7).
Now suppose that

$$(\omega\tau)/2 = (\pi/2) + j\pi, \qquad j \in \mathbb{N}^0.$$

In this case, $A = C$ and $\omega^2 = B + E - D$, and thus $\mathcal{D}(\omega) = \mathcal{F}(\omega) = 0$. Hence, ω^2 satisfies Eq. (5.8) in this case as well. The next lemma (see [4]) provides the relation between solutions of the characteristic polynomial (5.1) and the roots of (5.8).

Lemma 3 *If $\pm i\omega$, $(\omega > 0)$ is a pair of purely imaginary roots of the characteristic equation (5.1), then ω^2 is a positive root of the quartic polynomial equation (5.8).*

The next lemma (see [4]) provides the algorithm of solving the critical delay values for purely imaginary roots of (5.1).

Lemma 4 *If Eq. (5.8) has a positive root ω_N^2 $(\omega_N > 0)$ and $\mathcal{D}(\omega_N) \neq 0$, then system (5.3) has a unique real root*

$$\theta_N = \frac{\mathcal{F}(\omega_N)}{\mathcal{D}(\omega_N)},$$

when $\omega = \omega_N$. Hence, the characteristic equation (5.1) has a pair of purely imaginary roots $\pm i\omega_N$ when

$$\tau = \tau_N^j = \frac{2\tan^{-1}(\theta_N) + 2j\pi}{\omega_N}.$$

Following again strategy in [4], the transversality condition for the roots moving across the imaginary axis can be formulated as follows.

Proposition 4 *Let*

$$\mathcal{G}(\omega,\theta) = \left[D(1+\theta^2) + 2E(1-\theta^2)\right][2\omega(1-\theta^2) + 2A\theta] - [C\omega(1+\theta^2) - 4E\theta][A(1-\theta^2) - 4\omega\theta + C(1+\theta^2)].$$

If $\mathcal{G}(\omega_N,\theta_N) \neq 0$, then $i\omega_N$ is a simple root of the characteristic equation (5.1) for $\tau = \tau_N^j$ and there exists $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ which is the unique root for $\tau \in (\tau_N^j - \varepsilon, \tau_N^j + \varepsilon)$ for some small $\varepsilon > 0$ satisfying

$$\alpha(\tau_N^j) = 0, \quad \omega(\tau_N^j) = \omega_N, \quad \text{and} \quad \alpha'(\tau_N^j) > 0.$$

Moreover,

$$[dRe(\lambda)/d\tau]_{\tau=\tau_N^j} > 0, \qquad \mathcal{G}(\omega_N,\theta_N) > 0,$$

and

$$[dRe(\lambda)/d\tau]_{\tau=\tau_N^j} < 0, \qquad \mathcal{G}(\omega_N,\theta_N) < 0.$$

Hence, from the above discussion, we can establish the following result.

Theorem 3 *If the equilibrium (x_*, y_*) of system (2.1) is locally asymptotically stable (resp. unstable) when $\tau = 0$ and one has that*

$$\mathcal{G}(\omega_N,\theta_N) > 0 \qquad (\text{resp.} \quad \mathcal{G}(\omega_N,\theta_N) < 0),$$

then there exists $\tau_* > 0$ such that the equilibrium (x_*, y_*) is locally asymptotically stable (resp. unstable) when $\tau \in [0, \tau_*)$ and it is unstable (resp. locally asymptotically stable) when $\tau \in (\tau_*, \tau_* + \varepsilon)$ for $\varepsilon > 0$ and small. Furthermore, system (2.1) undergoes a Hopf bifurcation at (x_*, y_*) when $\tau = \tau_*$. As τ increases from $\tau = \tau_*$, stability switches may occur.

6. Numerical simulations

In this section we present some numerical simulations in order to illustrate the results of the present work. In particular we will see how it is possible to recover the complexity of the dynamics of the discrete original model.

6.1. Case $\tau_1 = \tau_2$.

We start with the case in which delays parameters are equal and positive. We consider the following values of the parameters:

$$\sigma = 0.1, \qquad a = 1.4, \qquad b = 0.3.$$

Using Lemma 4.2 we obtain the following value of the delay parameter:

$$\tau_0 = 0.102701$$

for which

$$\mathcal{G}(\omega, \theta) = 90756.8 > 0.$$

Then from Theorem 4.4 we should observe a stability switch for the point P^+. We consider the following values of τ:

$$\tau = 0.1, \quad 0.18, \quad 0.4$$

In figure 2 we represent the first case, since $\tau < \tau_0$ the fixed point is stable and the solutions converges to it. We recall that in this case the positive fixed point i is $P^+ \approx (0.63, 0.19)$.

