PULLBACK ATTRACTOR FOR A DYNAMIC BOUNDARY NON-AUTONOMOUS PROBLEM WITH INFINITE DELAY

R. A. Samprogna[1] and T. Caraballo[2]

[1]Departamento de Matemática
Centro de Ciências Exatas e de Tecnologia
Universidade Federal de São Carlos
Caixa Postal 676, 13.565-905 São Carlos SP, Brazil

[2]Departamento de Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla
Apdo. de Correos 1160, 41080-Sevilla, Spain

Abstract. In this work we prove the existence of solution for a p-Laplacian non-autonomous problem with dynamic boundary and infinite delay. We ensure the existence of pullback attractor for the multivalued process associated to the non-autonomous problem we are concerned. Finally, we also prove the existence of a more general attractor for the problem known as $\mathscr{D}$-pullback attractor.

1. Introduction

Let $\tau \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$ and $N \geq 3$, consider the following dynamical boundary conditions problem with infinite delay

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = f_1(t, u^t) + g_1(t,x), & (t,x) \in (\tau, +\infty) \times \Omega, \\ u_t + |\nabla u|^{p-2}\partial_{\overrightarrow{n}} u = f_2(t, u^t) + g_2(t,x), & (t,x) \in (\tau, +\infty) \times \Gamma, \\ u(\tau + s, x) = \Psi(s,x), \ s \in (-\infty, 0], x \in \overline{\Omega} \end{cases} \tag{P}$$

where $\overrightarrow{n}$ is the outer normal to Γ, $p \in [2, +\infty)$ and Δ_p denotes the p-Laplacian operator, defined by $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$. The external forces g_i, $i = 1, 2$, satisfy assumptions that will be stated later, Ψ is a given function defined in the interval $(-\infty, 0]$ and the external force field f_i containing some hereditary characteristic denoted by u^t, which is a function defined on $(-\infty, 0)$ by the relation $u^t(s) = u(t + s)$, $s \in (-\infty, 0)$.

The interest for problems with dynamic boundary conditions has been growing over the last forty years, see [8, 14, 16]. Motivated by mathematicians' interests and physical applications, the authors of [10] and [11] studied an autonomous version of Problem (P). After that, some works emerged of this problem, with a non-autonomous term just in perturbations g_i can be found in [15, 24] and [25], where the authors have established the existence of a uniform attractor and pullback attractor for the problems, respectively. In [22] the authors considered a non-autonomous term in perturbations f_i and ensured the existence of solution as well as

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the existence of $\mathscr{D}$-pullback attractor for the generalized process associated with a similar problem to (P) without uniqueness of solution.

The delay terms appear naturally in many applications as velocity field in wind tunnel and population growth, e.g., [17]. The study of the asymptotic behaviour of problems with finite delay with uniqueness or in multivalued contexts can be found in [5], a version with infinite delays can be found in [4], both works consider autonomous and non-autonomous problems. In the work [26] the authors developed a theory of pullback attractors for multivalued process associated with infinite delay problems and they established conditions to guarantee the existence of an invariant pullback attractor for this multivalued process. Our work in this paper will be based on these results. Another thing that motivates us is that there are only a few delay problems related to operator Δ_p which is a very good example of a nonlinear maximal monotone operator.

We organize this work as follows. In the next section, we recall some notations, definitions and properties of suitable spaces for the study of Problem (P). In Section 3 we present some definitions and a result that ensures the existence of the pullback attractor in multivalued context developed in [26]. In Section 4 we prove the existence of weak solution for Problem (P). Finally, Section 5 is devoted to ensure the existence of pullback attractor for our problem and then, in Section 6, we just choose a more general universe of sets to be attracted and show the existence of a more general attractor known as $\mathscr{D}$-pullback attractor.

2. Preliminaries

In this section, following [9] we define the appropriate spaces to study Problem (P).

Consider the Lebesgue space

$$L^r(\Gamma) = \{v : \|v\|_{L^r(\Gamma)} < \infty\},$$

where $\|v\|_{L^p(\Gamma)} = \left(\int_\Gamma |v|^p dS\right)^{1/p}$, for $p \in [1,\infty)$, dS is the surface measure on Γ induced by dx and $\|v\|_{L^\infty(\Gamma)} = \inf\{C;\ |v(x)| \le C \ \text{ a.e. in } \Gamma\}$.

The phase space to be considered is given by

$$\mathbb{X}^p := L^p(\Omega, dx) \times L^p(\Gamma, dS) = \{F = (f,g); f \in L^p(\Omega) \text{ and } g \in L^p(\Gamma)\},$$

with the norm

$$\|F\|_{\mathbb{X}^p} = \left(\int_\Omega |f|^p dx + \int_\Gamma |g|^p dS\right)^{1/p},$$

for $1 \le p < \infty$, and

$$\|F\|_{\mathbb{X}^\infty} := \max\left\{\|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Gamma)}\right\},$$

for $p = +\infty$. This space can be identified with $L^p(\overline{\Omega}, d\mu)$ where $d\mu = dx \oplus dS$, i.e., if $A \subset \overline{\Omega}$ is $\mu-$measurable, then $\mu(A) = |A \cap \Omega| + S(A \cap \Gamma)$.

Note that the space $\mathbb{X}^2$, with the following inner product

$$\langle \cdot, \cdot \rangle_{\mathbb{X}^2} := \langle \cdot, \cdot \rangle_{L^2(\Omega)} + \langle \cdot, \cdot \rangle_{L^2(\Gamma)},$$

is a separable Hilbert space.

For $p \in (1,\infty)$ we define the fractional order Sobolev space

$$W^{1-\frac{1}{p},p} := \left\{u \in L^p(\Gamma) :\ \int_\Gamma \int_\Gamma \frac{|u(x)-u(y)|^p}{|x-y|^{p+N-2}} dS_x dS_y < \infty\right\}.$$

Consider the vector subspace of $W^{1,p}(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma)$, given by

$$\mathbb{V}^p = \{U = (u, v); u \in W^{1,p}(\Omega) \text{ and } v = \gamma(u)\},$$

where $\gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\Gamma)$ is the continuous trace operator. In $\mathbb{V}^p$, we can consider the usual norm $\|U\|_{\mathbb{V}^p} = \|u\|_{W^{1,p}(\Omega)} + \|\gamma(u)\|_{W^{1-\frac{1}{p},p}(\Gamma)}$. The space $\mathbb{V}^p$ is densely and compactly contained in the Hilbert space $\mathbb{X}^2$ for $2 \leq p < +\infty$, as can be seen in [10].

Note that we can identify $u \in W^{1,p}$ as a couple $U = (u, \gamma(u)) \in \mathbb{V}^p$. The continuity of γ ensures the equivalence between the norms of $W^{1,p}(\Omega)$ and $\mathbb{V}^p$. We can show that $\mathbb{V}^p$ is a reflexive and separable space for $1 < p < \infty$. Furthermore,

$$\mathbb{V}^p \subset\subset \mathbb{X}^2 \subset (\mathbb{V}^p)^* \text{ for } 2 \leq p < +\infty. \tag{2.1}$$

3. Abstract Results

In this section we present a summary of definitions and results from [26], where the authors developed a theory of invariant pullback attractors in a multivalued context.

Let (X, ρ) be a complete metric space. For $x \in X$, $A, B \subset X$ and $\varepsilon > 0$ we define

$$\rho(x, A) := \inf_{a\in A}\{\rho(x, a)\}; \qquad dist(A, B) := \sup_{a\in A}\inf_{b\in B}\{\rho(a, b)\};$$
$$\mathcal{O}_\varepsilon(A) := \{z \in X; \rho(z, A) < \varepsilon\}.$$

Denote by $\mathcal{P}(X)$ the nonempty subsets of X.

