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PhD Dissertation

Analysis of infinite dimensional dynamical systems associated to partial functional differential equations

Análisis de sistemas dinámicos infinito-dimensionales asociados a ecuaciones en derivadas parciales funcionales

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Dedicado a "mi arma", Tomás Caraballo!!!

Contents

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Abstract

The main objective of this thesis is to study the long time behavior of several kinds of infinite dimensional dynamical systems associated to partial differential equations containing some kinds of hereditary characteristics (such as variable delay, distributed delay or memory, etc) in terms of pullback/random attractors and the stability analysis of stationary (steady-state) solutions. The thesis consists of three parts, and each part consists of two chapters.

Chapter 1 is devoted to the dynamics of an integer order stochastic reaction-diffusion equation with thermal memory when the nonlinear term is subcritical or critical. First of all, instead of the classic Galerkin approximations, a semigroup method together with the Lax-Milgram theorem is used to prove the existence, uniqueness and continuity of mild solutions. Then, the dynamics of solutions is analyzed by a priori estimates, and the existence of pullback random attractors is established. Besides, we prove that this pullback random attractors cannot "explode", a property known as upper semicontinuity.

It is well-known that integer order reaction-diffusion has been extensively applied in physics, biomedical and chemical sciences. Nevertheless, this integer order reaction-diffusion equation is inadequate to model some real situations, for instance, anisotropic diffusion, anomalous diffusion. But fractional order reaction-diffusion equation can model these phenomena successfully.

Hence, in Chapter 2, we focus on the asymptotical behavior of a fractional stochastic reaction-diffusion equation with memory, which is also called fractional integro-differential equation. Existence and uniqueness of mild solutions is proved by using the Lumer-Phillips theorem. Next, under appropriate assumptions on the memory kernel and on the magnitude of the nonlinearity, the existence of random attractor is achieved by obtaining some uniform estimates. Moreover, the random attractor is shown to have finite Hausdorff dimension, which means the asymptotic behavior of the system is determined by only a finite number of degrees of freedom, though the random attractor is a subset of an infinite-dimensional phase space.

As we can see, the first two chapters consider an important partial function differential equations with infinite distributed delay. However, partial functional differential equations include more than only distributed delays; for instance, also variable delay terms can be considered. Therefore, in the next chapter, we consider another significant partial functional differential equation but with variable delay.

In Chapter 3, we discuss the stability of stationary solutions to 2D Navier-Stokes equations when the external force contains unbounded variable delay. Above all, the existence and uniqueness of solutions is proved by Galerkin approximations and the energy method. The existence of stationary solutions is then established by means of the Lax-Milgram theorem and the Schauder fixed point the-

orem. Afterward, the local stability analysis of stationary solutions is carried out by several different approaches: the classical Lyapunov function method, the Razumikhin-Lyapunov technique and by constructing appropriate Lyapunov functionals. Nevertheless, by these methods, the best result we can obtain is the asymptotical stability of stationary solutions by constructing a suitable Lyapunov functionals. Fortunately, we could obtain polynomial stability of the steady-state in a particular case of unbounded variable delay, namely, the proportional delay.

However, the exponential stability of stationary solutions to Navier-Stokes equation with unbounded variable delay still seems an open problem. We can also wonder about the stability of stationary solutions to 2D Navier-Stokes equations with unbounded delay when it is perturbed by random noise.

Hence, in Chapter 4, a stochastic 2D Navier-Stokes equation with unbounded delay is analyzed in the phase space of continuous bounded functions with limits at $-\infty$. The existence and uniqueness of solutions in the case of infinite delay is first proved by using the classical Galerkin approximations. Next, the local stability analysis of constant solutions (equilibria) is carried out by exploiting two methods. Namely, the Lyapunov function method and by constructing appropriate Lyapunov functionals. Although it is not possible, in general, to establish the exponential convergence of the stationary solutions, the polynomial convergence towards the stationary solutions, in a particular case of unbounded variable delay can be proved. We would also like to mention that exponential stability of other special cases of infinite delay remains as an open problem for both the deterministic and stochastic cases.

Notice that Chapter 3 and Chapter 4 are both concerned with delayed Navier-Stokes equations, which is a very important Newtonian fluids, and it is extensively applied in physics, chemistry, medicine, etc. However, there are also many important fluids, such as blood, polymer solutions, and biological fluids, etc, whose motion cannot be modeled precisely by Newtonian fluids but by non-Newtonian fluids. Hence, in the next two chapters, we are interested in the long time behavior of an incompressible non-Newtonian fluids with delay.

In Chapter 5, we study the dynamics of non-autonomous incompressible non-Newtonian fluids with finite delay. The existence of global solution is showed by classical Galerkin approximations and the energy method. Actually, we also prove the uniqueness of solutions as well as the continuous dependence of solutions on the initial value. Then, the existence of pullback attractors for the non-autonomous dynamical system associated to this problem is established under a weaker condition in space $C([-h,0];H^2)$ rather than space $C([-h,0];L^2)$, and this improves the available results that worked on non-Newtonian fluids.

Finally, in Chapter 6, we consider the exponential stability of an incompressible non-Newtonian fluids with finite delay. The existence and uniqueness of stationary solutions are first established, and this is not an obvious and straightforward work because of the nonlinearity and the complexity of the term $N(u)$. The exponential stability of steady-state solutions is then analyzed by means of four different approaches. The first one is the classical Lyapunov function method, which requires the differentiability of the delay term. But this may seem a very restrictive condition. Luckily, we could use a Razumikhin type argument to weaken this condition, but allow for more general types of delay. In fact, we could obtain a better stability result by this technique. Then, a method relying on the construction of Lyapunov functionals and another one using a Gronwall-like lemma are also exploited to study the stability, respectively. We would like to emphasize that by using a Gronwall-like lemma, only the measurability of delay term is demanded, but still ensure the exponential stability.

The results of our investigation in these six chapters are included in the following papers:

- L. Liu and T. Caraballo, Analysis of a stochastic 2d-Navier-Stokes model with infinite delay, J. Dyn. Diff. Equ. (submitted).
- L. Liu and T. Caraballo, Well-posedness and dynamics of a fractional stochastic integro- differential equation, Phys. D, 355 (2017), pp. 45–57.
- L. Liu, T. Caraballo, and X. Fu, Exponential stability of an incompressible non-newtonian fluids with delay, Accepted by Discr. Cont. Dyn. Syst. B.
- L. Liu, T. Caraballo, and X. Fu, Dynamics of a non-autonomous incompressible non- Newtonian fluid with delay, Dyn. Partial Differ. Equ., 14 (2017), pp. 375–402.
- L. Liu, T. Caraballo, and P. Kloeden, Long time behavior of stochastic parabolic problems with white noise in materials with thermal memory, Rev. Mat. Complut., 30 (2017), pp. 687–717.
- L. Liu, T. Caraballo, and P. Marín-Rubio, Asymptotic behavior of 2d-Navier-Stokes equations with infinite delay, J. Diff. Equ. (to appear), (2018).

Resumen

En esta tesis estudiamos el comportamiento asintótico de sistemas dinámicos infinito-dimensionales asociados a ecuaciones en derivadas parciales funcionales que contienen algunos términos con propiedades hereditarias (tales como retraso variable, retraso distribuido o memoria, etc.). En concreto analizamos la existencia de atractores de tipo “pullback” y aleatorios, estudio de su estructura y carácter finito dimensional, así como el estudio de la estabilidad de soluciones estacionarias. La tesis consta de tres partes y cada parte contiene dos capítulos.

El Capítulo 1 está dedicado a estudiar el comportamiento asintótico de una ecuación de reacción-difusion estocástica con memoria térmica en los casos en que el término no lineal sea subcrítico y crítico. En lugar de la aproximación clásica de Galerkin, se usa un método de semigrupo junto con el teorema de Lax-Milgram para demostrar la existencia, unicidad, regularidad y continuidad de las soluciones débiles. Posteriormente, analizamos la dinámica de las soluciones mediante estimaciones a priori, y establecemos la existencia de atractores aleatorios para el sistema dinámico aleatorio asociado al problema. Además, demostramos que este atractor aleatorio no puede “explotar”, una propiedad conocida como semi-continuidad superior. Las ecuaciones en derivadas parciales de tipo estándar (en las que aparece el operador de Laplace $(-\Delta)^{\alpha}$ con $\alpha = 1$) se aplican ampliamente en problemas de física, biología y química. Sin embargo, en algunos casos, el proceso de partículas salta de un punto a otro con probabilidad de decaimiento en la ley de potencia, que no podría caracterizarse por estas ecuaciones estándar sino por ecuaciones fraccionarias (es decir, el operador de Laplace $(-\Delta)^{\alpha}$ con $0 < \alpha < 1$).

Por lo tanto, en el Capítulo 2 estudiamos una ecuación fraccionaria de reacción-difusión estocástica que describe un proceso de reacción-difusión con memoria que depende de la temperatura. Más precisamente, investigamos el buen planteamiento y la dinámica de la ecuación fraccionaria de reacción-difusión estocástica con memoria térmica, que también se denomina ecuación integro-diferencial. La existencia y la singularidad de las soluciones de esta ecuación integro-diferencial se demuestran usando el teorema de Lumer-Phillips. Luego, bajo hipótesis apropiadas sobre el núcleo del término de memoria y sobre la magnitud de la no linealidad, demostramos la existencia del atractor aleatorio y logramos obtener algunas estimaciones uniformes que permiten concluir que el atractor aleatorio tiene dimensión de Hausdorff finita, lo que significa que el comportamiento asintótico del sistema está determinado un número finito de grados de libertad, aunque el atractor aleatorio es un subconjunto de un espacio de fase de dimensión infinita.

Como acabamos de exponer, en la primera parte hemos considerado una clase de ecuaciones en derivadas parciales con retardo distribuido infinito, la existencia de atractor aleatorio, así como su semi-continuidad superior. Sin embargo, las ecuaciones en derivadas parciales funcionales incluyen

muchos más tipos de términos con retraso que los distribuidos, por ejemplo, también los retrasos de tipo variable tienen bastante interés. Por esta razón, en la Parte II, consideramos otro tipo de ecuación en derivadas parciales funcional que incluye el caso de retraso variable.

En el Capítulo 3, estudiamos la estabilidad de las ecuaciones 2D de Navier-Stokes cuando la fuerza externa contiene características hereditarias en el espacio de fase de funciones continuas y acotadas con límite en $-\infty$. Primero, demostramos la existencia, unicidad y regularidad de las soluciones mediante las aproximaciones de Galerkin y el método de la energía. La existencia de soluciones estacionarias la establecemos mediante el teorema Lax-Milgram y el teorema del punto fijo de Schauder. La estabilidad de las soluciones la analizamos mediante varios enfoques diferentes: el método clásico de función Lyapunov, la técnica de Razumikhin-Lyapunov y la construcción de funcionales de Lyapunov apropiados. Sin embargo, mediante estos métodos, el mejor resultado que podemos obtener es la estabilidad asintótica de la solución estacionaria mediante la construcción de una funciones de Lyapunov apropiada. Vale la pena mencionar que podríamos demostrar la estabilidad polinómica de la solución estacionaria en un caso particular de retraso variable no acotado. Sin embargo, también nos preguntamos sobre el comportamiento asintótico de las ecuaciones 2D de Navier-Stokes con retraso no acotado cuando es el modelo contiene una perturbación estocástica.

Por lo tanto, en el Capítulo 4 analizamos algunos resultados de ecuaciones bidimensionales estocásticas de Navier-Stokes con retraso ilimitado. La existencia y unicidad de las soluciones en el caso de un retraso ilimitado (infinito) se demuestran primero utilizando la técnica clásica de las aproximaciones de Galerkin. El análisis de la estabilidad local de las soluciones constantes (equilibrios) también se lleva a cabo explotando varios enfoques. A saber, el método de las funciones de Lyapunov, y la construcción de apropiados funcionales Lyapunov. Aunque no es posible, en general, establecer condiciones que garanticen el comportamiento asintótico exponencial de las soluciones, algunas condiciones suficientes aseguran la estabilidad polinómica de la solución estacionaria en un caso particular de retraso variable ilimitado, mientras que la estabilidad exponencial para otros casos especiales con retraso infinito permanece como un problema abierto tanto para el caso determinista como para el caso estocástico.

Observemos que en los capítulos 3 y 4 investigamos fluidos Newtonianos, en concreto, las ecuaciones de Navier-Stokes. Sin embargo, hay muchos fluidos importantes, como por ejemplo la sangre, los plásticos fundidos, las fibras sintéticas, las pinturas y grasas, las soluciones de polímeros, suspensiones, adhesivos, tintes, barnices y fluidos biológicos, etc., cuyos movimientos no pueden modelarse con fluidos Newtonianos de forma precisa, pero sí que pueden hacerse usando fluidos no Newtonianos. Por lo tanto, en los próximos dos capítulos, analizamos el comportamiento límite de los fluidos no Newtonianos con retraso.

En el Capítulo 5 investigamos la dinámica de un fluido no Newtoniano incompresible no autónomo con retraso. La existencia de una solución global se obtiene mediante la aproximación clásica de Galerkin y el método de la energía. En realidad, también demostramos la unicidad de la solución y la dependencia continua respecto de los valores iniciales. Posteriormente analizamos el comportamiento límite del sistema dinámico asociado al fluido incompresible no Newtoniano. Finalmente, establecemos la existencia de atractores de tipo pullback para el sistema dinámico no autónomo asociado al problema.

Finalmente, en el Capítulo 6 consideramos la estabilidad exponencial de un fluido no Newtoniano incompresible. La existencia y la regularidad de las soluciones estacionarias se establecen primero. La estabilidad exponencial de las soluciones estacionarias se analiza a continuación por medio de cuatro enfoques diferentes. El primero es el método clásico de las funciones de Lyapunov, mientras que el segundo se basa en un argumento tipo Razumikhin. Luego, utilizamos un método que se basa en la construcción de funcionales de Lyapunov y otro que usa un lema de tipo Gronwall que permiten estudiar la estabilidad, respectivamente.

Los resultados de estos seis capítulos forman parte del contenido de los siguientes trabajos:

- L. Liu and T. Caraballo, Analysis of a stochastic 2d Navier-Stokes model with infinite delay, J. Dyn. Diff. Equ. (sometido).
- L. Liu and T. Caraballo, Well-posedness and dynamics of a fractional stochastic integro- differential equation, Phys. D, 355 (2017), pp. 45–57.
- L. Liu, T. Caraballo, and X. Fu, Exponential stability of an incompressible non-newtonian fluids with delay, Discr. Cont. Dyn. Syst. B. (2018) (aceptado)
- L. Liu, T. Caraballo, and X. Fu, Dynamics of a non-autonomous incompressible non- Newtonian fluid with delay, Dyn. Partial Differ. Equ., 14 (2017), pp. 375–402.
- L. Liu, T. Caraballo, and P. Kloeden, Long time behavior of stochastic parabolic problems with white noise in materials with thermal memory, Rev. Mat. Complut., 30 (2017), pp. 687–717.
- L. Liu, T. Caraballo, and P. Marín-Rubio, Asymptotic behavior of 2d-Navier-Stokes equations with infinite delay, J. Diff. Equ. (por aparecer), (2018).

Introduction

Background

The study of dynamical systems could date back to the late 19th century. In 1881, H. Poincare worked on celestial mechanics [140], and introduced the theory of qualitative differential equations, which is known as the geometric theory of differential equations. Thanks to this theory, we could investigate the asymptotic behavior of solutions directly through the equation itself without obtaining the explicit solutions to the equation. Actually, it is almost impossible to obtain explicit solutions to most differential equations. Almost at the same time, the Soviet mathematician Lyapunov [124] made a huge and pioneering work on the qualitative theory of differential equations. He studied the stability of solutions, the existence and regression of periodic orbits, which later became the founding of dynamical systems. Then, from 1912 onwards, Birkhoff expanded the study of dynamical systems in the context of the three-body problem, including his ergodic theorem, Birkhoff [13, 14]. There are many great works on dynamical systems since then, such as D. Ruelle [144] , S.Smale [151]and F. Takens [156], Liao Shantao, Wen Lan et al. [46, 62, 68, 114, 172].

Classified by the dimension of the phase space, dynamical systems can be classified into finite dimensional dynamical systems and infinite dimensional dynamical systems. Research on finite-dimensional dynamical systems has been undergoing at least 50 years old, but the problem of dynamical systems is far from being limited to finite-dimensional situations. In fact, many realistic problems belong to the framework of infinite dimensional dynamical systems, as they are modeled by partial differential equations. For instance, the problem of flow past body in flow mechanics, namely, vortex appears when the water flows past the object that is settled in the water. Another famous example is Benard's convection problem, i.e., if we heat a closed container full of liquid, and when its bottom temperature equals the top temperature, then the convection, even chaos occurs. Besides, some dissipative partial differential equations, such as, reaction-diffusion equation[101], Navier-Stokes equation[99], non-Newtonian fluids, Kuramoto-Sivashinskey equation, Cahn-Hilliard equation as well as Ginzburg-Landan equation, etc, have showed similar chaos phenomena. In addition, compared with finite dimensional dynamical systems, infinite dimensional dynamical systems display new trait, it could exhibit not only time chaos but also spatial chaos, which is related tightly to our daily life. All of this demonstrate that it is necessary and very important for us to study infinite dimensional dynamical systems. Even more, it is possible that we may find a new way to the study of turbulence by investigating infinite dimensional dynamical systems. This is why physicists, mathematicians and mechanics are devoted to the investigation of infinite dimensional dynamical systems.

It is worth mentioning that infinite-dimensional dynamical systems not only interacts with other fields of dynamical systems (such as differential dynamical systems, Hamiltonian dynamical systems, topological dynamical systems, complex dynamical systems, ergodic theory, stochastic dynamical systems, etc.), but also penetrates closely into many aspects such as physics, mechanics and even biology, biomedicine, economics, ocean-atmosphere, and engineering technology.

On the other hand, in many case, the change of systems depends not only on the present state, but also on the previous history. In fact, the time lag in dynamical systems is usually unavoidable, even if the information is transmitted at the speed of light. In this case, partial functional differential equations instead of partial differential equations can describe the evolution of the systems much more better. Moreover, partial functional differential equations are also increasingly appearing in the disciplines of population ecology, cell biology, and biomolecular chemistry. Needless to say, the study of infinite dimensional dynamic systems generated by partial functional differential equations has great significance. It attracts many mathematicians to start working in this area.

All of these indicate that it is imperative to study the asymptotic behavior of infinite dimensional dynamical systems produced by partial functional differential equations, which has important theoretical significance. First of all, it can help to complete the theory of infinite dimensional dynamical systems, promote the development of disciplines, such as functional differential equations, dynamical systems as well as numerical mathematics etc. Moreover, this study has huge practical application value. We benefit of understanding biology, medicine, chemistry, physics, control engineering, atmosphere, and ocean phenomena better, especially chaos, so that we could understand and master the law of these disciplines. Then we could serve the world better with these knowledges, which, in return, gives inexhaustible vitality of studying infinite dimensional dynamical systems associated to partial functional differential equations.

Research state

In recent decades, the theory of infinite dimensional dynamical systems has made tremendous developments and produced many important achievements. Guo et al. [83, 86], Zhong et al. [155, 182], Zhou et al. [183], Ladyzhenskaya [111], Temam [75, 160], M.Vishik [164], Hale [89], Robinson [143], Chueshov [48, 49] and Sell [145, 146], Haraux [94, 95], Zelik [9, 163, 176] study the global attractors and their dimensions of some dissipative nonlinear evolution equations, the existence of inertial manifolds and the problems of inertial manifolds. A.Babin and M.Vishik[6], V. Chepyzhov and M.Vishik [47, 165], T. Caraballo [23, 28], P. Kloeden [106, 107], Duan [154], Moise [134], Cui and Langa [58], Miranville et al. [132, 133], investigate the existence of pullback attractors for non-autonomous infinite dimensional dynamical systems, meanwhile [51, 57, 74, 148] focus on the existence of random attractors to stochastic infinite dimensional dynamical systems, and Han et al. [10, 34, 92, 93, 169] studied the existence of attractor for lattice dynamical systems. For more information about attractors, please refer [54, 96, 98, 159]. And for exponential attractor, trajectory attractor, dimension of attractor, inertial manifolds, readers are referred to [43, 60, 166].

As far as we know, compared to infinite dimensional dynamical systems associated to PDEs, there are less work on infinite dimensional dynamical systems associated to PFDEs at present. Xu et al. [112, 168] and Caraballo et al. [19] mainly focus on the global exponential stability of dynamical systems and the existence of attractors. In this thesis, we will study the long time behavior of infinite dimensional dynamical systems associated to several kinds of partial functional differential equations in terms of pullback attractor and stability of stationary solutions. Consequently, we will structure our work in three parts.

Part I: Parabolic problems with thermal memory

As an important mathematical-physical model, reaction-diffusion equations have extensive applications. For instance, we use reaction-diffusion equations to model nonlinear heat transport, or to describe the motion of fluids, such as water, oil and gas in porous media, or to model the electron of semiconductors. In fact, the reaction-diffusion equations have also been applied in biological mathematics, ecological environment, biomedical, chemical, physical problems. Besides, reaction-diffusion equations have also been used to study both the reaction of Belousov-Zhabotinskii and the metabolism of enzymes.

And there are many significant works on reaction-diffusion equations, for example, Pata et al. analyzed the long time behavior of a deterministic reaction-diffusion equation with memory in [52, 53] and studied the thermal equation in [78, 137] and its existence of attractors. Li [113] proved the existence of uniform attractors for parabolic problems with memory in the cases that the nonlinearities are subcritical and critical. Nevertheless, as far as we know, most of those models are considered in deterministic case, namely, they did not take into account white noise effects. Therefore, in the first chapter of Part I, we focus on the long time behavior of a stochastic reaction-diffusion equation with thermal memory. The existence and uniqueness of mild solutions instead of weak solutions are proved by a semigroup method. Then the existence and uniqueness of pullback random attractor is established in the critical and subcritical cases.

Notice that, the mentioned references deal with integer order reaction-diffusion equation. However, it has been proved that sometimes, especially when self-orgnization phenomena occurs, a fractional order differential equation can model this phenomena more precisely. Thus, in Chapter 2 of Part I, we study a fractional stochastic reaction-diffusion equation with thermal memory. First of all, the well-posedness is proved by Lumer-Philips theorem. Then Sobolev embedding theorem is used to prove the existence of random attractor with finite Hausdorff dimension. We would like to mention that [85, 121] studied the existence and ergodicity of random attractors in fractional stochastic reaction-diffusion equations without memory.

All the results of Part I are new. In some extent, these results improve the available corresponding work in the literature. The results in Chapter 1 have been published in [119] (*L. Liu, T. Caraballo, and P. Kloeden, Long time behavior of stochastic parabolic problems with white noise in materials with thermal memory, Rev. Mat. Complut., 30 (2017), pp. 687–717*), while the work [116] (*L. Liu and T. Caraballo, Well-posedness and dynamics of a fractional stochastic integro-differential equation, Phys. D, 355 (2017), pp. 45–57*) contains the results proved in Chapter 2.

Part II: Navier-Stokes equation with infinite delay

This part is devoted to a well-known Newtonian fluids, i.e., 2D–Navier-Stokes fluids. Navier-Stokes model is one of the most important mathematical physics equation, and is more widely used in real life. For instance, in aeronautics and astronautics, the Navier-Stokes model can simulate the helicopter hovering aerodynamic performance. On the other hand, the Navier-Stokes equation can simulate the movement of small-scale water in offshore engineering. Studying the Navier-Stokes equation also helps us to understand the oceans, benefit the development and utilization of marine resources and develop the marine economy and industry.

Caraballo et al. [27, 29, 30, 131] discuss the existence of solutions for 2D/3D Navier-Stokes equations with time delays, existence and regularity of attractors, [35] analyze the existence and uniqueness as well as exponential stability of the fixed point of the Navier-Stokes equations with time delay. Marín-Rubio et al. [130] analyzed a case with distributed unbounded delay. However, there is no work on Navier-Stokes equations with unbounded variable delay. Thus, in Chapter 3 we analyze the asymptotic behavior of Navier-Stokes equations with unbounded variable delay. The existence and uniqueness of weak solutions is obtained by the theory of partial functional differential equations. Then by constructing a suitable Lyapunov functionals, the asymptotical stability of stationary solutions is proved. Besides, the polynomial stability is established under proportional delay case. We would like to point out that in this case of unbounded delay, polynomial stability is the best result we were able to obtain, but it remains as an open problem the point of analyzing whether there may be cases in which we can prove exponential stability.

On the other hand, a natural question also appears. We wonder about the behavior of our model when some noise may appear. There are some previous work already done concerning stochastic 2D-Navier-Stokes equations with finite delay. Here, in Chapter 4, we extend our previous analysis carried out in Chapter 3 to the stochastic framework. In [157, 158] Taniguchi studied the existence and uniqueness of solutions and the exponential stability of solutions to the stochastic 2D-Navier-Stokes equations. In Chapter 4, we discuss a type of stochastic 2D-Navier-Stokes equation with unbounded delay, and taking into account that our random term is not linear, which means Ornstein-Uhlenbeck transformation can not transform our stochastic problem into a random one, we cannot use the theory of random dynamical systems, and this brings additional difficulties to our proofs. Furthermore, the classical way to prove the uniqueness of solution is not enough now. A technical lemma is introduced to prove the existence and uniqueness of weak solutions. Then the asymptotic stability of stationary (steady-state) solutions is proved as well as the polynomial stability.

However, as we have already mentioned, the exponential stability of other special cases of infinite delay remains as an open problem for the deterministic and the stochastic cases.

The results in Chapter 3 are contained in the paper [120] (*L. Liu, T. Caraballo, and P. Marín-Rubio, Asymptotic behavior of 2d-Navier-Stokes equations with infinite delay, J. Diff. Equ. (to appear), (2018)*), while the ones in Chapter 4 are in [115] (*L. Liu and T. Caraballo, Analysis of a stochastic 2d Navier-Stokes model with infinite delay, J. Dyn. Diff. Equ. (submitted)*).

Part III: Non-Newtonian models with delay

Even though there are many Newtonian fluids in our real world, such as water, alcohol and most of the pure liquid, low molecular weight compounds, etc, we cannot ignore non-Newtonian fluids, since many other very important fluids, such as blood, cornstarch, cytoplasm, polyethylene, oil, mud, applesauce, agar, etc, are non-Newtonian fluids. Actually, non-Newtonian fluids have various application in medicine, chemical industry and environmental protection. In medicine, for example, human blood belongs to non-Newtonian fluids, mastery of non-Newtonian viscous features and hemodynamics of blood, which is beneficial for observation and control the blood viscosity, but also helps to diagnose and treat cardiovascular disease; and because artherosclerosis arises in the arterial wall shear stress is closely related. In chemical industry, making full use of the viscous characteristics of non-Newtonian fluids can be applied to wastewater treatment, which is very conductive to environmental protection.

There is a wide literature already published on non-Newtonian fluids. The existence and uniqueness of solutions of non-Newtonian flow without delay is studied in [7, 12], while a maximal compact attractor of a non-Newtonian system in an unbounded channel is obtained in [15]. Zhao et al. [178–180] studied the existence and regularity of pullback attractors for non-Newtonian fluids problems with time-delay, while [103] focused on pullback attractor of a non-autonomous non-Newtonian equation with bounded variable delays.

However, in these works, the existence and stability of stationary solutions to non-Newtonian fluids with delay are seldom discussed. Hence, Part III is devoted to analyzing the asymptotic behavior of non-Newtonian fluids problems with delay. First, in Chapter 5 we generalize the result of [103] to a more general delay case, in other words, our results hold true for variable and distributed delays but with weaker condition on the forcing terms. The existence and uniqueness of weak solutions are showed by the energy method. And the existence of pullback attractor in the phase space $C([-h,0];H^2)$ is established by using the pullback ω-limit compactness and a priori estimates. These results improve the corresponding ones in [103] .

Then, in Chapter 6, we analyze the existence and exponential stability of stationary solutions. Though [85] investigated the stationary solution to a non-Newtonian fluids without delay, there is no any detailed proof for the existence of stationary solutions. Therefore, the first goal of this chapter is to prove the existence and uniqueness of stationary solutions, which is not a trivial task at all. Finally, four different approaches are used to verify the exponential stability of stationary solutions.

The results in Chapter 5 are contained in the paper [118] (*L. Liu, T. Caraballo, and X. Fu, Dynamics of a non-autonomous incompressible non- Newtonian fluid with delay, Dyn. Partial Differ. Equ., 14 (2017), pp. 375–402.*), while the results in Chapter 6 are contained in [117] (*L. Liu, T. Caraballo, and X. Fu, Exponential stability of an incompressible non-Newtonian fluids with delay, Discr. Cont. Dyn. Syst. B, accepted*).

Open Problems

This work focuses on the asymptotic behavior of infinite dimensional dynamical systems associated to several kinds of partial functional differential equations. In particular, stochastic reaction-diffusion equations with memory, 2D Navier-Stokes equations with unbounded delay as well as non-Newtonian with bounded delay. Also the existence of pullback/random attractors is proved in some cases and the asymptotic stability, either polynomial or exponential stability are obtained.

However, there are still many problems in these fields that need further study. As for reaction-diffusion equation with memory, which has been proved to possess pullback attractor. But the dimension of the attractor and the existence of inertial manifolds is still unsolved, the existence and uniqueness of stationary solutions and its exponential stability is unknown, either.

When it comes to the fractional reaction-diffusion equation with memory, we still wonder the low bound of this random attractors, and the existence of inertial manifolds as well as their morse decomposition. Besides, the long time behavior of time-fractional reaction-diffusion equation and fractional Brownian motion are still unknown.

For Navier-Stokes equations with unbounded delay, we have shown the polynomial stability of fixed points under the case of unbounded variable delay. Nevertheless, we wonder whether we can obtain the exponential stability of stationary solutions and existence of attractor. Especially, we are interested in the pantograph equation, which is a typical but simple unbounded variable delayed differential equation. We believe that the study of pantograph equation can help us to improve our knowledge about 2D–Navier-Stokes equations with unbounded delay.

To the end, we still studied non-Newtonian fluids with finite delay, the existence and uniqueness of pullback attractor is established, and the exponential stability of stationary solutions are proved as well. However, we still would like to analyze the Hausdorff dimension or fractal dimension of the pullback attractor, as well as the existence of inertial manifolds and morse decomposition. Furthermore, we also would like to discuss the dynamics of stochastic non-Newtonian fluids with both finite delay and infinite delay. All the problems deserve our attraction, and actually, these are our forthcoming work.

Part I

Parabolic problems with thermal memory

Part I focuses on a kind of stochastic parabolic problems with thermal memory. Two chapters are included in this part, i.e., Chapter 1 and Chapter 2. First of all, we recall some basic concepts and the theory of random dynamical systems, and introduce Ornstein-Uhlenbeck process, which is one of the keys to solve our problems in both chapters.

Then, in Chapter 1, we analyze the dynamics of a stochastic parabolic problem with memory which describes the heat flow in a rigid, isotropic, homogeneous heat conductor with linear memory. The nonlinear source term satisfies subcritical and critical growth conditions. Such a nonlinear heat supply might describe, for instance, temperature-dependent radiative phenomena (see, e.g. [174]). In addition, a non-Fourier constitutive law for the heat flux is considered in this chapter. The resulting linearized model is derived in the framework of the well-established theory of hear flow with memory due to Coleman and Gurtin [50]. More precisely, we study the existence and uniqueness of solutions for this model, and then the existence and upper-semicontinuity of the random attractor is established.

Notice that equations with fractional derivative are becoming a focus of interest since the fractional derivative and fractional integral have a wide range of applications in physics, biology, chemistry, population dynamics, geophysical fluid dynamics, finance and other fields of applied sciences. One meets them in the theory of systems with chaotic dynamics (see [147]), dynamics in a complex or porous medium [65, 150]; random walks with a memory and flights [79] and many other situations. In Chapter 2, we focus on the asymptotic behavior of a fractional stochastic reaction-diffusion equation in materials with memory. The well-posedness is proved by a Lumper-Phillips theorem, and existence of random attractor is obtained by a priori estimates, as well as the finite Hausdorff dimension of the corresponding random attractor are showed.

Preliminaries

Now we are in a position to recall some notations about random dynamical systems as well as some theory of pullback random attractors, see [2, 25, 28] for more information. We begin with the concepts of parametric dynamical system, see [57]. Let X be a separable Banach space. To define a cocycle for a non-autonomous stochastic equation in X, we need to use two parametric spaces, say, Ω_1 and Ω, where Ω_1 is responsible for non-autonomous deterministic external terms and Ω for stochastic terms. We may take Ω_1 either as the collection of translations of deterministic time dependent terms [25, 63] or simply as the collection of initial times [167]. In this paper, we choose Ω_1 as the collection of initial times and write $\Omega_1 = \mathbb{R}$. For random parameters, we will choose the standard probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel $\sigma-$algebra induced by the compact open topology of Ω, and P is the Wiener measure on $(\Omega, \mathcal{F})$. There is a group $\{\theta_t\}_{t\in\mathbb{R}}$ of mappings acting on $(\Omega, \mathcal{F}, P)$ defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega \text{ and } t \in \mathbb{R}. \tag{0.0.1}$$

In terms of (0.0.1), one may define a new group $\{\tilde{\theta}_t\}_{t\in\mathbb{R}}$ on the product space $\mathbb{R} \times \Omega := \tilde{\Omega}$ given by

$$\tilde{\theta}_t(\tau, \omega) = (\tau + t, \theta_t\omega), \quad \text{for all } (\tau, \omega) \in \tilde{\Omega},\ t \in \mathbb{R}. \tag{0.0.2}$$

Hereafter we write $\tilde{\omega} = (\tau, \omega)$ with $(\tau, \omega) \in \tilde{\Omega}$.

A cocycle of non-autonomous random dynamical systems is defined as follows.

Definition 0.0.1. *A mapping* $\Phi : \mathbb{R}^+ \times \tilde{\Omega} \times X \to X$ *is called a continuous cocycle on* X *over* $\mathbb{R}$ *and* $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ *if for all* $t, s \in \mathbb{R}^+$ *and* $\tilde{\omega} \in \tilde{\Omega}$, *the following conditions are satisfied:*

(i) $\Phi(\cdot, (\tau, \cdot), \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ *is* $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$*-measurable;*

(ii) $\Phi(0, \tilde{\omega}, \cdot)$ *is the identity on X;*

(iii) $\Phi(t + s, \tilde{\omega}, \cdot) = \Phi(t, \tilde{\theta}_s\tilde{\omega}, \cdot) \circ \Phi(s, \tilde{\omega}, \cdot)$*;*

(iv) $\Phi(t, \tilde{\omega}, \cdot) : X \to X$ *is continuous.*

Definition 0.0.2. *A family* $D = \left\{D(\tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\right\}$ *of nonempty bounded subsets of* X *is said to be tempered if for any* $c > 0$

$$\lim_{t\to+\infty} e^{-ct} \sup\left\{ \| x \|_X : x \in D(\tilde{\theta}_{-t}\tilde{\omega})\right\} = 0.$$

From now on, we use $\mathcal{D}$ to denote the collection of all tempered families of nonempty bounded subsets of X.

Definition 0.0.3. *Let* $K = \left\{K(\tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\right\} \in \mathcal{D}$. *Then* K *is called a* $\mathcal{D}$*-pullback absorbing set for a cocycle* Φ *on* X, *if for every* $B \in \mathcal{D}$ *and all* $\tilde{\omega} \in \tilde{\Omega}$, *there exists* $T = T(\tilde{\omega}, B) > 0$ *such that*

$$\Phi\left(t, \tilde{\theta}_{-t}\tilde{\omega}, B(\tilde{\theta}_{-t}\tilde{\omega})\right) \subset K(\tilde{\omega}) \text{ for all } t \geqslant T.$$

Definition 0.0.4. *Let* $B = \{B(\tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\} \in \mathcal{D}$. *Then* Φ *is said to be* $\mathcal{D}-$*pullback asymptotically compact in* X *if for all* $\tilde{\omega} \in \tilde{\Omega}$, *the sequence*

$$\left\{\Phi(t_n, \tilde{\theta}_{-t_n}\tilde{\omega}, x_n) : x_n \in B(\tilde{\theta}_{-t_n}\tilde{\omega})\right\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X \text{ when } t_n \to +\infty.$$

Definition 0.0.5. *A family* $\mathcal{A} = \{\mathcal{A}(\tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\} \in \mathcal{D}$ *is called a pullback random attractor for* Φ *in* X *if the following conditions are fulfilled:*

(i) For each $\tau \in \mathbb{R}$, $\mathcal{A}(\tau, \cdot)$ *is measurable with respect to the P-completion of* $\mathcal{F}$ *in* Ω *and* $\mathcal{A}(\tilde{\omega})$ *is compact for all* $\tilde{\omega} \in \tilde{\Omega}$.

(ii) $\mathcal{A}$ *is invariant, that is, for every* $\tilde{\omega} \in \tilde{\Omega}$,

$$\Phi(t, \tilde{\omega}, \mathcal{A}(\tilde{\omega})) = \mathcal{A}(\tilde{\theta}_t\tilde{\omega}) \text{ for all } t \geqslant 0.$$

(iii) $\mathcal{A}$ *attracts every member of* $\mathcal{D}$, *that is, for every* $B = \{B(\tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\} \in \mathcal{D}$, *and for every,* $\tilde{\omega} \in \tilde{\Omega}$,

$$\lim_{t\to+\infty} dist_X\Big(\Phi\left(t, \tilde{\theta}_{-t}\tilde{\omega}, B(\tilde{\theta}_{-t}\tilde{\omega})\right), \mathcal{A}(\tilde{\omega})\Big) = 0,$$

where $dist_X(\cdot, \cdot)$ *denotes the Hausdorff semi-distance under the norm of* X, *i.e., for two nonempty sets* $A, B \subset X$,

$$dist_X(A, B) := \sup_{a\in A} dist_X(a, B) = \sup_{a\in A} \inf_{b\in B} \| a - b \|_X .$$

Next we turn to introduce the definitions concerning u.s.c. for a family of sets.

Definition 0.0.6. *([28]) Let Z and I be metric spaces. A family of sets $\{A_\epsilon\}_{\epsilon\in I}$ in Z is said to be upper semi-continuous (u.s.c.) at $\epsilon_0 \in I$ if*

$$\lim_{\epsilon\to\epsilon_0} dist_Z(A_\epsilon, A_{\epsilon_0}) = 0.$$

The following propositions can be found in [28, 51, 56].