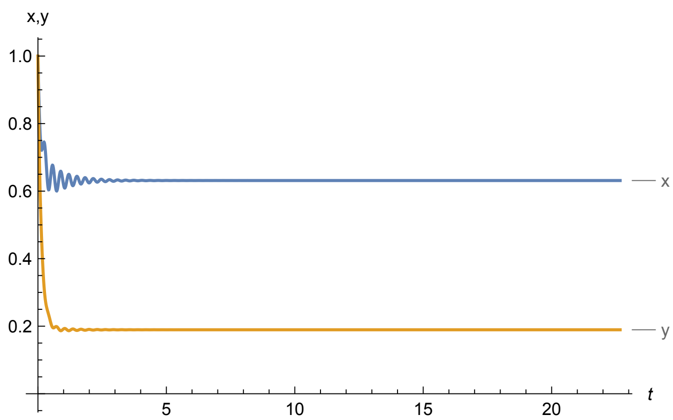


Figure 2. The case $\tau = 0.1 < \tau_0$. Solutions converges to the fixed point P^+.

Then we consider a value of the delay parameter bigger than τ_0, thanks to Theorem 4.4 a stability switch occurs and both fixed points $P^\pm$ are unstable. Then we expect the existence of an attractor. In figure 3 we represent the case $\tau = 0.18$, we observe that a stable limit cycle appears and solutions converge to it. This is generated by a Hopf bifurcation.

If we increase the values of τ we find limit cycles with higher period, in figure 4 we represent period 2 and 3 corresponding to $\tau = 0.33$ and $\tau = 0.39$ respectively.
If we consider higher values of the parameter τ the limit cycle turns into the a chaotic attractor very similar to the famous "Hénon Attractor" (see figure 5) which has been obtained for the original discrete model.

In this case we observe that we have obtained a high variety of asymptotic behavior with respect to the model without time delay.

6.2. Case, $\tau_1, \tau_2 > 0$

In this case we have fixed the parameters of the system as follows

$$a = 1.4, \quad b = 0.3, \quad \sigma = 1, \tau_1 = 3, \tag{6.1}$$

and let the second time delay parameters varies in the set:

$$\tau_2 \in \{0, 10, 13, 14, 18, 36.5\}. \tag{6.2}$$

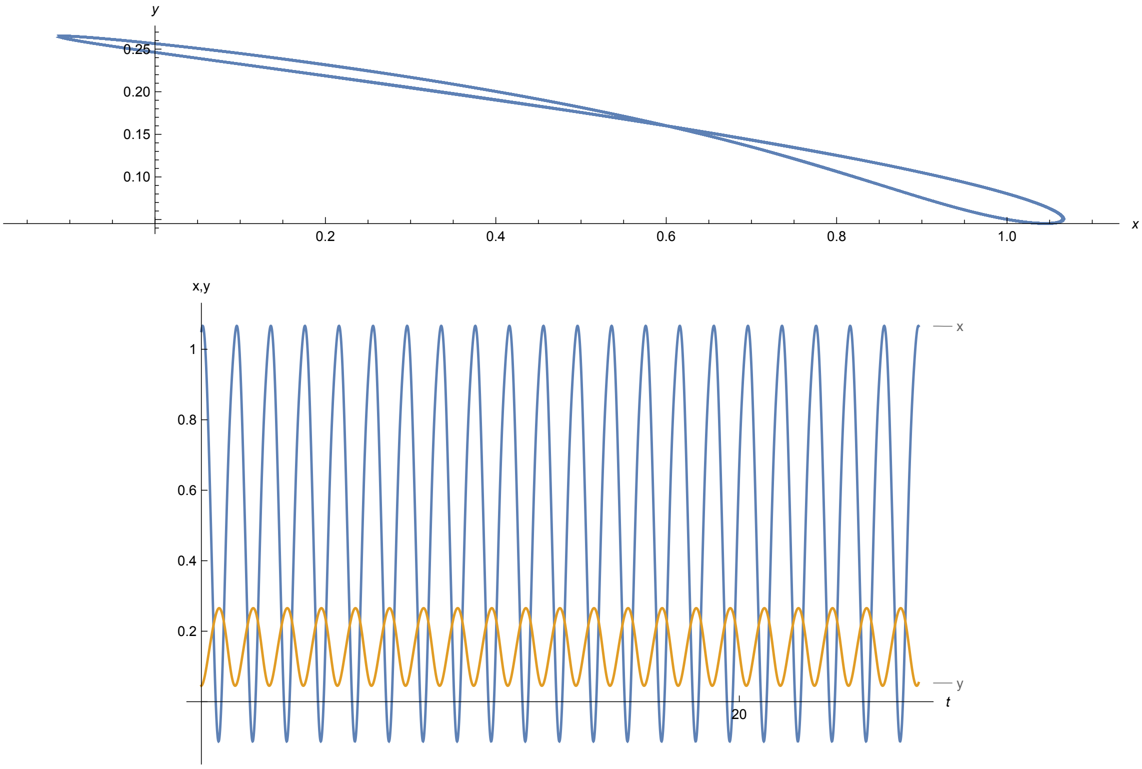


Figure 3. The case $\tau = 0.18 > \tau_0$. A stable limit cycle appears produced by a Hopf bifurcation. We represents also both $x(t)$ and $y(t)$.