Definition 3.1. *A family of mappings $U(t, \tau) : X \to \mathcal{P}(X)$, $t \geq \tau$, $\tau \in \mathbb{R}$, is said to be a* ***multivalued process*** *if*

(1) $U(\tau, \tau)x = \{x\}, \forall\ \tau \in \mathbb{R},\ x \in X$;
(2) $U(t, s)U(s, \tau)x = U(t, \tau)x,\ \forall\ t \geq s \geq \tau,\ \tau \in \mathbb{R},\ x \in X$.

Definition 3.2. *Let $\{U(t, \tau)\}$ be a multivalued process on X. We say that $\{U(t, \tau)\}$ is*

(1) ***pullback dissipative****, if there exists a family of bounded sets $D = \{D(t)\}_{t\in\mathbb{R}}$ in X such that for any bounded set $B \subset X$ and each $t \in \mathbb{R}$, there exists a $\tau_0 = \tau_0(B, t) \in \mathbb{R}$ such that*
$$U(t, \tau)B \subset D(t),\ \forall\ \tau \leq \tau_0.$$
The family of sets D is known as ***pullback absorbing*** *family;*
(2) ***pullback asymptotically upper semicompact*** *in X if for each fixed $t \in \mathbb{R}$ and $B \subset X$ bounded, any sequence $\{\tau_n\}$ with $\tau_n \to -\infty$, $\{x_n\} \subset B$, and $\{y_n\}$ with $y_n \in U(t, \tau_n)x_n$, this last sequence $\{y_n\}$ is precompact in X.*

Definition 3.3. *A family of nonempty compact subsets $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ of X is said to be a* ***pullback attractor*** *for the multivalued process $\{U(t, \tau)\}$ if*

(1) $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ *is* ***invariant****, i.e.,*
$$U(t, \tau)A(\tau) = A(t),\ \forall\ t \geq \tau,\ \tau \in \mathbb{R};$$
(2) $\mathcal{A}$ *is* ***pullback attracting****, i.e., for every bounded set B of X and any fixed $t \in \mathbb{R}$,*
$$\lim_{\tau\to-\infty} dist(U(t, \tau)B, A(t)) = 0.$$

Definition 3.4. *Let $\{U(t,\tau)\}$ be a multivalued process on X. We say that $U(t,\tau)$ is* ***upper semicontinuous (or U.S.C.)*** *in x for fixed $t \geq \tau$, $\tau \in \mathbb{R}$, if $x_n \to x$, then for any $y_n \in U(t,\tau)x_n$, there exist a subsequence $y_{n_k} \in U(t,\tau)x_{n_k}$ and $y \in U(t,\tau)x$ such that $y_{n_k} \to y$ in X.*

Theorem 3.5. *(*[26, Theorem 7, p. 88]*) Let X be a Banach space and let $\{U(t,\tau)\}$ be a pullback dissipative, pullback asymptotically upper semicompact and upper semicontinuous multivalued process on X with $\cup_{\tau\leq t}D(\tau)$ bounded for all $t \in \mathbb{R}$, where $D = \{D(t)\}_{t\in\mathbb{R}}$ is a absorbing family. Then $\{U(t,\tau\}$ possesses a minimal pullback attractor $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$.*

4. Existence of Solution

Let $\lambda > 0$ be fixed and H a Hilbert space. One possibility to deal with infinite delays is to consider the space:

$$C_\lambda(H) = \left\{\varphi \in C((-\infty,0];H) : \exists \lim_{s\to-\infty} e^{\lambda s}\varphi(s) \in H\right\},$$

which is a Banach space with the norm

$$\|\varphi\|_\lambda := \sup_{s\in(-\infty,0]} e^{\lambda s}||\varphi(s)||_H.$$

This space was considered in [18, 26], the properties of this space that will allow us to deal with infinite delays can be found in [12]. Later we will set a more appropriate λ to our particular problem.

Let $f_i : \mathbb{R} \times C_\lambda(L_i) \to L_i$, for $i = 1, 2$, where $L_1 = L^2(\Omega)$ and $L_2 = L^2(\Gamma)$, and satisfies the following assumptions:

(F1) for all $\xi \in C_\lambda(L_i)$, the mapping $\mathbb{R} \ni t \to f_i(t,\xi) \in L_i$ is mensurable;
(F2) for each $t \in \mathbb{R}$, $f_i(t,0) = 0$;
(F3) there exists $K_i > 0$ such that $\forall\, t \in \mathbb{R}$, $\forall \xi,\eta \in C_\lambda(L_i)$,

$$\|f_i(t,\xi) - f_i(t,\eta)\|_{L_i} \leq K_i\|\xi - \eta\|_{C_\lambda(L_i)}.$$

See [20] for examples of functions with these properties. And for $g_{i,s}$ we have the following assumption:

(G1) let $g_1 \in L^{p'}_{loc}(\mathbb{R};L^{p'}(\Omega))$, $g_2 \in L^{p'}_{loc}(\mathbb{R};L^{p'}(\Gamma))$ where p' denotes the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 4.1. *Let $\Psi \in C_\lambda(\mathbb{X}^2)$, then notice that there exists $\psi(s) \in L^2(\Omega)$ and $\phi(s) \in L^2(\Gamma)$ for each $s \in (-\infty,0]$ such that $\Psi = (\psi,\phi)$. Moreover,*

$$\begin{aligned}\|\psi\|^2_{C_\lambda(L_1)} + \|\phi\|^2_{C_\lambda(L_2)} &= \sup_{s\in(-\infty,0]}\left(e^{2\lambda s}\|\psi(s)\|^2_{L^2(\Omega)}\right) + \sup_{s\in(-\infty,0]}\left(e^{2\lambda s}\|\phi(s)\|^2_{L^2(\Gamma)}\right)\\ &= \sup_{s\in(-\infty,0]}\left(e^{2\lambda s}\left(\|\psi(s)\|^2_{L^2(\Omega)} + \|\phi(s)\|^2_{L^2(\Gamma)}\right)\right)\\ &= \sup_{s\in(-\infty,0]} e^{2\lambda s}\|\Psi(s)\|^2_{\mathbb{X}^2} = \|\Psi\|^2_{C_\lambda(\mathbb{X}^2)}.\end{aligned}$$

Remark 4.2. *For $\xi \in C_\lambda(W^{1,p}(\Omega))$ and each $s \in [-\infty,0]$ we have $\gamma(\xi)(s) = \gamma(\xi(s))$. Then, from the continuity of trace,*

$$\gamma(\xi) \in C_\lambda\left(W^{1-\frac{1}{p},p}(\Gamma)\right).$$

Definition 4.3 (Weak Solution to Problem (P))**.** *Given $\Psi = (\psi, \phi) \in C_\lambda(\mathbb{X}^2)$, $\tau \in \mathbb{R}$, the couple $U(t) = (u(t), w(t))$ is said to be a weak solution to Problem (P) if $w(t) = \gamma(u(t))$ a.e. in (τ, T) for each $T > \tau$, and U satisfies*

(i)
$$U \in C([\tau, +\infty); \mathbb{X}^2) \cap L^\infty(\tau, +\infty; \mathbb{X}^2);$$