Proposition 0.0.7. *Let Φ be a continuous RDS on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ according to Definition 0.0.1. If Φ has a compact measurable (w.r.t $\mathcal{F}$) $\mathcal{D}-$pullback attracting set K in $\mathcal{D}$, then Φ has a unique $\mathcal{D}-$pullback attractor $\mathcal{A}$ in $\mathcal{D}$ given by*

$$\mathcal{A}(\tilde\omega) = \bigcap_{r\geqslant 0} \overline{\bigcup_{t\geqslant r} \Phi\left(t, \tilde\theta_{-t}\tilde\omega, K(\tilde\theta_{-t}\tilde\omega)\right)}, \ \textit{for each } \tilde\omega \in \tilde\Omega.$$

Now, we still need to introduce the Ornstein-Uhlenbeck transformation, writing

$$z^*(\omega) = -\int_{-\infty}^{0} e^s \omega(s) ds, \tag{0.0.3}$$

it is easy to check that $\bar z(t,\omega) = z^*(\theta_t\omega)$ is an Ornstein-Uhlenbeck stationary process which solves the Itô equation

$$d\bar z + \bar z dt = dW.$$

Therefore, if we denote $z(\omega)(x) = z^*(\omega)h(x)$, then the real-valued stochastic process $z(\theta_t\omega)(x) = z^*(\theta_t\omega)h(x)$ is a solution to

$$dz + zdt = h(x)dW. \tag{0.0.4}$$

Let us now recall that (see Proposition 4.3.3 in [2]) that there exists $r_1(\omega) > 0$ tempered s.t.

$$|z^*(\omega)|^2 + |z^*(\omega)|^p + |(-\Delta)^{\frac{\alpha}{2}} z^*(\omega)|^2 + |(-\Delta)^{\alpha} z^*(\omega)|^2 \leqslant r_0(\omega), \text{ where } r_0(\theta_t\omega) \leqslant e^{\frac{\lambda}{2}|t|} r_0(\omega),$$

and λ will be specified later.

Then, it is straightforward to check that

$$|z(\omega)|^2 + |z(\omega)|^p + |(-\Delta)^{\frac{\alpha}{2}} z(\omega)|^2 + |(-\Delta)^{\alpha} z(\omega)|^2 \leqslant r(\omega), \tag{0.0.5}$$

where $r(\omega)$ satisfies the same as $r_0(\omega)$.

Then it follows from (0.0.5) that, for P-a.e. $\omega \in \Omega$,

$$|z(\theta_t\omega)|^2 + |z(\theta_t\omega)|^p + |(-\Delta)^{\frac{\alpha}{2}} z(\theta_t\omega)|^2 + |(-\Delta)^{\alpha} z(\theta_t\omega)|^2 \leqslant e^{\frac{\lambda}{2}|t|} r(\theta_t\omega), \ t \in \mathbb{R}. \tag{0.0.6}$$

Chapter 1

Long time behavior of stochastic parabolic problems with thermal memory

A large class of physical phenomena in which delay effects occur, such as viscoelasticity, population dynamics or heat flow in real conductors is modeled by equations in materials with memory, where the dynamics is influenced by the past history of the state variables. This is because that materials with memory have the property that the mathematical-physical description of their state at a given point of time includes such states in which the materials have been at earlier points of time. Here, in this chapter we study the following stochastic parabolic equation in materials with thermal memory with subcritical and critical nonlinearity

$$\begin{aligned}&\frac{\partial u}{\partial t} - \frac{\partial}{\partial t}\int_{-\infty}^{t}\mu_1(t-s)u(x,s)ds - \lambda\Delta u - \int_{-\infty}^{t}\mu_2(t-s)\Delta u(x,s)ds + f(u)\\&\quad = g(x,t) + \epsilon h(x)\frac{dW}{dt}, \quad x\in\mathcal{O},\ t\geqslant\tau,\end{aligned} \tag{1.0.1}$$

with initial and boundary values

$$u(x,\tau) = u_\tau(x), \quad x\in\mathcal{O},\ u(x,t) = 0, \quad x\in\partial\mathcal{O},\ t\geqslant\tau, \tag{1.0.2}$$

where $\mathcal{O}\subset\mathbb{R}^n, n\geqslant 3$ is a bounded domain with smooth boundary, $\lambda > 0$ and ϵ are constants. In addition, $u(x,t)$ is the unknown function, $\mu_1,\mu_2 : \mathbb{R}^+ \to \mathbb{R}$ are the heat flux memory kernels, f is the nonlinear heat supply satisfying some dissipativeness and growth conditions, $g(x,t)$ is time-dependent forcing term, $h\in H^2(\mathcal{O})\cap W^{2,p}(\mathcal{O})$ and W is real valued two-sided Wiener process on some probability space which will be specified later.

Equations with memory have received increasing interest in recent years. The authors of [19, 76, 78, 81] studied the existence of pullback attractor, global attractors, uniform attractors and exponential stability of heat equation (1.0.1) with $\mu_1 = 0$. Damped wave equations with memory were investigated in [44, 61, 138, 180], while hyperbolic phase-field systems with memory were considered in [73, 82]. Li [113] proved the existence of uniform attractors for parabolic problems with memory in the cases that the nonlinearities term is subcritical and critical. Nevertheless, as far as we know, most of those models are considered in deterministic case, namely they did not take into account white noise effects. But the authors of [110, 162] have demonstrated that, under certain circumstances, the noise can benefit the system in some way. This is an interesting phenomenon because noise is generally considered as a nuisance to systems. To the best of our knowledge, no work has been reported on the existence and uniqueness of mild solution and limit behavior of solutions for equation (1.0.1) with critical nonlinear term.

Motivated by the above considerations, we will analyze the dynamics of solutions to (1.0.1) when the nonlinear heat supply f has a subcritical growth exponent and a critical growth exponent. More precisely, we will focus on (1.0.1) in three aspects: (i) Existence, uniqueness and continuity of mild solutions will be studied by a semigroup method (see [139]). (ii) The existence and uniqueness of pullback random attractor will be proved by a priori estimates and solution decomposition method. (iii) The upper semi-continuity of pullback random attractor will also be checked. We mention that Caraballo [16] considered the existence and asymptotic behavior for a stochastic heat equation with multiplicative noise in materials with memory, mean-square random attractors of stochastic delay differential equations with random delay were studied in [173]. Readers are referred to [17, 18, 41] for more information about stochastic partial differential equations with memory or delay.

To this end, the framework of this chapter is as follows. In the next Section 1.1, we recall some definitions and basic theory of random dynamical systems. Then in Section 1.2, we show the well-posedness of problem Eq.(1.0.1), and Section 1.3 establishes the existence of pullback random attractor. Finally, Section 1.4 contains the upper semi-continuity of the random attractor that is obtained in Section 1.3.

1.1 Definitions and Basic Theory

We have already recalled some definitions at the beginning of Part I. But in order to improve the completion and readability, we prefer to present some abstract spaces, which particularly fit this chapter.

Let $A = -\Delta$ with domain $D(A) = H_0^1(O) \cap H^2(O)$. Denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the $L^2(O)$ inner product and the norm, respectively. Consider the family of Hilbert spaces $D(A^{s/2})$, $s \in \mathbb{R}$, whose inner products and norms are given by

$$(\cdot,\cdot)_{D(A^{s/2})} = (A^{s/2}\cdot, A^{s/2}\cdot) \ \text{ and } \ \|\cdot\|_{D(A^{s/2})} = \|A^{s/2}\cdot\|.$$

Then one has the compact and dense injections,

$$D(A^{s/2}) \hookrightarrow D(A^{r/2}), \quad \forall\, s > r,$$

and the continuous embedding,

$$D(A^{s/2}) \hookrightarrow L^{2n/(n-2s)}(O), \quad \forall\, s \in [0, \frac{n}{2}).$$

Recall the following interpolation results: let $\alpha \geqslant \beta$. For every ϑ, $0 \leqslant \vartheta \leqslant 1$, there is a constant $C = C(\alpha, \beta, \vartheta)$, s.t.

$$\|A^{\nu/2}u\| \leqslant C\|A^{\alpha/2}u\|^{\vartheta}\|A^{\beta/2}u\|^{1-\vartheta}, \quad \forall\, u \in D(A^{\alpha/2}),$$

where $\nu = \vartheta\alpha + (1-\vartheta)\beta$. For convenience, denote by

$$\mathcal{H}_s = D(A^{s/2}) \text{ with norm } \|\cdot\|_{\mathcal{H}_s} = \|A^{s/2}\cdot\|.$$

Then, $\mathcal{H}_0 = L^2(\mathcal{O})$, $\mathcal{H}_1 = H_0^1(\mathcal{O})$, and $\mathcal{H}_2 = H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$.

In order to deal with the memory term of (1.0.1), we introduce the family of weighted spaces. In view of (H1) and (H2), we consider the weighted Hilbert spaces $L^2_{\nu_i}(\mathbb{R}^+;\mathcal{H}_r)$, $i = 1,2$, endowed with the inner products and norms, respectively,

$$(\phi_1,\phi_2)_{\nu_i,\mathcal{H}_r} = \int_0^\infty \nu_i(s)(\phi_1(s),\phi_2(s))_{\mathcal{H}_r}ds, \quad \|\phi\|^2_{\nu_i,\mathcal{H}_r} = \int_0^\infty \nu_i(s)\|\phi(s)\|^2_{\mathcal{H}_r}ds, \ i = 1,2.$$

As in [77, 80], we introduce the Hilbert spaces,

$$Q^r_{\nu_1,\nu_2} = L^2_{\nu_1}(\mathbb{R}^+;\mathcal{H}_r) \cap L^2_{\nu_2}(\mathbb{R}^+;\mathcal{H}_{r+1}),$$

endowed with the inner products,

$$(\eta_1,\eta_2)_{Q^r_{\nu_1,\nu_2}} = \int_0^\infty \nu_1(s)\left(A^{r/2}\eta_1(s), A^{r/2}\eta_2(s)\right)ds + \int_0^\infty \nu_2(s)\left(A^{(r+1)/2}\eta_1(s), A^{(r+1)/2}\eta_2(s)\right)ds,$$

and the norms

$$\|\eta\|^2_{Q^r_{\nu_1,\nu_2}} = (\eta,\eta)_{Q^r_{\nu_1,\nu_2}} = \int_0^\infty \nu_1(s)\|A^{r/2}\eta(s)\|^2 ds + \int_0^\infty \nu_2(s)\|A^{(r+1)/2}\eta(s)\|^2 ds.$$

Finally, we define the product spaces,

$$\mathcal{M}_r = \mathcal{H}_r \times Q^r_{\nu_1,\nu_2},$$

where

$$\mathcal{H}_r = D(A^{\frac{r}{2}}), \quad Q^r_{\nu_1,\nu_2} = L^2_{\nu_1}(\mathbb{R}^+;\mathcal{H}_r) \cap L^2_{\nu_2}(\mathbb{R}^+;\mathcal{H}_{r+1}),$$

that endowed with the norms,

$$\|z\|^2_{\mathcal{M}_r} = \|(u,\eta)\|^2_{\mathcal{M}_r} = \|u\|^2_{\mathcal{H}_r} + \|\eta\|^2_{Q^r_{\nu_1,\nu_2}}, \quad z = (u,\eta) \in \mathcal{M}_r.$$

For the upper semi-continuity of a family of parameterized pullback attractors, we borrow the following results from [24, 26].

Proposition 1.1.1. *Let I be an interval of $\mathbb{R}$. Given $\epsilon \in I$, let $\{\Phi^\epsilon(t,\tilde\omega)\}_{\epsilon\in I}$ be a family of continuous RDSs on X over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$. Suppose that*

(i) there exists a map $R_{\epsilon_0} : \tilde\omega \to \mathbb{R}$ such that $B = \left\{B(\tilde\omega) = \{x \in X : \|x\|_X \leqslant R_{\epsilon_0}(\tilde\omega)\} : \tilde\omega \in \tilde\Omega\right\} \in \mathcal{D}$,

(ii) for each $\epsilon \in I$, Φ^ϵ has a pullback attractor $\mathcal{A}^\epsilon$ and a pullback absorbing set D_ϵ such that for all $\tilde\omega \in \tilde\Omega$, $\limsup_{\epsilon\to\epsilon_0} \|D_\epsilon(\tilde\omega)\|_X \leqslant R_{\epsilon_0}(\tilde\omega)$,

(iii) $\bigcup_{\epsilon\in I} \mathcal{A}^\epsilon(\tilde\omega)$ is precompact in X for each $\tilde\omega \in \tilde\Omega$,

(iv) there exists $\epsilon_0 \in I$ such that $\lim_{n\to+\infty} \Phi^{\epsilon_n}(t,\tilde\omega,x_n) = \Phi^{\epsilon_0}(t,\tilde\omega,x)$ for every $t \in \mathbb{R}^+$, $\tilde\omega \in \tilde\Omega$, ϵ_n, ϵ_0 with $\epsilon_n \to \epsilon_0$, and x_n, x with $x_n \to x$.

Then for each $\tilde\omega \in \tilde\Omega$, $d_H(\mathcal{A}^\epsilon(\tilde\omega), \mathcal{A}^{\epsilon_0}(\tilde\omega)) \to 0$ as $\epsilon \to \epsilon_0$.

1.2 Well-posedness

Now we prove the existence of solutions by a semigroup method and the Lax-Milgram theorem. Before stating the problem in a suitable framework, we enumerate the assumptions on the term in which the delay is present. Hereafter, we suppose that the nonlinear heat supply $f(u)$ satisfies

(f1) $f \in C^1(\mathbb{R}),\ \ f(0) = 0$;

(f2) $f(s)s \geqslant \alpha_1 |s|^{p+1} - \alpha_2,\ s \in \mathbb{R}$;

(f3) $|f'(s)| \leqslant \alpha_3(1 + |s|^{p-1}),\ s \in \mathbb{R}$,

where $1 < p \leqslant 1 + \frac{4}{n}$, α_i, $i = 1, 2, 3$, are positive numbers. In order to study the dynamical behavior of (1.0.1) with critical nonlinearity, we also impose the assumption as in [42, 113],

(f4) $\lim\limits_{|s|\to\infty} \frac{|f'(s)|}{|s|^{\frac{4}{n}}} = 0$,

which implies that for any given $\nu > 0$, there is a positive constant C_ν such that

$$|f(s_1) - f(s_2)| \leqslant |s_1 - s_2|(C_\nu + \nu|s_1|^{\frac{4}{n}} + \nu|s_2|^{\frac{4}{n}}). \tag{1.2.1}$$

Remark 1.2.1. *(i) From (f3), it is not difficult to check that $|f(s)| \leqslant \alpha_4 + \alpha_5 |s|^p$ holds for any $s \in \mathbb{R}$, where α_4, α_5 are positive constants.*

(ii) As it is pointed out in [113], the lack of bound from below for f' is the reason for $1 + \frac{4}{n}$ to be the critical exponent for the nonlinearity f. And in case of (1.2.1), we call f is an almost critical nonlinearity.

Assume that $\mu_1'(\infty) = \mu_2(\infty) = \mu_1(\infty) = 0$. Let $\nu_1(s) = \mu_1''(s)$ and $\nu_2(s) = -\mu_2'(s)$ satisfy

(H1) $\nu_i \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+),\ \ \nu_i(s) \geqslant 0,\ \ \nu_i'(s) \leqslant 0,\ \ i = 1, 2,\ \forall s \in \mathbb{R}^+$,

(H2) $\nu_i'(s) + \delta_i \nu_i(s) \leqslant 0,\ \ i = 1, 2,\ \forall s \in \mathbb{R}^+$,

where δ_i are positive constants, $i = 1, 2$.

Denote $\mu_1(0) = \mu_0$. Along the lines of the procedure suggested by Dafermos in his pioneering work [59], we introduce the new variable

$$\eta^t(x, s) = \int_0^s u^t(x, r)dr = \int_{t-s}^t u(x, r)dr, \quad s \geqslant 0, \tag{1.2.2}$$

where

$$u^t(x, s) = u(x, t - s), \quad s \geqslant 0.$$

Then the original equation (1.0.1)-(1.0.2) can be transformed into the following equivalent system by (1.2.2):

$$\begin{aligned} &\frac{\partial u}{\partial t} - \mu_0 u - \lambda \Delta u + \int_0^\infty \nu_1(s)\eta^t(s)ds - \int_0^\infty \nu_2(s)\Delta\eta^t(s)ds + f(u) = g(x, t) + \epsilon h(x)\frac{dW}{dt}, \\ &\partial_t \eta^t(x, s) + \partial_s \eta^t(x, s) = u, \quad x \in \mathcal{O},\ \ s > 0,\ \ t \geqslant \tau, \end{aligned} \tag{1.2.3}$$

with the initial and boundary values

$$u(x,\tau) = u_\tau(x),\ \eta^\tau(x,s) = \eta_\tau(x,s),\ x \in \mathcal{O}, \quad u(x,t) = 0,\ \eta^t(x,s) = 0\ \ x \in \partial\mathcal{O},\ s > 0,\ t \geqslant \tau. \quad (1.2.4)$$

Note that Eq.(1.2.3) is stochastic equation, and we need to transfer (1.2.3) into a deterministic one only with random parameter.

Set $v(t) = u(t) - \epsilon z(\theta_t\omega)$. Problem (1.2.3)-(1.2.4) can be transformed into a pathwise deterministic problem by (0.0.4)

$$\begin{aligned}&\frac{\partial v}{\partial t} - \mu_0 v - \lambda\Delta v + \int_0^\infty \nu_1(s)\eta^t(s)ds - \int_0^\infty \nu_2(s)\Delta\eta^t(s)ds + f(u) = g(x,t) + \epsilon(\mu_0+1)z + \epsilon\lambda\Delta z,\\ &\partial_t\eta^t + \partial_s\eta^t = v + \epsilon z, \quad x \in \mathcal{O},\ t \geqslant \tau,\end{aligned} \quad (1.2.5)$$

with the initial and boundary values

$$\begin{aligned}&v(x,\tau) = u(x,\tau) - \epsilon z(\theta_\tau\omega) = v_\tau(x),\ \eta^\tau(x,s) = \eta_\tau(x,s),\ x \in \mathcal{O},\ s \geqslant 0,\\ &v(x,t) = 0,\ \ \eta^t(x,s) = 0,\ \ x \in \partial\mathcal{O},\ s \geqslant 0,\ t \geqslant \tau.\end{aligned} \quad (1.2.6)$$

In order to present our results, we write the system (1.2.5)-(1.2.6) as a Cauchy problem

$$\frac{d\phi}{dt} = L\phi + F(\phi, \theta_t\omega, t), \quad (1.2.7)$$

defined in the phase space

$$\mathcal{M}_0 = L^2(\mathcal{O}) \times Q^0_{\nu_1,\nu_2}$$

with norms

$$\|\phi\|^2 = \|(v,\eta^t)\|^2 = \|v\|^2 + \|\eta^t\|^2_{Q^0_{\nu_1,\nu_2}} = \|v\|^2 + \|\eta^t\|^2_{L^2_{\nu_1}(\mathbb{R}^+;L^2(\mathcal{O}))} + \|\eta^t\|^2_{L^2_{\nu_2}(\mathbb{R}^+;H^1_0(\mathcal{O}))}.$$

Also take $\phi = (v(t), \eta^t) \in \mathcal{M}_0$. Then system (1.2.5) is equivalent to the Cauchy problem (1.2.7) with

$$L\phi = (\mu_0 v + \lambda\Delta v - \int_0^\infty \nu_1(s)\eta^t(s)ds + \int_0^\infty \nu_2(s)\Delta\eta^t(s)ds, v - \partial_s\eta^t)$$

and

$$F(\phi, \theta_t\omega, t) = (-f(v+\epsilon z) + g + \epsilon(\mu_0+1)z + \epsilon\lambda\Delta z, \epsilon z)\,. \quad (1.2.8)$$

It is proved in [138] that

$$\partial_t\eta^t = -\partial_s\eta^t + v + \epsilon z,\ \eta^t(0) = 0,$$

can be considered as $\partial_t\eta^t = T\eta^t + v + \epsilon z$, where

$$T\eta^t = -\partial_s\eta^t, \quad \eta^t \in D(T),$$

is the generator of a translation semigroup with domain

$$D(T) = \left\{\eta^t \in Q^0_{\nu_1,\nu_2} | \partial_s \eta^t \in Q^0_{\nu_1,\nu_2}, \eta^t(0) = 0\right\}.$$

Since the domain of L is defined by

$$D(L) = \{\phi \in \mathcal{M}_0 | L\phi \in \mathcal{M}_0\},$$

we have

$$D(L) = \left\{(v, \eta^t) \in \mathcal{M}_0 | v \in H^1_0(O), \eta^t \in D(T), \mu_0 v + \lambda \Delta v - \int_0^\infty \nu_1(s)\eta^t(s)ds + \int_0^\infty \nu_2(s)\Delta\eta^t(s)ds \in L^2(O)\right\}.$$

For the coefficient λ in (1.0.1), we assume that $\lambda\lambda_1 - 2\mu_0 > 0$, where λ_1 is the first eigenvalue of A in $H^1_0(O)$. From now on, we denote by c a generic positive number which may change its value from line to line or even in the same line.

Theorem 1.2.2. *(Well-posedness) Assume that hypotheses (f1)-(f2) are satisfied, $g \in L^2_{loc}(\mathbb{R}; L^2(O))$ and the initial data $(v_\tau, \eta_\tau) \in \mathcal{M}_0$. Then, problem (1.2.7) possesses a unique mild solution with*

$$v \in C([\tau, \infty); L^2(O)) \text{ and } \eta^t \in C([\tau, \infty); Q^0_{\nu_1,\nu_2}). \tag{1.2.9}$$

If the initial data $(v_\tau, \eta_\tau) \in D(L)$, then the solution is regular, namely,

$$v \in C([\tau, \infty); H^1_0(O)) \text{ and } \eta^t \in C([\tau, \infty); Q^1_{\nu_1,\nu_2}).$$

In addition, if $\phi = (v, \eta^t)$, $\bar{\phi} = (\bar{v}, \bar{\eta}^t)$ are two mild solutions of (1.2.7), then for any $T > \tau$,

$$\|\phi(t) - \bar{\phi}(t)\|^2_{\mathcal{M}_0} \leqslant e^{c_0 T}\|\phi(\tau) - \bar{\phi}(\tau)\|^2_{\mathcal{M}_0}, \quad \tau \leqslant t \leqslant T, \tag{1.2.10}$$

where c_0 is a positive constant depending on the initial data.

Proof. The proof is split into three steps.

Step 1: We show that the operator L is the infinitesimal generator of a C^0−semigroup of contraction e^{Lt} in $\mathcal{M}_0$, that is, L is m-dissipative in $\mathcal{M}_0$. By the definition of $L\phi$,

$$\begin{aligned}
(L\phi, \phi)_{\mathcal{M}_0} &= \left(\mu_0 v + \lambda\Delta v - \int_0^\infty \nu_1(s)\eta^t(s)ds + \int_0^\infty \nu_2(s)\Delta\eta^t(s)ds, v\right)_{L^2(O)} + (v - \partial_s\eta^t, \eta^t)_{Q^0_{\nu_1,\nu_2}} \\
&= \mu_0\|v\|^2 - \lambda\|\nabla v\|^2 - \int_0^\infty \nu_1(s)\int_O \partial_s\eta^t \cdot \eta^t dxds - \int_0^\infty \nu_2(s)\int_O \partial_s\nabla\eta^t \cdot \nabla\eta^t dxds \\
&\leqslant \mu_0\|v\|^2 - \lambda\|\nabla v\|^2 - \frac{\delta_1}{2}\|\eta^t\|_{L^2_{\nu_1}(\mathbb{R}^+;L^2(O))} - \frac{\delta_2}{2}\|\eta^t\|_{L^2_{\nu_2}(\mathbb{R}^+;H^1_0(O))} \\
&\leqslant (\mu_0 - \lambda_1\lambda)\|v\|^2 - \frac{\delta_1}{2}\|\eta^t\|_{L^2_{\nu_1}(\mathbb{R}^+;L^2(O))} - \frac{\delta_2}{2}\|\eta^t\|_{L^2_{\nu_2}(\mathbb{R}^+;H^1_0(O))} \leqslant 0, \quad \text{for all } \phi \in D(L),
\end{aligned}$$

which shows that L is dissipative in $\mathcal{M}_0$.

Now we show that L is maximal, i.e., for each $F \in \mathcal{M}_0$, there exists a solution $\phi \in D(L)$ of

$$(I - L)\phi = F.$$

Equivalently, for each $F = (f_1, f_2) \in \mathcal{M}_0$, there exists $\phi = (v, \eta^t) \in D(L)$ such that

$$\begin{aligned} &v - \mu_0 v - \lambda \Delta v + \int_0^\infty \nu_1(s)\eta^t(s)ds - \int_0^\infty \nu_2(s)\Delta\eta^t(s)ds = f_1, \\ &\eta^t - v + \partial_s \eta^t = f_2. \end{aligned} \tag{1.2.11}$$

To solve the above systems, we begin with multiplying (1.2.11)$_2$ by e^s and then integrate over $(0, s)$,

$$\eta^t(s) = v(1 - e^{-s}) + \int_0^s e^{\tau - s} f_2(\tau)d\tau. \tag{1.2.12}$$

Substituting (1.2.12) into (1.2.11)$_1$ and denoting $k_1 = \int_0^\infty \nu_1(s)(1 - e^{-s})ds$, $k_2 = \int_0^\infty \nu_2(s)(1 - e^{-s})ds$, we obtain

$$(1+k_1-\mu_0)v-(\lambda+k_2)\Delta v = -\int_0^\infty \nu_1(s)\int_0^s e^{\tau-s}f_2(\tau)d\tau ds+\int_0^\infty \nu_2(s)\int_0^s e^{\tau-s}\Delta f_2(\tau)d\tau ds+f_1. \tag{1.2.13}$$

In order to solve (1.2.13), we define the bilinear form

$$a(w_1, w_2) = (1 + k_1 - \mu_0)\int_{\mathcal{O}} w_1 w_2 dx + (\lambda + k_2)\int_{\mathcal{O}} \nabla w_1 \nabla w_2 dx, \quad w_1, w_2 \in H_0^1(\mathcal{O}).$$

It is easy to check that $a(w_1, w_2)$ is continuous and coercive in $H_0^1(\mathcal{O})$. Also we have

$$H_0^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \hookrightarrow H^{-1}(\mathcal{O}).$$

We are going to apply the Lax-Milgram theorem. It suffices to prove that the right-hand side of (1.2.13) is an element of $H^{-1}(\mathcal{O})$. Obviously,

$$f_1 \in L^2(\mathcal{O}) \hookrightarrow H^{-1}(\mathcal{O}).$$

Let $f^* = -\int_0^\infty \nu_1(s)\int_0^s e^{\tau-s}f_2(\tau)d\tau ds + \int_0^\infty \nu_2(s)\int_0^s e^{\tau-s}\Delta f_2(\tau)d\tau ds$. We only need to verify that $f^* \in H^{-1}(\mathcal{O})$. We use similar arguments used by Giorgi et al. [77]. For $w \in H_0^1(\mathcal{O})$ with $\|\nabla w\| \leqslant 1$, it is not difficulty to check that

$$\left|(f^*, w)_{H^{-1}, H_0^1}\right| = \left|-\int_0^\infty \nu_1(s)\int_0^s e^{\tau-s}\int_{\mathcal{O}} f_2(\tau) w dx d\tau ds + \int_0^\infty \nu_2(s)\int_0^s e^{\tau-s}\int_{\mathcal{O}} \nabla f_2(\tau)\nabla w dx d\tau ds\right| < \infty,$$

which implies that $f^* \in H^{-1}(\mathcal{O})$. Then, by the Lax-Milgram theorem, equation (1.2.13) has a weak solution

$$\tilde{v} \in H_0^1(\mathcal{O}).$$

In view of (1.2.12), we obtain

$$\tilde{\eta}^t(s) = \tilde{v}(1-e^{-s}) + \int_0^s f_2(\tau)e^{\tau-s}d\tau$$

and need to show that $\tilde{\eta}^t \in Q^0_{\mu_1,\mu_2}$. From (1.2.12) and the fact that $\tilde{v} \in H^1_0(O)$, we find

$$\|\nabla\tilde{\eta}^t\|^2 \leqslant \|\nabla\tilde{v}\|^2 + \int_0^s e^{\tau-s}\|\nabla f_2(\tau)\|^2 d\tau, \quad \|\tilde{\eta}^t\|^2 \leqslant \|\tilde{v}\|^2 + \int_0^s e^{\tau-s}\|f_2(\tau)\|^2 d\tau.$$

Then

$$\begin{aligned}&\int_0^\infty \nu_1(s)\|\tilde{\eta}^t(s)\|^2 ds + \int_0^\infty \nu_2(s)\|\nabla\tilde{\eta}^t(s)\|^2 ds\\ &\quad\leqslant k_1\|\tilde{v}\|^2 + k_2\|\nabla\tilde{v}\|^2 + \int_0^\infty \nu_2(\tau)\|\nabla f_2(\tau)\|^2 d\tau < \infty,\end{aligned}$$

and hence $\tilde{\eta}^t \in Q^0_{\nu_1,\nu_2}$. It follows that

$$\tilde{\phi} = (\tilde{v}, \tilde{\eta}^t) \in \mathcal{M}_0$$

is a weak solution of (1.2.11).

To complete the proof of the maximality of L, we still need to show that $\tilde{\phi} \in D(L)$. Indeed, from $(1.2.11)_2$, we see that

$$\partial_s\tilde{\eta}^t = f_2 + \tilde{v} - \tilde{\eta}^t \in Q^0_{\nu_1,\nu_2}.$$

Since $\tilde{\eta}^t(0) = 0$, we conclude that $\tilde{\eta}^t \in D(T)$. By inspection $(1.2.11)_1$, we find that

$$-\mu_0\tilde{v} - \lambda\Delta\tilde{v} + \int_0^\infty \nu_1(s)\tilde{\eta}^t(s)ds - \int_0^\infty \nu_2(s)\Delta\tilde{\eta}^t(s)ds = -\tilde{v} + f_1 \in L^2(O).$$

Therefore $(\tilde{v}, \tilde{\eta}^t) \in D(L)$.

Step 2: We are going to prove that the operator $F(\phi, \theta_t\omega, t)$ defined in (1.2.8) is locally Lipschitz with respect to ϕ from $\mathcal{M}_0$ into $\mathcal{M}_0$ for $\omega \in \Omega$, and that $F(\phi, \theta_t\omega, t)$ is continuous in (ϕ, t) and measurable in ω w.r.t. $\mathcal{F}$. Let B be a bounded set in $\mathcal{M}_0$ and ϕ, $\bar{\phi} \in B$. Writing $\phi = (v, \eta^t)$, $\bar{\phi} = (\bar{v}, \bar{\eta}^t)$, then

$$\|F(\phi, \theta_t\omega, t) - F(\bar{\phi}, \theta_t\omega, t)\|^2_{\mathcal{M}_0} = \int_O |f(\bar{u}) - f(u)|^2 dx. \tag{1.2.14}$$

Since $f \in C^1(\mathbb{R})$, for any $N > 0$, there exists $L_f(N) > 0$ such that for all $|s_1| \leqslant N, |s_2| \leqslant N$, we have

$$|f(s_1) - f(s_2)| \leqslant L_f(N)|s_1 - s_2|,$$

which along with (1.2.14) yields

$$\|F(\phi, \theta_t\omega, t) - F(\bar{\phi}, \theta_t\omega, t)\|^2_{\mathcal{M}_0} = \int_O |f(\bar{u}) - f(u)|^2 dx \leqslant L_f^2(B)\|v - \bar{v}\|^2 \leqslant L_f^2(B)\|v - \bar{v}\|^2_{\mathcal{M}_0}.$$

From Step 1, Step 2 and the Lumer-Phillips theorem (see for instance [139, Theorem 6.1.4 and 6.1.5]), problem (1.2.7) has a unique local mild solution

$$\phi(t,\tau,\omega,\phi_\tau) = e^{Lt}\phi_\tau(\omega) + \int_\tau^t e^{L(t-r)}F(\phi(r,\tau,\omega,\phi_\tau),\theta_r\omega,r)dr \tag{1.2.15}$$

defined on $[\tau, T]$. Next, in Step 3, we will prove that the local mild solution, in fact, is global solution, i.e., $T = +\infty$.

Step 3: Set $\delta_0 = \min\{\delta_1, \delta_2\}$. Taking the inner product of $(1.2.7)_1$ with v in $L^2(O)$, and $(1.2.7)_2$ with η^t in $Q^0_{\nu_1,\nu_2}$, then adding the two results gives

$$\frac{d}{dt}(\|v\|^2 + \|\eta^t\|^2_{Q^0_{\nu_1,\nu_2}}) + (\lambda\lambda_1 - 2\mu_0)\|v\|^2 + \delta_0\|\eta^t\|^2 + \alpha_1\|u\|^{p+1}_{p+1} \leqslant c + c\|g\|^2 + c\epsilon(\|z\|^2 + \|z\|^{p+1}_{p+1} + \|\nabla z\|^2).$$

Hence with $\delta = \min\{\lambda\lambda_1 - 2\mu_0, \frac{\delta_0}{2}\}$ we have

$$\frac{d}{dt}(\|v\|^2 + \|\eta^t\|^2_{Q^0_{\nu_1,\nu_2}}) + \delta(\|v\|^2 + \|\eta^t\|^2_{Q^0_{\nu_1,\nu_2}}) \leqslant c + c\|g\|^2 + c\epsilon(\|z\|^2 + \|z\|^{p+1}_{p+1} + \|\nabla z\|^2). \tag{1.2.16}$$

By the Gronwall Lemma, we obtain, for any $t \in [\tau, T]$,

$$\begin{aligned}\|v\|^2 + \|\eta^t\|^2_{Q^0_{\nu_1,\nu_2}} &\leqslant e^{-\delta(t-\tau)}(\|v_\tau\|^2 + \|\eta_\tau\|^2_{Q^0_{\nu_1,\nu_2}}) + c\int_\tau^t e^{\delta(s-t)}ds + c\int_\tau^t e^{\delta(s-t)}\|g(s)\|^2 ds \\ &\quad + c\epsilon\int_\tau^t e^{\delta(s-t)}(\|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|^{p+1}_{p+1} + \|\nabla z(\theta_s\omega)\|^2)ds < \infty,\end{aligned}$$

where we use the fact that $z(\theta_t\omega)$ is continuous in t, for any fixed $T > \tau$ and $t \in [\tau, T]$. Then,

$$\|\phi(t,\tau,\omega,\phi_\tau(\omega))\|^2_{\mathcal{M}_0} = \|v\|^2 + \|\eta^t\|^2_{Q^0_{\nu_1,\nu_2}} < \infty,$$

which means that the local mild solution we obtained above cannot blow up in finite time, i.e., $T = \infty$. Hence, problem (1.2.7) has a unique global mild solution $\phi \in C([\tau,\infty); \mathcal{M}_0)$ for all $t \geqslant \tau$, so (1.2.9) holds. Moreover, the continuity with respect to initial data, namely, (1.2.10), follows from the representation formula and the locally Lipschitz property of F. □

1.3 Existence of pullback random attractor in $\mathcal{M}_0$

We now establish the existence of a pullback attractor in phase space $\mathcal{M}_0$. From Theorem 1.2.2, we know that $\phi = (v, \eta^t)$ is a global solution to problem (1.2.7), define $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$:

$$\Phi : \mathbb{R}^+ \times \tilde{\Omega} \times \mathcal{M}_0 \to \mathcal{M}_0, \ \ (t, \tilde{\omega}, \phi_\tau) \to \Phi(t, \tilde{\omega}, \phi_\tau),$$

for the stochastic problem (1.2.7). Given $t \in \mathbb{R}^+$, $(\tau, \omega) \in \tilde{\Omega}$ and $\phi_\tau \in \mathcal{M}_0$, set

$$\Phi(t,(\tau,\omega),\phi_\tau) = \phi(t+\tau,\tau,\theta_{-\tau}\omega,\phi_\tau(\theta_{-\tau}\omega)) = \left(v(t+\tau,\tau,\theta_{-\tau}\omega, v_\tau(\theta_{-\tau}\omega)), \eta^t(t+\tau,\tau,\theta_{-\tau}\omega,\eta_\tau(\theta_{-\tau}\omega))(s)\right), \tag{1.3.1}$$

where $\eta^t(t+\tau,\tau,\theta_{-\tau}\omega,\eta_\tau(\theta_{-\tau}\omega))(s) = \int_0^s u(t+\tau-r,\tau,\theta_{r-\tau}\omega,u_\tau(\theta_{r-\tau}\omega))dr$.

Hence, $\psi = (u,\eta^t)$ is a global solution to problem (1.0.1). Then the solution $\psi = (u,\eta^t) \in C([\tau,\infty);\mathcal{M}_0)$ defines a continuous random dynamical system over $\mathbb{R}$ and $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$:

$$\Psi : \mathbb{R}^+ \times \tilde{\Omega} \times \mathcal{M}_0 \to \mathcal{M}_0, \quad (t,\tilde{\omega},\psi_\tau) \to \Psi(t,\tilde{\omega},\psi_\tau).$$

Given $t \in \mathbb{R}^+$, $(\tau,\omega) \in \tilde{\Omega}$ and $\psi_\tau \in \mathcal{M}_0$, set

$$\Psi(t,(\tau,\omega),\psi_\tau) = \psi(t+\tau,\tau,\theta_{-\tau}\omega,\psi_\tau(\theta_{-\tau}\omega)) = \phi(t+\tau,\tau,\theta_{-\tau}\omega,\phi_\tau(\theta_{-\tau}\omega)) + (\epsilon z(\theta_t\omega),0). \tag{1.3.2}$$

Obviously, Φ and Ψ defined by (1.3.1) and (1.3.2), respectively, satisfy all conditions (i)-(iii) in Definition 0.0.1. On the other hand, we can see that

$$\Psi(t,(\tau,\omega),\psi_\tau) = T(\theta_t\omega)\Phi(t,(\tau,\omega),\phi_\tau),$$

where $T(\omega)(a,b)^\top = (a+\epsilon z(\omega),0)^\top$ is an homeomorphism of $\mathcal{M}_0$. Hence, Φ and Ψ are equivalent. In what follows, we establish uniform estimates for the solutions to problem (1.2.7) and prove the existence and upper semi-continuity of a pullback random attractor for RDS Φ based on Proposition 0.0.7 and Proposition 1.1.1. To this end, we specify a collection $\mathcal{D}_\delta$ of families of subsets of $\mathcal{M}_0$.