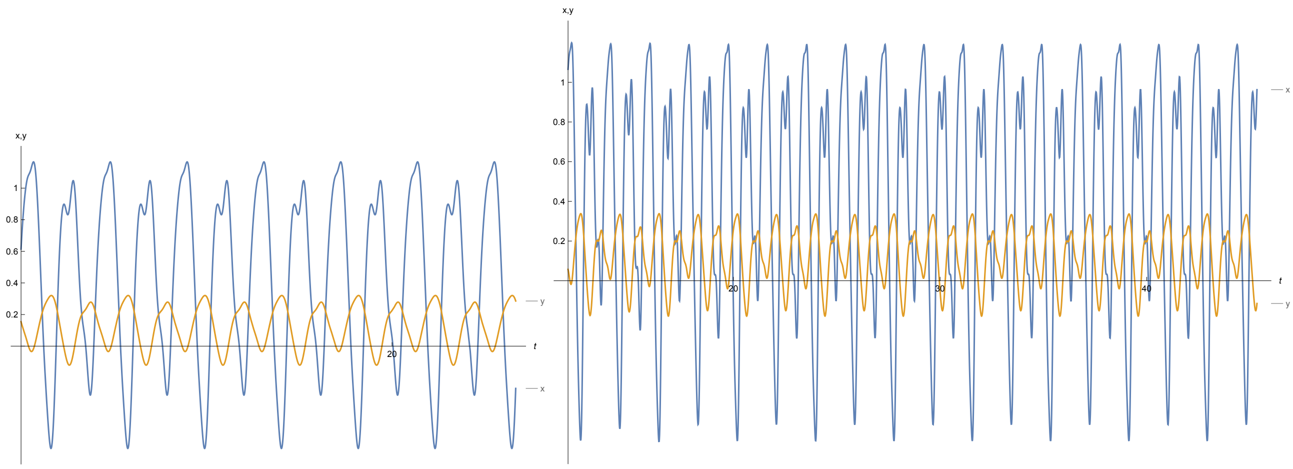


Figure 4. The solutions for $\tau = 0.33$ and $\tau = 0.39$ respectively. We observe period 2 and period 3 respectively.

In figure 6 below we represent the solutions of the system, while in figure 7 we represent their time series for the parameters value as listed above.

In some intervals of the parameter we observe limit cycles with increasing period and possibly a chaotic attractor with different shape with respect to that of the Hénon map.