(ii)
$$\partial_t U \in L^{p'}_{loc}(\tau, +\infty; (\mathbb{V}^p)^*);$$

(iii) *for all $V = (v, \gamma(v)) \in \mathbb{V}^p$,*

$$\begin{aligned} \langle \partial_t U, V\rangle_{\mathbb{X}^2} + \left\langle |\nabla u|^{p-2}\nabla u, \nabla v\right\rangle_{L^2(\Omega)} + \left\langle |u|^{p-2}u, v\right\rangle_{L^2(\Omega)} = \left\langle f_1(t, u^t), v\right\rangle_{L^2(\Omega)} \\ + \left\langle f_2(t, \gamma(u^t)), \gamma(v)\right\rangle_{L^2(\Gamma)} + \langle g_1(t), v\rangle_{L^2(\Omega)} + \langle g_2(t), \gamma(v)\rangle_{L^2(\Gamma)} \end{aligned} \tag{4.1}$$

a.e. in (τ, T), for each $T > \tau$;

(iv) *$U^\tau = \Psi$ in $C_\lambda(\mathbb{X}^2)$, which means, $u^\tau = \psi$ in $C_\lambda(L^2(\Omega))$ and $w^\tau = \phi$ in $C_\lambda(L^2(\Gamma))$.*

Before showing the existence of a weak solution to Problem (P), we obtain a priori estimates for a weak solution in the space $\mathbb{X}^2$.

Lemma 4.4. *Assume hypotheses (F1)-(F3) and (G1) are satisfied and let $U(t) = (u(t), \gamma(u)(t))$ be a weak solution to Problem (P) with initial delay condition $\Psi \in C_\lambda(\mathbb{X}^2)$ in $\tau \in \mathbb{R}$. Then, there is a finite constant $K(t, \tau, \Psi)$ such that*

$$\begin{aligned} \|U(t)\|_{\mathbb{X}^2} + \Theta \int_\tau^t \|U\|^p_{\mathbb{V}^p} ds \le \|\Psi\|^2_{C_\lambda(\mathbb{X}^2)} \\ + C_1 \int_\tau^t \left(\|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right) ds + K(t, \tau, \Psi), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \|U^t\|^2_{C_\lambda(\mathbb{X}^2)} \le e^{C(t-\tau)} \left(\|\Psi\|_{C_\lambda(\mathbb{X}^2)} + \tilde{C}(t-\tau)\right) \\ + C_\varepsilon \int_\tau^t e^{2C(t-s)} \left(\|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right) ds, \end{aligned} \tag{4.3}$$

for all $t \ge \tau$, with C, C_ε, C_1, $\tilde{C}$ and Θ positive constants independent of τ and t.

Proof: Let U be a weak solution of Problem (P). Take $V = U$ in (4.1), and from Hölder's and Young's inequalities we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} + \|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} &= \left\langle f_1(t, u^t), u\right\rangle_2 + \left\langle f_2(t, \gamma(u^t)), \gamma(u)\right\rangle_{2,\Gamma} \\ &\quad + \langle g_1(t), u\rangle_2 + \langle g_2(t), \gamma(u)\rangle_{2,\Gamma} \\ &\le C_\varepsilon \|f_1(t, u^t)\|^{p'}_{p'} + C_\varepsilon \|f_2(t, \gamma(u^t))\|^{p'}_{p',\Gamma} + C_\varepsilon \|g_1(t)\|^{p'}_{p'} + C_\varepsilon \|g_2(t)\|^{p'}_{p',\Gamma} \\ &\quad + 2\varepsilon \|u\|^p_p + 2\varepsilon \|\gamma(u)\|^p_{p,\Gamma} \\ &\le C_\varepsilon \left(\|f_1(t, u^t)\|^{p'}_{p'} + \|f_2(t, \gamma(u^t))\|^{p'}_{p',\Gamma} + \|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right) + 2\varepsilon \|U\|^p_{\mathbb{V}^p}. \end{aligned}$$

Then, as the norm of $\mathbb{V}^p$ is equivalent to the norm of $W^{1,p}(\Omega)$, there is a constant M_Ω such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} + M_\Omega\|U\|^p_{\mathbb{V}^p}\\ &\quad\leq C_\varepsilon\left(\|f_1(t,u^t)\|^{p'}_{p'} + \|f_2(t,\gamma(u^t))\|^{p'}_{p',\Gamma} + \|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right) + 2\varepsilon\|U\|^p_{\mathbb{V}^p}.\end{aligned}$$

Take $\varepsilon > 0$ such that

$$\Theta := \frac{1}{2}(M_\Omega - 2\varepsilon) > 0, \tag{4.4}$$

and multiplying by 2, incorporating the constants, and integrating between τ to t

$$\begin{aligned}&\|U(t)\|^2_{\mathbb{X}^2} + \Theta\int_\tau^t \|U\|^p_{\mathbb{V}^p}ds \leq \|U(\tau)\|^2_{\mathbb{X}^2}\\ &+ C_\varepsilon\int_\tau^t\left(\|f_1(t,u^t)\|^{p'}_{p'} + \|f_2(t,\gamma(u^t))\|^{p'}_{p',\Gamma} + \|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right)ds.\end{aligned}$$

Thus, from Lemma 2.1 of [22], (F2) and (F3), there are $\kappa_1, \kappa_2 > 0$ and $C_{\kappa_1}, C_{\kappa_2} > 0$, such that

$$\begin{aligned}&\|U(t)\|^2_{\mathbb{X}^2} + \Theta\int_\tau^t \|U\|^p_{\mathbb{V}^p}ds \leq \|U(\tau)\|^2_{\mathbb{X}^2}\\ &+ C_\varepsilon\int_\tau^t\left(\kappa_1\|f_1(t,u^t)\|^2_2 + \kappa_2\|f_2(t,\gamma(u^t))\|^2_{2,\Gamma} + \|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right)ds\\ &\qquad\qquad + C_\varepsilon(t-\tau)(C_{\kappa_1} + C_{\kappa_2})\\ &\leq \|U(\tau)\|^2_{\mathbb{X}^2} + C_\varepsilon\int_\tau^t\left(\kappa_1K_1^2\|u^t\|^2_{C_\lambda(L_1)} + \kappa_2K_2^2\|\gamma(u^t)\|^2_{C_\lambda(L_2)}\right)ds\\ &\qquad + C_\varepsilon\int_\tau^t\left(\|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right)ds + C_\varepsilon(t-\tau)(C_{\kappa_1} + C_{\kappa_2}).\end{aligned}$$

Take $K := \max\{\kappa_1K_1^2, \kappa_2K_2^2\}$ and let

$$C := C_\varepsilon K \tag{4.5}$$

and $\tilde{C} := C_\varepsilon\,(C_{\kappa_1} + C_{\kappa_2})$. From Remark 4.1 we have

$$\begin{aligned}\|U(t)\|^2_{\mathbb{X}^2} + \Theta\int_\tau^t \|U\|^p_{\mathbb{V}^p}ds &\leq \|U(\tau)\|^2_{\mathbb{X}^2}\\ &+ C_\varepsilon\int_\tau^t\left(\|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right)ds + C\int_\tau^t\|U^s\|^2_{C_\lambda(\mathbb{X}^2)}ds + \tilde{C}(t-\tau).\end{aligned} \tag{4.6}$$

for $t \geq \tau$.