Suppose $D = \{D(\tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\}$ is a family of bounded nonempty subsets of $\mathcal{M}_0$ satisfying, for every $\tilde{\omega} \in \tilde{\Omega}$,

$$\lim_{s\to-\infty} e^{\delta s}\|D(\tilde{\theta}_s\tilde{\omega})\|^2_{\mathcal{M}_0} = 0, \tag{1.3.3}$$

where the positive number $\delta = \min\{\lambda\lambda_1 - 2\mu_0\}$. Denote by $\mathcal{D}_\delta$ the collection of all tempered families of tempered nonempty subsets of $\mathcal{M}_0$ which fulfil condition (1.3.3), i.e.,

$$\mathcal{D}_\delta = \Big\{D = \{D(\tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\} : \ D \text{ satisfies } (1.3.3)\Big\}. \tag{1.3.4}$$

1.3.1 Existence of pullback absorbing set in $\mathcal{M}_0$

This subsection is devoted to obtaining a pullback absorbing set for the cocycle Φ in $\mathcal{M}_0$. Henceforth, we assume that $g \in C_b(\mathbb{R},L^2(O))$, where $C_b(\mathbb{R},L^2(O))$ denotes the set of continuous bounded functions from $\mathbb{R}$ into $L^2(O)$. We begin with the following lemma.

Lemma 1.3.1. *Assume that (f1)-(f3) and $(H1)-(H2)$ hold. Let $B = \{B(\tau,\omega) : (\tau,\omega) \in \tilde{\Omega}\} \in \mathcal{D}_\delta$. Then*

$$\|v(\tau,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})\|^2 + \|\eta^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta_{\tau-t})\|^2_{Q^0_{\nu_1,\nu_2}} \leqslant R(\omega) \tag{1.3.5}$$

for any $\phi_{\tau-t} = (v_{\tau-t},\eta_{\tau-t}) \in B(\tilde{\theta}_{-t}(\tau,\omega))$, where $R(\omega) = \gamma_1 + \gamma_1(\epsilon^2+\epsilon^{p+1})r(\omega)$.

Proof. By a similar procedure as to Step 3 in Section 3, we have

$$\begin{aligned}
&\frac{d}{dt}(\|v\|^2 + \|\eta^t\|^2_{Q^0_{\nu_1,\nu_2}}) + \delta(\|v\|^2 + \|\eta^t\|^2_{Q^0_{\nu_1,\nu_2}}) + \alpha_1\|u\|^{p+1}_{p+1} \\
&\leqslant \epsilon^2\left(1 + \frac{4\lambda_1(\mu_0+1)^2}{\lambda} + \frac{2k_0}{\delta_0}\right)\|z\|^2 + \left(\frac{\epsilon\alpha_4}{2(p+1)}\right)^{p+1}(\frac{\alpha_1}{p})^{-p}\|z\|^{p+1}_{p+1} \\
&\quad + \epsilon^2(2\lambda + \frac{2k_0}{\delta_0})\|\nabla z\|^2 + \frac{4\lambda_1}{\lambda}\|g\|^2 + 2\alpha_3|O|.
\end{aligned} \tag{1.3.6}$$

Multiplying (1.3.6) by $e^{\delta t}$ and then integrating over $[\tau - t, \tau]$ with $t \geqslant 0$, we obtain for every $\omega \in \Omega$,

$$\begin{aligned}
&\|v(\tau, \tau - t, \omega, v_{\tau-t})\|^2 + \|\eta^t(\tau, \tau - t, \omega, \eta_{\tau-t})\|^2_{Q^0_{\nu_1,\nu_2}} \\
&\leqslant e^{-\delta t}(\|v_{\tau-t}\|^2 + \|\eta_{\tau-t}\|^2_{Q^0_{\nu_1,\nu_2}}) + \gamma_0 \int_{\tau-t}^{\tau} e^{\delta(s-\tau)}(1 + \|g\|^2)ds \\
&\quad + \gamma_0\epsilon^2 \int_{\tau-t}^{\tau} e^{\delta(s-\tau)} \left(\|z(\theta_s\omega)\|^2 + \|\nabla z(\theta_s\omega)\|^2\right) ds + \gamma_0\epsilon^{p+1} \int_{\tau-t}^{\tau} e^{\delta(s-\tau)}\|z(\theta_s\omega)\|^{p+1}_{p+1}ds,
\end{aligned} \tag{1.3.7}$$

where $\gamma_0 = \max\left\{\frac{4\lambda_1}{\lambda}, 2\alpha_3|O|, 1 + \frac{4\lambda_1(\mu_0+1)^2}{\lambda} + \frac{2k_0}{\delta_0}, \left(\frac{\alpha_4}{2(p+1)}\right)^{p+1} (\frac{\alpha_1}{p})^{-p}, 2\lambda + \frac{2k_0}{\delta_0}\right\}$.

Recall that $z(\theta_t\omega) = h(x)z^*(\theta_t\omega)$. Then we have

$$\|z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|^{p+1}_{p+1} + \|\nabla z(\theta_t\omega)\|^2 \leqslant r(\theta_t\omega)$$

where $r(\theta_t\omega)$ satisfies

$$r(\theta_t\omega) \leqslant e^{\frac{\delta}{2}|t|}r(\omega), \quad t \in \mathbb{R}.$$

Replacing ω by $\theta_{-\tau}\omega$ in (1.3.7), we obtain

$$\begin{aligned}
&\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \|\eta^t(\tau, \tau - t, \theta_{-\tau}\omega, \eta_{\tau-t})\|^2_{Q^0_{\nu_1,\nu_2}} \\
&\leqslant e^{-\delta t}(\|v_{\tau-t}\|^2 + \|\eta_{\tau-t}\|^2_{Q^0_{\nu_1,\nu_2}}) + \gamma_0 \int_{\tau-t}^{\tau} e^{\delta(s-\tau)}\|g\|^2 ds + \gamma_0 \int_{\tau-t}^{\tau} e^{\delta(s-\tau)}ds \\
&\quad + \gamma_0\epsilon^2 \int_{\tau-t}^{\tau} e^{\delta(s-\tau)} \left(\|z(\theta_{s-\tau}\omega)\|^2 + \|\nabla z(\theta_{s-\tau}\omega)\|^2\right) ds + \gamma_0\epsilon^{p+1} \int_{\tau-t}^{\tau} e^{\delta(s-\tau)}\|z(\theta_{s-\tau}\omega)\|^{p+1}_{p+1}ds \\
&\leqslant e^{-\delta t}(\|v_{\tau-t}\|^2 + \|\eta_{\tau-t}\|^2_{Q^0_{\nu_1,\nu_2}}) + \gamma_0 \int_{-t}^{0} e^{\delta s}(1 + \|g\|^2)ds + \gamma_0(\epsilon^2 + \epsilon^{p+1}) \int_{-t}^{0} e^{\frac{\delta}{2}s}r(\omega)ds.
\end{aligned} \tag{1.3.8}$$

Since $(v_{\tau-t}, \eta_{\tau-t}) \in B(\tilde{\theta}_{-t}(\tau, \omega))$, there exists $T(\tau, \omega, B) > 0$ such that for all $t > T(\tau, \omega, B)$,

$$e^{-\delta t}(\|v_{\tau-t}\|^2 + \|\eta_{\tau-t}\|^2_{Q^0_{\nu_1,\nu_2}}) \leqslant \frac{\gamma_0(1 + \|g\|^2)}{\delta}.$$

Therefore, for all $t > T(\tau, \omega, B)$,

$$\begin{aligned}
&\|v(\tau, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \|\eta^t(\tau, \tau - t, \theta_{-\tau}\omega, \eta_{\tau-t})\|^2_{Q^0_{\nu_1,\nu_2}} \\
&\leqslant \gamma_1 + \gamma_1(\epsilon^2 + \epsilon^{p+1})r(\omega) := R(\omega),
\end{aligned}$$

where $\gamma_1 = \max\{\frac{2\gamma_0(1+\|g\|^2)}{\delta}, \gamma_0\}$ and $\|g\|^2 = \sup_{r\in\mathbb{R}} \|g(\cdot, r)\|^2 < \infty$. The proof is finished. □

Remark 1.3.2. *Denote $v(r) = v(r, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})$ and $\eta^t(r) = \eta^t(r, \tau - t, \theta_{-\tau}\omega, \eta_{\tau-t})(s)$, we can prove that there exist a positive constant ρ_0 and a tempered variable $r(\omega)$ such that*

$$\|v(r, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \|\eta(r, \tau - t, \theta_{-\tau}\omega, \eta_{\tau-t})\|^2_{Q^0_{\nu_1,\nu_2}} \leqslant \rho_0 + \rho_0(\epsilon^2 + \epsilon^{p+1})e^{-\frac{\delta}{2}(r-\tau)}r(\omega).$$

Define

$$D(\tau,\omega) := D_\omega = \left\{\phi \in \mathcal{M}_0 : \|\phi(\tau,\tau-t,\theta_{-\tau}\omega,\phi_{\tau-t}(\theta_{-\tau}\omega))\|^2_{\mathcal{M}_0} \leqslant R(\omega)\right\}. \tag{1.3.9}$$

Let D be the family consisting of these sets given by (1.3.9), i.e.,

$$D = \left\{D(\tau,\omega) :\ D(\tau,\omega) \text{ is defined by (1.3.9)},\ (\tau,\omega) \in \tilde{\Omega}\right\}. \tag{1.3.10}$$

It is clear that D given by (1.3.10) belongs to $\mathcal{D}_\delta$.

Next, we prove that the random dynamical system Φ associated to problem (1.2.7) has a compact measurable pullback attracting set.

1.3.2 Decomposition of solutions

In this subsection, we decompose the solution of (1.2.7) into a sum of two parts, of which, one part decays exponentially and the other one is bounded in a "higher regular" space by using the method in [38, 81], and obtain some a priori estimates for the solutions, which are the basis for constructing a compact measurable attracting set for RDS Φ.

For any $(\tau,\omega) \in \tilde{\Omega}$, set

$$D_1(\tau,\omega) = \bigcup_{t \geqslant T(\tau,\omega,D)} \phi(\tau,\tau-t,\theta_{-\tau}\omega, D(\tau-t,\theta_{-t}\omega)) \subset D(\tau,\omega), \tag{1.3.11}$$

then by (1.3.9),

$$\Phi(t,\tau-t,\theta_{-t}\omega, D_1(\tau-t,\theta_{-t}\omega)) = \phi(\tau,\tau-t,\theta_{-\tau}\omega, D_1(\tau-t,\theta_{-t}\omega)) \subset D_1(\tau,\omega) \subset D(\tau,\omega), \quad t \geqslant 0. \tag{1.3.12}$$

For any $(\tau,\omega) \in \tilde{\Omega}$ and $t \geqslant 0$, let $\phi(r) = \phi(r,\tau-t,\theta_{-\tau}\omega,\phi_{\tau-t}(\theta_{-\tau}\omega))$ $(r \geqslant \tau-t)$ be a mild solution of system (1.2.7) with the initial value $\phi_{\tau-t}(\theta_{-\tau}\omega) = (v_{\tau-t},\eta_{\tau-t}) \in D_1(\tau-t,\theta_{-t}\omega) \subset D(\tau-t,\theta_{-t}\omega)$, then it follows from (1.3.12) that $\phi(r) \in D(r-\tau,\theta_{r-\tau}\omega)$ for all $r \geqslant \tau-t$. We decompose $\phi(r)$ into $\phi(r) = \phi_L(r) + \phi_N(r)$, where $\phi_L(r) = (v_L(r),\eta^t_L(r))$ and $\phi_N(r) = (v_N(r),\eta^t_N(r))$ satisfying, respectively,

$$\begin{aligned} &\partial_t v_L - \mu_0 v_L - \lambda\Delta v_L + \int_0^\infty \nu_1(s)\eta^t_L(s)ds + \int_0^\infty \nu_2(s)\Delta\eta^t_L(s)ds + f(v_L) + Kv_L = 0,\\ &\partial_t\eta^t_L + \partial_s\eta^t_L = v_L, \quad x \in \mathcal{O},\ s > 0,\ r \geqslant \tau-t, \end{aligned} \tag{1.3.13}$$

with the initial and boundary values

$$v_L(x,t) = 0,\ \eta^t_L(x,s) = 0,\ x \in \mathcal{O},\ v_L(x,\tau) = v_\tau(x),\ \eta^\tau_L(x,s) = \eta_\tau(x,s),\ x \in \partial\mathcal{O}, s > 0,\ r < \tau-t. \tag{1.3.14}$$

and

$$\begin{aligned} &\partial_t v_N - \mu_0 v_N - \lambda\Delta v_N + \int_0^\infty \nu_1(s)\eta^t_N(s)ds + \int_0^\infty \nu_2(s)\Delta\eta^t_N(s)ds + f(u) - f(v_L)\\ &\qquad = Kv_L + g + \epsilon(\mu_0+1)z + \epsilon\lambda\Delta z,\\ &\partial_t\eta^t_N + \partial_s\eta^t_N = v_N + \epsilon z, \quad x \in \mathcal{O},\ s > 0,\ r \geqslant \tau-t, \end{aligned} \tag{1.3.15}$$

with then initial and boundary values

$$v_L(x,t)=0,\ \eta_L^t(x,s)=0,\ x\in\mathcal{O},\ v_L(x,\tau)=0,\ \eta_L^\tau(x,s)=0,\ x\in\partial\mathcal{O}, s>0,\ r<\tau-t. \qquad (1.3.16)$$

Obviously, system (1.3.13) is a deterministic (non-random) non-autonomous system independent of ω. Notice that assumption $(f2)$ implies that there exists $K_0>0$ such that $f(u)u\geqslant -K_0|u|^2$. Set $K>K_0$. In order to estimate the component of ϕ_L, we start with the estimate of v_L.

Lemma 1.3.3. *Suppose that assumptions of Lemma 1.3.1 hold. Then the solution of* (1.3.13) *satisfies*

$$\|v_L(\tau,\tau-t,v_{L,\tau-t})\|^2+\|\eta_L^t(\tau,\tau-t,\eta_{L,\tau-t})\|^2_{Q^0_{\nu_1,\nu_2}}\leqslant e^{-\delta t}R_0(\omega).$$

Proof. Multiplying $(1.3.13)_1$ by v_L and integrating over $\mathcal{O}$ in $L^2(\mathcal{O})$, multiplying $(1.3.13)_2$ by η_L^t and integrating over $\mathcal{O}$ in $Q^0_{\nu_1,\nu_2}$, then adding the results, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|v_L\|^2+\|\eta_L^t\|^2_{Q^0_{\nu_1,\nu_2}})-\mu_0\|v_L\|^2+\lambda\|\nabla v_L\|^2+\int_0^\infty \nu_1(s)\int_{\mathcal{O}}\partial_s\eta_L^t\cdot\eta_L^t dxds\\&+\int_0^\infty\nu_2(s)\int_{\mathcal{O}}\partial_s\nabla\eta_L^t\cdot\nabla\eta_L^t dxds+\int_{\mathcal{O}}f(v_L)v_Ldx=0.\end{aligned}\qquad (1.3.17)$$

Some computations then yield

$$\frac{d}{dt}(\|v_L\|^2+\|\eta_L^t\|^2_{Q^0_{\nu_1,\nu_2}})+\delta(\|v_L\|^2+\|\eta_L^t\|^2_{Q^0_{\nu_1,\nu_2}})+(K-K_0)\|v_L\|^2\leqslant 0. \qquad (1.3.18)$$

By the Gronwall Lemma, we conclude that there exits a tempered variable $R_0(\omega)>0$ such that

$$\|v_L(\tau,\tau-t,v_{L,\tau-t})\|^2+\|\eta_L^t(\tau,\tau-t,\eta_{L,\tau-t})\|^2_{Q^0_{\nu_1,\nu_2}}\leqslant e^{-\delta t}(\|v_{\tau-t}\|^2+\|\eta_{\tau-t}\|^2_{Q^0_{\nu_1,\nu_2}})\leqslant e^{-\delta t}R_0(\omega). \qquad (1.3.19)$$

This finishes the proof. □

Hereafter, denote $R_i(\xi,\tau,\omega)=\rho_i+\rho_i(\epsilon^2+\epsilon^p)^{l_i}e^{-\beta_i(\xi-\tau)}r(\omega)^{n_i}$, $R_i(\omega):=R(\tau,\tau,\omega)=\rho_i+\rho_i(\epsilon^2+\epsilon^p)^{l_i}r(\omega)^{n_i}$ for $\rho_i,l_i,\beta_i,n_i>0$, $i=1,2,3,\cdots$, and $\xi\geqslant\tau-t$.

Lemma 1.3.4. *Assume that $(f1)-(f3)$ hold with $1<p<1+4/n$, or $(f1)-(f2)$ and $(f4)$ hold with $p=1+4/n$, then the solution of* (1.3.15) *satisfies the inequality*

$$\|v_N(\tau,\tau-t,\theta_{-\tau}\omega,0)\|^2_\sigma+\|\eta_N^t(\tau,\tau-t,\theta_{-\tau}\omega,0)\|^2_{Q^\sigma_{\nu_1,\nu_2}}\leqslant R_4(\omega),$$

where $0<\sigma<\min\{1,\frac{2p-np+2}{2}\}$.

Proof. Taking the inner product of $(1.3.15)_1$ with $A^\sigma v_N$ in $L^2(\mathcal{O})$, $(1.3.15)_2$ with $A^\sigma\eta_N^t$ in $Q^0_{\nu_1,\nu_2}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|A^{\frac{\sigma}{2}}v_N\|^2+\|\eta_N^t\|^2_{Q^\sigma_{\nu_1,\nu_2}})-\mu_0\|A^{\frac{\sigma}{2}}v_N\|^2+\lambda\|A^{\frac{1+\sigma}{2}}v_N\|^2+\frac{\delta_0}{2}\|\eta^t\|^2_{Q^\sigma_{\nu_1,\nu_2}}+\int_{\mathcal{O}}(f(u)-f(v_L))A^\sigma v_Ndx\\&\leqslant\int_{\mathcal{O}}g\cdot A^\sigma v_Ndx+\int_{\mathcal{O}}Kv_L\cdot A^\sigma v_Ndx+\epsilon(\mu_0+1)\int_{\mathcal{O}}z\cdot A^\sigma v_Ndx+\epsilon\lambda\int_{\mathcal{O}}\Delta z\cdot A^\sigma v_Ndx\\&\quad+\int_0^\infty\nu_1(s)\int_{\mathcal{O}}A^{\frac{\sigma}{2}}(v_N+\epsilon z)\cdot A^{\frac{\sigma}{2}}\eta_N^tdxds+\int_0^\infty\nu_2(s)\int_{\mathcal{O}}A^{\frac{1+\sigma}{2}}(v_N+\epsilon z)\cdot A^{\frac{1+\sigma}{2}}\eta_N^tdxds.\end{aligned}\qquad (1.3.20)$$

By the Young's inequality, we have

$$\begin{aligned}&\int_O g\cdot A^\sigma v_N dx+\int_O Kv_L\cdot A^\sigma v_N dx+\epsilon(\mu_0+1)\int_O z\cdot A^\sigma v_N dx+\epsilon\lambda\int_O \Delta z\cdot A^\sigma v_N dx\\&\leqslant \frac{\lambda\lambda_1}{4}\|A^\sigma v_N\|^2+c(\|g\|^2+\|v_L\|^2)+c\epsilon^2(\|z\|^2+\|\Delta z\|^2).\end{aligned}\tag{1.3.21}$$

Note that if $p<1+4/n$, then $\frac{p-1}{4}n-\frac{1-\sigma}{2}<\frac{1+\sigma}{2}$, and by Lemma 1.3.1 we know that

$$\begin{aligned}&\left|\int_O (f(u)-f(v_L))A^\sigma v_N dx\right|\\&\leqslant c\int_O (1+|u|^{p-1}+|v_L|^{p-1})|v_N+\epsilon z||A^\sigma v_N|dx\\&\leqslant c\int_O (1+|u|^{p-1}+|v_L|^{p-1})|v_N||A^\sigma v_N|dx+c\epsilon\int_O(1+|u|^{p-1}+|v_L|^{p-1})|z||A^\sigma v_N|dx\\&\leqslant c\left(1+(\int_O|u|^2dx)^{(p-1)/2}+(\int_O|v_L|^2dx)^{(p-1)/2}\right)\left(\int_O|v_N|^{2n/(2n-np+2(1-\sigma))}dx\right)^{(2n-np+2(1-\sigma))/2n}\\&\quad\times\left(\int_O|A^\sigma v_N|^{2n/(n-2(1-\sigma))}dx\right)^{(n-2(1-\sigma))/2n}+c\epsilon\left(1+(\int_O|u|^2dx)^{(p-1)/2}+(\int_O|v_L|^2dx)^{(p-1)/2}\right)\\&\quad\times\left(\int_O|z|^{2n/(2n-np+2(1-\sigma))}dx\right)^{(2n-np+2(1-\sigma))/2n}\left(\int_O|A^\sigma v_N|^{2n/(n-2(1-\sigma))}dx\right)^{(n-2(1-\sigma))/2n}\\&\leqslant c\|A^{(1+\sigma)/2}v_N\|\cdot\|v_N\|_{L^{2n/[n-2(\frac{p-1}{2})n-(1-\sigma)]}}(1+\|u\|^{p-1}+\|v_L\|^{p-1})\\&\quad+c\epsilon\|A^{(1+\sigma)/2}v_N\|\cdot\|z\|_{L^{2n/[n-2(\frac{p-1}{2})n-(1-\sigma)]}}(1+\|u\|^{p-1}+\|v_L\|^{p-1})\\&\leqslant c\|A^{(1+\sigma)/2}v_N\|\cdot\|A^{\frac{p-1}{4}n-\frac{1-\sigma}{2}}v_N\|(1+\|u\|^{p-1}+\|v_L\|^{p-1})\\&\quad+c\epsilon\|A^{(1+\sigma)/2}v_N\|\cdot\|A^{\frac{p-1}{4}n-\frac{1-\sigma}{2}}z\|(1+\|u\|^{p-1}+\|v_L\|^{p-1})\\&\leqslant c\|A^{(1+\sigma)/2}v_N\|\cdot\|v_N\|^{1-\vartheta}\|A^{(1+\sigma)/2}v_N\|^{\vartheta}(1+\|u\|^{p-1}+\|v_L\|^{p-1})\\&\quad+c\epsilon\|A^{(1+\sigma)/2}v_N\|\cdot\|A^{(1+\sigma)/2}z\|(1+\|u\|^{p-1}+\|v_L\|^{p-1})\\&\leqslant\frac{\lambda}{4}\|A^{(1+\sigma)/2}v_N\|^2+c(1+\|u\|^{p-1}+\|v_L\|^{p-1})^{\frac{2}{1-\vartheta}}\|v_N\|^2+c\epsilon^2(1+\|u\|^{p-1}+\|v_L\|^{p-1})^2\|A^{\frac{1+\sigma}{2}}z\|^2.\end{aligned}\tag{1.3.22}$$

On the other hand, if $p = 1 + 4/n$, then

$$
\begin{aligned}
&\left|\int_{\mathcal{O}}(f(u) - f(v_L))A^\sigma v_N dx\right| \\
&\leqslant \int_{\mathcal{O}}(C_\nu + \nu|u|^{\frac{4}{n}} + \nu|v_L|^{\frac{4}{n}})|v_N + \epsilon z| \cdot |A^\sigma v_N| dx \\
&\leqslant \int_{\mathcal{O}}(C_\nu + \nu|u|^{\frac{4}{n}} + \nu|v_L|^{\frac{4}{n}})|v_N| \cdot |A^\sigma v_N| dx + \epsilon\int_{\mathcal{O}}(C_\nu + \nu|u|^{\frac{4}{n}} + \nu|v_L|^{\frac{4}{n}})|z| \cdot |A^\sigma v_N| dx \\
&\leqslant C_\nu(\int_{\mathcal{O}}|v_N|^{2n/(n+2(1-\sigma))}dx)^{(n+2(1-\sigma))/2n}(\int_{\mathcal{O}}|A^\sigma v_N|^{2n/(n-2(1-\sigma))}dx)^{(n-2(1-\sigma))/2n} + \nu\left((\int_{\mathcal{O}}|u|^2dx)^{2/n} + (\int_{\mathcal{O}}|v_L|^2dx)^{2/n}\right) \\
&\quad \times(\int_{\mathcal{O}}|v_N|^{2n/[n-2(1+\sigma)]}dx)^{[n-2(1+\sigma)]/2n}(\int_{\mathcal{O}}|A^\sigma v_N|^{2n/[n-2(1-\sigma)]}dx)^{[n-2(1-\sigma)]/2n} \\
&\quad + C_\nu\epsilon(\int_{\mathcal{O}}|z|^{2n/(n+2(1-\sigma))}dx)^{(n+2(1-\sigma))/2n}(\int_{\mathcal{O}}|A^\sigma v_N|^{2n/(n-2(1-\sigma))}dx)^{(n-2(1-\sigma))/2n} + \nu\epsilon\left((\int_{\mathcal{O}}|u|^2dx)^{2/n} + (\int_{\mathcal{O}}|v_L|^2dx)^{2/n}\right) \\
&\quad \times(\int_{\mathcal{O}}|z|^{2n/[n-2(1+\sigma)]}dx)^{[n-2(1+\sigma)]/2n}(\int_{\mathcal{O}}|A^\sigma v_N|^{2n/[n-2(1-\sigma)]}dx)^{[n-2(1-\sigma)]/2n} \\
&\leqslant C_\nu\|v_N\|_{L^{2n/(n+2(1-\sigma))}}\|A^\sigma v_N\|_{L^{2n/(n-2(1-\sigma))}} + \nu C(\|u\|^{4/n} + \|v_L\|^{4/n})\|v_N\|_{L^{2n/n-2(1+\sigma)}}\|A^\sigma v_N\|_{L^{n-2(1-\sigma)}} \\
&\quad + C_\nu\epsilon\|z\|_{L^{2n/(n+2(1-\sigma))}}\|A^\sigma v_N\|_{L^{2n/(n-2(1-\sigma))}} + \nu\epsilon C(\|u\|^{4/n} + \|v_L\|^{4/n})\|z\|_{L^{2n/n-2(1+\sigma)}}\|A^\sigma v_N\|_{L^{n-2(1-\sigma)}} \\
&\leqslant \frac{\lambda}{4}\|A^{\frac{1+\sigma}{2}}v_N\|^2 + cC_\nu\|v_N\|^2 + \nu C_\nu(\|u\|^{\frac{4}{n}} + \|v_L\|^{\frac{4}{n}})\|A^{\frac{1+\sigma}{2}}v_N\|^2 \\
&\quad + cC_\nu^2\epsilon^2\|z\|^2 + c\nu^2\epsilon^2(\|u\|^{\frac{4}{n}} + \|v_L\|^{\frac{4}{n}})^2\|A^{\frac{1+\sigma}{2}}z\|^2.
\end{aligned} \tag{1.3.23}
$$

If $p = 1 + \frac{4}{n}$, then by Lemmas 1.3.1 and 1.3.3, we can choose ν small enough such that, for every $(\tau, \omega) \in \tilde{\Omega}$,

$$
\nu C_\nu(\|u\|^{\frac{4}{n}} + \|v_L\|^{\frac{4}{n}})\|A^{\frac{1+\sigma}{2}}v_N\|^2 \leqslant \frac{\lambda}{4}\|A^{\frac{1+\sigma}{2}}v_N\|^2,\ c\nu^2\epsilon^2(\|u\|^{\frac{4}{n}} + \|v_L\|^{\frac{4}{n}})^2\|A^{\frac{1+\sigma}{2}}z\|^2 \leqslant c\epsilon^2\|A^{\frac{1+\sigma}{2}}z\|^2. \tag{1.3.24}
$$

From (1.3.20)-(1.3.24), we have

$$
\begin{aligned}
&\frac{d}{dt}(\|A^{\frac{\sigma}{2}}v_N\|^2 + \|\eta_N^t\|^2_{Q^\sigma_{\nu_1,\nu_2}}) + \delta(\|A^{\frac{\sigma}{2}}v_N\|^2 + \|\eta_N^t\|^2_{Q^\sigma_{\nu_1,\nu_2}}) \\
&\leqslant c(1 + \|u\|^{p-1} + \|v_L\|^{p-1})^{\frac{2}{1-\theta}}\|v_N\|^2 + c\epsilon^2(1 + \|u\|^{p-1} + \|v_L\|^{p-1})^2\|A^{\frac{1+\sigma}{2}}z\|^2 \\
&\quad + c\epsilon^2(\|z\|^2 + \|\Delta z\|^2) + c(\|g\|^2 + e^{-\delta(r-\tau+t)}R_0(\omega)) \\
&\leqslant R_1(r, \tau, \omega) + c\epsilon^2(1 + R_2(r, \tau, \omega))(\|z(\theta_{r-\tau}\omega)\|^2 + \|A^{\frac{1+\sigma}{2}}z(\theta_{r-\tau}\omega)\|^2 + \|\Delta z(\theta_{r-\tau}\omega)\|^2) \\
&\quad + c(1 + e^{-\delta(r-\tau+t)}R_0(\omega)).
\end{aligned} \tag{1.3.25}
$$

Applying the Gronwall lemma to (1.3.25)c, it follows that for t large enough,

$$
\|A^{\frac{\sigma}{2}}v_N(\tau, \tau - t, \theta_{-\tau}\omega, 0)\|^2 + \|\eta_N^t(\tau, \tau - t, \theta_{-\tau}\omega, 0)\|^2_{Q^\sigma_{\nu_1,\nu_2}} \leqslant R_3(\omega).
$$

This completes the proof. □

Lemma 1.3.5. *Let the assumption of Lemma 1.3.4 hold. Then for any* $B = \{B(\tau,\omega) : (\tau,\omega) \in \tilde{\Omega}\}(\in \mathcal{D}_\delta) \subset \mathcal{M}_\sigma$ *and for any* $(v_{\tau-t}, \eta_{\tau-t}) \in B(\tilde{\theta}_{-t}(\tau,\omega))$,

$$\|v(\tau, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|_\sigma^2 + \|\eta^t(\tau, \tau-t, \theta_{-\tau}\omega, \eta_{\tau-t})\|^2_{Q^\sigma_{\nu_1,\nu_2}} \leqslant R_4(\omega).$$

Proof. Taking the inner product of $(1.2.5)_1$ with $A^\sigma v$ in $L^2(\mathcal{O})$ and $(1.2.5)_2$ with $A^\sigma \eta^t$ in $Q^0_{\mu_1,\mu_2}$. Then we can finish the proof similarly to the proof of Lemma 1.3.1. □

On the basis of the above lemmas, we have the following results.

Lemma 1.3.6. *For* $0 < \sigma < \frac{1}{2}$ *and* $\sigma \leqslant s \leqslant 1$, *we have*

$$\|A^{\frac{s}{2}} v_N(\tau, \tau-t, \theta_{-\tau}\omega, 0)\|^2 + \|\eta^t_N(\tau, \tau-t, \theta_{-\tau}\omega, 0)\|^2_{Q^s_{\nu_1,\nu_2}} \leqslant R_5(\omega).$$

Proof. Multiplying $(1.3.15)_1$ by $A^s v_N$, and $(1.3.15)_2$ by $A^s \eta^t_N$, then sum the results to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|A^{\frac{s}{2}} v_N\|^2 + \|\eta^t_N\|^2_{Q^s_{\nu_1,\nu_2}}) + \frac{\delta_0}{2}\|\eta^t_N\|^2_{Q^s_{\mu_1,\mu_2}} - \mu_0\|A^{\frac{s}{2}} v_N\|^2 + \lambda\|A^{\frac{1+s}{2}} v_N\|^2 + \int_{\mathcal{O}}(f(u) - f(v_L))A^s v_N dx \\
&\leqslant \int_{\mathcal{O}} g \cdot A^s v_N dx + \int_{\mathcal{O}} K v_L \cdot A^s v_N dx + \epsilon \int_{\mathcal{O}}((\mu_0+1)z + \lambda\Delta z)\cdot A^s v_N dx + \epsilon\int_0^\infty \nu_1(s)\int_{\mathcal{O}} z\cdot A^s\eta^t_N dxds \\
&\quad + \epsilon\int_0^\infty \nu_2(s)\int_{\mathcal{O}} A^{\frac{1+s}{2}} z \cdot A^{\frac{1+s}{2}}\eta^t_N dxds.
\end{aligned} \tag{1.3.26}$$

If $n \geqslant 4$, by straightforward computations we have

$$2n > (n-2\sigma)p + 2(s+\sigma-1), \tag{1.3.27}$$

and if $n = 3$, we can choose σ close to $1/2$ such that (1.3.27) holds. Hence,

$$\begin{aligned}
&\left|\int_{\mathcal{O}}(f(u) - f(v_L))A^s v_N dx\right| \\
&\leqslant c\int_{\mathcal{O}}(1 + |u|^{p-1} + |v_L|^{p-1})|v_N + \epsilon z|\cdot|A^s v_N| dx \\
&\leqslant c\int_{\mathcal{O}}(1 + |u|^{p-1} + |v_L|^{p-1})|v_N|\cdot|A^s v_N| dx + c\epsilon\int_{\mathcal{O}}(1 + |u|^{p-1} + |v_L|^{p-1})|z|\cdot|A^s v_N| dx \\
&\leqslant c\left\{1 + (\int_{\mathcal{O}}|u|^{\frac{2n}{n-2\sigma}}dx)^{\frac{n-2\sigma}{2n}(p-1)} + (\int_{\mathcal{O}}|v_L|^{\frac{2n}{n-2\sigma}}dx)^{\frac{n-2\sigma}{2n}(p-1)}\right\}(\int_{\mathcal{O}}|v_N|^{\frac{2n}{\tilde{n}}}dx)^{\frac{\tilde{n}}{2n}} \\
&\quad \cdot(\int_{\mathcal{O}}|A^s v_N|^{\frac{2n}{n-2(1-s)}}dx)^{\frac{n-2(1-s)}{2n}} + c\epsilon\left\{1 + (\int_{\mathcal{O}}|u|^{\frac{2n}{n-2\sigma}}dx)^{\frac{n-2\sigma}{2n}(p-1)} + (\int_{\mathcal{O}}|v_L|^{\frac{2n}{n-2\sigma}}dx)^{\frac{n-2\sigma}{2n}(p-1)}\right\} \\
&\quad (\int_{\mathcal{O}}|z|^{\frac{2n}{\tilde{n}}}dx)^{\frac{\tilde{n}}{2n}}\cdot(\int_{\mathcal{O}}|A^s v_N|^{\frac{2n}{n-2(1-s)}}dx)^{\frac{n-2(1-s)}{2n}} \\
&\leqslant c\left\{1 + \|A^{\frac{\sigma}{2}} u\|^{p-1} + \|A^{\frac{\sigma}{2}} v_L\|^{p-1}\right\}\|v_N\|_{L^{\frac{2n}{\tilde{n}}}}\|A^{\frac{1+s}{2}} v_N\| \\
&\quad + c\epsilon\left\{1 + \|A^{\frac{\sigma}{2}} u\|^{p-1} + \|A^{\frac{\sigma}{2}} v_L\|^{p-1}\right\}\|z\|_{L^{\frac{2n}{\tilde{n}}}}\|A^{\frac{1+s}{2}} v_N\|,
\end{aligned} \tag{1.3.28}$$

where $\tilde{n} = 2n - [(n-2\sigma)p + 2(s+\sigma-1)]$.

Let $s' = [-n+(n-2\sigma)p+2(s+\sigma-1)]/2$. Since $p \leqslant 1+\frac{4}{n}$, we can choose $p > 0$ such that $s' > 0$. By calculation, we get that $0 < s' < 1+s$. Thus, using interpolation inequality, we obtain

$$\|v_N\|_{L^{\frac{2n}{\tilde{n}}}} = \|v_N\|_{\frac{2n}{n-2[-n+(n-2\sigma)p+2(s+\sigma-1)]/2}} = \|v_N\|_{L^{\frac{2n}{n-2s'}}} \leqslant c\|A^{\frac{s'}{2}}v_N\| \leqslant c\|v_N\|^{1-\vartheta}\|A^{\frac{1+s}{2}}v_N\|^{\vartheta}, 0 < \vartheta < 1,$$

which together with (1.3.28) implies that

$$\begin{aligned} |\int_{\mathcal{O}}(f(u)-f(v_L))A^s v_N dx| \leqslant{} & \frac{\lambda}{4}\|A^{\frac{1+s}{2}}v_N\|^2 + c(1+\|A^{\frac{\sigma}{2}}u\|^{p-1}+\|A^{\frac{\sigma}{2}}v_L\|^{p-1})^{\frac{2}{1-\vartheta}}\|v_N\|^2 \\ & + c\epsilon^2(1+\|A^{\frac{\sigma}{2}}u\|^{p-1}+\|A^{\frac{\sigma}{2}}v_L\|^{p-1})^{\frac{2}{1-\vartheta}}\|A^{\frac{s'}{2}}z\|^2. \end{aligned} \tag{1.3.29}$$

On the other hand, thanks to the Young inequality,

$$\begin{aligned} & \int_{\mathcal{O}} g\cdot A^s v_N dx + \int_{\mathcal{O}} Lv_L\cdot A^s v_N dx + c\epsilon\int_{\mathcal{O}}(z+\Delta z)\cdot A^s v_N dx \\ & \leqslant \frac{\lambda}{4}\|A^{\frac{1+s}{2}}v_N\|^2 + c(\|g\|^2+\|v_L\|^2) + c\epsilon^2(\|z\|^2+\|A^{\frac{1+s}{2}}z\|^2). \end{aligned} \tag{1.3.30}$$

and

$$\epsilon\int_0^{\infty}\nu_1(s)\int_{\mathcal{O}} zA^s\eta_N^t dxds + \epsilon\int_0^{\infty}\nu_2(s)\int_{\mathcal{O}} A^{\frac{1+s}{2}}zA^{\frac{1+s}{2}}\eta_N^t dxds \leqslant \frac{\delta_0}{4}\|\eta_N^t\|^2_{Q^s_{\nu_1,\nu_2}} + c\epsilon^2(\|A^{\frac{s}{2}}z\|^2+\|A^{\frac{1+s}{2}}z\|^2). \tag{1.3.31}$$

Therefore, it follows from (1.3.26) and (1.3.29)-(1.3.31) that

$$\begin{aligned} & \frac{d}{dt}(\|A^{\frac{s}{2}}v_N\|^2+\|\eta_N^t\|^2_{Q^s_{\nu_1,\nu_2}}) + \delta(\|A^{\frac{s}{2}}v_N\|^2+\|\eta_N^t\|^2_{Q^s_{\nu_1,\nu_2}}) \\ & \leqslant c(1+\|g\|^2) + c(1+\|A^{\frac{\sigma}{2}}u\|^{p-1}+\|A^{\frac{\sigma}{2}}v_L\|^{p-1})^{\frac{2}{1-\vartheta}}\|v_N\|^2 \\ & + c\epsilon^2(1+\|A^{\frac{\sigma}{2}}u\|^{p-1}+\|A^{\frac{\sigma}{2}}v_L\|^{p-1})^{\frac{2}{1-\vartheta}}\|A^{\frac{s}{2}}z\|^2 + c\epsilon^2(\|z\|^2+\|A^{\frac{1+\sigma}{2}}z\|^2). \end{aligned}$$

Applying Lemma 1.3.5 and the Gronwall lemma to the above inequality gives the desired result. □

1.3.3 Existence of the pullback random attractor

Now, we prove the compactness of the memory term. Note that for any $(\tau,\omega)\in\tilde{\Omega}$, $t \geqslant 0$,

$$\eta_N^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta_{N,\tau-t}(\theta_{-\tau}\omega))(s) = \begin{cases} \int_0^s u_N(\tau-r,\tau-t,\theta_{r-\tau}\omega,u_{N,\tau-t}(\theta_{r-\tau}\omega))dr, & 0 < s \leqslant t, \\ \int_0^t u_N(\tau-r,\tau-t,\theta_{r-\tau}\omega,u_{N,\tau-t}(\theta_{r-\tau}\omega))dr, & s > t. \end{cases} \tag{1.3.32}$$

Lemma 1.3.7. *Under the assumption of Lemma 1.3.6. For every given $(\tau,\omega)\in\tilde{\Omega}$, let*

$$E(\tau,\omega) := E(\tau,\omega)(s) = \overline{\bigcup_{(v_{\tau-t},\eta_{\tau-t})\in D_1(\tau-t,\theta_{-t}\omega)}\ \bigcup_{t\geqslant 0}\eta_N^t(\tau,\tau-t,\theta_{-\tau}\omega,\eta_{N,\tau-t}(\theta_{-\tau}\omega))(s)} \subset Q^0_{\nu_1,\nu_2},$$

where $\phi = (v,\eta^t)$ is the solution of (1.2.7). Then $E(\tau,\omega)$ is relatively compact in $Q^0_{\nu_1,\nu_2}$.