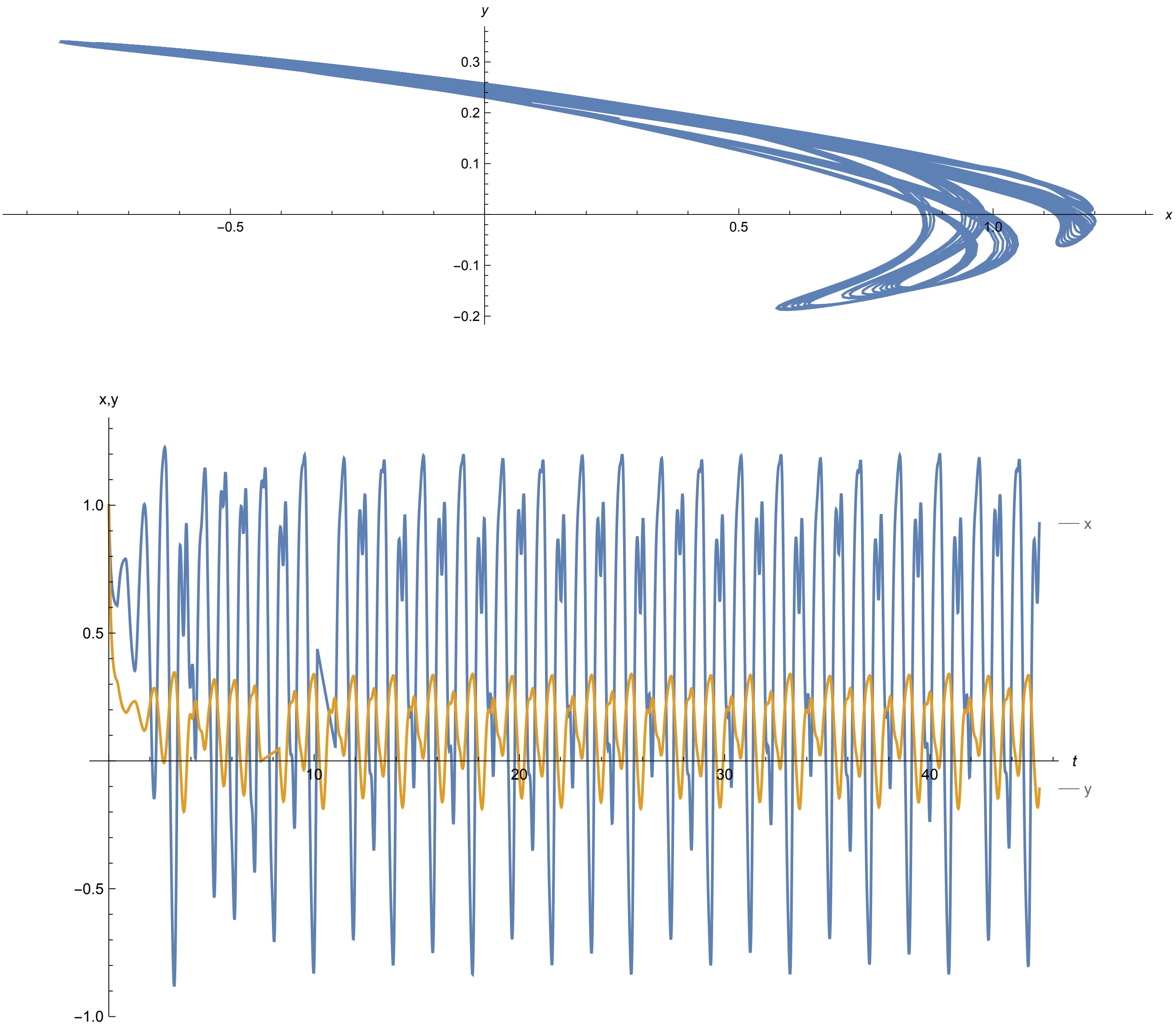


Figure 5. The case $\tau = 0.4 > \tau_0$. Both fixed point are unstable and the attractor is chaotic and resembles in shape the famous "Hénon Attractor" obtained for the original discrete model.