Further

$$\begin{aligned}\|U^t\|^2_{C_\lambda(\mathbb{X}^2)} \leq \max\Bigg\{ & \sup_{l\in(-\infty,\tau-t]} e^{2\lambda l}\|\Psi(l+t-\tau)\|^2_{\mathbb{X}^2}, \\ & \sup_{l\in(\tau-t,0]}\Bigg[e^{2\lambda l}\Bigg(\|U(\tau)\|^2_{\mathbb{X}^2} + C_\varepsilon\int_\tau^{t+l}\left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma}\right)ds \\ & \qquad +C\int_\tau^{t+l}\|U^s\|^2_{C_\lambda(\mathbb{X}^2)}ds + \tilde{C}(t-\tau)\Bigg)\Bigg]\Bigg\} \\ \leq \max\Bigg\{ & \sup_{l\in(-\infty,\tau-t]} e^{2\lambda l}\|\Psi(l+t-\tau)\|^2_{\mathbb{X}^2}, \\ & \|U(\tau)\|^2_{\mathbb{X}^2} + C_\varepsilon\int_\tau^{t}\left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma}\right)ds \\ & \qquad +C\int_\tau^{t}\|U^s\|^2_{C_\lambda(\mathbb{X}^2)}ds + \tilde{C}(t-\tau)\Bigg\},\end{aligned}$$

and, note that

$$\sup_{l\in(-\infty,\tau-t]} e^{2\lambda l}\|\Psi(l+t-\tau)\|^2_{\mathbb{X}^2} = \sup_{l\leq 0} e^{2\lambda(l-(t-\tau))}\|\Psi(l))\|^2_{\mathbb{X}^2} = e^{-2\lambda(t-\tau)}\|\Psi\|_{C_\lambda(\mathbb{X}^2)} \leq \|\Psi\|_{C_\lambda(\mathbb{X}^2)},$$

and $\|U(\tau)\|_{\mathbb{X}^2} = \|\Psi(0)\|_{\mathbb{X}^2} \leq \|\Psi\|_{C_\lambda(\mathbb{X}^2)}$. From Gronwall's Lemma

$$\|U^t\|^2_{C_\lambda(\mathbb{X}^2)} \leq e^{C(t-\tau)}\left(\|\Psi\|_{C_\lambda(\mathbb{X}^2)} + \tilde{C}(t-\tau)\right) + C_\varepsilon e^{C(t-\tau)}\int_\tau^t\left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma}\right)ds,$$

ensuring estimate (4.3) for all $t \geq \tau$, with this estimate and (4.6) we can deduce estimate (4.2).

■

Theorem 4.5. *Let $\Psi \in C_\lambda(\mathbb{X}^2)$ and $\tau \in \mathbb{R}$. Assume (F1)-(F3) and (G1) hold true. Then there exists at least one weak solution for Problem (P) with initial delay condition Ψ in τ.*

Proof: We will define some appropriate operators to reformulate expression (4.1) in order to have a simpler functional formulation of our problem, see [10] and [22] for examples of the same method. Then let, for $U, V \in \mathbb{V}^p$, the following operator

$$\beta_p(U,V) = \left\langle |\nabla u|^{p-2}\nabla u, \nabla v\right\rangle_2 + \left\langle |u|^{p-2}u, v\right\rangle_2.$$

For each $U \in \mathbb{V}^p$ we have $\beta_p U := \beta_p(U,\cdot) \in (\mathbb{V}^p)^*$ and the operator $\beta_p : \mathbb{V}^p \to (\mathbb{V}^p)^*$ is a maximal monotone operator, see [22].

And we define

$$\mathcal{F}(t,U^t) = \begin{pmatrix} f_1(t,u^t) \\ f_2(t,\gamma(u^t)) \end{pmatrix}, G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \text{ and } \partial_t U = \begin{pmatrix} u_t \\ \gamma(u)_t \end{pmatrix}$$

in the usual way, see [22] for more details.

In this way, finding a weak solution of Problem (P) is equivalent to find a function U with regularities of weak solution definition, and satisfying the following functional equation

$$\partial_t U + \beta_p U = \mathcal{F}(t,U^t) + G(t) \tag{4.7}$$

in $L^{p'}(0,T;(\mathbb{V}^p)^*)$, see Remark 4.6 in [22].

In order to find a weak solution to Problem (P), we use the Faedo-Galerkin approximation. Since $\mathbb{X}^2$ is separable and $\mathbb{V}^p$ is dense in $\mathbb{X}^2$, there is a orthonormal basis of $\mathbb{X}^2$ contained in $\mathbb{V}^p$. We denote such basis by $\{\Phi_n = (\phi_n, \psi_n) \in \mathbb{X}^2; n \in \mathbb{N}\}$.

Let

$$K_n = span\{\Phi_1, ..., \Phi_n\}, \quad K_\infty = \cup_{n=1}^{\infty} K_n,$$

and $Pr_n : \mathbb{X}^2 \to K_n$ be the orthogonal projection.

Given $\Psi \in C_\lambda(\mathbb{X}^2)$ and $T > \tau$ we want to find a solution $U_n = \sum_{i=1}^n d_i(t)\Phi_i \in K_n$ for an $n-$dimensional version of problem (4.7), which is equivalent to find a solution of the following system of ordinary differential equations

$$\begin{cases} \langle \partial_t U_n, \Phi_i\rangle + \langle \beta_p U_n, \Phi_i\rangle = \langle \mathcal{F}(t, U_n^t), \Phi_i\rangle + \langle G(t), \Phi_i\rangle \\ \langle U_n(\tau + s), V\rangle = \langle Pr_n\Psi(s), V\rangle\,, \text{ for } s \in (-\infty, 0], \end{cases}$$

for all $1 \le i \le n$ and a.e. in $[\tau, T]$, where $\langle\cdot,\cdot\rangle$ denote the dual product between $(\mathbb{V}^p)^*$ and $\mathbb{V}^p$.

The above system of ordinary functional differential equations with infinite delay fulffils the conditions for existence and uniqueness of local solution established in Theorem 1.1 of [13]. A priori estimates ensure that solutions do exist for all time in $[\tau, T]$.

Estimate (4.3) of Lemma 4.4 ensures that for $\Psi \in C_\lambda(\mathbb{X}^2)$ and $R > 0$ such that $\|\Psi\|_{C_\lambda(\mathbb{X}^2)} \le R$, there exists a constant $C = C(\tau, T, R)$, but independent of n and $t \in (\tau, T)$, such that

$$\|U_n^t\|^2_{C_\lambda(\mathbb{X}^2)} \le C(\tau, T, R). \tag{4.8}$$

In particular, the previous limit and estimate (4.2) imply the existence of another constant (relabelled the same) $C = C(\tau, T, R)$ such that

$$\begin{cases} \|U_n(t)\|_{L^\infty(\tau,T;\mathbb{X}^2)} \le C \\ \|U_n(t)\|_{L^p(\tau,T;\mathbb{V}^p)} \le C. \end{cases} \tag{4.9}$$

Then, this guarantees that $\beta_p U_n$ is bounded in $L^{p'}(\tau, T; (\mathbb{V}^p)^*)$, see [22] for more details. Hypotheses (F2), (F3), (4.8) and recalling that $\mathbb{X}^2 \subset (\mathbb{V}^p)^*$ continuously imply that $\mathcal{F}(t, U_n^t)$ is bounded in $L^{p'}(\tau, T; (\mathbb{V}^p)^*)$. Note that,

$$\partial_t U_n = -\beta_p U_n + \mathcal{F}(t, U_n^t) + G(t) \text{ in } L^{p'}(\tau, T; (\mathbb{V}^p)^*). \tag{4.10}$$