Proof. By Lemma A.1.10, we need to verify two conditions:

(i) $E(\tau,\omega)$ is bounded in $L^2_{\nu_1}(\mathbb{R}^+;\mathcal{H}_1)\cap H^1_{\nu_1}(\mathbb{R}^+;\mathcal{H}_0)$ and $L^2_{\nu_2}(\mathbb{R}^+;\mathcal{H}_2)\cap H^1_{\nu_2}(\mathbb{R}^+;\mathcal{H}_1)$;

(ii) $\sup_{\eta^t\in E(\tau,\omega)}(\|\eta^t\|^2_{\mathcal{H}_0}+\|\eta^t\|^2_{\mathcal{H}_1})\leqslant h(s)$.

From Lemma 1.3.6, we know that $E(\tau,\omega)$ is bounded in $L^2_{\nu_1}(\mathbb{R}^+;\mathcal{H}_1)\cap L^2_{\nu_2}(\mathbb{R}^+;\mathcal{H}_2)$. By (1.3.32), we have

$$\partial_s\eta^t_N(\tau,\tau-t,\theta_{-\tau}\omega,\eta_{N,\tau-t}(\theta_{-\tau}\omega))(s)=\begin{cases}u_N(\tau-s,\tau-t,\theta_{s-\tau}\omega,u_{N,\tau-t}(\theta_{s-\tau}\omega)), & 0<s\leqslant t,\\ 0, \quad s>t.\end{cases}\tag{1.3.33}$$

By (H1), we know that $\nu_1,\ \nu_2\in C^1(\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$, which along with Lemma 1.3.6 we find that $E(\tau,\omega)$ is bounded in $H^1_{\nu_1}(\mathbb{R}^+;\mathcal{H}_0)\cap H^1_{\nu_2}(\mathbb{R}^+;\mathcal{H}_1)$. Indeed, we have

$$\begin{aligned}&\int_0^\infty \nu_1(s)\|\partial_s\eta^t_N\|^2ds+\int_0^\infty \nu_2(s)\|\nabla\partial_s\eta^t_N\|^2ds\\ &=\int_0^t \nu_1(s)\|u_N(\tau-s)\|^2ds+\int_0^t \nu_2(s)\|\nabla u_N(\tau-s)\|^2ds\\ &=\int_0^t \nu_1(\tau-s)\|u_N(s)\|^2ds+\int_0^t \nu_2(\tau-s)\|\nabla u_N(s)\|^2ds<\infty,\end{aligned}$$

which implies (i) holds.

On the other hand, by (1.3.32) and using Lemma 1.3.6 again, we obtain that

$$\begin{aligned}&\sup_{\eta^t\in E(\tau,\omega),s\in\mathbb{R}^+}(\|\eta^t(s)\|_{\mathcal{H}_0}+\|\eta^t(s)\|_{\mathcal{H}_1})\\ &=\sup_{t\geqslant 0}\ \sup_{(\nu_{\tau-t},\eta_{\tau-t})\in D_1(\tau-t,\theta_{-t}\omega),s\in\mathbb{R}^+}\Big(\|\eta^t_N(\tau,\tau-t,\theta_{-\tau}\omega,\eta_{N,\tau-t}(\theta_{-\tau}\omega))(s)\|_{\mathcal{H}_0}+\|\eta^t_N(\tau,\tau-t,\theta_{-\tau}\omega,\eta_{N,\tau-t}(\theta_{-\tau}\omega))(s)\|_{\mathcal{H}_1}\Big)\\ &\leqslant s^2(R(\omega)+R_5(\omega)):=h(s).\end{aligned}$$

By (H2), we know that ν_1 and ν_2 decay exponentially, so it is easy to check that $h(s)\in L^1_{\nu_1}\cap L^1_{\nu_2}$. Then (ii) holds. By Lemma A.1.10, the proof is complete. □

We can now state our main result about the existence of pullback random attractor for the RDS Φ.

Theorem 1.3.8. *Assume that either $(f1)-(f3)$ hold with $1<p<1+\frac{4}{n}$, or $(f1)$, $(f2)$ and $(f4)$ hold with $p=1+\frac{4}{n}$. Let $(H1)-(H2)$ hold and $g\in C_b(\mathbb{R};L^2(\mathcal{O}))$. Then the RDS Φ associated with (1.2.7) possesses a compact measurable $\mathcal{D}$−pullback attracting set $\Lambda(\tau,\omega)\subset\mathcal{M}_0$ and possesses a $\mathcal{D}_\delta$−pullback random attractor $\mathcal{A}(\tau,\omega)\subseteq\Lambda(\tau,\omega)\cap D(\tau,\omega)$ for any $(\tau,\omega)\in\tilde{\Omega}$.*

Proof. For any $(\tau,\omega)\in\tilde{\Omega}$, in view of Lemma 1.3.6, let $B_s(\tau,\omega)$ be the closed ball of $\mathcal{H}_s$ of radius $R_5(\omega)$, where $0<s\leqslant 1$. Setting

$$\Lambda(\tau,\omega)=B_s(\tau,\omega)\times E(\tau,\omega),\tag{1.3.34}$$

then $\Lambda(\tau,\omega)\in\mathcal{D}_\delta(\mathcal{M}_0)$. Since the embedding $\mathcal{H}_s\hookrightarrow L^2(\mathcal{O})$ is compact, $B_s(\tau,\omega)$ is compact in $L^2(\mathcal{O})$. We have proved in Lemma 1.3.7 that $E(\tau,\omega)$ is compact in $Q^0_{\nu_1,\nu_2}$, so $\Lambda(\tau,\omega)$ is compact in $\mathcal{M}_0(:=L^2(\mathcal{O})\times Q^0_{\nu_1,\nu_2})$.

Now we show the following attraction property of $\Lambda(\tau,\omega)$, namely, for every $B_0\in\mathcal{D}_\delta(\mathcal{M}_0)$,

$$\lim_{t\to+\infty} d_{\mathcal{M}_0}\Big(\Phi(t,\tau-t,\theta_{-t}\omega,B_0(\tau-t,\theta_{-t}\omega)),\Lambda(\tau,\omega)\Big)=0. \tag{1.3.35}$$

By Lemma 1.3.1, there exists $t_*=t_*(\tau,\omega,B_0)\geqslant 0$ such that

$$\phi(\tau,\tau-t,\theta_{-\tau}\omega,B_0(\tau-t,\theta_{-t}\omega))\subseteq D(\tau,\omega),\quad \forall t>t_*. \tag{1.3.36}$$

Let $t>t_*$ and $t_0=t-t_*>T(\tau,\omega,B_0)$. Using the cocycle property (iii) in Definition 0.0.1, we have

$$\begin{aligned}&\phi(\tau,\tau-t,\theta_{-\tau}\omega,B_0(\tau-t,\theta_{-t}\omega))\\ &=\phi(\tau,\tau-t_0-t_*,\theta_{-\tau}\omega,B_0(\tau-t_0-t_*,\theta_{-t}\omega))\\ &=\phi(\tau,\tau-t_0,\theta_{-\tau}\omega,\phi(\tau-t_0,\tau-t_0-t_*,\theta_{-\tau}\omega,B_0(\tau-t_0-t_*,\theta_{-t}\omega)))\\ &\subseteq\phi(\tau,\tau-t_0,\theta_{-\tau}\omega,D(\tau-t_0,\theta_{-t_0}\omega))\subseteq D_1(\tau,\omega).\end{aligned} \tag{1.3.37}$$

Take any $\phi(\tau,\tau-t,\theta_{-\tau}\omega,\phi_{\tau-t}(\theta_{-\tau}\omega))\in\phi(\tau,\tau-t,\theta_{-\tau}\omega,B_0(\tau-t,\theta_{-t}\omega))$ for $t>t_*+T(\tau,\omega,B_0)$, where $\phi_{\tau-t}(\theta_{-\tau}\omega)\in B_0(\tau-t,\theta_{-t}\omega)$. From (1.3.37) and Lemma 1.3.6, we have

$$\phi_N(\tau,\tau-t,\theta_{-\tau}\omega,\phi_{\tau-t}(\theta_{-\tau}\omega))=\phi(\tau,\tau-t,\theta_{-\tau}\omega,\phi_{\tau-t}(\theta_{-\tau}\omega))-\phi_L(\tau,\tau-t,\theta_{-\tau}\omega,\phi_{L,\tau-t}(\theta_{-\tau}\omega))\in\Lambda(\tau,\omega). \tag{1.3.38}$$

Thus, by Lemma 1.3.3, we obtain

$$\begin{aligned}\inf_{\chi\in\Lambda(\tau,\omega)}\|\phi(\tau,\tau-t,\theta_{-\tau}\omega,\phi_{\tau-t}(\theta_{-\tau}\omega))-\chi\|^2_{\mathcal{M}_0}&\leqslant\|\phi_L(\tau,\tau-t,\theta_{-\tau}\omega,\phi_{L,\tau-t}(\theta_{-\tau}\omega))\|^2_{\mathcal{M}_0}\\ &\leqslant R_0(\omega)e^{-\delta t},\quad\forall t>t_*+T(\tau,\omega,B_0).\end{aligned} \tag{1.3.39}$$

It follows that

$$d_{\mathcal{M}_0}\Big(\Phi(t,\tau-t,\theta_{-t}\omega,B_0(\theta-t,\theta_{-t}\omega)),\Lambda(\tau,\omega)\Big)\leqslant R_0(\omega)e^{-\delta t}\to 0\ \text{ as } t\to+\infty, \tag{1.3.40}$$

which means (1.3.35) holds. By Proposition 0.0.7, RDS Φ associated with (1.2.7) possesses a $\mathcal{D}_\delta-$pullback random attractor $\mathcal{A}(\tau,\omega)\subseteq\Lambda(\tau,\omega)\cap D(\tau,\omega)$. The proof is completed. □

1.4 Upper semi-continuity of pullback random attractor

In this subsection, we regard the coefficient $\epsilon\in\mathbb{R}$ as a parameter in system (1.2.7). In view of Theorem 1.2.2 and 1.3.8, we can define a family of random dynamical system $\{\Phi^\epsilon(t,(\tau,\omega))\}_{\epsilon\in\mathbb{R}}$ associated to (1.2.7), and know that $\{\Phi^\epsilon(t,(\tau,\omega))\}_{\epsilon\in\mathbb{R}}$ possess a corresponding family of pullback random attractors $\{\mathcal{A}^\epsilon(\tau,\omega)\}_{\epsilon\in\mathbb{R}}$. Here let us consider the upper semi-continuity of pullback random attractors $\{\mathcal{A}^\epsilon(\tau,\omega)\}_{\epsilon\in\mathbb{R}}$ as $\epsilon\to\epsilon_0$ by Proposition 1.1.1.

Based on Proposition 1.1.1 and the results in Section 4, we have the following upper semi-continuity of pullback random attractors $\{\mathcal{A}^\epsilon(\tau,\omega)\}_{\epsilon\in\mathbb{R}}$ for $\{\Phi^\epsilon(t,(\tau,\omega))\}_{\epsilon\in\mathbb{R}}$.

Theorem 1.4.1. *Suppose that the conditions in Theorem 1.3.8 hold. Then for any* $(\tau,\omega)\in\tilde{\Omega}$,

$$\lim_{\epsilon\to\epsilon_0} d_{\mathcal{M}_0}(\mathcal{A}^{\epsilon}(\tau,\omega),\mathcal{A}^{\epsilon_0}(\tau,\omega)) = \sup_{\phi\in\mathcal{A}^{\epsilon}(\tau,\omega)}\ \inf_{\tilde{\phi}\in\mathcal{A}^{\epsilon_0}(\tau,\omega)} \|\phi-\tilde{\phi}\|_{\mathcal{M}_0} = 0. \tag{1.4.1}$$

Proof. Let us check that conditions (i)-(iv) of Proposition 1.1.1 are fulfilled.
(i) It is trivial to verify that for any $\epsilon_0\in\mathbb{R}$, there exists $F=\{F(\tau,\omega)=\{\phi^{\epsilon_0}\in\mathcal{M}_0 : \|\phi^{\epsilon_0}\|^2_{\mathcal{M}_0}\leqslant R_{\epsilon_0}(\tau,\omega)\} : (\tau,\omega)\in\tilde{\Omega}\}\in\mathcal{D}_\delta$ with $R_{\epsilon_0}(\tau,\omega)=2\gamma_1+\gamma_1(\epsilon_0^2+\epsilon_0^{p+1})r(\omega)$.
(ii) By Lemma 1.3.1 and Theorem 1.3.8, we know that for any $(\tau,\omega)\in\tilde{\Omega}$ and $\epsilon\in\mathbb{R}$, the pullback random attractor $\mathcal{A}^{\epsilon}(\tau,\omega)$ for $\Phi^{\epsilon}(t,(\tau,\omega))$ is included in the absorbing ball $D(\epsilon,\omega)=\{\phi^{\epsilon}\in\mathcal{M}_0 : \|\phi^{\epsilon}\|^2_{\mathcal{M}_0}\leqslant R_{\epsilon}(\tau,\omega)\}$, i.e., $\mathcal{A}^{\epsilon}(\tau,\omega)\subseteq D(\tau,\omega)\subset\mathcal{M}_0$, where $R_{\epsilon}(\tau,\omega)=\gamma_1+\gamma_1(\epsilon^2+\epsilon^{p+1})r(\omega)$. We can check that

$$\limsup_{\epsilon\to\epsilon_0} R_{\epsilon}(\tau,\omega)\leqslant R_{\epsilon_0}(\tau,\omega). \tag{1.4.2}$$

(iii) Let $|\epsilon|\leqslant 1$. For every $(\tau,\omega)\in\tilde{\Omega}$, using Theorem 1.3.8 once again, we find that $\mathcal{A}^{\epsilon}(\tau,\omega)\subseteq\Lambda^{\epsilon}(\tau,\omega)\subset\mathcal{M}_0$. Note that $R_5(\omega)$ and $R_{\epsilon}(\tau,\omega)$ are both increasing functions in $|\epsilon|$. By the construction of $\Lambda^{\epsilon}(\tau,\omega)$ in (1.3.34), we can choose the compact set $\Lambda^{\epsilon}(\tau,\omega)$ satisfying

$$\Lambda^{\epsilon}(\tau,\omega)\subset\Lambda^{1}(\tau,\omega),\quad \forall\,|\epsilon|\leqslant 1. \tag{1.4.3}$$

Hence,

$$\bigcup_{|\epsilon|\leqslant 1}\mathcal{A}^{\epsilon}(\tau,\omega)\subseteq\bigcup_{|\epsilon|\leqslant 1}\Lambda^{\epsilon}(\tau,\omega)\subseteq\Lambda^{1}(\tau,\omega)\subset\mathcal{M}_0. \tag{1.4.4}$$

Thus, $\bigcup_{|\epsilon|\leqslant 1}\mathcal{A}^{\epsilon}(\tau,\omega)$ is precompact in $\mathcal{M}_0$.

(iv) Let $|\epsilon|\leqslant 1$. For every $t\geqslant 0$, $(\tau,\omega)\in\tilde{\Omega}$, let $\phi^{\epsilon}(t,(\tau,\omega),\phi^{\epsilon}_{\tau}(\omega))$ and $\phi^{\epsilon_0}(t,(\tau,\omega),\phi^{\epsilon_0}_{\tau}(\omega))$ be the solutions of (1.2.7) with ϵ and ϵ_0, initial data $\phi^{\epsilon}_{\tau}(\omega)$ and $\phi^{\epsilon_0}_{\tau}(\omega)$, respectively. Set $U=\phi^{\epsilon}-\phi^{\epsilon_0}=(w,\xi)=(v^{\epsilon}-v^{\epsilon_0},\eta^t_{\epsilon}-\eta^t_{\epsilon_0})$, then

$$\dot{U}=LU+F(U),\quad U_\tau=\phi^{\epsilon}_{\tau}(\omega)-\phi^{\epsilon_0}_{\tau}(\omega), \tag{1.4.5}$$

where

$$\begin{aligned}
LU &= \left(\mu_0 w+\lambda\Delta w-\int_0^\infty \nu_1(s)\xi^t(s)ds+\int_0^\infty \nu_2(s)\Delta\xi^t(s)ds,\ w-\partial_s\xi^t\right),\\
F(U) &= F^{\epsilon}(\phi^{\epsilon},\theta_t\omega,t)-F^{\epsilon_0}(\phi^{\epsilon_0},\theta_t\omega,t)\\
&= \left((\epsilon-\epsilon_0)(\mu_0+1)z(\theta_t\omega)+(\epsilon-\epsilon_0)\lambda\Delta z(\theta_t\omega)+f(u^{\epsilon_0})-f(u^{\epsilon}),(\epsilon-\epsilon_0)z(\theta_t\omega)\right),\\
U_\tau(\omega) &= \left(v^{\epsilon}_{\tau}-v^{\epsilon_0}_{\tau},\eta^{\epsilon}_{\tau}-\eta^{\epsilon_0}_{\tau}\right).
\end{aligned}$$

Take the inner product of (1.4.5) with U in $\mathcal{M}_0$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|^2_{\mathcal{M}_0}=(LU,U)_{\mathcal{M}_0}+(F(U),U)_{\mathcal{M}_0}. \tag{1.4.6}$$

By (H2), we have

$$(LU,U)_{\mathcal{M}_0}\leqslant(\mu_0-\lambda_1\lambda)\|w\|^2-\frac{\delta_0}{2}\|\xi^t\|^2_{\mathcal{M}_0}\leqslant-\delta\|U\|^2_{\mathcal{M}_0}. \tag{1.4.7}$$

$$
\begin{aligned}
(F(U), U)_{\mathcal{M}_0} = & (\epsilon - \epsilon_0)(\mu_0 + 1) \int_{\mathcal{O}} zw dx - (\epsilon - \epsilon_0)\lambda \int_{\mathcal{O}} \nabla z \nabla w dx + (\epsilon - \epsilon_0) \int_0^\infty \nu_1(s) \int_{\mathcal{O}} z\xi^t dx ds \\
& + (\epsilon - \epsilon_0) \int_0^\infty \nu_2(s) \int_{\mathcal{O}} \nabla z \nabla \xi^t dx ds + \int_{\mathcal{O}} (f(u^{\epsilon_0}) - f(u^\epsilon)) w dx \\
& \leqslant c(\epsilon - \epsilon_0)^2 (\|z\|^2 + \|\nabla z\|^2) + c(\|w\|^2 + \|\nabla w\|^2) + \frac{\delta_0}{4} \|\xi^t\|^2_{Q^0_{\nu_1,\nu_2}} + \int_{\mathcal{O}} (f(u^{\epsilon_0}) - f(u^\epsilon)) w dx
\end{aligned}
\tag{1.4.8}
$$

For the last term in (1.4.8), we have

$$
\int_{\mathcal{O}} (f(u^{\epsilon_0}) - f(u^\epsilon)) w dx \leqslant R_6(r, \tau, \omega)(\|w\|^2 + \|\nabla w\|^2), \quad \text{if } p < 1 + \frac{4}{n}, \tag{1.4.9}
$$

and

$$
\int_{\mathcal{O}} (f(u^{\epsilon_0}) - f(u^\epsilon)) w dx \leqslant R_7(r, \tau, \omega) \|\nabla w\|^2, \quad \text{if } p = 1 + \frac{4}{n}. \tag{1.4.10}
$$

It follows from (1.4.6)-(1.4.10) that

$$
\frac{d}{dt} \|U\|^2_{\mathcal{M}_0} \leqslant R_8(r, \tau, \omega) \|U\|^2_{\mathcal{M}_0} + c(\epsilon - \epsilon_0)^2 (\|z(\theta_t \omega)\|^2 + \|\nabla z(\theta_t \omega)\|^2), \quad r \geqslant \tau - t. \tag{1.4.11}
$$

Apply the Gronwall lemma to (1.4.11) with ω replaced by $\theta_{-\tau}\omega$ to find

$$
\begin{aligned}
& \|U(\tau, \tau - t, \theta_{-\tau}\omega, U_{\tau-t})\|^2_{\mathcal{M}_0} \\
& = \|v^\epsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\epsilon_{\tau-t}) - v^{\epsilon_0}(\tau, \tau - t, \theta_{-\tau}\omega, v^{\epsilon_0}_{\tau-t})\|^2 + \|\eta^t_\epsilon(\tau, \tau - t, \theta_{-\tau}\omega, \eta^\epsilon_{\tau-t}) - \eta^t_{\epsilon_0}(\tau, \tau - t, \theta_{-\tau}\omega, \eta^{\epsilon_0}_{\tau-t})\|^2_{Q^0_{\nu_1,\nu_2}} \\
& \leqslant \|\phi^\epsilon(\tau, \tau - t, \theta_{-\tau}\omega, \phi^\epsilon_{\tau-t}) - \phi^{\epsilon_0}(\tau, \tau - t, \theta_{-\tau}\omega, \phi^{\epsilon_0}_{\tau-t})\|^2_{\mathcal{M}_0} e^{\int_{\tau-t}^{\tau} \rho_8 + \rho_8(\epsilon^2 + \epsilon^p)^{l_8} e^{-\beta_8(s-\tau)} r(\omega)^{n_8} ds} \\
& \quad + c(\epsilon - \epsilon_0)^2 \int_{\tau-t}^{\tau} e^{\int_\tau^r \rho_8 + \rho_8(\epsilon^2 + \epsilon^p)^{l_8} e^{-\beta_8(s-\tau)} r(\omega)^{n_8} ds} (\|z(\theta_{r-\tau}\omega)\|^2 + \|\nabla z(\theta_{r-\tau}\omega)\|^2) dr.
\end{aligned}
\tag{1.4.12}
$$

From (1.4.12), we see that for any $(\tau, \omega) \in \tilde{\Omega}$, $t \geqslant 0$, $\epsilon_n \to \epsilon_0$, and $\phi^{\epsilon_n}_{\tau-t}, \phi^{\epsilon_0}_{\tau-t} \in \mathcal{M}_0$ with $\phi^{\epsilon_n}_{\tau-t} \to \phi^{\epsilon_0}_{\tau-t}$, it holds that:

$$
\begin{aligned}
\lim_{n\to\infty} \Big(& \|v^{\epsilon_n}(\tau, \tau - t, \theta_{-\tau}\omega, v^{\epsilon_n}_{\tau-t}) - v^{\epsilon_0}(\tau, \tau - t, \theta_{-\tau}\omega, v^{\epsilon_0}_{\tau-t})\|^2 \\
& + \|\eta^t_{\epsilon_n}(\tau, \tau - t, \theta_{-\tau}\omega, \eta^{\epsilon_n}_{\tau-t}) - \eta^t_{\epsilon_0}(\tau, \tau - t, \theta_{-\tau}\omega, \eta^{\epsilon_0}_{\tau-t})\|^2_{Q^0_{\nu_1,\nu_2}} \Big) = 0.
\end{aligned}
\tag{1.4.13}
$$

Up to now, all of the conditions (i)-(iv) of Proposition 1.1.1 are satisfied. The proof is finished. □

Remark 1.4.2. *We would like to mention that in [113], Li proved the existence of uniform attractor for (1.0.1) with $\epsilon = 0$, but did not consider the pullback set up for the asymptotic behavior, which is our motivation for our paper.*

Up to now, we have studied stochastic standard parabolic problem with memory as well as subcritical and critical nonlinearity, namely, Eq. (1.0.1). The existence of random attractor as well as upper semi-continuity of the random attractor are established. However, as we said at the very beginning of this Part that the fractional derivative equation can exhibit significant self-organization phenomena or asymmetry, which is very different from the classic derivative equation. Besides, the fractional differential equations have a wide range of applications in physics, biology, finance and so on. Therefore, in the next chapter, we discuss a stochastic fractional parabolic problem with memory.

Chapter 2

Dynamics of a fractional stochastic reaction-diffusion equation with thermal memory

In this chapter we investigate the well-posedness and dynamics of a fractional stochastic integro-differential equation describing a reaction process depending on the temperature itself, and which is derived in the framework of the well-established theory of heat flows with memory (see [50]) on $\mathcal{O} \subset \mathbb{R}^3$, which is a bounded domain with smooth boundary $\partial\mathcal{O}$,

$$\frac{\partial u}{\partial t} + \beta(1-\gamma)(-\Delta)^\alpha u + \int_0^\infty \mu(s)(-\Delta)^\alpha u(t-s)ds + f(u) = k(x) + h(x)\frac{dW}{dt}, \quad x \in \mathcal{O}, \ t > 0, \tag{2.0.1}$$

with boundary condition

$$u(x,t) = 0, \quad x \in \partial\mathcal{O}, \quad t > 0, \tag{2.0.2}$$

and initial condition

$$u(x,t) = u_0(x,t), \quad x \in \mathcal{O}, \ t \leqslant 0. \tag{2.0.3}$$

Here, $\alpha \in (0,1)$, $\beta \in (0,+\infty)$ and $\gamma \in (0,1)$, $u(x,t)$ is the unknown function, while μ is a decreasing and non-negative memory kernel; f is a nonlinear reaction term (for instance, $f(u) = u^3 - u$), $k(\cdot) \in L^2(\mathcal{O})$ and $h(\cdot) \in H^{2\alpha}(\mathcal{O})$ are given functions. W is a two-sided real-valued Wiener process on a probability space which will be specified later. In the present case, the dynamics of u relies on the past history of the diffusion term, that is, $\int_0^\infty \mu(s)(-\Delta)^\alpha u(t-s)ds$.

Problem (2.0.1) with $\alpha = 1$ as well as $h(x) = 0$ is well known and has been extensively studied (see [22, 55]), and can be interpreted as a model of heat diffusion with memory which also accounts for a reaction process depending on the temperature itself (see [76] and related references therein). Namely, if $u(t)$ represents the temperature of a material occupying $\mathcal{O}$ for any time t, as in [50], we can consider the following heat flux law

$$\vec{q}(x,t) = -\beta(1-\gamma)\nabla u(x,t) - \int_0^\infty \mu(s)\nabla u(x,t-s)ds,$$

where $\beta(1-\gamma)$ is the instantaneous heat conductivity and $\mu(s)$ is a memory or relaxation kernel. Then, assuming the total energy is proportional to u (with proportionality constant 1 for simplicity), the standard semilinear heat equation with memory, i.e.,

$$\frac{\partial u}{\partial t} - \beta(1-\gamma)\Delta u - \int_0^\infty \mu(s)\Delta u(t-s)ds + f(u) = k(x), \tag{2.0.4}$$

could be recovered from the energy balance

$$u_t + \nabla \cdot \overrightarrow{q} = k - f(u),$$

(see [38] for a more detailed explanation and more references on the topic). This kind of equation can also be proposed to describe many different phenomena, such as the evolution of the velocity of certain viscoelastic fluids [64], the thermo mechanical behavior of polymers [87, 142], the diffusion of the chemical potential of a penetrant in polymers near the glass transition [102], and some models in population dynamics [67]. Concerning equation (2.0.4) (which is a deterministic heat equation with memory) existence, uniqueness, and asymptotic behavior results can be found in [55, 78, 81]. In particular, equation (2.0.4) is shown to have a uniform attractor, which has finite Hausdorff dimension (see [78]), whereas in [77] the existence of absorbing sets in suitable function spaces is achieved.

Observe that the aforementioned literature mainly dealt with versions of Eq. (2.0.4) in a deterministic context. But, it is sensible to assume that the models of certain phenomena from the real world are more realistic if some kind of uncertainty, for instance, some randomness or environmental noise, is also considered in the formulation. In fact, the random perturbations are intrinsic effects in a variety of settings and spatial scales. They may be most obviously influential at the microscopic and smaller scales but indirectly they play an important role in macroscopic phenomena. We will take into account an additive noise in our model which we interpret as the environmental noisy effect produced on the system, and will exploit the theory of random dynamical systems (see [2, 11]) to obtain information on the dynamics of our model, in particular we will be able to prove the existence of random attractor. When $\alpha = 1$, problem (2.0.1) reduces to a standard stochastic heat equation with memory. In this case, a similar stochastic equation with additive noise in materials with memory is studied in [38], and the existence of pullback attractors is also established, while in [16], the existence and stability of solutions for stochastic heat equations with multiplicative noise in materials with memory is proved.

Nevertheless, the previously cited references are concerned with equations with standard Laplace operator, namely, $\alpha = 1$ in equation (2.0.1). However, it is mentioned in [8] that some research on classical diffusion equation may be inadequate to model many real situations, for instance, a particle plume spreads faster than that predicted by the classical model, and may exhibit significant self-organization phenomena or asymmetry, see details in [152]. In this case, these situations are called anomalous diffusion. One popular model for anomalous diffusion is the fractional diffusion equation, where the usual second derivative operator in space, i.e., the Laplacian operator $-\Delta$, is replaced by a fractional derivative operator $(-\Delta)^\alpha$ with $0 < \alpha < 1$. Indeed, equations with fractional derivative are becoming a focus of interest since the fractional derivative and fractional integral have a wide range of applications in physics, biology, chemistry, population dynamics, geophysical fluid dynamics, finance and other fields of applied sciences. One meets them in the theory of systems with chaotic dynamics (see [147, 175]); dynamics in a complex or porous medium [65, 150]; random walks with a memory and flights [79] and many other situations. When $\mu = 0$, this is the case of no memory term, (2.0.1) reduces to a fractional stochastic parabolic equation with noise. In this case, the ergodicity of a stochastic fractional reaction-diffusion equation with additive noise is studied in [85], whereas the existence of random attractor for a fractional stochastic reaction-diffusion equation is proved in [121] under the assumption of $\alpha \in [\frac{1}{2}, 1)$. However, as far as we know, there are no works dealing with fractional stochastic reaction-diffusion equations with both white noise term and memory terms, and this is the reason of the current investigation in this paper.

Inspired by [38, 85], we are devoted to investigating a stochastic fractional integro-differential equation. More precisely, in this work, we analyze the well-posedness and dynamics of a fractional stochastic reaction-diffusion equation with memory term, which is expressed by convolution integrals and represent the past history of one or more variables. The main features of the present paper work are summarized as follows: Both the fractional diffusion term (instead of standard diffusion term, i.e., $-\Delta u$) and the memory term are considered. Besides, the well-posedness is analyzed by a semigroup method (see [139] for more information), which is different from the classical Faedo-Galerkin method (see [160]). Then the existence of random attractor is established by a priori estimates and solutions decomposition. Moreover, by using the method introduced by Debussche in [60], we obtain that the random attractor has finite Hausdorff dimension.

2.1 Definitions and Basic Theory

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with the Borel $\sigma-$algebra $\mathcal{B}(X)$, and $\{\theta_t : \Omega \to \Omega,\ t \in \mathbb{R}\}$ be a family of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, P)$.

In this chapter we need some theory of random dynamical systems. Since we introduced them in Preliminaries I, all we need to do is to replace $\tilde{\omega}$ by ω in Definition 0.0.1-0.0.4 as well as Proposition 0.0.7. Besides, we need to replace "cocycles" by "random dynamical systems" in this chapter, since we only work on stochastic equation without time-dependent forcing term. We would also like to mention that we only recall some notations and propositions that are particular for this chapter in Section 2.1.

Definition 2.1.1. *Let X be a metric space with a metric d. A set-valued map $\omega \to B(\omega)$ taking values in the closed/compact subsets of X is said to be a random closed/compact set in X if the mapping $\omega \longmapsto dist(x, B(\omega))$ is measurable for all $x \in X$, where $d(x, D) := \inf_{y \in D} d(x, y)$. A set-valued map $\omega \mapsto U(\omega)$ taking values in the open subsets of X is said to be a random open set if $\omega \mapsto U^c(\omega)$ is a random closed set, where $U^c(\omega)$ denotes the complement of U, i.e., $U^c := X \backslash U$.*

Definition 2.1.2. *Let A be a linear operator on a Hilbert space X. For any $m \in \mathbb{N}$, the $m-$dimensional trace of A is defined as*

$$Tr_m(A) = \sup_Q \sum_{j=1}^{m} (Au_j, u_j)_X,$$

where the supremum ranges over all possible orthogonal projections Q in X on the $m-$dimensional space QX belonging to the domain of A, and $\{u_1, u_2, \cdots, u_m\}$ is an orthonormal basis of QX.

Proposition 2.1.3. *(See [122]) Let $\mathcal{A}(\omega)$ be a compact measurable set which is invariant under a random map $\Psi(\omega)(\cdot), \omega \in \Omega$, for some ergodic metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. Assume that the following conditions are satisfied.*

(i) $\Psi(\omega)(\cdot)$ is almost surely uniformly differentiable on $\mathcal{A}(\omega)$, that is, for every $u, u+h \in \mathcal{A}(\omega)$, there exists $D\Psi(\omega, u)$ in $\mathcal{L}(X)$, the space of the bounded linear operators from X to X, such that

$$\|\Psi(\omega)(u+h) - \Psi(\omega)(u) - D\Psi(\omega, u)h\| \leqslant \bar{k}(\omega)\|h\|^{1+\rho},$$

where $\rho > 0$ and $\bar{k}(\omega)$ is a random variable satisfying $\bar{k}(\omega) \geqslant 1$ and $E(\ln \bar{k}) < \infty$.

(ii) $\omega_d(D\Psi(\omega, u)) \leqslant \overline{\omega}_d(\omega)$ holds when $u \in \mathcal{A}(\omega)$ and there is some random variable $\overline{\omega}_d(\omega)$ satisfying $E(\ln(\overline{\omega}_d)) < 0$, where

$$\omega_d(D\Psi(\omega, u)) = \alpha_1(D\Psi(\omega, u)) \cdots \alpha_d(D\Psi(\omega, u)),$$

$$\alpha_d(D\Psi(\omega, u)) = \sup_{G\subset X, dim G\leqslant d} \inf_{v\in G, \|v\|_X=1} \|D\Psi(\omega, u)v\|.$$

(iii) $\alpha_1(D\Psi(\omega, u)) \leqslant \overline{\alpha_1}(\omega)$ holds when $u \in \mathcal{A}(\omega)$ and there is a random variable $\overline{\alpha_1}(\omega) \geqslant 1$ with $E(\ln \overline{\alpha_1}) < \infty$.

Then the Hausdorff dimension $d_H(\mathcal{A}(\omega))$ of $\mathcal{A}(\omega)$ is less than d almost surely.

Throughout the work, we denote by $A = (-\Delta)^\alpha$ $(0 < \alpha < 1)$ the fractional Laplace operator with domain $D(A) = H^{2\alpha}(\mathcal{O})$. With usual notation, we introduce the space L^p, H^k and H_0^k acting on $\mathcal{O}$. Let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product on the real Hilbert space $L^2(\mathcal{O})$, respectively, and let $\|\cdot\|_p$ denote the L^p-norm. With abuse of notation, we use $(\cdot,\cdot)$ to denote also duality between L^p and its dual space L^q.

The inner products on $H^\alpha(\mathcal{O})$, $H^{2\alpha}(\mathcal{O})$ can be defined in the following manner:

$$(u, v)_{H^\alpha(\mathcal{O})} = ((-\Delta)^{\frac{\alpha}{2}}u, (-\Delta)^{\frac{\alpha}{2}}v)$$

and

$$(u, v)_{H^{2\alpha}(\mathcal{O})} = ((-\Delta)^\alpha u, (-\Delta)^\alpha v).$$

Assuming $\mu(\infty) = 0$, set

$$g(s) = -\mu'(s). \tag{2.1.1}$$

In what follows, we take $\beta = 2, \gamma = \frac{1}{2}$ for simplicity, and the following set of hypotheses are required:

(H1) $g(\cdot) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $g(s) \geqslant 0$, $g'(s) \leqslant 0$, $g'(s) + \delta g(s) \leqslant 0$, $\forall\ s \in \mathbb{R}^+$ and some $\delta > 0$;

(H2) $f(\cdot) \in C^1(\mathbb{R}^+)$, $f(u)u \geqslant \alpha_1|u|^p - \alpha_2$, $f'(u) > -\alpha_3$, $|f(u)| \leqslant \alpha_4(1 + |u|^{p-1})$, where α_i $(i = 1, 2, 3, 4)$, $p \geqslant 1$ are positive numbers.