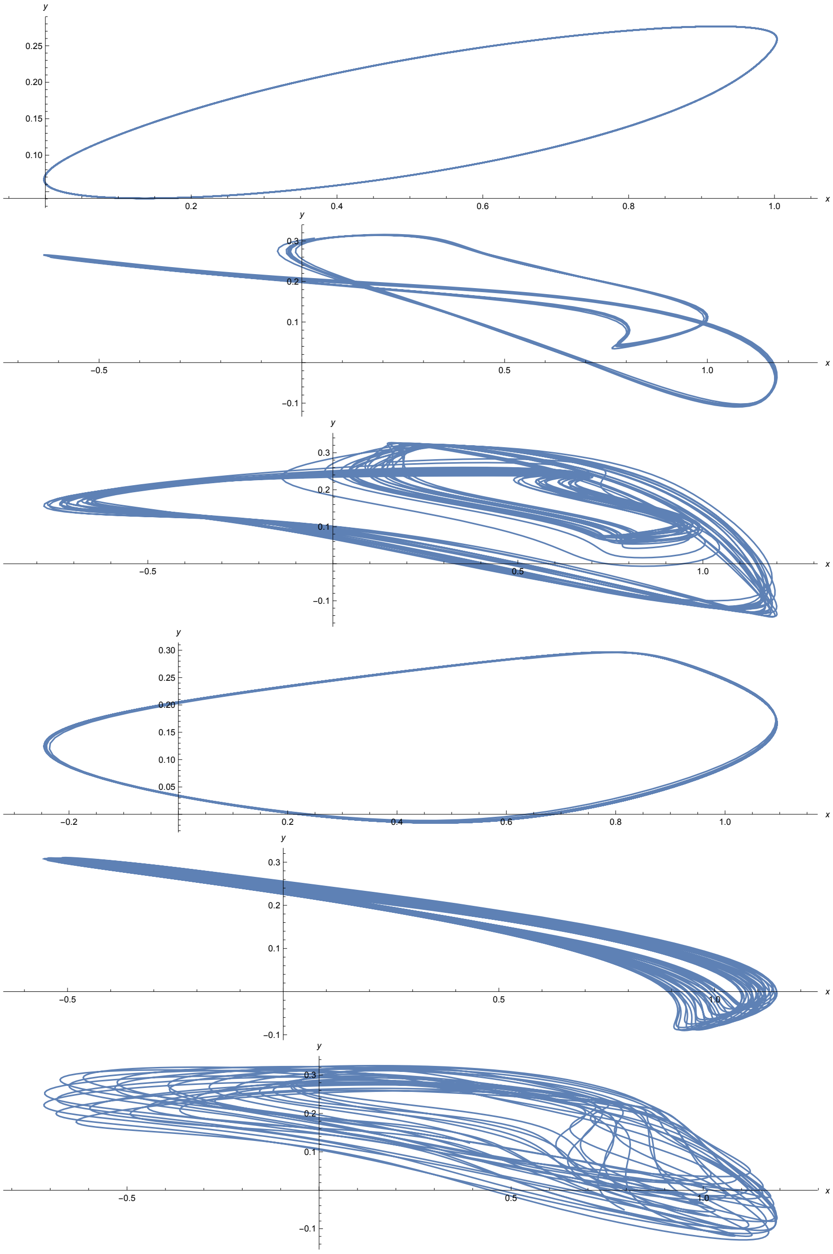
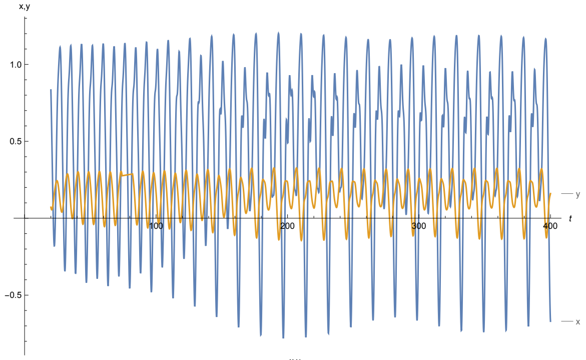
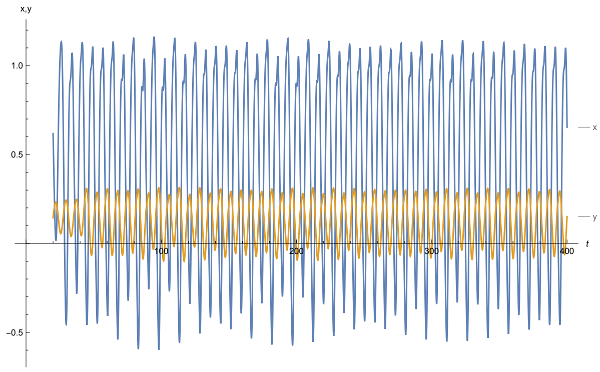


Figure 6. The solutions of the system with parameters in (6.1) and τ_2 as in (6.2).





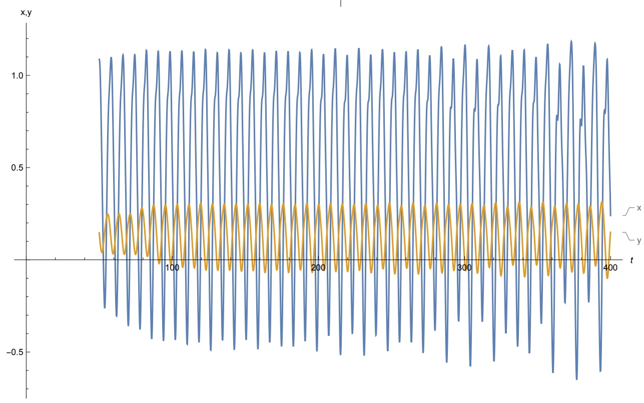


Figure 7. Time series for $\tau_2 \in \{13, 18, 36.5\}$.

7. Existence of attractor

In this section we consider the problem of the existence of a local and a global attractor for the model with and without delay respectively, in other words, a compact invariant set which attracts any bounded set in $\mathbb{R}^2$ (see Caraballo and Han [3] for the definitions and main results ensuring the existence of global attractors).

We start with the following preliminary result:

Proposition 5 *The set*

$$\Sigma = \left\{(x, y) \in \mathbb{R}^2 : \qquad x \geq x_*^-, \quad y \geq y_*^-\right\}.$$

is positively invariant for the solutions of the system (2.1).

Proof. In order to proof the statement is sufficient to observe that the vector field point inward or it is tangent to $\partial\Sigma$ on $\partial\Sigma$.