Therefore, the limits of $\beta_p U_n$ and $\mathcal{F}(t, U_n^t)$ ensure that there exists a constant (relabelled the same) $C(\tau, T, R)$ such that

$$\|\partial_t U_n\|_{L^{p'}(\tau,T;(\mathbb{V}^p)^*)} \le C. \tag{4.11}$$

The limits in (4.9) and (4.11) ensure that there is a subsequence (which we relabel the same) $\{U_n\}$, and an element $U \in L^\infty(\tau, T; \mathbb{X}^2) \cap L^p(\tau, T; \mathbb{V}^p)$ with $\partial_t U \in L^{p'}(\tau, T; (\mathbb{V}^p)^*)$, such that

$$\begin{cases} U_n \overset{*}{\rightharpoonup} U \text{ in } L^\infty(\tau, T; \mathbb{X}^2), \\ U_n \rightharpoonup U \text{ in } L^p(\tau, T; \mathbb{V}^p), \\ \partial_t U_n \overset{*}{\rightharpoonup} \partial_t U \text{ in } L^{p'}(\tau, T; (\mathbb{V}^p)^*). \end{cases} \tag{4.12}$$

From compactness results, see Theorems 1.4 and 1.5 page 32 of [7], the sequences in fact have the following convergences

$$\begin{cases} U_n \to U \text{ in } L^p(\tau, T; \mathbb{X}^2), \\ U_n \to U \text{ in } C([\tau, T]; \mathbb{X}^2). \end{cases} \tag{4.13}$$

Note that,

$$Pr_n\Psi \to \Psi \text{ in } C_\lambda(\mathbb{X}^2),$$

and thanks to the strong convergence in $C([\tau, T]; \mathbb{X}^2)$ yield that

$$U_n^t \to U^t \text{ in } C_\lambda(\mathbb{X}^2) \ \forall\ t \leq T,$$

see, for instance, [18, 19] and [20] for details about both convergences.

The above convergence and hypotheses (F2) and (F3) imply that

$$\mathcal{F}(t, U_n^t) \to \mathcal{F}(t, U^t) \text{ in } L^{p'}(\tau, T; \mathbb{X}^2),$$

which together with convergences (4.12) and the theory of maximal monotone operators allow us to deduce that

$$\beta_p U_n \overset{*}{\rightharpoonup} \beta_p U \text{ in } L^{p'}(\tau, T; (\mathbb{V}^p)^*),$$

see [22] for details.

Therefore, U is solution of the limit equation of (4.10) in the weak star topology of $L^{p'}(\tau, T; (\mathbb{V}^p)^*)$. This ensures that U is a weak solution of Problem (P) in the interval $(-\infty, T]$ with initial condition $U^\tau = \Psi$.

■

The existence of solution allows us define the multivalued process $\{U(t, \tau)\}$ on $C_\lambda(\mathbb{X}^2)$ by

$$U(t, \tau)\Psi = \left\{U^t \ | U(\cdot) \text{ is a solution of Problem } (P) \text{ with } U^\tau = \Psi \in C_\lambda(\mathbb{X}^2)\right\}.$$

Indeed, item (2) of Definition 3.1 follows from concatenation and translation of solutions, see [4] and [5] for details.

Lemma 4.6. *The multivalued process $\{U(t, \tau)\}$ is upper-semicontinuous in $C_\lambda(\mathbb{X}^2)$.*

Proof: Let $\tau \in \mathbb{R}$, $\{\Psi_n\}_{n\in\mathbb{N}}$ and Ψ such that $\Psi_n \to \Psi$ in $C_\lambda(\mathbb{X}^2)$, and let $\{Y_n\}_{n\in\mathbb{N}}$ such that $Y_n^\cdot \in U(\cdot, \tau)\Psi_n$.

Given $T > \tau$, observe that, as $\Psi_n \to \Psi$ in $C_\lambda(\mathbb{X}^2)$, given $R > 0$, except for a finite number of elements, we have that $\{\Psi_n\} \subset B_{C_\lambda(\mathbb{X}^2)}(\Psi, R)$. Then, from Lemma 4.4 the sequence $\{Y_n\}$ is bounded in $L^\infty(\tau, T; \mathbb{X}^2)$ and $L^p(\tau, T; \mathbb{V}^p)$.

Then, similarly to the proof of Theorem 4.5, we can ensure the existence of an element $Y^\cdot \in U(\cdot, \tau)\Psi$ such that $Y_n^t \to Y^t$ in $C_\lambda(\mathbb{X}^2)$ for all $t \leq T$.

Therefore, as $T > \tau$ is arbitrary, it follows that $\{U(t, \tau)\}$ is upper-semicontinuous.

■

5. Pullback Attractor for Problem (P)

In this section we develop some estimates to show that the multivalued process generated by solutions of Problem (P) possesses a pullback absorbing family and it is pullback asymptotically upper semicompact. Therefore, we can ensure the existence of pullback attractor for the problem.

First we summary some aspects of the constants that appeared in the development of Lemma 4.4 and we will develop some technical property with these constants to make easy our study and understanding of the reader. Consider in this section $p > 2$.

We choose $\varepsilon > 0$ and define $\Theta = \frac{1}{2}(M_\Omega - 2\varepsilon) > 0$, see (4.4). Another constant that we need is $C > 0$ defined in (4.5).

Note that there is a constant $\tilde{M} > 0$ such that $\|U\|_{\mathbb{X}^2} \leq \tilde{M}\|U\|_{\mathbb{X}^p}$. From Young's inequality, see [2] page 92, we can choose a constant $\delta > 0$ such that

$$\|U\|^2_{\mathbb{X}^2} \leq \delta \left(\tilde{M}^2\|U\|^2_{\mathbb{X}^p}\right)^{\frac{p}{2}} + C_\delta.1^{\left(\frac{p}{2}\right)'},$$

with $C_\delta > 0$. Then we take $\delta > 0$ such that $\Theta > \delta\tilde{M}^p C$, and define the following constant

$$\sigma := 2\left(\frac{\Theta}{\delta\tilde{M}^p} - C\right) > 0. \tag{5.1}$$

Remark 5.1. *Note that*

$$\|U\|^p_{\mathbb{X}^p} \leq \|U\|^p_{\mathbb{V}^p}, \quad \textit{and} \quad \frac{1}{\delta\tilde{M}^p}\|U\|^2_{\mathbb{X}^2} - \frac{C_\delta}{\delta\tilde{M}^p} \leq \|U\|^p_{\mathbb{V}^p}.$$

Then, consider the following additional assumption:

(G2) $\sup\limits_{r\leq 0} e^{-\sigma r}\displaystyle\int_{-\infty}^{r} e^{\sigma s}\left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma}\right)ds < +\infty,$

Remark 5.2. *If we assume (G1) and (G2) we have*

$$e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma}\right)ds < +\infty, \ \forall\ t \in \mathbb{R}.$$

Lemma 5.3. *If $\lambda > \frac{\Theta}{\delta\tilde{M}^p}$, then for $\tau \in \mathbb{R}$, $\Psi \in C_\lambda(\mathbb{X}^2)$ and U a weak solution of Problem (P) with assumptions (F1)-(F3) and (G1)-(G2), we have that*

$$\|U^t\|^2_{C_\lambda(\mathbb{X}^2)} \leq e^{\sigma(\tau-t)}\|\Psi\|_{C_\lambda(\mathbb{X}^2)} + \frac{2\tilde{C}}{\sigma} + 2C_\varepsilon\int_{-\infty}^{t} e^{\sigma(s-t)}\left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma}\right)ds \tag{5.2}$$

for each $t \geq \tau$, with C_ε and $\tilde{C}$ positive constants independent of t and τ.