Note that (H1) implies the exponential decay of $g(\cdot)$. Nevertheless, it allows $g(\cdot)$ to have a singularity at $s = 0$, whose order is less than 1, since $g(\cdot)$ is a non-negative L^1-function.

Now, let $L^2_g(\mathbb{R}^+, L^2(\mathcal{O}))$ be the Hilbert space of L^2-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$(\eta_1, \eta_2)_{L^2_g(\mathbb{R}^+, L^2(\mathcal{O}))} = \int_0^\infty g(s) \int_{\mathcal{O}} \eta_1(s, x) \cdot \eta_2(s, x) dx ds.$$

Similarly on $M := L^2_g(\mathbb{R}^+, H^\alpha(\mathcal{O}))$ and $M_1 := L^2_g(\mathbb{R}^+, H^{2\alpha}(\mathcal{O}))$, respectively, we have the inner products

$$(\eta_1, \eta_2)_{L^2_g(\mathbb{R}^+, H^\alpha(\mathcal{O}))} = \int_0^\infty g(s) \int_{\mathcal{O}} (-\Delta)^{\alpha/2} \eta_1(s, x) \cdot (-\Delta)^{\alpha/2} \eta_2(s, x) dx ds$$

and

$$(\eta_1, \eta_2)_{L^2_g(\mathbb{R}^+, H^{2\alpha}(\mathcal{O}))} = \int_0^\infty g(s) \int_{\mathcal{O}} (-\Delta)^{\alpha} \eta_1(s, x) \cdot (-\Delta)^{\alpha} \eta_2(s, x) dx ds,$$

where operators $(-\Delta)^{\alpha/2}$ and $(-\Delta)^\alpha$ are considered with respect the spatial variable $x \in \mathcal{O}$. In the sequel, we will omit the variable x when no confusion is possible.

Finally, we introduce the Hilbert spaces

$$\mathcal{H} = L^2(\mathcal{O}) \times L^2_g(\mathbb{R}^+, H^\alpha(\mathcal{O}))$$

and

$$\mathcal{V} = H^\alpha(\mathcal{O}) \times L^2_g(\mathbb{R}^+, H^{2\alpha}(\mathcal{O})).$$

Using (2.1.1) and the classic variable change (1.2.2) (which is from Chapter 1), Eq. (2.0.1)-(2.0.3) could be transform into

$$\frac{\partial u}{\partial t} + (-\Delta)^\alpha u + \int_0^\infty g(s)(-\Delta)^\alpha \eta^t(s) ds + f(u) = k(x) + h(x)\frac{dW}{dt}, \quad x \in \mathcal{O}, \ t > 0, \tag{2.1.2}$$

$$\partial_t \eta^t = -\partial_s \eta^t + u, \quad x \in \mathcal{O}, \quad t > 0, \ s > 0, \tag{2.1.3}$$

with boundary condition

$$u(x, t) = 0, \quad x \in \partial\mathcal{O}, \quad t > 0, \tag{2.1.4}$$

and initial condition

$$u(x, t) = u_0(x, t), \quad \eta^0(x, s) = \eta_0(x, s), \quad x \in \mathcal{O}, \ t \leqslant 0, \ s > 0. \tag{2.1.5}$$

And the term

$$\eta^0(x, s) = \int_0^s u^0(x, r) dr = \int_{-s}^0 u(x, r) dr, \ x \in \mathcal{O}, \quad s \geqslant 0,$$

is the prescribed initial integral past history of $u(x, t)$, which does not depend on $u_0(x, t)$, and is assumed to vanish on $\partial\mathcal{O}$, as well as $u(x, t)$. As a consequence it follows that

$$\eta^t(x, s) = 0, \quad x \in \partial\mathcal{O}, \quad t > 0 \text{ and } s > 0.$$

Indeed, the above assertion is obvious if $t \geqslant s$, and if $t < s$ we can write

$$\eta^t(x,s) = \eta_0(x, s-t) + \int_0^t u(x,r)dr.$$

In order to present our results, let us write system (2.1.2)-(2.1.5) as a Cauchy problem. Denote $w(t) = (u(t), \eta^t)$, $w_0 = (u_0, \eta_0)$, and set

$$Lw = (-(-\Delta)^\alpha u - \int_0^\infty g(s)(-\Delta)^\alpha \eta^t(s)ds, u - \partial_s \eta^t).$$

and

$$F(w, \theta_t\omega) = (k - f(u) + h\frac{dW}{dt}, 0).$$

Problem (2.1.2)-(2.1.5) can be written

$$\frac{dw}{dt} = Lw + F(w, \theta_t\omega) \tag{2.1.6}$$

$$w(x,t) = 0, \quad x \in \partial\mathcal{O}, \ t > 0, \tag{2.1.7}$$

$$w(x,t) = w_0(x,t), \ x \in \mathcal{O}, \ t \leqslant 0. \tag{2.1.8}$$

Now we present our main results of this paper.

Theorem 2.2.4 *Assume that hypotheses (H1)-(H2) are satisfied and initial data* $(u_0, \eta_0) \in \mathcal{H}$*. Then, problem* (2.1.6)-(2.1.8) *possesses a unique mild solution in the class*

$$u \in C([0,\infty); L^2(\mathcal{O})), \ \text{and} \ \eta^t \in C([0,\infty); M). \tag{2.1.9}$$

If initial data $(u_0, \eta_0) \in D(L)$*, then the solution is more regular, i.e.,* $u \in C([0,\infty); H^\alpha(\mathcal{O}))$*, and* $\eta^t \in C([0,\infty); M_1)$*. In addition, if* $w(t) = (u, \eta^t)$ *and* $\bar{w}(t) = (\bar{u}, \bar{\eta}^t)$ *are two mild solutions of* (2.1.6)-(2.1.8)*, then for any* $T > 0$*,*

$$\|w(t) - \bar{w}(t)\|^2_{\mathcal{H}} \leqslant e^{cT}\|w(0) - \bar{w}(0)\|^2_{\mathcal{H}}, \quad 0 \leqslant t \leqslant T, \tag{2.1.10}$$

where $c > 0$ *is a constant independent of the initial data.*

The proof of Theorem 2.2.4 is presented in Section 3 by means of semigroup arguments.

The next main result of our paper concerns the generation of a random dynamical system, the existence of the corresponding random attractor and its finite Hausdorff dimension. These are included in Theorems 2.3.7 and 2.4.2 which are the content included in the theorem below (see Sections 4 and 5).

Theorem (See Theorem 2.3.7 and 2.4.2) *Assume that* $k(\cdot) \in L^2(\mathcal{O})$ *and that hypotheses (H1)-(H2) hold with* $\alpha \in [\frac{1}{2}, 1)$ *and* $p \in [2, 1 + \frac{3}{3-2\alpha})$*. Then the random dynamical system* Φ *generated by* (2.1.6)-(2.1.8) *possesses a random attractor* $\mathcal{A}$ *in* $\mathcal{H}$*. Moreover, if the second derivative of* f *is bounded, then the random attractor has finite Hausdorff dimension.*

2.2 Well-posedness

In this section, we show the existence, uniqueness and continuous dependence of mild solutions of the problem (2.1.6)-(2.1.8).

Formally, if u solves Eq. (2.1.2), then the variable $v(t) = u(t) - z(\theta_t\omega)$ should satisfy

$$\frac{\partial v}{\partial t} + (-\Delta)^\alpha v + \int_0^\infty g(s)(-\Delta)^\alpha \eta^t(s)ds + f(v+z) = k(x) + z - (-\Delta)^\alpha z.$$

with boundary condition and initial condition:

$$v(x,t) = 0,\ \eta^t(x,s) = 0,\ \ x \in \partial\mathcal{O},\ t > 0,\ \ v(x,t) = v_0(x,t),\ \eta^0(x,s) = \eta_0(x,s), x \in \mathcal{O}, s \geqslant 0, t \leqslant 0,$$

where we use Ornstein-Uhlenbeck transformation, and $z(\theta_t\omega)$ is from (0.0.4).

Similarly, we can write the above system as a Cauchy problem. To this end, denote $\varphi(t,\omega,\varphi_0) = (v(t,\omega,v_0), \eta^t(\omega,\eta_0(\cdot)))$ with $v_0 = u_0 - z(\omega)$, $\eta^0 = \eta_0(\cdot)$, we have the following compact form

$$\begin{aligned} \frac{d\varphi}{dt} &= L\varphi + F(\varphi,\theta_t\omega) \\ \varphi(0,\omega,\varphi_0) &= (v_0,\eta_0(\cdot)) := \varphi_0, \end{aligned} \tag{2.2.1}$$

where

$$L\varphi = \left(-(-\Delta)^\alpha v - \int_0^\infty g(s)(-\Delta)^\alpha \eta^t(s)ds, -\partial_s\eta^t + v\right) \tag{2.2.2}$$

and

$$F(\varphi,\theta_t\omega) = \Big(k - f(u) - (-\Delta)^\alpha z(\theta_t\omega) + z(\theta_t\omega), z(\theta_t\omega)\Big). \tag{2.2.3}$$

By the proof in [138], we obtain that the domain of T is

$$D(T) = \{\eta^t \in M | \partial_s\eta^t \in M,\ \eta(0) = 0\}.$$

Since the domain of L is defined by

$$D(L) = \{\varphi \in \mathcal{H} | L\varphi \in \mathcal{H}\},$$

one has

$$D(L) = \Big\{(v,\eta^t) \in \mathcal{H} \mid v \in L^2(\mathcal{O}),\ \eta^t \in D(T),\ -(-\Delta)^\alpha v - \int_0^\infty g(s)(-\Delta)^\alpha \eta^t(s)ds \in L^2(\mathcal{O})\Big\}.$$

We begin with the following lemma, which is an important step to prove the existence of mild solution of problem (2.2.1).

Lemma 2.2.1. *Operator L is the infinitesimal generator of a C^0-semigroup of contractions e^{Lt} in $\mathcal{H}$.*

Proof. We show that L is m-dissipative in $\mathcal{H}$. By (H1) and the definition of $L\varphi$, we infer that

$$(L\varphi, \varphi) = -\|(-\Delta)^{\frac{\alpha}{2}} v\|^2 + \frac{1}{2}\int_0^\infty g'(s)\|(-\Delta)^{\frac{\alpha}{2}}\eta^t(s)\|^2 ds \leqslant 0,$$

for all $\varphi = (v, \eta^t) \in D(L)$. This proves that L is dissipative in $\mathcal{H}$.

Next we show that L is maximal, that is, for each $F \in \mathcal{H}$, there exists a solution $\varphi \in D(L)$ of

$$(I - L)\varphi = F.$$

Equivalently, for each $F = (f_1, f_2) \in \mathcal{H}$, there exists $\varphi = (v, \eta^t) \in D(L)$ such that

$$v + (-\Delta)^\alpha v + \int_0^\infty g(s)(-\Delta)^\alpha \eta^t(s) ds = f_1, \tag{2.2.4}$$

$$\eta^t - v + \partial_s \eta^t = f_2. \tag{2.2.5}$$

To solve system (2.2.4)-(2.2.5), we first multiply (2.2.5) by e^s and integrate over $(0, s)$. Then,

$$\eta^t = v(1 - e^{-s}) + \int_0^s e^{\tau - s} f_2(\tau) d\tau. \tag{2.2.6}$$

Including (2.2.6) into (2.2.4) we obtain, by denoting $k_1 = \int_0^\infty g(s)(1 - e^{-s}) ds$,

$$v + (-\Delta)^\alpha v + k_1(-\Delta)^\alpha v = f_1 - \int_0^\infty g(s) \int_0^s e^{\tau - s}(-\Delta)^\alpha f_2(\tau) d\tau ds. \tag{2.2.7}$$

In order to solve equation (2.2.7) we define the bilinear form

$$a(w_1, w_2) = \int_{\mathcal{O}} w_1 w_2 dx + \int_{\mathcal{O}} (-\Delta)^{\frac{\alpha}{2}} w_1 \cdot (-\Delta)^{\frac{\alpha}{2}} w_2 dx + k_1 \int_{\mathcal{O}} (-\Delta)^{\frac{\alpha}{2}} w_1 \cdot (-\Delta)^{\frac{\alpha}{2}} w_2 dx, \; w_1, w_2 \in H^\alpha(\mathcal{O}).$$

It is easy to check that $a(w_1, w_2)$ is continuous and coercive in $H^\alpha(\mathcal{O})$. And we have

$$H^\alpha(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \hookrightarrow H^{-\alpha}(\mathcal{O}).$$

We now aim at applying the Lax-Milgram theorem. It suffices to prove that the right hand side of (2.2.7) is an element of $H^{-\alpha}(\mathcal{O})$. Obviously,

$$f_1 \in H^{-\alpha}(\mathcal{O}).$$

Let f^* denote the last term in (2.2.7), and we only need to show that $f^* \in H^{-\alpha}(\mathcal{O})$. We apply arguments similar to those used by Giorgi [76]. For $w \in H^\alpha(\mathcal{O})$ with $\|(-\Delta)^{\frac{\alpha}{2}} w\| \leqslant 1$,

$$\begin{aligned}
\left|(f^*, w)_{H^{-\alpha}, H^\alpha}\right| &= \left|\int_0^\infty g(s) \int_0^s e^{\tau - s}\left(\int_{\mathcal{O}} (-\Delta)^{\frac{\alpha}{2}} f_2(\tau)(-\Delta)^{\frac{\alpha}{2}} w dx\right) d\tau ds\right| \\
&\leqslant \int_0^\infty e^\tau \|(-\Delta)^{\frac{\alpha}{2}} f_2(\tau)\| \int_\tau^\infty g(s) e^{-s} ds d\tau \\
&\leqslant \int_0^\infty e^\tau g(\tau)\|(-\Delta)^{\frac{\alpha}{2}} f_2(\tau)\| \int_\tau^\infty e^{-s} ds d\tau \\
&= \int_0^\infty g(\tau)\|(-\Delta)^{\frac{\alpha}{2}} f_2(\tau)\| d\tau < \infty,
\end{aligned}$$

which implies that $f^* \in H^{-\alpha}(O)$. Then, thanks to Lax-Milgram's theorem, Eq. (2.2.7) has a weak solution

$$\tilde{v} \in H^{\alpha}(O).$$

Now, in view of (2.2.5), it follows

$$\tilde{\eta}^t(s) = \tilde{v}(1 - e^{-s}) + \int_0^s f_2(\tau) e^{\tau - s} d\tau.$$

Let us show that $\tilde{\eta}^t \in M$. From (2.2.6), taking into account that $\tilde{v} \in H^{\alpha}(O)$, we obtain

$$\|(-\Delta)^{\frac{\alpha}{2}} \tilde{\eta}^t(s)\|^2 \leqslant \|(-\Delta)^{\frac{\alpha}{2}} \tilde{v}\|^2 + \int_0^s e^{\tau - s} \|(-\Delta)^{\frac{\alpha}{2}} f_2(\tau)\|^2 d\tau.$$

Then, as above in the proof of f^*,

$$\begin{aligned}
\int_0^{\infty} g(s) \|(-\Delta)^{\frac{\alpha}{2}} \tilde{\eta}^t(s)\|^2 ds &\leqslant k_0 \|(-\Delta)^{\frac{\alpha}{2}} \tilde{v}\|^2 + \int_0^{\infty} g(s) \int_0^s e^{\tau - s} \|(-\Delta)^{\frac{\alpha}{2}} f_2(\tau)\|^2 d\tau ds \\
&\leqslant k_0 \|(-\Delta)^{\frac{\alpha}{2}} \tilde{v}\|^2 + \int_0^{\infty} g(\tau) \|(-\Delta)^{\frac{\alpha}{2}} f_2(\tau)\|^2 d\tau \\
&\leqslant k_0 \|(-\Delta)^{\frac{\alpha}{2}} \tilde{v}\|^2 + \|f_2(\tau)\|_M^2 < \infty,
\end{aligned}$$

and hence $\tilde{\eta}^t \in M$. It follows that

$$\tilde{\varphi} = (\tilde{v}, \tilde{\eta}^t) \in \mathcal{H}$$

is a weak solution of (2.2.4)-(2.2.5).

To complete the proof of maximality of L we prove that $\tilde{\varphi} \in D(L)$. Indeed, from (2.2.5) we see that

$$\partial_s \tilde{\eta}^t = f_2 + \tilde{v} - \tilde{\eta}^t \in M.$$

Obviously $\tilde{\eta}(0) = 0$, we conclude that $\tilde{\eta}^t \in D(T)$. By inspection of (2.2.4) we find that

$$(-\Delta)^{\alpha} \tilde{v} + \int_0^{\infty} g(s) (-\Delta)^{\alpha} \tilde{\eta}^t(s) ds = -\tilde{v} + f_1 \in L^2(O).$$

Therefore $(\tilde{v}, \tilde{\eta}) \in D(L)$. □

Lemma 2.2.2. *The operator $F : \mathcal{H} \to \mathcal{H}$ defined in (2.2.3) is locally Lipschitz continuous.*

Proof. Let B be a bounded set in $\mathcal{H}$ and $\varphi, \bar{\varphi} \in B$. Writing $\varphi = (v, \eta^t)$, $\bar{\varphi} = (\bar{v}, \bar{\eta}^t)$ and using (H2), one obtains

$$\begin{aligned}
\|F(\varphi, \theta_t \omega) - F(\bar{\varphi}, \theta_t \omega)\|_{\mathcal{H}}^2 &= \|f(\bar{\varphi}) - f(\varphi)\|_2^2 \\
&= \int_O |\, f(\bar{v} + z) - f(v + z)\,|^2 \, dx \\
&= \int_O |\, -f' \cdot (v - \bar{v})\,|^2 \, dx \\
&\leqslant \int_O |\, \alpha_3 (v - \bar{v})\,|^2 \, dx \\
&\leqslant \alpha_3^2 \|v - \bar{v}\|^2 \\
&\leqslant \alpha_3^2 \|\varphi - \bar{\varphi}\|_{\mathcal{H}}^2.
\end{aligned}$$

□

To complete the existence of solution, we still need the following lemma.

Lemma 2.2.3. *Assume that (H1)-(H2) hold. Then for any fixed $T > 0$, the solution φ of problem* (2.2.1) *satisfies the following inequality:*

$$\|\varphi(t,\omega,\varphi_0)\|^2_{\mathcal{H}} \leqslant \|\varphi_0\|^2_{\mathcal{H}} + c\int_0^T e^{\lambda s}(\|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|_p^p + \|(-\Delta)^{\frac{\alpha}{2}}z(\theta_s\omega)\|^2)ds$$
$$+ c(e^{\lambda T} - 1), \quad \forall t \in [0,T].$$

Proof. Taking the inner product of (2.2.1) with φ in $\mathcal{H}$ yields

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2_{\mathcal{H}} = (L\varphi,\varphi)_{\mathcal{H}} + (F(\varphi,\theta_t\omega),\varphi)_{\mathcal{H}}, \tag{2.2.8}$$

where

$$(L\varphi,\varphi)_{\mathcal{H}} = -\|(-\Delta)^{\frac{\alpha}{2}}v\|^2 - \int_0^\infty g(s)\int_{\mathcal{O}}(-\Delta)^{\frac{\alpha}{2}}\eta^t\cdot(-\Delta)^{\frac{\alpha}{2}}v\,dxds + (-\partial_s\eta^t + v,\eta^t)_M. \tag{2.2.9}$$

From (H1), we have

$$\begin{aligned}(-\partial_s\eta^t + v,\eta^t)_M &= \frac{1}{2}\int_0^\infty g'(s)\|(-\Delta)^{\frac{\alpha}{2}}\eta^t\|^2ds + \int_0^\infty g(s)\int_{\mathcal{O}}(-\Delta)^{\frac{\alpha}{2}}v\cdot(-\Delta)^{\frac{\alpha}{2}}\eta^t dxds\\ &\leqslant -\frac{\delta}{2}\int_0^\infty g(s)\|(-\Delta)^{\frac{\alpha}{2}}\eta^t\|^2ds + \int_0^\infty g(s)\int_{\mathcal{O}}(-\Delta)^{\frac{\alpha}{2}}v\cdot(-\Delta)^{\frac{\alpha}{2}}\eta^t dxds.\end{aligned} \tag{2.2.10}$$

On the other hand,

$$(F(\varphi,\theta_t\omega),\varphi)_{\mathcal{H}} = \int_{\mathcal{O}}\Big(k - f(u) - (-\Delta)^\alpha z + z\Big)v\,dx + (z,\eta^t)_M. \tag{2.2.11}$$

By Hölder's inequality and Young's inequality, we obtain

$$(z,\eta^t)_M = \int_0^\infty g(s)\int_{\mathcal{O}}(-\Delta)^{\frac{\alpha}{2}}z\cdot(-\Delta)^{\frac{\alpha}{2}}\eta^t dxds \leqslant \frac{\delta}{4}\|\eta^t\|^2_M + c\|(-\Delta)^{\frac{\alpha}{2}}z\|^2. \tag{2.2.12}$$

From (H2), and Young's inequality,

$$\begin{aligned}-\int_{\mathcal{O}}f(u)v\,dx &\leqslant -\frac{\alpha_1}{2}\|u\|_p^p + c(1 + \|z\|^2 + \|z\|_p^p),\\ \int_{\mathcal{O}}kv\,dx &\leqslant \frac{\lambda_1}{8}\|v\|^2 + \frac{2\|k\|^2}{\lambda_1},\\ (-(-\Delta)^\alpha z, v) &\leqslant \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}v\|^2 + \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}z\|^2,\\ (z,v) &\leqslant \frac{\lambda_1}{8}\|v\|^2 + \frac{2}{\lambda_1}\|z\|^2.\end{aligned} \tag{2.2.13}$$

It follows from (2.2.8)-(2.2.13) that

$$\frac{d}{dt}\|\varphi\|_{\mathcal{H}}^2 + \|(-\Delta)^{\frac{\alpha}{2}}v\|^2 + \frac{\delta}{2}\|\eta^t\|_M^2 + \alpha_1\|u\|_p^p \leqslant \frac{\lambda_1}{2}\|v\|^2 + c(1 + \|z\|^2 + \|z\|_p^p + \|(-\Delta)^{\frac{\alpha}{2}}z\|^2).$$

By Young's inequality with $\frac{1}{p/2} + \frac{1}{p/(p-2)} = 1$, we obtain

$$\lambda_1\|v\|^2 \leqslant \frac{\alpha_1}{2^p}\|v\|_p^p + c|\mathcal{O}| \leqslant \alpha_1\|u\|_p^p + c(1 + \|z\|_p^p).$$

Take $\lambda = \min\{\frac{\lambda_1}{2}, \frac{\delta}{2}\}$, then

$$\frac{d}{dt}\|\varphi\|_{\mathcal{H}}^2 + \lambda\|\varphi\|_{\mathcal{H}}^2 + \|(-\Delta)^{\frac{\alpha}{2}}v\|^2 \leqslant c(1 + \|z\|^2 + \|z\|_p^p + \|(-\Delta)^{\frac{\alpha}{2}}z\|^2). \tag{2.2.14}$$

By the Gronwall lemma,

$$\begin{aligned}\|\varphi(t,\omega,\varphi_0(\omega))\|_{\mathcal{H}}^2 \leqslant e^{-\lambda t}\|\varphi_0(\omega)\|_{\mathcal{H}}^2 + c\int_0^t e^{\lambda(s-t)}\Big(1 + \|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|_p^p \\ + \|(-\Delta)^{\frac{\alpha}{2}}z(\theta_s\omega)\|^2\Big)ds.\end{aligned} \tag{2.2.15}$$

Notice that $z(\theta_t\omega)$ is continuous in t, for any fixed $T > 0$ and $t \in [0, T]$. Then, we obtain

$$\begin{aligned}\|\varphi(t,\omega,\varphi_0(\omega))\|_{\mathcal{H}}^2 \leqslant \|\varphi_0(\omega)\|_{\mathcal{H}}^2 + c\int_0^T e^{\lambda s}\Big(\|z(\theta_s\omega)\|^2 + \|z(\theta_s\omega)\|_p^p \\ + \|(-\Delta)^{\frac{\alpha}{2}}z(\theta_s\omega)\|^2\Big)ds + c(e^{\lambda T} - 1) < \infty.\end{aligned}$$

The proof is completed. □

Theorem 2.2.4. *(Well-posedness) Assume that hypotheses (H1)-(H2) are satisfied and initial data $(u_0, \eta_0) \in \mathcal{H}$. Then, problem (2.1.6)-(2.1.8) possesses a unique mild solution in the class*

$$u \in C([0,\infty); L^2(\mathcal{O})), \ \text{and}\ \eta^t \in C([0,\infty); M). \tag{2.2.16}$$

If initial data $(u_0, \eta_0) \in D(L)$, then the solution is more regular, i.e., $u \in C([0,\infty); H^\alpha(\mathcal{O}))$, and $\eta^t \in C([0,\infty); M_1)$. In addition, if $w(t) = (u, \eta^t)$ and $\bar{w}(t) = (\bar{u}, \bar{\eta}^t)$ are two mild solutions of (2.1.6)-(2.1.8), then for any $T > 0$,

$$\|w(t) - \bar{w}(t)\|_{\mathcal{H}}^2 \leqslant e^{cT}\|w(0) - \bar{w}(0)\|_{\mathcal{H}}^2, \quad 0 \leqslant t \leqslant T, \tag{2.2.17}$$

where $c > 0$ is a constant independent of the initial data.

Proof. From Lemma 2.2.1 and 2.2.2, and Lumer-Phillip's theorem (see for instance Pazy [139], Theorem 6.1.4 and 6.1.5), problem (2.2.1) has a unique local mild solution

$$\varphi(t,\omega,\varphi_0) = e^{Lt}\varphi_0(\omega) + \int_0^t e^{L(t-r)}F(\varphi(r,\omega,\varphi_0),\theta_r\omega)dr \tag{2.2.18}$$

defined in $[0, T]$.

Let us prove that $T = \infty$. Indeed, Lemma 2.2.3 implies that the local solution (v, η^t) cannot blow-up in finite time and thus $T = \infty$. Hence, problem (2.2.1) has a global solution $\varphi(\cdot, \omega, \varphi_0) \in C([0, \infty), \mathcal{H})$ with $\varphi(0, \omega, \varphi_0) = \varphi_0(\omega)$ for all $t \geqslant 0$. Then, (2.2.16) holds. Moreover, the continuity with respect to initial data, i.e. Eq. (2.2.17), follows from the representation formula (2.2.18) and the Lipschitz property of F. □

Note that $u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t\omega)$. Then the process $\phi = (u, \eta^t)$ is the solution of problem (2.0.1)-(2.0.3). We now define a mapping $\Phi : \mathbb{R}^+ \times \Omega \times \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned} \Phi(t, \omega)\phi_0 &= \phi(t, \omega, \phi_0) \\ &= (u(t, \omega, u_0), \eta^t(\omega, \eta_0)) \\ &= (v(t, \omega, u_0 - z(\omega)) + z(\theta_t\omega), \eta^t(\omega, \eta_0)), \quad \text{for all } (t, \omega, \phi_0) \in \mathbb{R}^+ \times \Omega \times \mathcal{H}. \end{aligned} \tag{2.2.19}$$

It is not difficult to check that Φ is a continuous random dynamical system associated to Eq.(2.0.1). In the next section, we establish uniform estimates for the solutions of problem (2.0.1)-(2.0.3) and prove the existence of a random attractor for Φ.

2.3 Existence of random attractor

In this section we prove the existence of random attractor for our problem. We begin with the uniform estimates of solutions that will be necessary for our analysis.

2.3.1 A priori estimates

Now, we first prove the existence of random absorbing sets for the RDS Φ, which is necessary to establish the existence of random attractors. From now on, we always assume that $\mathcal{D}$ is the collection of all tempered subsets of $\mathcal{H}$ with respect to $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. The next lemma shows that Φ has a random absorbing set in $\mathcal{H}$.

Lemma 2.3.1. *Assume that $(H1) - (H2)$ hold. Then there exists a random absorbing set $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ for Φ in $\mathcal{H}$, i.e., for any $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $P - a.e.\ \omega \in \Omega$, there is $T_{1B}(\omega) > 0$ such that*

$$\Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega), \quad \forall\ t \geqslant T_{1B}(\omega).$$

Proof. The process is similar to that of Lemma 2.2.3 with slight modifications. We only sketch it. We first derive uniform estimates on $\varphi(t, \omega, \varphi_0) = (v(t, \omega, v_0), \eta^t(\omega, \eta_0)) = (u(t, \omega, u_0) - z(\theta_t\omega), \eta^t(\omega, \eta_0))$, from which the uniform estimates on $\phi = (u(t, \omega, u_0), \eta^t(\omega, \eta_0))$ follow immediately. Multiply (2.2.14) by $e^{\lambda t}$ and integrate over $[0, t]$ to obtain

$$\begin{aligned} &\|\varphi(t, \omega, \varphi_0)\|^2_{\mathcal{H}} + \int_0^t e^{\lambda(s-t)}\|(-\Delta)^{\frac{\alpha}{2}} v(s, \omega, v_0(\omega))\|^2 ds \\ &\leqslant e^{-\lambda t}\|\varphi_0(\omega)\|^2_{\mathcal{H}} + c\int_0^t e^{\lambda(s-t)}(1 + \beta(\theta_s\omega))ds, \end{aligned} \tag{2.3.1}$$

where

$$\beta(\theta_t\omega) := \|z(\theta_t\omega)\|^2 + \|z(\theta_t\omega)\|_p^p + \|(-\Delta)^{\frac{\alpha}{2}} z(\theta_t\omega)\|^2 \leqslant r(\theta_t\omega),$$

and $r(\theta_t\omega)$ satisfies

$$r(\theta_t\omega) \leqslant e^{\frac{\lambda}{2}|t|} r(\omega), \ \ t \in \mathbb{R}.$$

Replacing ω by $\theta_{-t}\omega$ in (2.3.1) yields

$$\begin{aligned}
&\|\varphi(t, \theta_{-t}\omega, \varphi_0(\theta_{-t}\omega))\|_{\mathcal{H}}^2 + \int_0^t e^{\lambda(s-t)} \|(-\Delta)^{\frac{\alpha}{2}} v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 ds \\
&\leqslant e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + c\int_0^t e^{\lambda(s-t)}(1 + \beta(\theta_{s-t}\omega))ds \\
&\leqslant e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + c\int_{-t}^0 e^{\lambda s} e^{-\frac{\lambda}{2}s} r(\omega)dr + c(1 - e^{-\lambda t}) \\
&\leqslant e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + \frac{cr(\omega)}{\lambda}(1 - e^{-\frac{\lambda}{2}t}) + c.
\end{aligned} \tag{2.3.2}$$

Note that $\Phi(t, \omega)\phi_0(\omega) = \phi(t, \omega, \phi_0(\omega)) = (v(t, \omega, u_0 - z(\omega)) + z(\theta_t\omega), \eta^t(\omega, \eta_0))$. Consequently, from (2.3.2), we have, for all $t \geqslant 0$,

$$\begin{aligned}
&\|\Phi(t, \theta_{-t}\omega)\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 \\
&= \|v(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega)) + z(\omega)\|^2 + \|\eta^t(\theta_{-t}\omega, \eta_0(\theta_{-t}\omega))\|_M^2 \\
&\leqslant 2\|v(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega))\|^2 + 2\|z(\omega)\|^2 + \|\eta^t(\theta_{-t}\omega, \eta_0(\theta_{-t}\omega))\|_M^2 \\
&\leqslant 2e^{-\lambda t}(\|u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega)\|^2 + \|\eta_0(\theta_{-t}\omega)\|_M^2) + cr(\omega) + c + 2\|z(\omega)\|^2 \\
&\leqslant 4e^{-\lambda t}(\|u_0(\theta_{-t}\omega)\|^2 + \|\eta_0(\theta_{-t}\omega)\|_M^2 + \|z(\theta_{-t}\omega)\|^2) + cr(\omega) + c + 2\|z(\omega)\|^2 \\
&= 4e^{-\lambda t}(\|\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + \|z(\theta_{-t}\omega)\|^2) + cr(\omega) + c + 2\|z(\omega)\|^2.
\end{aligned} \tag{2.3.3}$$

Since $\phi_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)(\in \mathcal{D})$ and $\|z(\omega)\|^2$ is tempered, there exists $T_{1B}(\omega) > 0$, such that for all $t \geqslant T_{1B}(\omega)$,

$$4e^{-\lambda t}(\|\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + \|z(\theta_{-t}\omega)\|^2) \leqslant cr(\omega) + c,$$

which along with (2.3.3) shows that, for all $t \geqslant T_{1B}(\omega)$,

$$\|\Phi(t, \theta_{-t}\omega)\phi_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 \leqslant c(1 + r(\omega) + \|z(\omega)\|^2) := R_0(\omega). \tag{2.3.4}$$

Given $\omega \in \Omega$, denote by

$$K(\omega) = \left\{\phi \in \mathcal{H} : \ \|\phi(t, \theta_{-t}\omega, \phi_0(\theta_{-t}\omega))\|_{\mathcal{H}}^2 \leqslant R_0(\omega)\right\}.$$

It is obviously that $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Further, (2.3.4) indicates that $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for Φ in $\mathcal{H}$, which completes the proof. $\square$

We next derive uniform estimates for u in $H^{\alpha}(\mathcal{O})$.

Lemma 2.3.2. *Assume that* $(H1)-(H2)$ *hold. Let* $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$. *Then there exists* $T_{2B}(\omega)>T_{1B}(\omega)$, *such that for all* $t\geqslant T_{2B}(\omega)$ *and* $P-a.e.\ \omega\in\Omega$, *it follows*

$$\int_t^{t+1} e^{\lambda(s-t)}\|(-\Delta)^{\frac{\alpha}{2}}u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|^2ds\leqslant cR_0(\omega),$$

where $R_0(\omega)$ *is defined as in Lemma 2.3.1.*

Proof. By a similar procedure as it was done in Lemma 4.3 in [11], we can obtain

$$\int_t^{t+1}\|(-\Delta)^{\frac{\alpha}{2}}v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2ds\leqslant c(1+r(\omega)).$$

On the other hand,

$$\begin{aligned}&\|(-\Delta)^{\frac{\alpha}{2}}u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|^2=\|(-\Delta)^{\frac{\alpha}{2}}v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))+(-\Delta)^{\frac{\alpha}{2}}z(\theta_{s-t-1}\omega)\|^2\\&\leqslant 2\|(-\Delta)^{\frac{\alpha}{2}}v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2+2\|(-\Delta)^{\frac{\alpha}{2}}z(\theta_{s-t-1}\omega)\|^2 \text{ for all } t\geqslant 0.\end{aligned}\tag{2.3.5}$$

Integrating inequality (2.3.5) with respect to s over $[0,t]$, one can check that there exists $T_{2B}(\omega)>T_{1B}(\omega)$, such that for all $t>T_{2B}(\omega)$ we have

$$\int_t^{t+1}\|(-\Delta)^{\frac{\alpha}{2}}u(s,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|^2ds\leqslant c(1+r(\omega)+\|z(\omega)\|^2).$$

The proof is therefore completed. □

In order to show the existence of random attractor for Φ associated with the problem (2.0.1)-(2.0.3), we need to prove the existence of compact measurable attracting set of Φ.

2.3.2 Asymptotic compactness

In this subsection, our main purpose is to obtain a random compact attracting set of Φ. To this end, we decompose the solution of (2.2.1) into a sum of two parts: one decays exponentially and the other is bounded in a "higher regular" space by using the method in [81], and obtain some a priori estimates for the solutions, which are the basis to construct a compact measurable attracting set for Φ. More precisely, we split the solution φ to (2.2.1) as the sum $\varphi=\varphi_L+\varphi_N$, where $\varphi_L=\varphi_L(t,\omega,\varphi_0)=(v_L,\eta_L^t)$ and $\varphi_N=\varphi_N(t,\omega,\varphi_0)=(v_N,\eta_N^t)$ satisfy, respectively,

$$\begin{cases}\partial_t\varphi_L=L\varphi_L,\\ \varphi_L(t,\omega,\varphi_0)=\varphi_{0L}(\omega)=(v_0,\eta_0),\quad s\geqslant 0,\end{cases}\tag{2.3.6}$$

and

$$\begin{cases}\partial_t\varphi_N=L\varphi_N+F(\varphi,\theta_t\omega),\\ \varphi_N(t,\omega,\varphi_0)=(0,0),\quad s\geqslant 0.\end{cases}\tag{2.3.7}$$

First we have to show that φ_L has an exponential decay, that is,

$$\|\varphi_L(t,\theta_{-t}\omega,\varphi_{0L}(\theta_{-t}\omega))\|^2_{\mathcal{H}} \leqslant e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|^2_{\mathcal{H}}, \quad \forall\ \varphi_0(\theta_{-t}\omega)\in\mathcal{H}. \tag{2.3.8}$$

It is apparent that the solution φ_L to (2.3.6) fulfills the estimates (2.3.2) with $c=0$, namely,

$$\|\varphi_L(t,\theta_{-t}\omega,\varphi_{0L}(\theta_{-t}\omega))\|^2_{\mathcal{H}} \leqslant e^{-\lambda t}\|\varphi_{0L}(\theta_{-t}\omega)\|^2_{\mathcal{H}} = e^{-\lambda t}\|\varphi_0(\theta_{-t}\omega)\|^2_{\mathcal{H}}. \tag{2.3.9}$$

Note that $\phi_L=\varphi_L$, we have

$$\|\phi_L(t,\theta_{-t}\omega,\phi_{0L}(\theta_{-t}\omega))\|^2_{\mathcal{H}} \leqslant e^{-\lambda t}\|\phi_0(\theta_{-t}\omega)\|^2_{\mathcal{H}}. \tag{2.3.10}$$

Since

$$\|\phi_N(t,\theta_{-t}\omega,0)\|^2_{\mathcal{H}} \leqslant 2\|\phi(t,\theta_{-t}\omega,\phi_0(\theta_{-t}\omega))\|^2_{\mathcal{H}} + 2\|\phi_L(t,\theta_{-t}\omega,\phi_{0L}(\theta_{-t}\omega))\|^2_{\mathcal{H}},$$

we also have

$$\|\phi_N(t,\theta_{-t}\omega,0)\|^2_{\mathcal{H}} \leqslant 10e^{-\lambda t}\|\phi_0(\theta_{-t}\omega)\|^2_{\mathcal{H}} + c(1+r(\omega)). \tag{2.3.11}$$

For further reference, we denote by $\eta^t_N(\omega,\eta_0)$ the second component of the solution ϕ_N to (2.3.6) at time t with initial time 0 and initial value $\phi(0,\omega,\phi_0)=\phi_0(\omega)$. Observe that η^t_N can be computed explicitly from the second component of (2.3.7) and the zero boundary data as follows:

$$\eta^t_N(\omega,\eta_0) = \begin{cases} \int_0^s u_N(t-r)dr, & 0<s\leqslant t,\\ \int_0^t u_N(t-r)dr, & s>t. \end{cases} \tag{2.3.12}$$

Our goal is to build a compact attracting set for the random dynamical system Φ.