7.1. The case without time delay

Then main ingredient for proving the existence of an attractor is the following theorem

Theorem 4 *There exists a positive constant M_a such that the set*

$$A_\eta = \{(x, y) \in \Sigma : \qquad x^2 + y^2 \leq \frac{2M_a}{\sigma(1-b)} + \eta\}$$

is a compact absorbing set for the solutions of system (2.1) *for any* $\eta > 0$.

Proof. We multiply the first equation of the system by x and the second by y, then we sum the results:

$$\frac{1}{2}\frac{d}{dt}(x^2 + y^2) + \frac{1}{\sigma}(x^2 + y^2) = \frac{1}{\sigma}x - \frac{a}{\sigma}x^3 + \frac{1}{\sigma}(b+1)yx$$

If $b \in (0, 1)$ we have:

$$\frac{1}{2}\frac{d}{dt}(x^2 + y^2) + \frac{1}{2\sigma}(1-b)(x^2 + y^2) \leq \frac{1}{\sigma}x(1 - ax^2).$$

we observe that for $x \in \Sigma$, the function

$$f(x) = x(1 - ax^2),$$

has its global maximum among the following values:

$$f(x_-^*) = \frac{(1-b)}{2a^2}\left\{2a + (1-b)\left[\sqrt{(b-1)^2 + 4a} + 1 - b\right]\right\}, \qquad f(\frac{1}{\sqrt{3a}}) = \frac{2}{3\sqrt{3a}}.$$

Thus we obtain the following inequality

$$\frac{1}{2}\frac{d}{dt}(x^2 + y^2) + \frac{1}{2\sigma}(1-b)(x^2 + y^2) \leq \frac{1}{\sigma}M_a. \tag{7.1}$$

where

$$M_a = \max\{f(x_-^*), f(\frac{1}{\sqrt{3a}})\}.$$

Multiplying (7.1) by $e^{\frac{1-b}{\sigma}t}$,

$$\frac{d}{dt}\left([x^2 + y^2]e^{\frac{1-b}{\sigma}t}\right) \leq \frac{2}{\sigma}M_a e^{\frac{1-b}{\sigma}t}.$$

Integrating the previous inequality in $[0, t]$ we obtain:

$$x^2 + y^2 \leq e^{-\frac{1-b}{\sigma}t}\left\{x_0^2 + y_0^2 - \frac{2M_a}{\sigma(1-b)}\right\} + \frac{2M_a}{\sigma(1-b)}$$

Then, for any bounded set in Σ of the form

$$D = \{(x, y) \in \Sigma : \qquad x^2 + y^2 \leq d^2\},$$

there exists a time $T(D) = -\frac{\sigma}{1-b}\ln\left(\frac{\eta}{d^2}\right)$ such that for $(x_0, y_0) \in D$ we have

$$x^2(t) + y^2(t) < \frac{2M_a}{\sigma(1-b)} + \eta, \qquad \text{for all } t \geq T(D),$$

which concludes the proof.

Theorem 5 *The system* (2.1) *possesses a compact attractor* $\mathcal{A} \subset \Sigma$.

Proof. Thanks to Theorem 2.7 in [3] we obtain the existence of a compact attractor by the existence of a compact absorbing set.

7.2. The case with delay

For simplicity, we only consider the case in which $\tau_1 = \tau_2$. In this situation we use a different approach, we will find a positive invariant set in which a local attractor lies and which attracts all the bounded sets of the positive invariant set.

First of all we observe that Proposition 5 is still valid. Next we prove the following preliminary result:

Proposition 6 *Suppose that $x(t)$ and $y(t)$ are bounded and suppose that the initial data satisfy*

$$x(\theta) \le \frac{1}{1-b}, \qquad y(\theta) \le \frac{b}{1-b}, \qquad \text{for all} \quad \theta \in [-\tau, 0].$$

Then

$$x_M = \max_{t\ge -\tau}\{x(t)\} \le \frac{1}{1-b},$$
$$y_M = \max_{t\ge -\tau}\{y(t)\} \le \frac{b}{1-b}.$$

Proof. From the second equation we have that

$$\dot{y} = \frac{b}{\sigma}x(t-\tau) - \frac{1}{\sigma}y \le \frac{b}{\sigma}x_M - \frac{1}{\sigma}y,$$

from which it follows immediately that

$$y_M \le b \cdot x_M.$$

Now suppose T is such that $x(T) = x_M$, then

$$\dot{x}(T) = 0,$$

and from the first equation of the system we obtain:

$$-\frac{1}{\sigma}x_M + \frac{1}{\sigma} - \frac{1}{\sigma}ax^2(t-\tau) + \frac{1}{\sigma}y(t-\tau) = 0.$$

Then

$$x_M = 1 - ax_\tau^2 + y_\tau \le 1 + y_M \le 1 + bx_M,$$

from which we obtain

$$x_M \le \frac{1}{1-b}.$$

The previous proposition provides bounds for solutions whenever they are bounded. In the next proposition we will prove that any solution is bounded.