Proof:

Following the proof of Lemma 4.4, see (4.6), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} + \Theta\|U\|^p_{\mathbb{V}^p}\\
&\quad\leq C_\varepsilon\left(\|f_1(t,u^t)\|^{p'}_{p'} + \|f_2(t,\gamma(u^t))\|^{p'}_{p',\Gamma} + \|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right)\\
&\quad\leq C_\varepsilon\left(\|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right) + C\left(\|u^t\|^2_{C_\lambda(L_1)} + \|\gamma(u)^t\|^2_{C_\lambda(L_2)}\right) + \tilde{C}
\end{aligned}$$

Then, from Remark 5.1, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} + \frac{\Theta}{\delta\tilde{M}^p}\|U\|^2_{\mathbb{X}^2}\\
&\quad\leq C_\varepsilon\left(\|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma}\right) + C\left(\|u^t\|^2_{C_\lambda(L_1)} + \|\gamma(u)^t\|^2_{C_\lambda(L_2)}\right) + \tilde{C} + \frac{C_\delta\Theta}{\delta\tilde{M}^p}
\end{aligned}$$

relabelled $\tilde{C} := \tilde{C} + \frac{C_\delta \Theta}{\delta \widetilde{M}^p}$, then multiply by 2 and take $\tilde{\Theta} := \frac{2\Theta}{\delta \widetilde{M}^p}$. Now multiplying by $e^{\tilde{\Theta} t}$, integrating from τ to t and multiplying the last expression by $e^{-\tilde{\Theta} t}$ we have

$$\|U(t)\|^2_{\mathbb{X}^2} \leq e^{\tilde{\Theta}(\tau - t)} \|U(\tau)\|^2_{\mathbb{X}^2}$$
$$+ 2C_\varepsilon \int_\tau^t e^{\tilde{\Theta}(s-t)} \left(\|g_1(t)\|^{p'}_{p'} + \|g_2(t)\|^{p'}_{p',\Gamma} \right) ds + 2C \int_\tau^t e^{\tilde{\Theta}(s-t)} \|U^t\|^2_{C_\lambda(\mathbb{X}^2)} ds + \int_\tau^t e^{\tilde{\Theta}(s-t)} 2\tilde{C} ds$$

Consequently,

$$\|U^t\|^2_{C_\lambda(\mathbb{X}^2)}$$
$$\leq \max \left\{ \sup_{l \in (-\infty, \tau - t]} e^{2\lambda l} \|\Psi(l + t - \tau)\|^2_{\mathbb{X}^2}, \right.$$
$$\sup_{l \in (\tau - t, 0]} \left[e^{2\lambda l} \left(e^{\tilde{\Theta}(\tau - (t+l))} \|U(\tau)\|^2_{\mathbb{X}^2} + 2C_\varepsilon \int_\tau^{t+l} e^{\tilde{\Theta}(s-t)} \left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma} \right) ds \right. \right.$$
$$\left. \left. + 2C \int_\tau^{t+l} e^{\tilde{\Theta}(s-t)} \|U^s\|^2_{C_\lambda(\mathbb{X}^2)} ds + \int_\tau^t e^{\tilde{\Theta}(s-t)} 2\tilde{C} ds \right) \right] \Bigg\}$$
$$\leq \max \left\{ \sup_{l \in (-\infty, \tau - t]} e^{2\lambda l} \|\Psi(l + t - \tau)\|^2_{\mathbb{X}^2}, \right.$$
$$e^{\tilde{\Theta}(\tau - t)} \|U(\tau)\|^2_{\mathbb{X}^2} + 2C_\varepsilon \int_\tau^t e^{\tilde{\Theta}(s-t)} \left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma} \right) ds$$
$$\left. + 2C \int_\tau^t e^{\tilde{\Theta}(s-t)} \|U^s\|^2_{C_\lambda(\mathbb{X}^2)} ds + \int_\tau^t e^{\tilde{\Theta}(s-t)} 2\tilde{C} ds \right\}.$$

Note that, as $\lambda \geq \frac{\Theta}{\delta \widetilde{M}^p}$, we have

$$\sup_{l \in (-\infty, \tau - t]} e^{2\lambda l} \|\Psi(l + t - \tau)\|^2_{\mathbb{X}^2} \leq e^{-\tilde{\Theta}(t - \tau)} \|\Psi\|_{C_\lambda(\mathbb{X}^2)},$$

and $e^{-\tilde{\Theta}(t-\tau)} \|U(\tau)\|^2_{\mathbb{X}^2} \leq e^{-\tilde{\Theta}(t-\tau)} \|\Psi\|^2_{C_\lambda(\mathbb{X}^2)}$, and then

$$\|U^t\|^2_{C_\lambda(\mathbb{X}^2)} \leq e^{\tilde{\Theta}(\tau - t)} \|\Psi\|_{C_\lambda(\mathbb{X}^2)} + \int_\tau^t e^{\tilde{\Theta}(s-t)} 2\tilde{C} ds$$
$$+ 2C_\varepsilon \int_\tau^t e^{\tilde{\Theta}(s-t)} \left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma} \right) ds + 2C \int_\tau^t e^{\tilde{\Theta}(s-t)} \|U^s\|^2_{C_\lambda(\mathbb{X}^2)} ds.$$

Note that $\sigma = \tilde{\Theta} - 2C$, and from Gronwall's Lemma

$$\|U^t\|^2_{C_\lambda(\mathbb{X}^2)} \leq e^{\sigma(\tau - t)} \|\Psi\|_{C_\lambda(\mathbb{X}^2)} + \frac{2\tilde{C}}{\sigma} + 2C_\varepsilon \int_{-\infty}^t e^{\sigma(s-t)} \left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma} \right) ds.$$

■

Lemma 5.3 ensures that the multivalued process $\{U(t, \tau)\}$ is pullback dissipative in $C_\lambda(\mathbb{X}^2)$, with the following pullback absorbing family.

Lemma 5.4. *Suppose hypotheses of Lemma 5.3 and for each $t \in \mathbb{R}$ define*

$$D(t) = \left\{ \Psi \in C_\lambda(\mathbb{X}^2); \|\Psi\|^2_{C_\lambda(\mathbb{X}^2)} \leq 1 + \frac{2\tilde{C}}{\sigma} + 2C_\varepsilon e^{-\sigma t} \int_{-\infty}^{t} e^{\sigma s} \left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma} \right) ds \right\},$$

then $D = \{D(t)\}_{t\in\mathbb{R}}$ is a pullback absorbing family for the multivalued process $\{U(t,\tau)\}$ on $C_\lambda(\mathbb{X}^2)$.

Note that when the constant σ in the estimate (5.2) is positive we can ensure the stability of the solution for $\tau \in \mathbb{R}$ small enough. This kind of property is proved in other works asking for some restrictions on constants like K_i's in (F3), see for instance [4, 5, 18, 19, 20] and [26]. Here, according to the handling presented at the beginning of the section, we obtain an appropriate σ for each choice of K_i's. It means that the existence of a pullback absorbing family is independent of the choice of K_i's.

Lemma 5.5. *Let $\lambda > \frac{\Theta}{\delta \widehat{M}^p}$, then the multivalued process $\{U(t,\tau)\}$ is pullback asymptotically upper-semicompact in $C_\lambda(\mathbb{X}^2)$.*

Proof: Let $t_0 \in \mathbb{R}$ fixed, and let $\{U_n(t_0,\tau_n,\Psi_n)\}_{n\in\mathbb{N}}$ be a sequence of weak solutions of Problem (P) with $\{\Psi_n\}_{n\in\mathbb{N}} \subset C_\lambda(\mathbb{X}^2)$ a sequence of initial conditions in $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, respectively, and $\tau_n \to -\infty$. Without loss of generality, we may assume that $\tau_n < t_0$ for all $n \in \mathbb{N}$.