Lemma 2.3.3. *Assume that (H1)-(H2) hold, $\alpha\in[\frac{1}{2},1)$ and $p\in[2,1+\frac{3}{3-2\alpha})$. Then there exists $T_{3B}(\omega)>T_{2B}(\omega)$, such that for all $t\geqslant T_{3B}(\omega)$ and P-a.e. $\omega\in\Omega$, it follows*

$$\|\phi_N(t,\theta_{-t}\omega,0)\|^2_{\mathcal{V}} + \frac{1}{2}\int_0^t e^{\lambda(s-t)}\|(-\Delta)^\alpha v_N(s,\theta_{-t-1}\omega,0)\|^2ds \leqslant R_1(\omega),$$

where $R_1(\omega):=c\Big(1+C(\lambda)\Big(R_0^2(\omega)+R_0(\omega)+r(\omega)+1\Big)\Big)$.

Proof. First we take the inner product of the first part of (2.3.7) with $(-\Delta)^\alpha v_N$ in $L^2(O)$ to deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{\frac{\alpha}{2}}v_N\|^2 = &-\|(-\Delta)^\alpha v_N\|^2 - \int_0^\infty g(s)\int_O(-\Delta)^\alpha\eta^t_N\cdot(-\Delta)^\alpha v_N dxds\\ &+\int_O(k-f(u)-(-\Delta)^\alpha z+z)\cdot(-\Delta)^\alpha v_N dx.\end{aligned} \tag{2.3.13}$$

Using (H2), Lemma A.1.7 and Young's inequality, we obtain

$$
\begin{aligned}
-\int_{\mathcal{O}} f(u)(-\Delta)^{\alpha} v_N dx &\leqslant \frac{1}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c\|1+|u|^{p-1}\|^2 \\
&\leqslant \frac{1}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c + c\|u\|_{2p-2}^{2p-2} \\
&\leqslant \frac{1}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c + c\|(-\Delta)^{\frac{\alpha}{2}} u\|^{\zeta}\|u\|^{1-\zeta} \\
&\leqslant \frac{1}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c + c(1+\|(-\Delta)^{\frac{\alpha}{2}} u\|^2)(1+\|u\|^2),
\end{aligned}
\tag{2.3.14}
$$

where $\zeta = \frac{3}{2\alpha}(\frac{p-2}{p-1})$.

On the other hand, by Young's inequality, we have

$$
\begin{aligned}
\int_{\mathcal{O}} (k(x) - (-\Delta)^{\alpha} z)\cdot(-\Delta)^{\alpha} v_N dx &\leqslant \frac{3}{8}\|(-\Delta)^{\alpha} v_N\|^2 + c(1+\|(-\Delta)^{\alpha} z\|^2), \\
\int_{\mathcal{O}} z\cdot(-\Delta)^{\alpha} v_N dx &\leqslant \frac{\lambda_1}{4}\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 + \frac{1}{\lambda_1}\|(-\Delta)^{\frac{\alpha}{2}} z\|^2.
\end{aligned}
$$

By the precedent inequalities,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 = \frac{\lambda_1}{4}\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 - \frac{1}{2}\|(-\Delta)^{\alpha} v_N\|^2 - \int_0^{\infty} g(s)\int_{\mathcal{O}} (-\Delta)^{\alpha}\eta_N^t\cdot(-\Delta)^{\alpha} v_N dxds \\
+ c(1+\|(-\Delta)^{\frac{\alpha}{2}} u\|^2)(1+\|u\|^2) + c(1+\|(-\Delta)^{\frac{\alpha}{2}} z\|^2 + \|(-\Delta)^{\alpha} z\|^2).
\end{aligned}
\tag{2.3.15}
$$

Taking now the inner product of the second part of (2.3.7) with $(-\Delta)^{2\alpha}\eta_N^t$, and thanks to similar computations as above,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{2\alpha}\eta_N^t\|_{\mathcal{M}_1}^2 &= -\int_0^{\infty} g(s)\int_{\mathcal{O}} \eta_{N,s}^t\cdot(-\Delta)^{2\alpha}\eta_N^t dxds + \int_0^{\infty} g(s)\int_{\mathcal{O}} v\cdot(-\Delta)^{2\alpha}\eta_N^t dxds \\
&\quad + \int_0^{\infty} g(s)\int_{\mathcal{O}} z\cdot(-\Delta)^{2\alpha}\eta_N^t dxds \\
&\leqslant -\frac{\delta}{4}\int_0^{\infty} g(s)\|(-\Delta)^{\alpha}\eta_N^t\|^2 ds - \int_0^{\infty} g(s)\int_{\mathcal{O}} (-\Delta)^{\alpha} v\cdot\Delta\eta_N^t dxds + c\|(-\Delta)^{\alpha} z\|^2.
\end{aligned}
\tag{2.3.16}
$$

Adding (2.3.15) and (2.3.16),

$$
\begin{aligned}
&\frac{d}{dt}\|\varphi_N\|_{\mathcal{V}}^2 + \frac{\delta}{2}\|(-\Delta)^{\alpha}\eta_N^t\|_{\mathcal{M}_1}^2 + \|(-\Delta)^{\alpha} v_N\|^2 \\
&\leqslant \frac{\lambda_1}{2}\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 + cp(\theta_t\omega) + c(1+\|(-\Delta)^{\frac{\alpha}{2}} u\|^2)(1+\|u\|^2) + c,
\end{aligned}
$$

Using Gagliardo-Nirenberg's inequality A.1.7,

$$
\lambda_1\|(-\Delta)^{\frac{\alpha}{2}} v_N\|^2 \leqslant \frac{1}{2}\|(-\Delta)^{\alpha} v_N\|^2 + c\|v_N\|^2.
$$

Taking $\lambda = \min\{\frac{\lambda_1}{2}, \frac{\delta}{2}\}$, by the previous inequalities we have

$$\frac{d}{dt}\|\varphi_N\|_{\mathcal{V}}^2 + \lambda\|\varphi_N\|_{\mathcal{V}}^2 + \frac{1}{2}\|(-\Delta)^{\alpha}v_N\|^2 \leqslant cp(\theta_t\omega) + c(1 + \|(-\Delta)^{\frac{\alpha}{2}}u\|^2)(1 + \|u\|^2) + c\|v_N\|^2 + c, \quad (2.3.17)$$

where $p(\theta_t\omega) = (1 + \|(-\Delta)^{\frac{\alpha}{2}}z(\theta_t\omega)\|^2 + \|(-\Delta)^{\alpha}z(\theta_t\omega)\|^2)$.

On the one hand, Lemma 2.3.1 and Lemma 2.3.2 ensure that there exists $T_{3B}(\omega) > T_{2B}(\omega)$ such that for all $t \geqslant T_{3B}(\omega)$,

$$\begin{aligned}&\int_t^{t+1}(1 + \|(-\Delta)^{\frac{\alpha}{2}}u(s, \theta_{s-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2)(1 + \|u(s, \theta_{s-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2)ds\\ &+ \int_t^{t+1}\|v_N(s, \theta_{s-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2 ds + \int_t^{t+1} p(\theta_{s-t-1}\omega)ds\\ &\leqslant cR_0^2(\omega) + cR_0(\omega) + cr(\omega) + c.\end{aligned}$$

Then, by Lemma A.1.7 and Lemma A.1.8 we can prove that for all $t \geqslant T_{3B}(\omega)$,

$$\begin{aligned}&\|\phi_N(t, \theta_{-t}\omega, 0)\|_{\mathcal{V}}^2 + \frac{1}{2}\int_0^t e^{\lambda(s-t)}\|(-\Delta)^{\alpha}v_N(s, \theta_{-t-1}\omega, 0)\|^2 ds\\ &\leqslant c\Big(1 + C(\lambda)\Big(R_0^2(\omega) + R_0(\omega) + r(\omega) + 1\Big)\Big) := R_1(\omega),\end{aligned}$$

as claimed. □

Remark 2.3.4. *Notice that, unlike the previous results, we are imposing now some restrictions on the values of α and p in Lemma 2.3.3. Indeed, the constant $\zeta = \frac{3}{2\alpha}(\frac{p-2}{p-1})$, appearing in (2.3.14), must belong to the interval $(0, 1)$, and this implies that, for a given $\alpha \in [\frac{1}{2}, 1)$, p has to belong to the interval $[2, 1 + \frac{3}{3-2\alpha})$ (see Figure 1 below). We would like to emphasize that the statement in Lemma 2.3.3 also holds true for $\alpha \in (0, \frac{1}{2})$, but as we will need to impose $\alpha \in [\frac{1}{2}, 1)$ in Lemma 2.3.6 to ensure asymptotic compactness of our random dynamical system, we prefer to state it in this way.*

Remark 2.3.5. *Caraballo et al. [38] proved the existence of a random attractor for random dynamical systems associated to (2.0.1) with $\alpha = 1$ and $p \geqslant 1$, while [55] investigated the deterministic version of (2.0.1) (i.e. $h(x) = 0$) with $\alpha = 1$ dealing with global attractors for the whole range $p < 4$. And in [121], authors considered (2.0.1) with $\mu = 0$ in the whole space $\mathbb{R}^n$, they assume that $p > 1$ and $\alpha \in [\frac{1}{2}, 1)$ hold, and proved random attractor in $L^2(\mathbb{R}^n)$.*

We now are in the position to finalize the proof of the existence of a random attractor.

Lemma 2.3.6. *Assume that $(H1) - (H2)$ hold, $\alpha \in [\frac{1}{2}, 1)$ and $p \in [2, 1 + \frac{3}{3-2\alpha})$. Denote by*

$$\mathcal{N} = \bigcup_{\eta_0 \in K(\theta_{-t}\omega)}\ \bigcup_{t \geqslant T_{3B}(\omega)}\ \bigcup_{\omega\in\Omega} \eta_N^t(\theta_{-t}\omega, \eta_0),$$

where $\{K(\omega)\}_{\omega\in\Omega}$ is defined in Lemma 2.3.1 and $T_{3B}(\omega)$ is defined in Lemma 2.3.3. Then $\mathcal{N}$ is relatively compact in $L_g^2(\mathbb{R}^+, H^{\alpha}(\mathcal{O}))$.

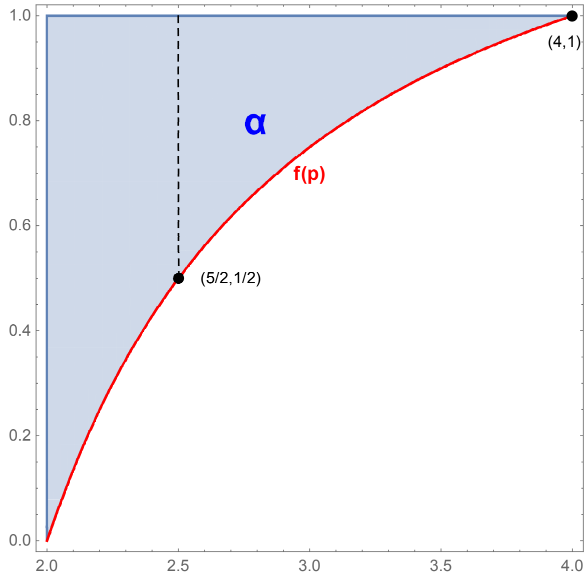

Figure 2.1: $\alpha > f(p) = \frac{3}{2}(1 - \frac{1}{p-1})$

Proof. It is clear from Lemma 2.3.3 that $\mathcal{N}$ is bounded in $L^2_g(\mathbb{R}^+, H^{2\alpha}(O))$. Let $\eta^t_N \in \mathcal{N}$. The derivative of (2.3.12) yields

$$\frac{\partial}{\partial s}\eta^t_N = \begin{cases} u_N(t-s), & 0 < s \leqslant t, \\ 0, & s > t. \end{cases} \tag{2.3.18}$$

Thus

$$\begin{aligned} \int_0^\infty g(s)\left\|\frac{\partial}{\partial s}\eta^t_N\right\|^2 ds &= \int_0^t g(s)\|u_N(t-s)\|^2 ds = \int_0^t g(t-s)\|u_N(s)\|^2 ds \\ &\leqslant g(0)\int_0^t e^{\lambda(s-t)}\|u_N(s)\|^2 ds < \infty. \end{aligned} \tag{2.3.19}$$

We then conclude that $\mathcal{N}$ is bounded in $L^2_g(\mathbb{R}^+, H^{2\alpha}(O)) \cap H^1_g(\mathbb{R}^+, L^2(O))$. Moreover, we can verify that, for every $\eta^t \in \mathcal{N}$,

$$\sup_{\eta^t\in\mathcal{N}, s\geqslant 0} \|\nabla\eta^t\|^2 = \begin{cases} s\cdot\int_{t-s}^t \|\nabla u_N(r)\|^2 dr, & 0 < s \leqslant t, \\ s\cdot\int_0^t \|\nabla u_N(r)\|^2 dr, & s > t. \end{cases}$$

By the embedding $H^{2\alpha}(O) \hookrightarrow H^1_0(O)$, we find that

$$\sup_{\eta^t\in\mathcal{N}, s\geqslant 0} \|\nabla\eta^t\|^2 \leqslant se^{\lambda s}\cdot\int_0^t e^{\lambda(r-t)}\|(-\Delta)^\alpha u_N(r)\|^2 dr := h(s), \quad t \geqslant 0.$$

Consequently, from Lemma 2.3.3 and the relation $u = v + z$, it is obvious that

$$\int_0^\infty g(s)\|\nabla\eta^t(s)\|^2 ds \leqslant \int_0^\infty sg(s)e^{\lambda s} ds\int_0^t e^{\lambda(r-t)}\|(-\Delta)^\alpha u_N(r)\|^2 dr < \infty,$$

which shows that $\mathcal{N} \subset L^2_g(\mathbb{R}^+, H^\alpha(\mathcal{O}))$ is a bounded set and $h(s) \in L^1_g(\mathbb{R}^+)$. Using A.1.10, the proof can be completed immediately. □

Now we restate our main result about existence of random attractor for the RDS Φ:

Theorem 2.3.7. *Assume that* $(H1)-(H2)$ *hold,* $\alpha \in [\frac{1}{2}, 1)$ *and* $p \in [2, 1+\frac{3}{3-2\alpha})$. *Then for every* $\omega \in \Omega$, *the random dynamical system* Φ *associated with Eq.* (2.0.1) *possesses a compact random attracting set* $\tilde{K}(\omega) \subset \mathcal{H}$ *and possesses a random attractor* $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ *with* $\mathcal{A}(\omega) = \tilde{K}(\omega) \cap K(\omega)$, *where* $K = \{K(\omega)\}_{\omega\in\Omega}$ *is defined in Lemma 2.3.1.*

Proof. Let $B_\mathcal{V}(\omega)$ be the closed ball in $\mathcal{V} = H^\alpha(\mathcal{O}) \times L^2_g(\mathbb{R}^+; H^{2\alpha}(\mathcal{O}))$ of radius $R_1(\omega)$. Setting $\tilde{K}(\omega) = B_\mathcal{V}(\omega) \times \overline{\mathcal{N}}$ with $\overline{\mathcal{N}}$ is the closure of $\mathcal{N}$, which is defined in Lemma 2.3.6. Since $H^\alpha(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$ is compact and $\mathcal{N}$ is compact in $L^2_g(\mathbb{R}^+; H^\alpha(\mathcal{O}))$. Thus, $\tilde{K}(\omega)$ is compact in $\mathcal{H}$ with $\mathcal{H} = L^2(\mathcal{O}) \times L^2_g(\mathbb{R}^+; H^\alpha(\mathcal{O}))$. Now we show the following attracting property of $\tilde{K}(\omega)$ holds for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, i.e.,

$$\lim_{t\to+\infty} \operatorname{dist}_\mathcal{H}\left(\Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \tilde{K}(\omega)\right) = 0. \tag{2.3.20}$$

By Lemma 2.3.1, there exits $t^* = t^*(B) > 0$ such that

$$\Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega), \quad \forall t \geqslant t^*, \tag{2.3.21}$$

where $K = \{K(\omega)\}_{\omega\in\Omega}$ is the absorbing set for Φ in $\mathcal{H}$.

Setting $t = \tilde{t} + t^* + t_1 > 0$, and using the cocycle properties, we deduce that

$$\begin{aligned} \Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) &= \Phi(t - t^* - t_1, \theta_{t^*+t_1}\theta_{-t}\omega) \circ \Phi(t^* + t_1, \theta_{-t}\omega)B(\theta_{-t}\omega) \\ &\subset \Phi(\tilde{t}, \theta_{-\tilde{t}}\omega)K(\omega). \end{aligned} \tag{2.3.22}$$

Pick any $\phi(t, \theta_{-t}\omega, \phi_0(\theta_{-t}\omega)) \in \Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)$ for $t \geqslant t^* + t_1 > 0$. Applying now Lemma 2.3.3 with $T_{3B}(\omega) = t^* + t_1$ implies

$$\|(-\Delta)^{\frac{\alpha}{2}} u_N\|^2 \leqslant \|\phi_N\|^2_\mathcal{V} \leqslant c(R_1(\omega) + \|(-\Delta)^{\frac{\alpha}{2}} z(\omega)\|^2).$$

It is then clear that $\phi_N = (u_N, \eta^t_N) \in \tilde{K}(\omega)$. Therefore, from (2.3.10),

$$\inf_{m\in\tilde{K}(\omega)} \|\phi(t) - m\|_\mathcal{H} \leqslant \|\phi_L\|_\mathcal{H} \leqslant e^{-\frac{\lambda}{2}t}\|\phi_0(\theta_{-t}\omega)\|_\mathcal{H}, \quad \forall\, t > t^* + t_1.$$

We conclude that

$$\operatorname{dist}_\mathcal{H}\left(\Phi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \tilde{K}(\omega)\right) \leqslant e^{-\frac{\lambda}{2}t}\|\phi_0(\theta_{-t}\omega)\|_\mathcal{H} \to 0, \quad \text{as} \quad t \to +\infty.$$

The proof follows immediately from Proposition 0.0.7. □

2.4 Finite Hausdorff dimension

In this section, we prove that the random attractor $\mathcal{A}(\omega)$, whose existence has been proved in Section 2.3, has finite Hausdorff dimension. To this end, we need the following condition on f:

$$|f''(u)| \leqslant \beta_1, \quad \text{for some } \beta_1 > 0. \tag{2.4.1}$$

Set $\Phi(\omega) = \Phi(1, \omega)$ and consider the following first variant equation of equation (2.1.6),

$$\frac{d\tilde{W}}{dt} = L\tilde{W} + F'(\tilde{W}, \theta_t\omega)\tilde{W}, \tag{2.4.2}$$

with

$$\tilde{W}(x,t) = \tilde{W}_0(x,t) = h, \ t \leqslant 0, \tag{2.4.3}$$

$$F^{'}(\tilde{W}, \theta_t\omega)\tilde{W} = (-f^{'}(u)U(t), 0) \tag{2.4.4}$$

and

$$L\tilde{W} = (-(-\Delta)^{\alpha}U - \int_0^{\infty} g(s)(-\Delta)^{\alpha}V(s)ds, U - V_s), \tag{2.4.5}$$

where $\tilde{W} = (U(t), V(t))$ with $U(t)$, $V(t)$ are the derivative of $u(t)$, η^t of problem (2.1.6), respectively.

Lemma 2.4.1. *Assume that $(H1) - (H2)$ hold, $\alpha \in [\frac{1}{2}, 1)$ and $p \in [2, 1 + \frac{3}{3-2\alpha})$, and (2.4.1) is fulfilled. Then the mapping $\Phi(\omega)$ is almost surely uniformly differentiable on $\mathcal{A}(\omega)$: P-a.e. $\omega \in \Omega$, for every $w \in \mathcal{A}(\omega)$, there exists a bounded linear operator $D\Phi(\omega, w)$ such that if w and $w + h$ are in $\mathcal{A}(\omega)$, there holds*

$$\|\Phi(\omega)(w+h) - \Phi(\omega)(w) - D\Phi(\omega, w)h\|_{\mathcal{H}} \leqslant \bar{k}(\omega)\|h\|_{\mathcal{H}}^{1+\rho},$$

where $\rho > 0$ and $\bar{k}(\omega)$ is a random variable such that

$$\bar{k}(\omega) \geqslant 1, \quad E(\ln \bar{k}) < \infty, \quad \omega \in \Omega.$$

Moreover, for any $w \in \mathcal{A}(\omega)$, $D\Phi(\omega, w)h = \tilde{W}(1)$, where $\tilde{W}(t)$ is the solution of Eq.(2.4.2).

Proof. Let $w = (u(t), \eta^t)$, $\bar{w} = (\bar{u}(t), \bar{\eta}^t)$ be solutions to Eq.(2.1.6) with initial data $w(0) = w_0$, $\bar{w}(0) = \bar{w}_0$ and $w_0 - \bar{w}_0 = h$. Then $Y = w - \bar{w}$ satisfies the following problem

$$\frac{dY}{dt} = LY + F(w, \theta_t\omega) - F(\bar{w}, \theta_t\omega) \tag{2.4.6}$$

with $F(w, \theta_t\omega) - F(\bar{w}, \theta_t\omega) = (f(\bar{u}) - f(u), 0)$ and $Y_0 = w_0 - \bar{w}_0 = h$.

Taking the inner product of (2.4.6) with Y in $\mathcal{H}$, we obtain

$$\frac{d}{dt}\|Y\|_{\mathcal{H}}^2 = 2(LY, Y)_{\mathcal{H}} + 2(F(w, \theta_t\omega) - F(\bar{w}, \theta_t\omega), Y)_{\mathcal{H}}. \tag{2.4.7}$$

Notice that

$$
\begin{aligned}
2(F(w,\theta_t\omega)-F(\bar{w},\theta_t\omega),Y)_{\mathcal{H}} &= 2(f(\bar{u})-f(u),u-\bar{u}) \\
&= -2(f'(u)(u-\bar{u}),u-\bar{u}) \\
&\leqslant 2\alpha_3\|u-\bar{u}\|^2 \\
&\leqslant 2\alpha_3\|Y\|^2_{\mathcal{H}}.
\end{aligned} \tag{2.4.8}
$$

and

$$
2(LY,Y)_{\mathcal{H}} = -2\|(-\Delta)^{\frac{\alpha}{2}}(u-\bar{u})\|^2 + \int_0^\infty g'(s)\|(-\Delta)^{\frac{\alpha}{2}}(\eta^t-\bar{\eta}^t)\|^2 ds \leqslant 0. \tag{2.4.9}
$$

Then, it follows from (2.4.7)-(2.4.9) that

$$
\frac{d}{dt}\|Y\|^2_{\mathcal{H}} \leqslant 2\alpha_3\|Y\|^2_{\mathcal{H}}. \tag{2.4.10}
$$

By Gronwall's lemma, we obtain

$$
\|Y(t,\omega,Y_0)\|^2_{\mathcal{H}} \leqslant e^{2\alpha_3 t}\|h\|^2_{\mathcal{H}}, \quad \forall\, 0\leqslant t\leqslant 1. \tag{2.4.11}
$$

Now, set $Z = Y - \tilde{W}$, then

$$
\frac{dZ}{dt} = LZ + F'(w,\theta_t\omega)Z + H(w,\bar{w}) \tag{2.4.12}
$$

with

$$
Z(x,t) = Z_0(x,t) = 0, \quad t\leqslant 0, \tag{2.4.13}
$$

where $Z = (u-\bar{u}-U, \eta^t-\bar{\eta}^t-V)$, $F'(w,\theta_t\omega)Z = (-f'(u)(u-\bar{u}-U),0)$, while $H(w,\bar{w}) = (f'(u)(u-\bar{u})-f(u)+f(\bar{u}),0)$.

Take the inner product of (2.4.12) with Z in $\mathcal{H}$ to get

$$
\frac{d}{dt}\|Z\|^2_{\mathcal{H}} = 2(LZ,Z)_{\mathcal{H}} + 2(F'(w,\theta_t\omega)Z,Z)_{\mathcal{H}} + 2(H(w,\bar{w}),Z)_{\mathcal{H}}. \tag{2.4.14}
$$

Note that

$$
2(LZ,Z)_{\mathcal{H}} = -2\|(-\Delta)^{\frac{\alpha}{2}}(u-\bar{u}-U)\|^2_{\mathcal{H}} + \int_0^\infty g'(s)\|(-\Delta)^{\frac{\alpha}{2}}(\eta^t-\bar{\eta}^t-V)\|^2 ds \leqslant 0, \tag{2.4.15}
$$

$$
2(F'(w,\theta_t\omega)Z,Z)_{\mathcal{H}} \leqslant 2\alpha_3\|u-\bar{u}-U\|^2 \leqslant 2\alpha_3\|Z\|^2_{\mathcal{H}}, \tag{2.4.16}
$$

and from (2.4.1) and Taylor's series, we derive

$$
\begin{aligned}
2(H(w,\bar{w}),Z)_{\mathcal{H}} &= 2(f''(u)(u-\bar{u})^2, u-\bar{u}-U) \\
&\leqslant c_1\|u-\bar{u}\|^4 + c\|u-\bar{u}-U\|^2 \leqslant c_1\|u-\bar{u}\|^4 + c\|Z\|^2_{\mathcal{H}}.
\end{aligned} \tag{2.4.17}
$$

It follows from (2.4.14)-(2.4.17) that

$$
\frac{d}{dt}\|Z\|^2_{\mathcal{H}} \leqslant c_2\|Z\|^2_{\mathcal{H}} + c_1\|u-\bar{u}\|^4. \tag{2.4.18}
$$

Therefore, by Gronwall's lemma, we find

$$\|Z\|_{\mathcal{H}}^2 \leqslant c_1 e^{c_2 t} \int_0^t \|u(s) - \bar{u}(s)\|^4 ds, \tag{2.4.19}$$

which together with (2.4.11) gives that

$$\|Z(1)\|_{\mathcal{H}} \leqslant C_1(\omega)\|h\|_{\mathcal{H}}^{1+\rho}, \tag{2.4.20}$$

where $C_1(\omega) = \sqrt{\frac{c_1 e^{c_2}}{4\alpha_3}(e^{4\alpha_3} - 1)}$ and $\rho = 1$. Choose $\bar{k}(\omega) = \max\{C_1(\omega), 1\}$. Hence, we obtain $E(\ln \bar{k}) < \infty$.

Therefore, $\Phi(\omega)$ is almost surely uniform differentiable on $\mathcal{A}(\omega)$. Furthermore, the differential of $\Phi(\omega)$ at w is $D\Phi(\omega, w)$. The proof is completed. □

Next, we check condition (iii) of Proposition 2.1.3. In fact, taking the inner product of (2.4.2) with $\tilde{W}$ in $\mathcal{H}$ and performing analogous calculations to those leading to (2.4.20), we obtain

$$\|\tilde{W}(1)\|_{\mathcal{H}}^2 \leqslant e^{2\alpha_3+\delta}\|\tilde{W}_0\|_{\mathcal{H}}^2. \tag{2.4.21}$$

Since $\alpha_1(D\Phi(\omega, w))$ is equal to the norm of $D\Phi(\omega, w) \in L(\mathcal{H})$, we choose

$$\overline{\alpha_1(\omega)} = \max\left\{e^{\alpha_3+\frac{\delta}{2}}, 1\right\}.$$

Then one has

$$\alpha_1(D\Phi(\omega, w)) \leqslant \overline{\alpha_1(\omega)},$$

and

$$E(\ln \overline{\alpha_1}) < \infty.$$

Theorem 2.4.2. *Assume that* $(H1)-(H2)$ *hold,* $\alpha \in [\frac{1}{2}, 1)$ *and* $p \in [2, 1 + \frac{3}{3-2\alpha})$, *and* (2.4.1) *is fulfilled. Then the random attractor* $\mathcal{A}(\omega)$ *has finite Hausdorff dimension.*

Proof. Now, we only need to verify condition (ii) of Proposition 2.1.3.

To this end, let $\tilde{W} = (U, V)$ be a unitary vector belonging to the domain of $L + F'(\tilde{W}, \theta_t\omega)$ with $F'(\tilde{W}, \theta_t\omega)\tilde{W} = (-f'(u)U, 0)$. Then

$$\left(\left(L + F'(\tilde{W}, \theta_t\omega)\right)\tilde{W}, \tilde{W}\right)_{\mathcal{H}} = (L\tilde{W}, \tilde{W})_{\mathcal{H}} - (f'(u)U, U)_{L^2}. \tag{2.4.22}$$

By means of direct calculations

$$(L\tilde{W}, \tilde{W})_{\mathcal{H}} \leqslant -\|(-\Delta)^{\frac{\alpha}{2}}U\|^2 - \frac{\delta}{2}\|V\|_M^2, \tag{2.4.23}$$

and

$$-(f'(u)U, U)_{L^2} \leqslant \alpha_3\|U\|^2. \tag{2.4.24}$$

Thus,

$$\left((L + F'(\tilde{W}, \theta_t\omega))\tilde{W}, \tilde{W}\right)_{\mathcal{H}} \leqslant -\|(-\Delta)^{\frac{\alpha}{2}} U\|^2 - \frac{\delta}{2}\|V\|_M^2 + \alpha_3\|U\|^2. \tag{2.4.25}$$

Therefore, we conclude that $L + F'(\tilde{W}, \theta_t\omega) \leqslant A$, where A is the diagonal operator acting on $L^2(O) \otimes L^2_g(\mathbb{R}^+, H^\alpha(O))$ defined by

$$\begin{pmatrix} -(-\Delta)^\alpha + \alpha_3 I & 0 \\ 0 & -\frac{\delta}{2}(-\Delta)^\alpha \end{pmatrix}$$

From the definition of Tr_m (Definition 2.1.2), it is clear that $Tr_m(L + F'(\tilde{W}, \theta_t\omega)) \leqslant Tr_m(A)$. Since A is diagonal, it is easy to see that

$$Tr_m(A) = \sup_Q \sum_{j=1}^m (A\tilde{W}_j, \tilde{W}_j)_{\mathcal{H}},$$

where the supremum is taken over the projections Q of the form $Q_1 \otimes Q_2$. This amounts to consider vectors $\tilde{W}_j$ where only one of the two components is non-zero (and in fact of norm one in its space). Choose then $m > \max\{\beta_1, \beta_2\} > 0$, and let n_1, n_2 be the numbers of vectors $\tilde{W}_j$ of the form $(U, 0)$ and $(0, V)$, respectively. Using Sobolev-Lieb-Thirring's inequality, we have

$$Tr_m(A) \leqslant -\beta_1|O|^\alpha n_1^{1+\alpha} + n_1 - \frac{\delta\beta_2}{2}|O|^\alpha n_2^{1+\alpha} + \frac{\delta}{2}n_2 + \alpha_3 n_1, \tag{2.4.26}$$

and from [60] we can deduce that

$$\omega_m(D\Phi(\omega, w)) \leqslant \exp\left\{-\beta_1|O|^\alpha n_1^{1+\alpha} + (1 + \alpha_3)n_1 - \frac{\delta\beta_2}{2}|O|^\alpha n_2^{1+\alpha} + \frac{\delta}{2}n_2\right\}.$$

Denote

$$\overline{\omega}_m(\omega) = \exp\left\{-\beta_1|O|^\alpha n_1^{1+\alpha} + (1 + \alpha_3)n_1 - \frac{\delta\beta_2}{2}|O|^\alpha n_2^{1+\alpha} + \frac{\delta}{2}n_2\right\}.$$

On the other hand, (2.4.26) gives that

$$q_m \leqslant -\beta_1|O|^\alpha n_1^{1+\alpha} + (1 + \alpha_3)n_1 - \frac{\delta\beta_2}{2}|O|^\alpha n_2^{1+\alpha} + \frac{\delta}{2}n_2.$$

Since as m goes to infinity either n_1 or n_2 (or both) goes to infinity, it is clear that there exists m_0 such that $q_{m_0} < 0$. Then we have $\omega_{m_0}(D\Phi(\omega, w)) \leqslant \overline{\omega}_{m_0}(\omega)$ and $E(\ln \overline{\omega}_{m_0}) < 0$. Thus the desired conclusion follows from Proposition 2.1.3.

□

As we have seen, in Part I we studied the long time behavior of a stochastic (fractional) parabolic problem with delay, in which the delay is expressed by memory. In the next Part II, we will analyze another kind of parabolic equation with variable delay, i.e., 2D–Navier-Stokes equations with infinite delay.

Part II

Navier-Stokes equation with infinite delay

This Part II includes Chapter 3 and Chapter 4. The asymptotic behavior of one type of Newtonian fluids, i.e., the Navier-Stokes equations with infinite delay, is studied here. Navier-Stokes equations are within the most important mathematical physics model, and which is more widely used in real life. For instance, in aeronautics and astronautics, the Navier-Stokes model can simulate the helicopter hovering aerodynamic performance. On the other hand, the Navier-Stokes equation can simulate the movement of small-scale water in offshore engineering. Studying the Navier-Stokes equation also helps us to understand the oceans, benefit the development and utilization of marine resources and develop the marine economy and industry.

Owing to the fact that Navier-Stokes equations provide a suitable model to describe the motion of several important fluids, such as water, oil, air, etc, the long-time behavior of Navier-Stokes models (and its variants) has been regarded as an interesting and important problem in the theory of fluid dynamics, and has been receiving much attention for many years (see [3–5, 51, 66, 160] and references therein). The structure of this Part II is as follows. Next, we recall some basic definitions and abstract spaces. Then in Chapter 3, we consider deterministic Navier-Stokes equation with infinite delay. To conclude, in Chapter 4 we consider stochastic Navier-Stokes equation with unbounded delay.

In order to increase the readability of this work, we present the Navier-Stokes equations with infinite delay as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) + g(t,u_t) \text{ in } (0,T)\times\mathcal{O}, \\ div\, u = 0 \text{ in } (0,T)\times\mathcal{O}, \\ u = 0 \text{ in } (0,T)\times\partial\mathcal{O}, \\ u(\theta,x) = \phi(\theta,x),\ \ \theta\in(-\infty,0],\ \ x\in\mathcal{O}, \end{cases}$$

where $\mathcal{O}\subset\mathbb{R}^2$ is an open and bounded set with boundary $\partial\mathcal{O}$, $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p denotes the pressure, ϕ is the initial datum, f is a nondelayed external force term, and g is the external force containing some hereditary characteristic.

Now we recall some definitions. To start, we consider the following usual abstract spaces,

$$\mathcal{V} = \left\{u\in(C_0^\infty(\mathcal{O}))^2 : div\, u\equiv 0\right\},$$

H = the closure of $\mathcal{V}$ in $(L^2(\mathcal{O}))^2$ with norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for $u,v\in(L^2(\mathcal{O}))^2$,

$$(u,v) = \sum_{j=1}^{2}\int_{\mathcal{O}} u_j(x)v_j(x)dx,$$

V = the closure of $\mathcal{V}$ in $(H_0^1(\mathcal{O}))^2$ with norm $\|\cdot\|$, and inner product $((\cdot,\cdot))$, where for $u,v\in(H_0^1(\mathcal{O}))^2$,

$$((u,v)) = \sum_{i,j=1}^{2}\int_{\mathcal{O}}\frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx.$$

Identifying H with its dual by the Riesz theorem, it follows that $V\subset H\equiv H'\subset V'$, where the injections are dense and compact. We use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality pairing between V and V'. Now we define $A : V \to V'$ by $\langle Au, v\rangle = ((u, v))$, and the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_O u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \forall\, u, v, w \in V.$$

Let us denote $B : V \times V \to V'$ the operator given by $\langle B(u, v), w\rangle = b(u, v, w)$, for all $u, v, w \in V$, and $B(u) = B(u, u)$.

We recall that

$$b(u, v, w) = -b(u, w, v) \quad \forall\, u, v, w \in V,$$

and, consequently,

$$b(u, v, v) = 0 \quad \forall\, u, v, w \in V.$$

Note that the trilinear form b satisfies the following inequalities which will be used later in proofs (e.g., cf. [72, 130])

$$\begin{aligned} |b(u, v, u)| &\leqslant \|u\|^2_{(L^4(O))^2}\|v\| \\ &\leqslant 2^{-1/2}|u|\|u\|\|v\| \quad \forall\, u, v \in V. \end{aligned} \tag{2.4.27}$$

There are several phase spaces which allow us to deal with infinite delays (e.g., cf. [91, 97, 135]). As commented in the introduction, we aim to establish well-posedness and stability results for 2D Navier-Stokes equations with infinite delay operators in

$$BCL_{-\infty}(H) = \left\{\varphi \in C((-\infty, 0]; H) : \lim_{\theta\to-\infty} \varphi(\theta) \text{ exists in } H\right\},$$

which is a Banach space equipped with the norm

$$\|\varphi\|_{BCL_{-\infty}(H)} = \sup_{\theta\in(-\infty,0]} |\varphi(\theta)|_H.$$

Similarly, we define

$$BCL_{-\infty}(V) = \left\{\varphi \in C((-\infty, 0]; V) : \|\varphi(\theta)\| \text{ is bounded on } (-\infty, 0] \text{ and } \lim_{\theta\to-\infty} \varphi(\theta) \text{ exists in } V\right\},$$

Let us introduce some notation and assumptions on the delay and delay operator. We will denote $\mathbb{R}_+ = [0, \infty)$. Let X be a Banach space and consider a fixed $T > 0$. Given $u : (-\infty, T) \to X$, for each $t \in (0, T)$, we denote by u_t the function defined on $(-\infty, 0]$ by

$$u_t(\theta) = u(t + \theta), \quad \theta \in (-\infty, 0].$$

Chapter 3

Navier-Stokes equation with infinite delay

In this chapter, we investigate the following Navier-Stokes problem with unbounded delay

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) + g(t,u_t) \text{ in } (0,T)\times\mathcal{O}, \\ div\ u = 0 \text{ in } (0,T)\times\mathcal{O}, \\ u = 0 \text{ in } (0,T)\times\partial\mathcal{O}, \\ u(\theta,x) = \phi(\theta,x),\ \theta\in(-\infty,0],\ x\in\mathcal{O}, \end{cases}$$

where $\mathcal{O}\subset\mathbb{R}^2$ is an open and bounded set with boundary $\partial\mathcal{O}$, $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p denotes the pressure, ϕ is the initial datum, f is a nondelayed external force term, and g is the external force containing some hereditary characteristic.