Proposition 7 *The solutions of system (2.1), with $\tau_1 = \tau_2$, are bounded.*

Proof. In order to simplify the computations we perform the following transformation

$$\tilde{x} = x - x_*^-, \qquad \tilde{y} = y - y_*^-,$$

which transforms P^- into $(0,0)$ and P^+ into $\left(\frac{\sqrt{\Delta}}{a}, \frac{b\sqrt{\Delta}}{a}\right)$, where

$$\Delta = (1-b)^2 + 4a.$$

Moreover, the positive invariant set for the associated problem is $\mathbb{R}^2_+$. We rewrite the system in the new coordinates without replacing the notation:

$$\begin{cases} \dot{x} &= -\frac{1}{\sigma}x + \frac{1}{\sigma}y_\tau - \frac{a}{\sigma}x_\tau^2 + \frac{2a}{\sigma}|x_-^*|x_\tau, \\ \dot{y} &= -\frac{1}{\sigma}y + \frac{b}{\sigma}x_\tau. \end{cases}$$

We sum both equations, term by term, and obtain:

$$\dot{x} + \dot{y} + \frac{1}{\sigma}(x+y) = \frac{1}{\sigma}(x_\tau + y_\tau) + \frac{a}{\sigma}x_\tau\left(\frac{\sqrt{\Delta}}{a} - x_\tau\right).$$

Observe that the function

$$f(q) := \frac{a}{\sigma}q\left(\frac{\sqrt{\Delta}}{a} - q\right)$$

is positive for $q \in (0, \frac{\sqrt{\Delta}}{a})$ and achieves its maximum for $q = \frac{\sqrt{\Delta}}{2a}$ with

$$\max f = \frac{\Delta}{4a\sigma} = \frac{1}{\sigma}\left[\frac{(1-b)^2}{4a} + 1\right] := \frac{1}{\sigma}F.$$

We consider two different cases:
Case 1: Suppose

$$x(t) + y(t) \geq x(t-\tau) + y(t-\tau), \tag{7.2}$$

then

$$\dot{x}(t) + \dot{y}(t) \leq \begin{cases} 0, & \text{if} \quad x(t-\tau) \geq \frac{\sqrt{\Delta}}{a}, \\ \frac{1}{\sigma}F, & \text{if} \quad x(t-\tau) < \frac{\sqrt{\Delta}}{a}. \end{cases}$$

If there exists a time T such that $x(T-\tau) = \frac{\sqrt{\Delta}}{a}$ then $x(t) + y(t)$ is decreasing in a right neighbourhood of T, and remains decreasing till $x(t-\tau) \geq \frac{\sqrt{\Delta}}{a}$ and (7.2) holds. Then $x + y$ is bounded in this time interval.
Suppose on the contrary that

$$x(t-\tau) < \frac{\sqrt{\Delta}}{a}, \qquad \text{for all} \quad t \geq T,$$

then

$$x(t) < \frac{\sqrt{\Delta}}{a}, \qquad \text{for all} \quad t \geq T + \tau.$$

From the second equation of the system we have that

$$\dot{y} \leq -\frac{1}{\sigma}y + \frac{1}{\sigma}b\frac{\sqrt{\Delta}}{a}, \qquad \text{for all} \quad t \geq T,$$

then we conclude that $y(t)$ is bounded for all $t \geq T$.
We observe that if $x(t-\tau) \leq \frac{\sqrt{\Delta}}{a}$ in the interval $[t_1, t_2]$, then

$$y(t_2) \leq y(t_1)e^{-\frac{1}{\sigma}(t_2-t_1)} + \frac{1}{\sigma}b\frac{\sqrt{\Delta}}{a}\frac{(t_2-t_1)}{e^{\frac{1}{\sigma}t_2}}.$$

From the second equation of the system, it is not possible to have a sequence of such intervals in which

$$y(t) \to \infty \qquad \text{being} \qquad x(t-\tau) < \frac{\sqrt{\Delta}}{a}.$$