Consider $\{U_n^{t_0}\}$ and we will show that such sequence is precompact in $C_\lambda(\mathbb{X}^2)$ in two steps. See [18, 20] for examples of the same technique.

Step 1: We will show that there exist a function $W : (-\infty, 0] \to \mathbb{X}^2$ and a subsequence of $\{U_n^{t_0}\}$, relabelled the same, such that $U_n^{t_0} \to W$ in $C([-T, 0]; \mathbb{X}^2)$ for every $T > 0$.

Let $T > 0$, there is $n_0(t_0, T)$ such that for all $n \geq n_0$ we have $\tau_n < t_0 - T$, and from estimate (5.2) there is $R(t_0, T) > 0$ such that

$$\|U_n^t\|^2_{C_\lambda(\mathbb{X}^2)} \leq R(t_0,T),\ \forall\ t \in [t_0 - T, T] \ \text{ and } n \geq n_0, \tag{5.3}$$

where

$$R(t_0,T) = 1 + \frac{2\tilde{C}}{\sigma} + 2C_\varepsilon e^{-\sigma(t_0-T)} \int_{-\infty}^{t_0} e^{\sigma s} \left(\|g_1(s)\|^{p'}_{p'} + \|g_2(s)\|^{p'}_{p',\Gamma} \right) ds, \tag{5.4}$$

and then

$$\|U_n(t)\|^2_{\mathbb{X}^2} \leq R(t_0,T),\ \forall\ t \in [t_0 - T, t_0] \ \text{ and } n \geq n_0. \tag{5.5}$$

Take

$$Y_n(t) = U_n(t + t_0 - T),\ \forall\ t \in [0, T],$$

and note that, from (5.5), the sequence $\{Y_n\}$ is bounded in $L^\infty(0, T; \mathbb{X}^2)$.

Note that Y_n is a solution of functional formulation (4.7) with the following replaced functions

$$\tilde{G}(t) = G(t + t_0 - T) \text{ and } \tilde{\mathcal{F}}(t) = \mathcal{F}(t + t_0 - T, \cdot),\ \forall\ t \in [0, T].$$

with $Y_n^0 = U_n^{t_0-T}$ and $Y_n^T = U_n^{t_0}$.

From (5.3) we have that

$$\|Y_n^0\|^2_{C_\lambda(\mathbb{X}^2)} \leq R(t_0,T),\ \forall\ n \geq n_0(t_0,T),$$

and, from a priori estimate (4.2) it is possible to find $K(t_0, T)$ such that

$$\|Y_n\|_{L^p(0,T;\mathbb{V}^p)} \leq K(t_0,T). \tag{5.6}$$

Thanks to these estimates there exists $Y \in L^\infty(0,T;\mathbb{X}^2) \cap L^2(0,T;\mathbb{V}^p)$ such that

$$\begin{cases} Y_n \overset{*}{\rightharpoonup} Y \text{ in } L^\infty(0,T;\mathbb{X}^2), \\ Y_n \rightharpoonup Y \text{ in } L^p(0,T;\mathbb{V}^p). \end{cases} \tag{5.7}$$

Note that, from hypothesis (F3) there exists $\tilde{K} = \tilde{K}(K_1,K_2) > 0$ such that

$$\|\tilde{\mathcal{F}}(t,Y_n^t)\|_{\mathbb{X}^2}^2 \leq \tilde{K}\|Y_n^t\|_{C_\lambda(\mathbb{X}^2)}^2, \ \forall\ t \in [0,T], \tag{5.8}$$

and (5.5) ensures that $\tilde{\mathcal{F}}(t,Y_n^t)$ is bounded in $L^{p'}(0,T;(\mathbb{V}^p)^*)$, and from (5.6) we have that operator $\beta_p Y_n$ is bounded in the same space. Then, as it was done in the proof of Theorem 4.5, there exists $\partial_t Y \in L^{p'}(0,T;(\mathbb{V}^p)^*)$ such that

$$\partial_t Y_n \overset{*}{\rightharpoonup} \partial_t Y \text{ in } L^{p'}(0,T;(\mathbb{V}^p)^*). \tag{5.9}$$

From (5.7), (5.9) and Theorems 1.4 and 1.5 in page 32 of [7], we have

$$\begin{cases} Y_n \to Y \text{ in } L^p(\tau,T;\mathbb{X}^2), \\ Y_n \to Y \text{ in } C([\tau,T];\mathbb{X}^2). \end{cases} \tag{5.10}$$

Take $W(s) := Y(s+T)$ for $s \in [-T,0]$. Then $U^{t_0}_{n|[-T,0]} \to W$ in $C(-T,0;\mathbb{X}^2)$. Repeating the same procedure for $2T$, $3T$, etc. for a diagonal subsequence we can obtain a function $W \in C((-\infty,0];\mathbb{X}^2)$ such that $U^{t_0}_{n|[-T,0]} \to W$ in $C([-T,0];\mathbb{X}^2)$ on every interval $[-T,0]$.

Moreover, from estimate (5.3), we have

$$\|W(s)\|_{C_\lambda(\mathbb{X}^2)}^2 \leq 1 + \frac{2\tilde{C}}{\sigma} + Me^{\sigma T} \ \forall\ s \in [-T,0], \text{ for any } T > 0, \tag{5.11}$$

where

$$M = 2C_\varepsilon e^{-\sigma t_0} \int_{-\infty}^{t_0} e^{\sigma s} \left(\|g_1(s)\|_{p'}^{p'} + \|g_2(s)\|_{p',\Gamma}^{p'} \right) ds.$$

Step 2: Now we prove that $U_n^{t_0}$ converges to W in $C_\lambda(\mathbb{X}^2)$. In fact, we will show that for every $\varepsilon > 0$ there exists n_ε such that

$$\sup_{s\in(-\infty,0]} e^{2\lambda s}\|U_n^{t_0}(s) - W(s)\|_{\mathbb{X}^2}^2 \leq \varepsilon \ \forall\ n \geq n_\varepsilon. \tag{5.12}$$

Let $T_\varepsilon > 0$ such that

$$\max\left\{ e^{-2\sigma T_\varepsilon}, e^\sigma M e^{[\sigma-2\lambda]T_\varepsilon}, e^{-2\lambda T_\varepsilon}\left(1 + \frac{2\tilde{C}}{\sigma} + M\right), e^{-2\lambda T_\varepsilon}\left(1 + \frac{2\tilde{C}}{\sigma} + e^{\sigma T_\varepsilon} M\right)\right\} < \frac{\varepsilon}{8},$$

note that $\sigma - 2\lambda < 0$, and take $n_\varepsilon \geq n(t_0,T_\varepsilon)$ such that $e^{2\sigma s}\|U_n^{t_0}(s) - W(s)\|_{\mathbb{X}^2}^2 < \varepsilon$ for all $s \in [-T_\varepsilon, 0]$, and $\tau_n \leq t_0 - T_\varepsilon$, for all $n \geq n_\varepsilon$. This last choice is possible thanks to Step 1.