Due to their importance in fluid dynamics and in turbulence theory, the Navier-Stokes equations with delay have also been extensively studied over the last years. The analysis of the Navier-Stokes equations with hereditary terms was initiated by Caraballo and Real in [37], and developed in [20, 21, 32, 35, 39, 40], where several issues have been investigated the existence and uniqueness of solution, stationary solution, the existence of attractors (global, pullback and random ones) and the local exponential stability of state-steady solution of Navier-Stokes models with several types of delay (constant, bounded variable delay as well as bounded distributed delay). In the papers [69, 72, 77, 127–129] the authors have discussed the asymptotic behavior and regularity of solutions of 2D Navier-Stokes equations (and 3D-variations of Navier-Stokes models) with delay (finite and infinite). Wei and Zhang [171] have obtained the exponential stability and almost surely exponential stability of the weak solution for stochastic 2D Navier-Stokes equations with bounded variable delays by using the approach proposed in [21, 37].

It is worth emphasizing that all the mentioned works deal with finite delay (constant delay, bounded variable delay or bounded distributed delay) in the phase spaces $C([-h,0];H)$ and $L^2(-h,0;H)$ or infinite distributed delay in

$$C_\gamma(H) = \{\varphi\in C((-\infty,0];H) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } H\}\ (\gamma>0).$$

In fact, a complete application of theory of attractors is carried out in [130] for a 2D Navier-Stokes model with infinite delay in $C_\gamma(H)$ under some assumptions relating the force f and the delay operator

g among others. The positive character of γ plays a key role in those arguments. Without the help of this extra exponential weight related to the γ parameter, we still aim to study long-time behavior of solutions to problem (P), but the techniques from [130] do not seem to fit. So we wonder what can be obtained if the space $C_\gamma(H)$ is replaced. To our best knowledge, there is no work about Navier-Stokes models with unbounded variable delay in the phase space

$$BCL_{-\infty}(H) = \{\varphi \in C((-\infty, 0]; H) : \lim_{\theta \to -\infty} \varphi(\theta) \text{ exists in } H\}.$$

Inspired by [37, 130], in this chapter we study the asymptotic behavior of solutions to 2D Navier-Stokes equations with unbounded variable delay in the phase space $BCL_{-\infty}(H)$. We discuss the existence, uniqueness of weak solution, stationary solution as well as stability of stationary solution by several different approaches. More precisely, the existence and uniqueness of solution is proved by the classic Galerkin approximation and energy method, while the existence of stationary solution is established by the Schauder fixed point theorem and the Lax-Milgram theorem. Then, three methods are considered to discuss the stability of the stationary solutions. First, we consider the local stability of stationary solution by using Lyapunov functions, in which the differentiability of the delay term is required, and this maybe seem as a strong condition in some situations. Fortunately, we can use the Lyapunov-Razumihkin method to weaken the differentiability condition on the delay term, and only the continuity of the operators of the model and continuity on the delay term are necessary, which allows to include more general types of delay. By this method, but dealing with strong solution, a better result can be obtained. Besides, by constructing Lyapunov functionals, we also show the stability of stationary solution, which improves the one we obtain by a direct approach. Moreover, we verify, by exhibiting a special case of unbounded variable delay, that it may not be possible, in general, to ensure exponential stability of the stationary solutions when dealing with variable delays. In fact, we are able to ensure the polynomial stability of stationary solutions in the particular case of proportional delays. Therefore, for general unbounded delay the polynomial decay is a sharp result. It is still an open problem to establish some sufficient conditions ensuring asymptotic convergence to the stationary solutions with an exponential rate in some other situations of unbounded variable delays.

The framework of this chapter is as follows: in the next Section, we make clear the assumption on delay term and present some examples of delay terms. And in Section 3.2, we prove the well-posedenss of problem (P), then three different methods are used to analyze the stability of stationary solution to problem (P) in Section 3.3. Finally, in Section 3.4, we verify the polynomial stability of stationary solution in case of unbounded variable delay.

3.1 Preliminaries

We now enumerate the assumptions on the delay term g. Assume that $g : [0, T] \times BCL_{-\infty}(H) \to (L^2(\mathcal{O}))^2$, then

$(g1)$ For any $\xi \in BCL_{-\infty}(H)$, the mapping $[0, T] \ni t \mapsto g(t, \xi) \in (L^2(\mathcal{O}))^2$ is measurable.

$(g2)$ $g(\cdot, 0) = 0$.

(g3) There exists a constant $L_g > 0$ such that, for any $t \in [0,T]$ and all $\xi, \eta \in BCL_{-\infty}(H)$,

$$|g(t,\xi) - g(t,\eta)| \leqslant L_g \|\xi - \eta\|_{BCL_{-\infty}(H)}.$$

Remark 3.1.1. *(i) As it is pointed out in [130], condition (g2) is not really a restriction, otherwise, if $|g(\cdot,0)| \in L^2(0,T)$, we could redefine $\hat{f}(t) = f(t) + g(t,0)$ and $\hat{g}(t,\cdot) = g(t,\cdot) - g(t,0)$. In this way, the problem is exactly the same, but $\hat{f}$ and $\hat{g}$ satisfy the required assumptions.*

(ii) Conditions (g2) and (g3) imply that, for any $\xi \in BCL_{-\infty}(H)$,

$$|g(t,\xi)| \leqslant L_g \|\xi\|_{BCL_{-\infty}(H)} \ \forall t \in [0,T],$$

and therefore $|g(\cdot,\xi)| \in L^\infty(0,T)$.

Now we provide some examples of (unbounded) delay forcing terms which can be set within our general set-up (see [69, 70, 72, 77]).

Example 3.1.2 (Forcing term with unbounded variable delay)**.** *Let $G : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a measurable function satisfying $G(t,0) = 0$ for all $t \in [0,T]$, and assume that there exists $M > 0$ such that*

$$|G(t,u) - G(t,v)|_{\mathbb{R}^2} \leqslant M|u - v|_{\mathbb{R}^2} \quad \forall u, v \in \mathbb{R}^2.$$

Consider a function $\rho : [0,T] \to \mathbb{R}_+$, which plays the role of the delay. Assume that $\rho(\cdot)$ is measurable and define $g(t,\xi)(x) = G(t,\xi(-\rho(t))(x))$ for each $\xi \in BCL_{-\infty}(H)$, $x \in \mathcal{O}$ and $t \in [0,T]$.

Obviously, g satisfies (g1) − (g2). Now we check that g satisfies assumption (g3), for any $\xi, \eta \in BCL_{-\infty}(H)$,

$$\begin{aligned}
|g(t,\xi) - g(t,\eta)|^2 &= \int_{\mathcal{O}} |G(t,\xi(-\rho(t))) - G(t,\eta(-\rho(t)))|^2 dx \\
&\leqslant M^2 \int_{\mathcal{O}} |\xi(-\rho(t)) - \eta(-\rho(t))|^2 dx \\
&\leqslant M^2 \sup_{\theta \leqslant 0} \int_{\mathcal{O}} |\xi(\theta) - \eta(\theta)|^2 dx \\
&= M^2 \|\xi - \eta\|^2_{BCL_{-\infty}(H)}.
\end{aligned}$$

Example 3.1.3. *The above example is using the mapping G via the Nemytskii operator to deal with an operator from $[0,T] \times H$ into $(L^2(\mathcal{O}))^2$. So it is a particular case of the following. Take a Lipschitz mapping (uniformly w.r.t. $[0,T]$) $G : [0,T] \times H \to (L^2(\mathcal{O}))^2$ and consider $g(t,\xi) := G(t,\xi(-\rho(t))$ for any measurable function $\rho : [0,T] \to \mathbb{R}_+$. This operator g also fulfils (g1) − (g3).*

Example 3.1.4 (Forcing term with distributed delay)**.** *Let $G : [0,T] \times (-\infty,0] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a measurable function satisfying $G(t,s,0) = 0$ for all $(t,s) \in [0,T] \times (-\infty,0]$, and suppose that there exists a function $\alpha \in L^1(-\infty,0)$ such that*

$$|G(t,s,u) - G(t,s,v)|_{\mathbb{R}^2} \leqslant \alpha(s)|u - v|_{\mathbb{R}^2} \quad \forall u, v \in \mathbb{R}^2, \ \forall t \in [0,T], \ a.e. \ s \in (-\infty,0).$$

Define $g(t,\xi)(x) = \int_{-\infty}^{0} G(t,s,\xi(s)(x))ds$ *for each* $\xi \in BCL_{-\infty}(H)$, $t \in [0,T]$, *and* $x \in \mathcal{O}$. *Then the delayed term* g *in our problem becomes*

$$g(t,\xi) = \int_{-\infty}^{0} G(t,s,\xi(s))ds.$$

It is easy to check that g *satisfies* $(g1)-(g2)$. *On the other hand, if* $\xi,\eta \in BCL_{-\infty}(H)$, *for each* $t \in [0,T]$, *we deduce*

$$\begin{aligned}
|g(t,\xi)-g(t,\eta)|^2 &\leqslant \int_{\mathcal{O}}\left(\int_{-\infty}^{0} |G(t,s,\xi(s)(x)) - G(t,s,\eta(s)(x))|_{\mathbb{R}^2}ds\right)^2 dx \\
&\leqslant \int_{\mathcal{O}}\left(\int_{-\infty}^{0} \alpha(s)|\xi(s)(x)-\eta(s)(x)|_{\mathbb{R}^2}ds\right)^2 dx \\
&\leqslant \int_{\mathcal{O}}\left(\int_{-\infty}^{0} \alpha(s)ds\right)\left(\int_{-\infty}^{0} \alpha(s)|\xi(s)(x)-\eta(s)(x)|^2_{\mathbb{R}^2}ds\right)dx \\
&\leqslant \|\alpha\|_{L^1(-\infty,0)}\int_{-\infty}^{0}\alpha(s)\int_{\mathcal{O}}|\xi(s)(x)-\eta(s)(x)|^2dxds \\
&\leqslant \|\alpha\|_{L^1(-\infty,0)}\int_{-\infty}^{0}\alpha(s)(\sup_{s\leqslant 0}\int_{\mathcal{O}}|\xi(s)(x)-\eta(s)(x)|^2dx)ds \\
&\leqslant \|\alpha\|^2_{L^1(-\infty,0)}\|\xi-\eta\|^2_{BCL_{-\infty}(H)}.
\end{aligned}$$

After introducing the above operators an equivalent abstract formulation to problem (P) is

$$\frac{du}{dt} + \nu Au + B(u) = f + g(t,u_t) \ \ \forall t \geqslant 0, \tag{3.1.1}$$

$$u_0 = \phi. \tag{3.1.2}$$

Next we give the definition of weak solution to problem (3.1.1)-(3.1.2).

Definition 3.1.5. *Given an initial datum* $\phi \in BCL_{-\infty}(H)$, *a weak solution* u *to (3.1.1)-(3.1.2) in the interval* $(-\infty,T]$ *is a function* $u \in C((-\infty,T];H)\cap L^2(0,T;V)$ *with* $u_0 = \phi(0)$ *such that, for all* $v \in V$,

$$\frac{d}{dt}(u(t),v) + \nu((u(t),v)) + b(u(t),u(t),v) = \langle f(t),v\rangle + (g(t,u_t),v),$$

where the equation must be understood in the sense of $\mathcal{D}'(0,T)$.

3.2 Well-posedness

In this section we establish the existence of weak solution to (3.1.1)-(3.1.2) by a compactness method using the classic Faedo-Galerkin scheme. Denote

$$\lambda_1 = \inf_{v\in V\setminus\{0\}} \frac{\|v\|^2}{|v|^2} > 0.$$

For the existence of weak solution we have the following result.

Theorem 3.2.1. *Consider $f \in L^2(0,T;V')$, $g : [0,T] \times BCL_{-\infty}(H) \to (L^2(\mathcal{O}))^2$ satisfying $(g1)-(g3)$ and $\phi \in BCL_{-\infty}(H)$ given. Then there exists a unique weak solution to (3.1.1)-(3.1.2). Furthermore, if $f \in L^2(0,T;(L^2(\mathcal{O}))^2)$ and $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$, then the weak solution is a strong solution, i.e., $u \in L^2(0,T;D(A)) \cap C([0,T];V)$.*

Proof. We split it into several steps.

Step 1. The Galerkin approximation. By the definition of A and the classical spectral theory of elliptic operators, it follows that A possesses a sequence of eigenvalues $\{\lambda_j\}_{j\geq 1}$ and a corresponding family of eigenfunctions $\{w_j\}_{j\geq 1} \subset V$, which form a Hilbert basis of H, dense on V. We consider the subspace $V_m = \text{span}\{w_1, w_2, \cdots, w_m\}$, and the projector $P_m : H \to V_m$ given by $P_m u = \sum_{j=1}^m (u, w_j) w_j$, and define $u^{(m)}(t) = \sum_{j=1}^m \gamma_{m,j}(t) w_j$, where the superscript m will be used instead of (m), for short, since no confusion is possible with powers of u, and where the coefficients $\gamma_{m,j}(t)$ are required to satisfy the Cauchy problem

$$\frac{d}{dt}(u^m(t), w_j) + \nu((u^m(t), w_j)) + b(u^m(t), u^m(t), w_j) = \langle f(t), w_j\rangle + (g(t, u_t^m), w_j), \quad 1 \leqslant j \leqslant m,$$
$$u^m(\theta) = P_m\phi(\theta), \ \theta \in (-\infty, 0]. \tag{3.2.1}$$

The above system of ordinary functional differential equations with infinite delay fulfills the conditions for the existence and uniqueness of a local solution (e.g., cf. [90, 97, 161]). Hence, we conclude that (3.2.1) has a unique local solution defined in $[0, t_m)$ with $0 \leqslant t_m \leq T$. Next, we will obtain a priori estimates and ensure that the solutions u^m do exist in the whole interval $[0, T]$.

Step 2. A priori estimates. Multiplying each equation of (3.2.1) by $\gamma_{m,j}(t)$, $j = 1, \ldots, m$, summing up, and using Cauchy-Schwartz and Young's inequalities, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 &\leqslant \|f(t)\|_*\|u^m(t)\| + L_g\|u_t^m\|_{BCL_{-\infty}(H)}|u^m(t)| \\ &\leqslant \frac{\nu}{2}\|u^m(t)\|^2 + \frac{\|f(t)\|_*^2}{2\nu} + L_g\|u_t^m\|^2_{BCL_{-\infty}(H)}.\end{aligned}$$

Hence,

$$|u^m(t)|^2 + \nu\int_0^t \|u^m(s)\|^2 ds \leqslant |u^m(0)|^2 + \frac{1}{\nu}\int_0^t \|f(s)\|_*^2 ds + 2L_g\int_0^t \|u_s^m\|^2_{BCL_{-\infty}(H)} ds. \tag{3.2.2}$$

Particularly, we have

$$\|u_t^m\|^2_{BCL_{-\infty}(H)} \leqslant \|\phi\|^2_{BCL_{-\infty}(H)} + \frac{1}{\nu}\int_0^t \|f(s)\|_*^2 ds + 2L_g\int_0^t \|u_s^m\|^2_{BCL_{-\infty}(H)} ds.$$

Applying the Gronwall lemma, we obtain

$$\|u_t^m\|^2_{BCL_{-\infty}(H)} \leqslant \left(\|\phi\|^2_{BCL_{-\infty}(H)} + \frac{1}{\nu}\int_0^t \|f(s)\|_*^2 ds\right) e^{2L_g t},$$

whence there exists a constant $C > 0$, depending on some constants of the problem (namely, ν, L_g and f), and on T and $R > 0$, such that

$$\|u_t^m\|^2_{BCL_{-\infty}(H)} \leqslant C(T,R) \ \ \forall t \in [0,T], \ \ \|\phi\|_{BCL_{-\infty}(H)} \leqslant R, \ \ \forall m \geqslant 1,$$

which also implies that $\{u^m\}$ is bounded in $L^\infty(0,T;H)$.

Now it follows from (3.2.2) and the above uniform estimates that

$$\nu \int_0^t \|u^m(s)\|^2 ds \leqslant |u^m(0)|^2 + \int_0^t \left(\frac{1}{\nu}\|f(s)\|_*^2 + 2L_g C(T,R)\right) ds.$$

We can conclude the existence of another constant (relabelled the same) $C(T,R)$ such that

$$\|u^m\|^2_{L^2(0,T;V)} \leqslant C(T,R) \ \ \forall m \geqslant 1.$$

From (2.4.27) and (3.2.1),

$$\|(u^m)'\|_* \leqslant \nu\|u^m\| + 2^{1/2}|u^m| \cdot \|u^m\| + \|f\|_* + \lambda_1^{-1/2}|g(t,u_t^m)|,$$

which, together with Remark 2.1(ii) and the above estimates imply that $\{(u^m)'\}$ is bounded in $L^2(0,T;V')$.

Step 3. Approximation of initial datum in $BCL_{-\infty}(H)$. Let us check

$$P_m\phi \to \phi \ \text{ in } BCL_{-\infty}(H). \tag{3.2.3}$$

Indeed, if not, there exist $\epsilon > 0$ and a subsequence $\{\theta_m\} \subset (-\infty, 0]$, such that

$$|P_m\phi(\theta_m) - \phi(\theta_m)| > \epsilon \ \ \forall m. \tag{3.2.4}$$

Assume that $\theta_m \to -\infty$. Otherwise, if $\theta_m \to \theta$, then $P_m\phi(\theta_m) \to \phi(\theta)$, since $|P_m\phi(\theta_m) - \phi(\theta)| \leqslant |P_m\phi(\theta_m) - P_m\phi(\theta)| + |P_m\phi(\theta) - \phi(\theta)| \to 0$ as $m \to \infty$. With $\theta_m \to -\infty$ as $m \to \infty$, if we denote $x = \lim_{\theta \to -\infty} \phi(\theta)$, we obtain

$$|P_m\phi(\theta_m) - \phi(\theta_m)| \leqslant |P_m\phi(\theta_m) - P_m x| + |P_m x - x| + |x - \phi(\theta_m)| \to 0,$$

which contradicts (3.2.4), so (3.2.3) holds true.

Step 4. Compactness results. Following the same lines as those in [130, Theorem 5, p. 2017] with slight modifications, we can prove that

$$u^m \to u \ \text{ in } C([0,T];H).$$

Then steps 3 and 4 imply that

$$u_t^m \to u_t \ \text{ in } \ BCL_{-\infty}(H) \ \forall t \leqslant T.$$

Actually,

$$\begin{aligned}\sup_{\theta \leqslant 0} |u^m(t+\theta) - u(t+\theta)| &= \max\left\{\sup_{\theta \leqslant -t} |P_m\phi(\theta+t) - \phi(\theta+t)|, \sup_{-t \leqslant \theta \leqslant 0} |u^m(t+\theta) - u(t+\theta)|\right\} \\ &\leqslant \max\left\{\|P_m\phi - \phi\|_{BCL_{-\infty}(H)}, \sup_{-t \leqslant \theta \leqslant 0} |u^m(t+\theta) - u(t+\theta)|\right\} \to 0.\end{aligned}$$

Therefore, combining $(g3)$, we can prove that

$$g(\cdot, u_\cdot^m) \to g(\cdot, u_\cdot) \ \text{ in } L^2(0, T; H).$$

Thus, we can finally pass to the limit in (3.2.1), concluding that u solves (P).

Step 5. Uniqueness of solution. The uniqueness of solution can be obtained by using the Gronwall Lemma (see [21, 130] for more details). □

3.3 Asymptotic behavior of solutions

In this section we analyze the long time behavior of solutions in a neighborhood of a stationary solution to (3.1.1) in some particular settings for the delay operator. First of all, we provide several results ensuring the existence of stationary solutions and establish sufficient conditions for uniqueness. Then we show various different methods that can be used to study the stability properties: the Lyapunov function, the Razumikhin technique as well as by the construction of appropriate Lyapunov functionals. All the cases will be related to model (3.1.1) with unbounded variable delays. We would also like to mention that for particular unbounded variable delays, the exponential stability of stationary solutions cannot be obtained. However, we will be able to obtain some polynomial stability in the case of proportional variable delays.

3.3.1 Existence and uniqueness of stationary solutions

In order to investigate the existence and properties of stationary solutions to (3.1.1), we need to make some extra assumptions. Namely, we assume that f is independent of time, i.e., $f(t) \equiv f \in V'$, and g, defined now for all positive times, also is *autonomous* somehow. Indeed, if we put directly g autonomous, then the delay should have a distributed or fixed form, but an infinite variable delay would not be possible, therefore the explicit presence of t in the operator should not removed if we aim to keep the variable delay case. Namely, we introduce a new assumption for g. Denote by i the trivial immersion $i : H \to BCL_{-\infty}(H)$ given by $i(u) = \tilde{u}$ with $\tilde{u}(t) = u$ for all $t \le 0$. We require now that g fulfills

$(g4)$ $g(s, \xi) = g(t, \xi)$ for any $s, t \in \mathbb{R}_+$ and $\xi \in i(H)$.

If $(g2) - (g4)$ holds, we trivially have that $\tilde{g} : H \to (L^2(O))^2$ defined as $\tilde{g}(u) = g(0, i(u)(0))$, i.e., $\tilde{g} = g|_{\mathbb{R}_+ \times i(H)}$, is of course autonomous, Lipschitz (with the same Lipschitz constant L_g) and $\tilde{g}(0) = 0$.

Example 3.3.1. *Combining Examples 3.1.2 and 3.1.3 it is obvious that given a measurable function* $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $G : H \to (L^2(O))^2$ *Lipschitz with* $G(0) = 0$, *then* $\mathbb{R}_+ \times BCL_\infty(H) \ni (t, \xi) \mapsto g(t, \xi) := G(\xi(-\rho(t))) \in (L^2(O))^2$ *fulfills* $(g1) - (g4)$.

A stationary solution to (3.1.1) is a function $u^* \in V$ such that

$$\nu A u^* + B(u^*) = f + \tilde{g}(u^*). \tag{3.3.1}$$

Notice that (3.3.1) is not related to any delay form, and has already been analyzed in some previous works (e.g., cf. [21, 37, 72, 130]). Existence, eventually uniqueness and regularity of stationary solutions can be obtained from

Theorem 3.3.2. *Suppose that g satisfies conditions* $(g2)-(g4)$ *and* $\nu > \lambda_1^{-1}L_g$. *Then,*

(a) for all $f \in V'$, there exists at least one solution to (3.3.1);

(b) if $f \in (L^2(\mathcal{O}))^2$, the solutions to (3.3.1) belong to $D(A)$;

(c) if $(\nu - \lambda_1^{-1}L_g)^2 > (2\lambda_1)^{-1/2}\|f\|_*$, then the solution to (3.3.1) is unique.

3.3.2 Local stability: a direct approach

In this section we prove the local stability of stationary solution obtained in Theorem 3.3.2 when g is close to that in Example 3.3.1 (with a smoother driving term ρ) by a straightforward way.

Theorem 3.3.3. *Consider $f \in (L^2(\mathcal{O}))^2$, the delay forcing term is given by $g(t, u_t) = G(u(t-\rho(t)))$ with $G : H \to (L^2(\mathcal{O}))^2$ a Lipschitz operator with Lipschitz constant M, $G(0) = 0$, and $\rho \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\rho_* = \sup_{t\geqslant 0}\rho'(t) < 1$. Then there exist two constants l_1, l_2, depending only on $\mathcal{O}$, such that if*

$$2\nu > \frac{(2-\rho_*)\lambda_1^{-1}M}{1-\rho_*} + \frac{l_1}{\nu - \lambda_1^{-1}M}|f| + \frac{l_2}{\nu^2(\nu-\lambda_1^{-1}M)^3}|f|^3, \tag{3.3.2}$$

then there exists at least one solution $u_\infty \in D(A)$ to (3.3.1), and there exists a positive constant C, such that for any $\phi \in BCL_{-\infty}(H)$, the solution u to (3.1.1)-(3.1.2) with $f(t) \equiv f$ satisfies

$$|u(t) - u_\infty|^2 \leqslant C\left(|\phi(0) - u_\infty|^2 + \|\phi - u_\infty\|^2_{L^2((-\rho(0),0);H)}\right) \ \forall t \geqslant 0.$$

Proof. Consider u the solution to (3.1.1)-(3.1.2) for $f(t) \equiv f$ and let $u_\infty \in D(A)$ be a solution to (3.3.1) (this is possible since (3.3.2) implies that $\nu > \lambda_1^{-1}M = \lambda_1^{-1}L_g$ so Theorem 3.3.2 applies). We set $w(t) = u(t) - u_\infty$, and observe that

$$\frac{d}{dt}w(t) + \nu Aw(t) + B(u(t)) - B(u_\infty) = G(u(t-\rho(t))) - G(u_\infty).$$

By standard computations,

$$\begin{aligned}\frac{d}{dt}|w(t)|^2 &= -2\nu\|w(t)\|^2 - 2(B(u(t)) - B(u_\infty), w(t)) + 2(G(u(t-\rho(t))) - G(u_\infty), w(t))\\ &\leqslant -2\nu\|w(t)\|^2 - 2b(w(t), u_\infty, w(t)) + 2M|u(t-\rho(t)) - u_\infty||w(t)|\\ &\leqslant (-2\nu + \lambda_1^{-1}M)\|w(t)\|^2 - 2b(w(t), u_\infty, w(t)) + M|w(t-\rho(t))|^2.\end{aligned} \tag{3.3.3}$$

By (2.4.27), and using Sobolev embeddings (introducing suitable constants c_0, c_0', c_1), we have

$$\begin{aligned}2|b(w(t), u_\infty, w(t))| &\leqslant 2|w(t)|^2_{(L^4(\mathcal{O}))^2}\|u_\infty\|\\ &\leqslant c_0 2^{1/2}\lambda_1^{-1/2}\|w(t)\|^2|Au_\infty|.\end{aligned}$$

Since u_∞ solves (3.3.1), we deduce

$$\begin{aligned}\nu|Au_\infty| &\leqslant |f| + |G(u_\infty)| + |B(u_\infty)|\\ &\leqslant |f| + M|u_\infty| + c_0'\|u_\infty\||u_\infty|_\infty\\ &\leqslant |f| + \lambda_1^{-1/2}M\|u_\infty\| + \frac{c_1^2\lambda_1^{-1/2}}{2\nu}\|u_\infty\|^3 + \frac{\nu}{2}|Au_\infty|,\end{aligned}$$

from which we obtain that

$$|Au_\infty| \leqslant \frac{2}{\nu}|f| + \frac{2\lambda_1^{-1/2}M}{\nu}\|u_\infty\| + \frac{c_1^2\lambda_1^{-1/2}}{\nu^2}\|u_\infty\|^3.$$

On the other hand,

$$\begin{aligned}\nu\|u_\infty\|^2 &= (f, u_\infty) + (G(u_\infty), u_\infty)\\ &\leqslant \lambda_1^{-1/2}|f|\|u_\infty\| + \lambda_1^{-1}M\|u_\infty\|^2,\end{aligned}$$

which implies

$$\|u_\infty\| \leqslant \frac{\lambda_1^{-1/2}|f|}{\nu - \lambda_1^{-1}M}.$$

Hence, from the above inequalities,

$$|Au_\infty| \leqslant \frac{2}{\nu}|f| + \frac{2\lambda_1^{-1}M}{\nu(\nu - \lambda_1^{-1}M)}|f| + \frac{c_1^2\lambda_1^{-2}}{\nu^2(\nu - \lambda_1^{-1}M)^3}|f|^3.$$

Now, thanks to the previous inequalities,

$$\frac{d}{dt}|w(t)|^2 \leqslant \Big(-2\nu + \lambda_1^{-1}M + \frac{l_1}{\nu - \lambda_1^{-1}M}|f| + \frac{l_2}{\nu^2(\nu - \lambda_1^{-1}M)^3}|f|^3\Big)\|w(t)\|^2 + M|w(t-\rho(t))|^2, \quad (3.3.4)$$

where

$$l_1 = c_0 2^{3/2}\lambda_1^{-1/2}, \quad l_2 = 2^{1/2}c_0c_1^2\lambda_1^{-5/2}. \quad (3.3.5)$$

Taking $\eta = s - \rho(s) = \tau(s)$,

$$M\int_0^t |w(s-\rho(s))|^2 ds \leqslant \frac{M}{1-\rho_*}\int_{-\rho(0)}^t |w(\eta)|^2 d\eta.$$

Therefore, integrating (3.3.4) over $[0, t]$,

$$\begin{aligned}|w(t)|^2 \leqslant |w(0)|^2 &+ \int_0^t \left(\frac{\lambda_1^{-1}M}{1-\rho_*} - 2\nu + \lambda_1^{-1}M + \frac{l_1}{\nu - \lambda_1^{-1}M}|f| + \frac{l_2}{\nu^2(\nu - \lambda_1^{-1}M)^3}|f|^3\right)\|w(s)\|^2 ds\\ &+ \frac{M}{1-\rho_*}\int_{-\rho(0)}^0 |w(s)|^2 ds,\end{aligned}$$

which, together with (3.3.2), yield

$$|w(t)|^2 \leqslant |w(0)|^2 + \frac{M}{1-\rho_*}\int_{-\rho(0)}^{0} |w(s)|^2 ds,$$

whence the statement follows taking $C = \max\{1, M/(1-\rho_*)\}$. □

Remark 3.3.4. *Notice that if we wish to obtain exponential stability by the Lyapunov method, we can multiply (3.3.3) by $e^{\lambda t}$. Then by a similar process with slight modification, we would obtain*

$$e^{\lambda t}|w(t)|^2 \leqslant |w(0)|^2 + \int_0^t \left(\lambda\lambda_1^{-1} - 2\nu + \lambda_1^{-1}M + \frac{l_1}{\nu - \lambda_1^{-1}M}|f| + \frac{l_2}{\nu^2(\nu - \lambda_1^{-1}M)^3}|f|^3\right) e^{\lambda s}\|w(s)\|^2 ds$$
$$+ M\int_0^t e^{\lambda s}|w(s-\rho(s))|^2 ds.$$

Now we estimate the delay term. Setting $\eta = s - \rho(s) = \tau(s)$, we have

$$\int_0^t e^{\lambda s}|w(s-\rho(s))|^2 ds \leqslant \frac{1}{1-\rho_*}\int_{-\rho(0)}^{t-\rho(t)} e^{\lambda\tau^{-1}(\eta)}|w(\eta)|^2 d\eta.$$

Assuming $\tau^{-1}(t) \leqslant t + h$ (what implies that necessarily $\rho(t) \in [0, h]$, $h > 0$) for all $t \geqslant -\rho(0)$,

$$\int_0^t e^{\lambda s}|w(s-\rho(s))|^2 ds \leqslant \frac{e^{\lambda h}}{1-\rho_*}\int_{-\rho(0)}^{t} e^{\lambda\eta}|w(\eta)|^2 d\eta.$$

Therefore,

$$e^{\lambda t}|w(t)|^2 \leqslant |w(0)|^2 + \int_0^t \left(\lambda\lambda_1^{-1} - 2\nu + \lambda_1^{-1}M + \frac{l_1}{\nu - \lambda_1^{-1}M}|f| + \frac{l_2}{\nu^2(\nu - \lambda_1^{-1}M)^3}|f|^3\right) e^{\lambda s}\|w(s)\|^2 ds$$
$$+ M\int_0^t e^{\lambda s}|w(s-\rho(s))|^2 ds$$
$$\leqslant |w(0)|^2 + \int_0^t \left(\lambda\lambda_1^{-1} - 2\nu + \lambda_1^{-1}M + \frac{l_1}{\nu - \lambda_1^{-1}M}|f| + \frac{l_2}{\nu^2(\nu - \lambda_1^{-1}M)^3}|f|^3\right) e^{\lambda s}\|w(s)\|^2 ds$$
$$+ \frac{Me^{\lambda h}}{1-\rho_*}\int_0^t e^{\lambda s}|w(s)|^2 ds + \frac{Me^{\lambda h}}{1-\rho_*}\int_{-\rho(0)}^{0} e^{\lambda\eta}|w(\eta)|^2 d\eta,$$

neglecting the first integral on the right hand side, which is negative for $0 < \lambda \ll 1$ thanks to (3.3.2), we have

$$|w(t)|^2 \leqslant e^{-\lambda t}\left(|w(0)|^2 + \frac{Me^{\lambda h}}{1-\rho_*}\int_{-\rho(0)}^{0} e^{\lambda\eta}|w(\eta)|^2 d\eta\right)$$
$$\leqslant Ce^{-\lambda t}\left(|w(0)|^2 + \|\phi - u_\infty\|_{L^2((-\rho(0),0);H)}\right),$$

where $C = \max\{1, Me^{\lambda h}/(1-\rho_*)\}$. However, as mentioned before, this argument requires that $\rho(t) \in [0, h]$ is bounded. In other words, we could not prove, in general, the exponential stability of stationary solution to (3.1.1) with unbounded variable delay by Theorem 3.3.3.

Remark 3.3.5. *Observe that there exists at least one stationary solution under assumptions of Theorem 3.3.3, but it might not be unique since the relation on the coefficients are different from Theorem 3.3.2 (c), which ensures the uniqueness of stationary solution. However, if* $2(2\lambda_1)^{-1/4}\|f\|_*^{1/2} \leqslant \frac{\lambda_1^{-1}M\rho_*}{1-\rho_*} + \frac{l_1}{\nu-\lambda_1^{-1}M}|f| + \frac{l_2}{\nu^2(\nu-\lambda_1^{-1}M)^3}|f|^3$*, then Theorem 3.3.2 (c) implies that the stationary solution is unique.*

3.3.3 A Razumikhin technique

In the previous paragraph we have showed the local stability of stationary solution when the delay operator g contains unbounded variable delay which is driven by a continuously differentiable function ρ. However, it is possible to relax this restriction and prove a result for more general delay forcing terms by using a different method, namely, the Razumikhin method, which is also widely used in dealing with the stability properties of delay differential equations. But it is worth mentioning that this approach requires some kind of continuity concerning both the operators in the model and the delay term, and we also need to work with strong solutions instead of weak ones.

Theorem 3.3.6. *Consider* $f \in (L^2(\mathcal{O}))^2$ *and* $g : \mathbb{R}_+ \times BCL_{-\infty}(H) \to (L^2(\mathcal{O}))^2$ *satisfying conditions* $(g1)-(g4)$ *(uniformly for any* $T > 0$*) and such that for all* $\xi \in BCL_{-\infty}(H)$ *the mapping* $\mathbb{R}_+ \ni t \mapsto g(t,\xi) \in (L^2(\mathcal{O}))^2$ *is continuous. If there exists a stationary solution* $u_\infty \in D(A)$ *to (3.1.1) such that*

$$
\begin{aligned}
&-\nu\langle A(\phi(0)-u_\infty), \phi(0)-u_\infty\rangle - \langle B(\phi(0)) - B(u_\infty), \phi(0)-u_\infty\rangle \\
&+ (g(t,\phi)-g(t,u_\infty), \phi(0)-u_\infty) < 0, \quad t \geqslant 0, \qquad (3.3.6)
\end{aligned}
$$

whenever $\phi \in BCL_{-\infty}(H)$ *with* $\phi(0) \in V$ *and* $\phi \neq u_\infty$ *satisfies*

$$\|\phi - u_\infty\|^2_{BCL_{-\infty}(H)} = |\phi(0) - u_\infty|^2, \qquad (3.3.7)$$

then, for such ϕ,

$$|u(t;\phi) - u_\infty|^2 < \|\phi - u_\infty\|^2_{BCL_{-\infty}(H)} \quad \forall\, t \geqslant 0. \qquad (3.3.8)$$

Proof. We argue by contradiction. Suppose there exists an initial datum $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$ and $\phi \neq u_\infty$, such that (3.3.8) is false. Then, denoting

$$\sigma = \inf\{t > 0 : |u(t,\phi) - u_\infty| \geq \|\phi - u_\infty\|^2_{BCL_{-\infty}(H)}\},$$

we obtain that for all $0 \leqslant t \leqslant \sigma$

$$|u(t;\phi) - u_\infty|^2 \leqslant |u(\sigma;\phi) - u_\infty|^2 = \|\phi - u_\infty\|^2_{BCL_{-\infty}(H)}, \qquad (3.3.9)$$

and there is a sequence $\{t_k\}_{k\geqslant 1} \subset [\sigma,\infty)$ such that $t_k \searrow \sigma$, as $k \to \infty$, and

$$|u(t_k;\phi) - u_\infty|^2 \geq |u(\sigma;\phi) - u_\infty|^2. \qquad (3.3.10)$$

On the other hand, by virtue of (3.3.9) it is easy to deduce that

$$\sup_{\theta\leqslant 0} |u(\sigma+\theta;\phi) - u_\infty|^2 = \|u_\sigma - u_\infty\|^2_{BCL_{-\infty}(H)} = |u(\sigma;\phi) - u_\infty|^2,$$

which, in view of assumption (3.3.6)-(3.3.7), immediately implies

$$\begin{aligned}&-\nu\langle A(u(\sigma;\phi)-u_\infty),u(\sigma;\phi)-u_\infty\rangle-\langle B(u(\sigma;\phi))-B(u_\infty),u(\sigma;\phi)-u_\infty\rangle\\&+(g(\sigma,u_\sigma(\cdot;\phi))-g(t,u_\infty),u(\sigma;\phi)-u_\infty)<0.\end{aligned}$$

By the continuity of the operators in the problem, there exists $\epsilon_* > 0$ such that for all $\epsilon \in (0, \epsilon_*]$ and $t \in [\sigma, \sigma + \epsilon]$

$$\begin{aligned}&-\nu\langle A(u(t;\phi)-u_\infty),u(t;\phi)-u_\infty\rangle-\langle B(u(t;\phi))-B(u_\infty),u(t;\phi)-u_\infty\rangle\\&+(g(t,u_t(\cdot;\phi))-g(t,u_\infty),u(t;\phi)-u_\infty)<0.\end{aligned}$$

Denoting $w(t) = u(t;\phi) - u_\infty$,

$$\frac{d}{dt}|w(t)|^2=-2\nu\langle Aw(t),w(t)\rangle-2\langle B(u(t,\phi))-B(u_\infty),w(t)\rangle+2(g(t,u_t)-g(t,u_\infty),w(t))$$

for all $t \in [\sigma, \sigma + \epsilon]$. Therefore we obtain

$$\begin{aligned}|w(\sigma+\epsilon;\phi)|^2-|w(\sigma;\phi)|^2=&-2\int_\sigma^{\sigma+\epsilon}\nu\langle Aw(t),w(t)\rangle-\langle B(u(t,\phi))-B(u_\infty),w(t)\rangle dt\\&+2\int_\sigma^{\sigma+\epsilon}(g(t,u_t)-g(t,u_\infty),w(t))dt<0.\end{aligned}$$

Thus $|w(\sigma + \epsilon;\phi)|^2 < |w(\sigma;\phi)|^2$, which contradicts (3.3.10). Hence (3.3.8) is true. □

Remark 3.3.7. *(i) The above result is valid even without uniqueness of stationary solution.*

(ii) In the spirit of Example 3.3.1, it can be applied when $g(t,\xi) := G(\xi(-\rho(t)))$ *for* $(t,\xi) \in \mathbb{R}_+ \times BCL_\infty(H)$, *with* $\rho \in C(\mathbb{R}_+;\mathbb{R}_+)$.