Case 2: Suppose

$$x(t) + y(t) < x(t-\tau) + y(t-\tau). \tag{7.3}$$

If the previous expression is valid in $(T, +\infty)$ we obtain that $x + y$ is bounded by $x(-\tau) + y(-\tau)$.
If, on the contrary, there exists a infinite sequence of intervals $[t_1^n, t_2^n]$ such that (7.3) holds, then $x + y$ is decreasing in each interval.
We have to exclude the case in which the sequence of local maxima of $x + y$ blow up in infinite time. Suppose that T_n is the sequence of time for which the function $x + y$ attains the local maxima with $T_n \to +\infty$. For any local maximum we have

$$x(T_n) + y(T_n) = x(T_n - \tau) + y(T_n - \tau) + ax(T_n - \tau)\left[\frac{\sqrt{\Delta}}{a} - x(T_n - \tau)\right].$$

If

$$x(t_n - \tau) > \frac{\sqrt{\Delta}}{a},$$

then

$$x(T_n) + y(T_n) < x(T_n - \tau) + y(T_n - \tau),$$

and, as a consequence, at $T_n - \tau$ there is a local maximum or there exists $\varepsilon > 0$ such that at $T_n - \tau - \varepsilon$ there is a local maximum. Then we repeat the same reasoning with T_n replaced by $T_n - \tau$ or $T_n - \tau - \varepsilon$ and we recursively bound each local maximum by a previous one.
If

$$x(T_n - \tau) < \frac{\sqrt{\Delta}}{a},$$

then

$$x(T_n) + y(T_n) > x(T_n - \tau) + y(T_n - \tau),$$

and the previous expression remains true in a small right neighbourhood of T_n, then we are in case 1. Then if T_{n_k} is the subsequence of local maximum such that the previous two inequalities are verified, we have that from the second equation of the system we cannot have that $y(T_{n_k}) \to \infty$. From the previous propositions we obtain that the set

$$\Sigma := \left[x_-^*, \frac{1}{1-b}\right] \times \left[y_-^*, \frac{b}{1-b}\right]$$

is positively invariant, that is, if solutions start in Σ, they remain inside Σ for all time. As a consequence a local attractor is also contained in it (see figure (8)).

Theorem 6 *The system* (2.1) *with* $\tau_1 = \tau_2$, *possesses a local attractor inside the set* Σ.

Proof. Since Σ is positively invariant then it contains the omega limit sets of the solutions starting on it and as a consequence Σ contains a local attractor.

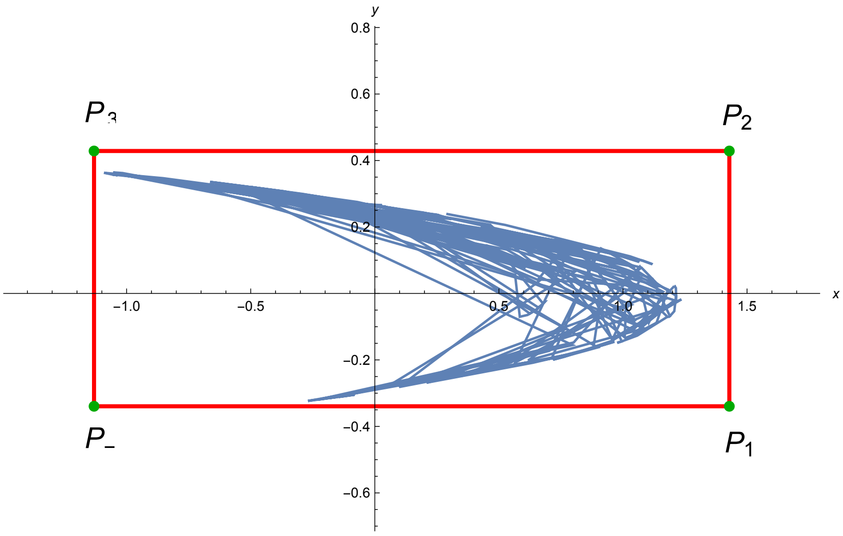


Figure 8. The attractor, for $a = 1.4$, $b = 0.3$, $\sigma = 0.1$, $\tau_1\tau_2 = 40$, is inside the set Σ. Here $P_- = (x_*^-, y_*^-)$, $P_1 = (\frac{1}{1-b}, y_*^-)$, $P_2 = (\frac{1}{1-b}, \frac{b}{1-b})$, $P_3 = (x_*^-, \frac{b}{1-b})$

8. Conclusions

In the present paper we have considered a continuous system derived from the Hénon map, in particular we have shown that the complex dynamics of the discrete model is not displayed by the continuous one. In order to recover the complexity of the solutions set it is necessary to consider time delay. Several interesting questions are left open and may be considered a natural continuation of the present work. Among them it is of interest to consider the problem of multi-stability, that is the coexistence of several local attractors (see [13] and references cited therein for a general discussion). The complicate conditions founded for stability of fixed points and for Hopf Bifurcation suggest that multi-stability is possible for the system, however it is far from the scope of the present article and we will be addressed in future research.

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