Then, in order to prove (5.12) we only need to check that

$$\sup_{s\in(-\infty,-T_\varepsilon]} e^{2\lambda s}\|U_n^{t_0}(s) - W(s)\|_{\mathbb{X}^2}^2 \leq \varepsilon \ \forall\ n \geq n_\varepsilon$$

From (5.11) and the choice of T_ε, for all $k \geq 0$ we have that for all $s \in [-(T_\varepsilon+k+1), -(T_\varepsilon+k)]$

$$\begin{aligned} e^{2\lambda s}\|W(s)\|_{\mathbb{X}^2}^2 &\leq e^{-2\lambda(T_\varepsilon+k)}\left(1 + Me^{\sigma(T_\varepsilon+k+1)}\right) \\ &= e^{-2\lambda T_\varepsilon}.e^{-2\lambda k} + Me^{\sigma(T_\varepsilon+k+1)-2\lambda(T_\varepsilon+k)} \\ &= e^{-2\lambda T_\varepsilon}.e^{-2\lambda k} + e^\sigma M e^{[\sigma-2\lambda]T_\varepsilon} e^{[\sigma-2\lambda]k} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \end{aligned}$$

Then, to finish, it suffices to prove that

$$\sup_{s\in(-\infty,-T_\varepsilon]} e^{2\lambda s}\|U_n^{t_0}(s)\|^2_{\mathbb{X}^2} \leq \frac{\varepsilon}{2} \ \forall\ n \geq n_\varepsilon.$$

We recall that

$$U_n^{t_0} = \begin{cases} \Psi_n(s+t_0-\tau_n), & \text{if } s \in (-\infty, \tau_n - t_0) \\ U_n(s+t_0), & \text{if } s \in [\tau_n - t_0, 0]. \end{cases} \tag{5.13}$$

Thus, the proof is finished if we prove that

$$\max\left\{ \sup_{s\in(-\infty,\tau_n-t_0)} e^{2\lambda s}\|\Psi_n(s+t_0-\tau_n)\|^2_{\mathbb{X}^2}, \sup_{s\in[\tau_n-t_0,-T_\varepsilon]} e^{2\lambda s}\|U_n(s+t_0)\|^2_{\mathbb{X}^2} \right\} \leq \frac{\varepsilon}{2}. \tag{5.14}$$

But observe that

$$\begin{aligned}
\sup_{s\in(-\infty,\tau_n-t_0)} e^{2\lambda s}\|\Psi_n(s+t_0-\tau_n)\|^2_{\mathbb{X}^2} &= \sup_{s\in(-\infty,\tau_n-t_0)} e^{2\lambda(s+t_0-\tau_n)}e^{2\lambda(\tau_n-t_0)}\|\Psi_n(s+t_0-\tau_n)\|^2_{\mathbb{X}^2} \\
&\leq e^{2\lambda(\tau_n-t_0)}\|\Psi_n\|^2_{\mathbb{X}^2} = e^{2\lambda(\tau_n-t_0)}\|U_n^{\tau_n}\|^2_{\mathbb{X}^2} \\
&\leq e^{2\lambda(\tau_n-t_0)}\left(1+\frac{2\tilde{C}}{\sigma}+C_\varepsilon e^{-\delta\tau_n}\int_{-\infty}^{\tau_n} e^{\delta s}\left(\|g_1(s)\|^{p'}_{p'}+\|g_2(s)\|^{p'}_{p',\Gamma}\right)ds\right) \\
&\leq e^{2\lambda(\tau_n-t_0)}\left(1+\frac{2\tilde{C}}{\sigma}+C_\varepsilon e^{-\delta t_0}\int_{-\infty}^{t_0} e^{\delta s}\left(\|g_1(s)\|^{p'}_{p'}+\|g_2(s)\|^{p'}_{p',\Gamma}\right)ds\right) \\
&\leq e^{2\lambda(\tau_n-t_0)}\left(1+\frac{2\tilde{C}}{\sigma}+M\right) \leq e^{-2\lambda T_\varepsilon}\left(1+\frac{2\tilde{C}}{\sigma}+M\right) \leq \frac{\varepsilon}{4},
\end{aligned}$$

thanks to the choice of n_ε and T_ε.

Finally, from (5.3) with $T = T_\varepsilon$, we also have

$$\begin{aligned}
\sup_{s\in[\tau_n-t_0,-T_\varepsilon]} e^{2\lambda s}\|U_n(s+t_0)\|^2_{\mathbb{X}^2} &= \sup_{s\in[\tau_n-t_0+T_\varepsilon,0]} e^{2\lambda(s-T_\varepsilon)}\|U_n(t_0-T_\varepsilon+s)\|^2_{\mathbb{X}^2} \\
&\leq e^{-2\lambda T_\varepsilon}\|U_n^{t_0-T_\varepsilon}\|^2_{\mathbb{X}^2} \leq e^{-2\lambda T_\varepsilon}R(t_0,T_\varepsilon) = e^{-2\lambda T_\varepsilon}\left(1+\frac{2\tilde{C}}{\sigma}+e^{\sigma T_\varepsilon}M\right) \leq \frac{\varepsilon}{4}.
\end{aligned}$$

The proof is completed.

■

Theorem 5.6. *Assume (F1)-(F3), (G1)-(G2) and also* $\lambda > \frac{\Theta}{2\delta\tilde{M}^p}$. *Then the multivalued process* $\{U(t,\tau)\}$ *defined in* $C_\lambda(\mathbb{X}^2)$ *associated with Problem* (P) *has the minimal pullback attractor* $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$.

Proof: The existence of minimal pullback attractor is a direct consequence of Theorem 3.5, Lemma 4.6, Lemma 5.4 and Lemma 5.5.

■

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Response for Reviewers

First of all, we want to thank the reviewers for his dedication and his great contributions to this work.

the interesting and helpful suggestions which allowed us to greatly improved the presentation of this paper.

One of the reviewers noted that we had not actually shown that the pullback absorbing family D defined in Lemma 5.4 have the property that $\cup_{\tau\leq t}D(\tau)$ is boundend for all $t\in\mathbb{R}$. To have this property we had to change our G2 hypothesis to

had

bounded

$$\text{(G2)}\quad \sup_{r\leq 0} e^{-\sigma r}\int_{-\infty}^{r} e^{\sigma s}\left(\|g_1(s)\|_{p'}^{p'}+\|g_2(s)\|_{p',\Gamma}^{p'}\right)ds<+\infty,$$

like in [1]. With that change the small final section loses its meaning and had to be removed. But the most important results of the work remain valid.

Furthermore, we have made other minor changes, which are pointed below:

1. page 10, line 3: We added the phrase: "Consider in this section $p>2$" at the end of the paragraph;
2. page 6 (line 10), page 8 (line -12) and page 9 (line 10): we have replaced "(H2) and (H3)" by "(F2) and (F3)";
3. page 6 (line 6) and page 10 (line 4):we have replaced "$\Theta := (M_\Omega - 2\varepsilon) > 0$" by "$\Theta := \frac{1}{2}(M_\Omega - 2\varepsilon) > 0$";
4. page 4, line 10: we have replaced "Banach space" by "Hilbert space"
5. page 12, at the beginning of the demonstration of Lemma 5.5: we have replaced "$U(\cdot;\tau_n,\Psi_n)$" by "$U(t_0;\tau_n,\Psi_n)$"

Proof

6. page 4, line 15: we removed the expressionwe "another spaces";
7. page 5, line 8: we removed the "{";
8. page 11, line -10: we have replaced "comma" by "period"

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[1] YEJUAN, W.; KLOEDEN, P. E. *Pullback attractors of a multi-valued process generated by parabolic differential equations with unbounded delays.* Nonlinear Analysis 90, 86–95, (2013).