A sufficient condition which implies (3.3.6) but easier to check in applications is given in the following

Corollary 3.3.8. *Suppose that f and g satisfy the assumptions of Theorem 3.3.3. If*

$$2\nu > 2\lambda_1^{-1}M + \frac{l_1}{\nu-\lambda_1^{-1}M}|f| + \frac{l_2}{\nu^2(\nu-\lambda_1^{-1}M)^3}|f|^3, \tag{3.3.11}$$

where l_1, l_2 are defined in (3.3.5), then there exists at least one solution $u_\infty \in D(A)$ to (3.3.1). Moreover, for all $\phi \in BCL_{-\infty}(H)$ with $\phi(0) \in V$ and $\phi \neq u_\infty$, the strong solution $u(t;\phi)$ to (3.1.1)-(3.1.2), satisfies (3.3.8).

Proof. Since $\nu > \lambda_1^{-1}M$, existence of stationary solution is guaranteed by Theorem 3.3.2 (a).

For the second statement we check that condition (3.3.11) implies the ones of Theorem 3.3.6. Indeed, suppose that $\phi \in BCL_{-\infty}(H)$, with $\phi(0) \in V$, is close to some stationary solution u_∞ (but not equal, otherwise it is trivial) and satisfies

$$\|\phi - u_\infty\|^2_{BCL_{-\infty}(H)} = |\phi(0) - u_\infty|^2.$$

Now we verify that (3.3.6) holds. Indeed

$$\begin{aligned}
&-\nu\langle A(\phi(0)-u_\infty),\phi(0)-u_\infty\rangle-\langle B(\phi(0))-B(u_\infty),\phi(0)-u_\infty\rangle+(g(t,\phi)-g(t,u_\infty),\phi(0)-u_\infty)\\
\leqslant&-\nu\|\phi(0)-u_\infty\|^2-b(\phi(0)-u_\infty,u_\infty,\phi(0)-u_\infty)+M\|\phi-u_\infty\|_{BCL_{-\infty}(H)}|\phi(0)-u_\infty|\\
\leqslant&-\nu\|\phi(0)-u_\infty\|^2+\lambda_1^{-1}M\|\phi(0)-u_\infty\|^2+|b(\phi(0)-u_\infty,u_\infty,\phi(0)-u_\infty)|.
\end{aligned}$$

By similar computations to those in the proof of Theorem 3.3.3

$$\begin{aligned}
&-\nu\langle A(\phi(0)-u_\infty),\phi(0)-u_\infty\rangle-\langle B(\phi(0))-B(u_\infty),\phi(0)-u_\infty\rangle+(g(t,\phi)-g(t,u_\infty),\phi(0)-u_\infty)\\
\leqslant&\left(-\nu+\lambda_1^{-1}M+\frac{l_1}{2(\nu-\lambda_1^{-1}M)}|f|+\frac{l_2}{2\nu^2(\nu-\lambda_1^{-1}M)^3}|f|^3\right)\|\phi(0)-u_\infty\|^2,
\end{aligned}$$

which is negative by (3.3.11). Thus, (3.3.6) holds and therefore (3.3.8) too. □

Remark 3.3.9. *Observe that the Razumihkin technique only requires continuity on the delay term and the operators of the model. Here (3.3.11) allows more choices for ν ensuring stability than the values provided by the condition (3.3.2). In other words, this latter result improves the former one.*

3.3.4 Stability via the construction of Lyapunov functionals

In this paragraph we analyze the stability of a very particular stationary solution to (3.1.1) by constructing Lyapunov functionals. Namely we assume that the stationary solution is the trivial one, of course modifying suitable the assumptions on f and g. We start by recalling a result, borrowed from [40], which is the key to prove the result concerning the construction of Lyapunov functionals. To this end, let us introduce an abstract problem; consider operators $\tilde{A}(t,\cdot):V\to V'$ and $\tilde{f}(t,\cdot):BCL_{-\infty}(H)\to(L^2(\mathcal{O}))^2$ with $\tilde{A}(t,0)=0$ and $\tilde{f}(t,0)=0$. Assume that the following problem is well-posed and the solution is continuous with respect to "t", i.e. $u(\cdot;\phi)\in C((-\infty,T];H)\cap L^2(0,T;V)$, for all $T>0$:

$$\begin{aligned}
&\frac{du}{dt}=\tilde{A}(t,u)+\tilde{f}(t,u_t)\quad\forall t>0,\\
&u(s)=\phi(s),\quad s\in(-\infty,0],
\end{aligned}\tag{3.3.12}$$

To prove the stability of the trivial solution to (3.3.12) by constructing Lyapunov functionals, we have the following result (cf. [40, Theorem 1.1] for a similar result).

Proposition 3.3.10. *Assume that there exists a functional $U:\mathbb{R}_+\times BCL_{-\infty}(H)\to\mathbb{R}_+$ such that, for any $\phi\in BCL_{-\infty}(H)$, the following conditions hold*

$$\begin{aligned}
&U(t,u_t)\geqslant\gamma_1|u(t)|^2\quad\forall t\geqslant0,\\
&U(0,u_0)\leqslant\gamma_2\|\phi\|^2_{BCL_{-\infty}(H)},\\
&\frac{d}{dt}U(t,u_t)\leqslant-\gamma_3|u(t)|^2,\quad t\geqslant0,
\end{aligned}$$

where $\gamma_1,\gamma_2,\gamma_3$ are positive numbers and $u(\cdot)=u(\cdot;\phi)$ is the solution to (3.3.12). Then the trivial solution of (3.3.12) is asymptotically stable.

Proof. From the first two conditions of Proposition 3.3.10, we have

$$\gamma_1|u(t)|^2 \leqslant U(t,u_t) \leqslant U(0,u_0) \leqslant U(0,_0) \leqslant \gamma_2\|\phi\|^2_{BCL_{-\infty}(H)},$$

which means

$$|u(t)|^2 \leqslant \frac{\gamma_2}{\gamma_1}\|\phi\|^2_{BCL_{-\infty}(H)}, \quad \forall t \geqslant 0. \tag{3.3.13}$$

Notice that $\frac{d}{dt}U(t,u_t) \leqslant -\gamma_3|u(t)|^2$, we deduce that

$$\int_0^\infty |u(s)|^2 ds \leqslant \frac{\gamma_2}{\gamma_3}\|\phi\|^2_{BCL_{-\infty}(H)}. \tag{3.3.14}$$

On the other hand, the solution of problem (3.3.12) is continuous respect to "t", which implies that the function $|u(t)|^2$ is continuous with respect to "t", which together with (3.3.13) and (3.3.14), we obtain that

$$\lim_{t\to+\infty} |u(t)|^2 = 0,$$

namely, the trivial solution of problem (3.3.12) asymptotically stable. □

We will apply the above result to the following equation, which is a particular case of (3.3.12).

$$\frac{du}{dt} = \tilde{A}(t,u) + F(u(t-\rho(t))), \tag{3.3.15}$$

where $\tilde{A}(t,\cdot) : V \to V'$ and $F : H \to (L^2(O))^2$ are appropriate operators. The following result is a slight variation of [40, Theorem 2.1].

Theorem 3.3.11. *Assume that operators in (3.3.15) satisfy*

$$\begin{aligned}
&\langle \tilde{A}(t,u), u\rangle \leqslant -\gamma\|u\|^2, \quad \gamma > 0,\\
&F : H \to (L^2(O))^2, \quad |F(u)| \leqslant \alpha|u|, \ u \in V,\\
&\rho \in C^1(\mathbb{R}_+;\mathbb{R}_+), \ \rho'(t) \leqslant \rho_* < 1.
\end{aligned}$$

Then the trivial solution of (3.3.15) is stable provided that

$$\gamma \geq \frac{\alpha}{\lambda_1\sqrt{1-\rho_*}}. \tag{3.3.16}$$

Proof. We construct $U : \mathbb{R}_+ \times BCL_{-\infty}(H) \to \mathbb{R}_+$ for (3.3.15) in the form

$$U(t,\phi) = |\phi(0)|^2 + \frac{c}{1-\rho_*}\int_{-\rho(t)}^0 |\phi(s)|^2 ds,$$

where $c > 0$ is a suitable constant, to be determined later on, such that U is a Lyapunov functional. Denoting by $U(t) = U(t, u_t(\cdot;\phi))$, where $u_t(\cdot;\phi)$ is the solution to (3.3.15) with initial value ϕ, we have $U(t) = |u(t)|^2 + \frac{c}{1-\rho_*}\int_{t-\rho(t)}^{t} \|u(s)\|^2 ds$. Consequently,

$$\begin{aligned}
\frac{d}{dt}U(t) &= 2\langle \tilde{A}(t,u(t)) + F(u(t-\rho(t))), u(t)\rangle + \frac{c}{1-\rho_*}|u(t)|^2 - \frac{c(1-\rho'(t))}{(1-\rho_*)}\|u(t-\rho(t))\|^2 \\
&\leqslant -2\gamma\|u(t)\|^2 + 2\alpha|u(t-\rho(t))\|u(t)| + \frac{c}{\lambda_1(1-\rho_*)}\|u(t)\|^2 - c|u(t-\rho(t))|^2 \\
&\leq \left(-2\gamma + \lambda_1^{-1}\left(\frac{c}{1-\rho_*} + \frac{\alpha^2}{c}\right)\right)\|u(t)\|^2,
\end{aligned}$$

where Poincaré and Young's inequalities have been used. Minimizing the coefficient in brackets in the right hand side, which is attained for $c = \alpha\sqrt{1-\rho_*}$, we conclude that

$$\frac{d}{dt}U(t) \leq 2\left(-\gamma + \frac{\alpha}{\lambda_1\sqrt{1-\rho_*}}\right)\|u(t)\|^2.$$

Then by (3.3.16) it arises $\frac{d}{dt}U(t) \leqslant 0$. As it is easy to check that $U(\cdot,\cdot)$ satisfies all the conditions in Proposition 3.3.10, the stability statement holds. □

Now going back to our original problem of this section, suppose that the origin is a stationary solution to (3.1.1), where we are assuming that $f \equiv 0$ and $g(t,u_t) = G(u(t-\rho(t)))$ with $G : H \to (L^2(O))^2$ a Lipschitz continuous function with Lipschitz constant $M > 0$ and $G(0) = 0$.

Corollary 3.3.12. *Consider the Navier-Stokes problem*

$$\frac{du}{dt} + \nu Au + B(u) = G(u(t-\rho(t))) \ \ \forall t \geqslant 0, \tag{3.3.17}$$

where $G : H \to (L^2(O))^2$ fulfills the above conditions and $\nu > \lambda_1^{-1}M$. Then, $u \equiv 0$ is the unique stationary solution. Moreover, it is stable provided that $\nu \geq M/(\lambda_1\sqrt{1-\rho_*})$.

Proof. The first part is a consequence of Theorem 3.3.2 (c).

Second statement follows from Theorem 3.3.11. Indeed (3.3.17) can be set in (3.3.15) by denoting $\tilde{A}(t,u) = -\nu Au - B(u)$ and $F(u(t-\rho(t))) = G(u(t-\rho(t)))$ taking $\gamma = \nu$ and $\alpha = M$. □

Remark 3.3.13. *Taking $f \equiv 0$ in Theorem 3.3.3, the trivial solution to (3.1.1) is stable if $\nu > \frac{(2-\rho_*)\lambda_1^{-1}M}{2(1-\rho_*)}$. Since $\frac{(2-\rho_*)\lambda_1^{-1}M}{2(1-\rho_*)} > \frac{M}{\lambda_1\sqrt{1-\rho_*}}$ for $\rho_* \in (0,1)$, Corollary 3.3.12 improves, for this case, the condition established in Theorem 3.3.3.*

3.4 Polynomial stability: a special unbounded variable delay case

As mentioned in the introduction, the main goal of this paper is to analyze the stability of stationary solutions to (3.1.1) in the unbounded variable delay case, less studied than finite delay cases. Three different methods have been used to study the stability of stationary solution in previous sections. However, instead of exponential stability, only local stability of the stationary solution of (3.1.1) is obtained. In fact, even for simple ordinary differential equations with unbounded variable delay, for instance, the pantograph equation, in which the delay term is given by $\rho(t) = (1-\lambda)t$ with $0 < \lambda < 1$, the exponential stability of stationary solution cannot be reached. But, fortunately, in this simple case the polynomial stability of stationary solution can be obtained, see [1, 104, 105] for details. Enlightened by [1], we show that it is still possible to prove the polynomial stability of stationary solution to Navier-Stokes equation with proportional delay, which is a particular case of unbounded variable delay. To this end, we first review the following pantograph equation and some technical lemmas that are used in this framework.

An example of the pantograph equation reads

$$x'(t) = ax(t) + bx(\lambda t) \;\; \forall t \geqslant 0, \;\; x(0) = x_0, \;\; \lambda \in (0,1), \tag{3.4.1}$$

which has been studied in [1, 104, 105] amongst many others.

The following lemma will be useful.

Lemma 3.4.1. *(Cf. [1, Lemma 3.4]) Let $a \in \mathbb{R}$, $b > 0$ and $\lambda \in (0,1)$. Assume x is the solution to (3.4.1) with $x(0) > 0$. Suppose $p \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$D^+ p(t) \leqslant ap(t) + bp(\lambda t), \quad t \geqslant 0,$$

with $0 < p(0) < x(0)$ and where D^+ denotes the Dini derivative. Then $p(t) \leqslant x(t)$ for all $t \geqslant 0$.

Lemma 3.4.2. *(Cf. [1, Lemma 3.5]) Let x be a solution to (3.4.1). If $a < 0$, $b \in \mathbb{R}$, then there exists $C = C(a, b, \lambda) > 0$ such that*

$$\limsup_{t\to\infty} \frac{|x(t)|}{t^\mu} = C|x(0)|,$$

where $\mu \in \mathbb{R}$ obeys

$$0 = a + |b|\lambda^\mu.$$

Then, for some (possible new) $C = C(a, b, \lambda) > 0$, we have

$$|x(t)| \leqslant C|x(0)|(1+t)^\mu, \quad t \geqslant 0. \tag{3.4.2}$$

Observe that if $\mu < 0$, then (3.4.2) implies polynomial stability of the trivial solution to (3.4.1). Next we use this idea to prove the polynomial stability of stationary solution to (3.1.1).

Theorem 3.4.3. *Consider (3.1.1) with $f \equiv 0$, $g(t, u_t) := L_g u(\lambda t)$ with $0 < \lambda < 1$, $L_g \in \mathbb{R}$ and $\nu > \lambda_1^{-1}|L_g|$. Then the origin is the unique stationary solution and any evolutionary solution u converges to zero polynomially, namely, there exist $C = C(\nu, L_g, \lambda) > 0$ and $\mu < 0$ such that*

$$|u(t)|^2 < C|u(0)|^2(1+t)^\mu \; \forall t \geqslant 0,$$

where μ satisfies $|L_g| - 2\nu\lambda_1 + |L_g|\lambda^\mu = 0$.

Proof. Taking the inner product of (3.1.1) with u in H, we obtain

$$\frac{d}{dt}|u(t)|^2 + 2\nu\|u(t)\|^2 = 2L_g(u(\lambda t), u(t)).$$

By Poincaré and Young's inequalities we have that

$$\frac{d}{dt}|u(t)|^2 + 2\lambda_1\nu|u(t)|^2 \leqslant |L_g||u(t)|^2 + |L_g||u(\lambda t)|^2.$$

Denoting by $w(t) = |u(t)|^2$,

$$w'(t) \leqslant (-2\lambda_1\nu + |L_g|)w(t) + |L_g|w(\lambda t).$$

By Lemmas 3.4.1 and 3.4.2, there exists $C = C(\nu, L_g, \lambda) > 0$ and $\mu \in \mathbb{R}$ such that

$$w(t) \leqslant Cw(0)(1+t)^{\mu},$$

Since $-2\lambda_1\nu + 2|L_g| < 0$, it holds that $\mu < 0$. Then the polynomial decay of solutions follows. □

Remark 3.4.4. *(i) From Theorem 3.4.3 we find that, as long as we have $\nu > \lambda_1^{-1}|L_g|$, any solution to (3.1.1) converges polynomially to zero. In this case, this result improves all the stability results established previously.*

(ii) In fact, our result can be extended to a more general case, namely, if the delay term is defined as $g(t, \phi) = G(\phi(-(1-\lambda)t))$, which is also Lipschitz.

(iii) From [1] we know that the convergence to equilibria needs not be at an exponential rate for equations with unbounded delay. Actually, for unbounded variable delay, even in the simplest case, i.e., the pantograph equation (3.4.1), only polynomial stability can be obtained because the solutions behave in polynomial way (cf. [104, Theorem 3] for more details). Consequently, it is still an open problem to obtain sufficient conditions for the exponential stability of solutions for equations with other types of unbounded variable delay. This will be investigated in future.

Chapter 4

Stochastic Navier-Stokes equation with infinite delay

In this chapter we generalize the results in Chapter 3 to the stochastic case. In other words, we will investigate the following stochastic Navier-Stokes equation with infinite delay

$$\frac{du}{dt} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) + g_1(t, u_t) + g_2(t, u_t)\frac{dW(t)}{dt}, \text{ in } (\tau, T)\times\mathcal{O}, \tag{4.0.1}$$

$$div\ u \equiv 0, \text{ in } (\tau, T)\times\mathcal{O}, \tag{4.0.2}$$

$$u = 0, \text{ in } (\tau, T)\times\partial\mathcal{O}, \tag{4.0.3}$$

$$u(\tau + s, x) = \phi(s, x),\ s\in(-\infty, 0],\ x\in\mathcal{O}, \tag{4.0.4}$$

where $\mathcal{O}\subset\mathbb{R}^2$ is a bounded open set with boundary $\partial\mathcal{O}$, $\nu > 0$ is the kinematic viscosity, u is the velocity field of the fluid, p is the pressure, ϕ is the initial datum, f is a nondelayed external force field, and g_1, g_2 are external forces containing some hereditary characteristic.

In [45, 171], authors established exponential stability to stochastic Navier-Stokes equations with bounded variable delay, respectively. And Taniguchi proved the existence and asymptotic behaviour of energy solutions to Navier-Stokes equations driven by Levy processes and external force terms with finite delay in [158].

However, as far as we know, there is no available work about stochastic Navier-Stokes equations with infinite delay, neither distributed delay nor unbounded variable delay. It is worth mentioning that many authors took the weighted space C_γ as the phase space when dealt with differential equations with infinite delay, and obtained exponential stability and convergence. Nevertheless, the methods that are used to prove exponential stability and the convergence of solutions only work for differential equations with distributed delay, and it does not work for unbounded variable delays, for example, the stochastic pantograph equation. Fortunately, authors in [120] solved this problem by choosing

$$BCL_{-\infty}(H) = \{\varphi\in C((-\infty, 0]; H) : \lim_{\theta\to-\infty}\varphi(\theta) \text{ exists in } H\}$$

as the phase space. But in this case, only asymptotic stability was established as they were not able to prove exponential stability. Yet under some special unbounded variable delay case, they obtained the polynomial stability of stationary solution.

Motivated by [1, 158], we study now a type of stochastic 2D-Navier-Stokes equations with infinite delay. More precisely, we will prove the existence and uniqueness of solution to Eq. (4.0.1), then focus on the stability analysis with unbounded variable delay. Besides, we study the polynomial stability of the stationary solution to Navier-Stokes equation with proportional delay. The classical Galerkin method will be used to prove the existence of solutions. However, the traditional technique to prove uniqueness of solution, used in the deterministic case, is not enough to verify the uniqueness of solution any more in the stochastic case. The main reason is that we cannot bound the trilinear term directly as we did in the deterministic case. Therefore, more technical skills are needed. Indeed, we use some auxiliary lemmas to solve this difficulty, and this is not trivial at all. And we can claim that the stability results proved for stochastic Navier-Stokes equations with unbounded variable delay are new.

4.1 Premilinaries

Let $(\Omega, P, \mathfrak{F})$ be a probability space on which an increasing and right continuous family $\{\mathfrak{F}_t\}_{t\in[0,\infty)}$ of complete sub-$\sigma-$algebra of $\mathfrak{F}$ is defined. Let $\beta_n(t)(n = 1, 2, 3, \cdots)$ be a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, P, \mathfrak{F})$. Set

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda'_n}\beta_n(t)e_n, \quad t \geqslant 0,$$

where $\lambda'_n(n = 1, 2, 3, \cdots)$ are nonnegative real numbers such that $\sum_{n=1}^{\infty} \lambda'_n < +\infty$, and $\{e_n\}(n = 1, 2, 3, \cdots)$ is a complete orthonormal basis in the real and separable Hilbert space K. Let $Q \in L(K, K)$ be the operator defined by $Qe_n = \lambda'_n e_n$. The above $K-$valued stochastic process $W(t)$ is called a $Q-$Wiener process. Given real numbers $a < b$, and a separable Hilbert space H we will denote by $I^2(a, b; H)$ the space of all processes $X \in L^2(\Omega \times (a, b), \mathfrak{F} \otimes \mathcal{B}((a, b)), dP \otimes dt; H)$ (where $\mathcal{B}((a, b))$ denotes the Borel $\sigma-$algebra on (a, b)) such that $X(t)$ is $\mathfrak{F}-$measurable a.e. $t \in (a, b)$. The space $I^2(a, b; H)$ is a closed subspace of $L^2(\Omega \times (a, b), \mathfrak{F} \otimes \mathcal{B}((a, b)), dP \otimes dt; H)$.

We will denote by $C(a, b; H)$ the Banach space of all continuous functions from $[a, b]$ into H equipped with sup norm. We will write $L^2(\Omega; C(a, b; H))$ instead of $L^2(\Omega, \mathfrak{F}, dP; C(a, b; H))$.

Let us also consider a real number $T > 0$. If we consider a function $x \in C(-\infty, T; H)$, for each $t \in [0, T]$ we will denote $x_t \in C(-\infty, 0; H)$ by $x_t(s) = x(t + s)$, $\forall s \in (-\infty, 0]$. Moreover, if $y \in L^2(-\infty, T; H)$, we will also denote $y_t \in L^2(-\infty, 0; H)$, for a.e. $t \in (0, T)$, by $y_t(s) = y(t + s)$ *a.e.* $s \in (-\infty, 0]$.

We now enumerate the assumptions on the delay terms g_1, g_2, we assume that $g_i : [\tau, T] \times BCL_{-\infty}(H) \to (L^2(\mathcal{O}))^2$, $i = 1, 2$.

(g1) For any $\xi \in BCL_{-\infty}(H)$, the mapping $[\tau, T] \ni t \mapsto g_i(t, \xi) \in (L^2(\mathcal{O}))^2$ is measurable.

(g2) $g_i(\cdot, 0) = 0$.

(g3) There exists a constants $L_{g_i} > 0$ such that, for any $t \in [\tau, T]$ and all $\xi, \eta \in BCL_{-\infty}(H)$,

$$|g_i(t,\xi) - g_i(t,\eta)| \leqslant L_{g_i} \|\xi - \eta\|_{BCL_{-\infty}(H)}.$$

Remark 4.1.1. *(i) As pointed out in [130], condition* (g2) *is not really a restriction, since otherwise, if* $|g_i(\cdot,0)| \in L^2(\tau,T)$*, we could redefine* $\hat{f}_i(t) = f_i(t) + g_i(t,0)$ *and* $\hat{g}_i(t,\cdot) = g_i(t,\cdot) - g_i(t,0)$*. In this way the problem is exactly the same,* $\hat{f}$ *and* $\hat{g}$ *satisfy the required assumptions.*
(ii) Conditions $(g2)$ *and* $(g3)$ *imply that*

$$|g_i(t,\xi)| \leqslant L_{g_i} \|\xi\|_{BCL_{-\infty}(H)},$$

so $|g_i(t,\xi)| \in L^\infty(\tau,T)$.

Examples of delay forcing term which satisfy $(g1) - (g3)$ could see Example 3.1.2, 3.1.3 and 3.1.4 in Chapter 3. Later on, to illustrate the different methods for the stability analysis, we focus on unbounded variable delays case. Readers are referred to [120] for details of the examples .
Next we give the definition of weak solution for problem (4.0.1).

Definition 4.1.2. *A stochastic process* $u(t), t \geqslant 0$*, is said to be a weak solution of Eq.* (4.0.1)*, if*

(1a) $u(t)$ *is* $\mathfrak{F}$*-adapted,*

(1b) $u(t) \in I^2(-\infty,T;V) \cap L^2(\Omega; C(-\infty,T;H))$*,*

(1c) The following equation holds as an identity in V'

$$u(t) = \phi(0) - \nu \int_0^t Au(s)ds - \int_0^t B(u(s))ds + \int_0^t (f(s) + g_1(s,u_s))\, ds + \int_0^t g_2(s,u_s)dW(s), \quad t \in [0,T].$$
$$u(t) = \phi(t), \quad t \in (-\infty, 0].$$

The following lemma (see Sritharan and Sundar [153]) is essential to prove the uniqueness of solution.

Lemma 4.1.3. *There exist a* $\lambda > 0$ *such that for any* $u, v \in V$*,*

$$-2(B(u) - B(v), u - v) - \nu(A(u-v), u-v) \leqslant \lambda \|v\|_4^4 |u-v|^2.$$

Denote by

$$\lambda_1 = \inf_{v \in V \setminus \{0\}} \frac{\|v\|^2}{|v|^2} > 0.$$

4.2 Existence and uniqueness of solutions

In this section, we establish existence and uniqueness of weak solution for Eq.(4.0.1). We begin with the uniqueness.

Lemma 4.2.1. *Assume that* $(g1)-(g3)$ *hold true. Then, for* $\phi \in I^2(-\infty,0;V)\cap L^2(\Omega;BCL_{-\infty}(H))$*, such that* $E[\sup_{-\infty<s\leqslant 0}|\phi(s)|^4]<\infty$*, there exists at most one weak solution to* (4.0.1)*.*

Proof. Let $u(t)$ and $v(t)$ be two solutions to (4.0.1) with the same initial value $u(s)=v(s)=\phi(s),\ s\leqslant 0$. Let $N>1$ be any fixed integer and

$$\tau_N := \inf\left\{t\leqslant T : \int_0^t |v(s)|_4^4 ds \geqslant N\right\}.$$

Without loss of generality we may assume that $E\int_0^T |v(s)|_4^4 ds<\infty$. Actually, this is a direct consequence of Lemma 4.2.4. Set

$$r(t) := \exp\left(-\lambda\int_0^t |v(s)|_4^4 ds\right),$$

where $\lambda>0$ is the one in Lemma 4.1.3. Hence,

$$r(t\wedge\tau_N)\geqslant \exp(-\lambda N).$$

Applying Itô formula to the function $r(t)|u(t)-v(t)|^2$, we have that

$$\begin{aligned}
r(t)|u(t)-v(t)|^2 = &-\lambda\int_0^t r(s)|v(s)|_4^4|u(s)-v(s)|^2 ds\\
&+2\int_0^t r(s)\left(u(s)-v(s),-\nu A(u(s)-v(s))-B(u(s))+B(v(s))\right)ds\\
&+2\int_0^t r(s)\left(u(s)-v(s),g_1(s,u_s)-g_1(s,v_s)\right)ds\\
&+2\int_0^t r(s)\left(u(s)-v(s),g_2(s,u_s)-g_2(s,v_s)\right)dW(s)\\
&+\int_0^t r(s)|g_2(s,u_s)-g_2(s,v_s)|^2 ds.
\end{aligned}$$

Thanks to Lemma 4.1.3, taking the supremum (w.r.t. t) and then taking expectation.

$$\begin{aligned}
&E\left[\sup_{0\leqslant l\leqslant t\wedge\tau_N} r(l)|u(l)-v(l)|^2\right]+\nu E\int_0^{t\wedge\tau_N} r(s)\|u(s)-v(s)\|^2 ds\\
&\leqslant 2E\left[\sup_{0\leqslant l\leqslant t\wedge\tau_N}\left|\int_0^l r(s)(u(s)-v(s),g_1(s,u_s)-g_1(s,v_s))ds\right|\right]\\
&\quad+2E\left[\sup_{0\leqslant l\leqslant t\wedge\tau_N}\left|\int_0^l r(s)(u(s)-v(s),g_2(s,u_s)-g_2(s,v_s))dW(s)\right|\right]\\
&\quad+E\left[\sup_{0\leqslant l\leqslant t\wedge\tau_N}\int_0^l r(s)|g_2(s,u_s)-g_2(s,v_s)|^2 ds\right].
\end{aligned}$$

The first term on the left-hand side of the above inequality can be bounded by

$$2E\left[\sup_{0\leqslant l\leqslant t\wedge\tau_N}|\int_0^l r(s)(u(s)-v(s),g_1(s,u_s)-g_1(s,v_s))ds|\right]$$
$$\leqslant \frac{1}{4}E\left[\sup_{0\leqslant l\leqslant t\wedge\tau_N} r(l)|u(l)-v(l)|^2\right]+4L_{g_1}^2\int_0^t Er(s\wedge\tau_N)|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2ds.$$

On the other hand, Burkholder-Davis-Gundy's inequality yields

$$2E\left[\sup_{0\leqslant l\leqslant t\wedge\tau_N}|\int_0^l r(s)(u(s)-v(s),g_2(s,u_s)-g_2(s,v_s))dW(s)|\right]$$
$$\leqslant 8E\left\{\sup_{0\leqslant l\leqslant t\wedge\tau_N} r^{1/2}(l)|u(l)-v(l)|\cdot[\int_0^t r(s)|g_2(s,u_s)-g_2(s,v_s)|^2_{L_2^0}ds]^{1/2}\right\}$$
$$\leqslant \frac{1}{4}E[\sup_{0\leqslant l\leqslant t\wedge\tau_N} r(l)|u(l)-v(l)|^2]+64L_{g_2}^2\int_0^t Er(s\wedge\tau_N)|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2ds.$$

And

$$E\left[\sup_{0\leqslant l\leqslant t\wedge\tau_N}\int_0^l r(s)|g_2(s,u_s)-g_2(s,v_s)|^2ds\right]\leqslant L_{g_2}^2\int_0^t Er(s\wedge\tau_N)|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2ds.$$

From the previous inequalities we have

$$E[\sup_{0\leqslant s\leqslant t}|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2]+2\nu E\int_0^t r(s\wedge\tau_N)\|u(s\wedge\tau_N)-v(s\wedge\tau_N)\|^2ds$$
$$\leqslant (8L_{g_1}^2+130L_{g_2}^2)e^{\lambda N}\int_0^t E\sup_{0\leqslant\tau\leqslant s}|u(\tau\wedge\tau_N)-v(\tau\wedge\tau_N)|^2ds.$$

By the Gronwall Lemma,

$$E[\sup_{0\leqslant s\leqslant t}|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2]=0.$$

Thus for any fixed $N>1$,

$$u(t\wedge\tau_N)=v(t\wedge\tau_N),\quad a.e.,\ \omega\in\Omega.$$

By Markov's inequality,

$$P(\tau_N<T)=P\left(\int_0^t|v(s)|_4^4ds\geqslant N\right)\leqslant\frac{E\int_0^t|v(s)|_4^4ds}{N},$$

since $E\int_0^t|v(s)|_4^4ds<\infty$, we obtain that $\tau_N\to T$ as $N\to\infty$. Consequently, $u(t)=v(t)$, a.e., $\omega\in\Omega$, for all $t\leqslant T$. The proof is completed.

Remark 4.2.2. *In above proof, we used Markov's inequality, and that is why we need the fourth moment of solutions be finite, see [141] for more details about Markov's inequality.*

□

Let $0 < T \leqslant \infty$ and thus $T = \infty$ means $[0, T] = [0, \infty)$. Let $\{w_j\}_{j=1}^{\infty} \subset D(A)$ be a complete orthonormal basis in $L^2(\mathcal{O})$. Now we use the Galerkin approximation method to prove the existence of weak solutions to Eq. (4.0.1). Set

$$u_n(t) = \sum_{j=1}^{n} \alpha_{nj}(t) w_j,$$

where $\alpha_{nj}(t)$ are determined by the following ordinary differential stochastic systems:

$$\begin{aligned}(u_n(t), w_j) = (u_{0n}, w_j) &+ \int_0^t (-\nu A u_n(s) + P_n B(u_n(s)) + P_n f(s), w_j) ds \\ &+ \int_0^t (P_n g_1(s, u_{ns}), w_j) ds + \int_0^t (P_n g_2(s, u_{ns}) dW(s), w_j), \quad j = 1, 2, \cdots, n,\end{aligned}$$

with an initial value $u_n(t) = P_n\phi(t)$, $t \in (-\infty, 0]$, where $u_{0n} = u_n(0) = P_n\phi(0) = \sum_{j=1}^{n} (u_0, w_j) w_j$.

Consider the next stochastic equation

$$\begin{aligned}&u_n(t) = u_{0n} + \int_0^t (-\nu A u_n(s) + B(u_n(s)) + P_n f(s)) ds + \int_0^t P_n g_1(s, u_{ns}) ds + \int_0^t P_n g_2(s, u_{ns}) dW(s). \\ &u_n(t) = P_n\phi(t), t \in (-\infty, 0],\end{aligned}$$

where $u_{0n} = u_n(0) = P_n\phi(0)$.

Lemma 4.2.3. *Assume that $(g_1) - (g_3)$ hold and $E \int_0^t |f(s)|^4 ds < \infty$. Then, for $\phi \in I^2(-\infty, 0; V) \cap L^2(\Omega; BCL_{-\infty}(H))$ such that $E[\sup_{-\infty < s \leqslant 0} |\phi(s)|^4] < \infty$, there exists a constant $c_0 > 0$ such that*

$$E[\sup_{0 \leqslant s \leqslant t} |u_n(s)|^2] + \int_0^t E\|u_n(s)\|^2 ds \leqslant c_0, \quad \textit{uniformly in } n \geqslant 1.$$

Proof. Use Itô's formula for $|u_n(t)|^2$,

$$\begin{aligned}|u_n(t)|^2 = |P_n u_0|^2 &+ 2\int_0^t (-\nu A u_n(s) - B(u_n(s)), u_n(s)) ds + 2\int_0^t (f(s) + g_1(s, u_{ns}), u_n(s)) ds \\ &+ 2\int_0^t (g_2(s, u_{ns}), u_n(s)) dW(s) + \int_0^t |g_2(s, u_{ns})|^2 ds.\end{aligned} \tag{4.2.1}$$

Note that $(u_n(t), -B(u_n(t))) = 0$. Taking supremum w.r.t t in (4.2.1) and expectation, we obtain

$$\begin{aligned}
&E[\sup_{0\leqslant s\leqslant t} |u_n(s)|^2] + 2\nu \int_0^t E\|u_n(s)\|^2 ds \\
&\leqslant E|u_0|^2 + 2E[\sup_{0\leqslant\tau\leqslant t} \int_0^\tau |u_n(s)||f(s)|ds] + 2E[\sup_{0\leqslant\tau\leqslant t} \int_0^\tau |u_n(s)||g_1(s, u_{ns})|ds] \\
&\quad + 2E[\sup_{0\leqslant\tau\leqslant t} |\int_0^\tau (u_n(s), g_2(s, u_{ns}))dW(s)|] + E[\sup_{0\leqslant\tau\leqslant t} \int_0^\tau |g_2(s, u_{ns})|^2 ds] \\
&= E|u_0|^2 + J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

$$J_1 = 2E[\sup_{0\leqslant\tau\leqslant t} \int_0^\tau |u_n(s)||f(s)|ds] \leqslant \int_0^t E[\sup_{0\leqslant\tau\leqslant s} |u_n(\tau)|^2]d\tau + E\int_0^t |f(s)|^2 ds.$$

$$J_2 = 2E[\sup_{0\leqslant\tau\leqslant t} \int_0^\tau |u_n(s)||g_1(s, u_{ns})|ds] \leqslant \frac{1}{4}E[\sup_{0\leqslant s\leqslant t} |u_n(s)|^2] + 4L_{g_1}^2 \int_0^t E[\sup_{0\leqslant\tau\leqslant s} |u_n(\tau)|^2]ds + 4L_{g_1}^2 E[\sup_{-\infty<s\leqslant 0} |\phi(s)|^4].$$

By Burkholder-Davis-Gundy's inequality,

$$\begin{aligned}
J_3 &= 2E[\sup_{0\leqslant\tau\leqslant t} |\int_0^\tau (u_n(s), g_2(s, u_{ns}))dW(s)|] \\
&\leqslant 8E[(\int_0^t |u_n(s)|^2 |g_2(s, u_{ns})|^2 ds)^{1/2}] \\
&\leqslant \frac{1}{4}E[\sup_{0\leqslant s\leqslant t} |u_n(s)|^2] + 64L_{g_2}^2 \int_0^t E[\sup_{0\leqslant\tau\leqslant s} |u_n(\tau)|^2]ds + 64L_{g_2}^2 E[\sup_{-\infty<s\leqslant 0} |\phi(s)|^4].
\end{aligned}$$

$$J_4 = E[\sup_{0\leqslant\tau\leqslant t} |\int_0^\tau |g_2(s, u_{ns})|^2 ds] \leqslant L_{g_2}^2 \int_0^t E[\sup_{0\leqslant\tau\leqslant s} |u_n(\tau)|^2]ds + L_{g_2}^2 E[\sup_{-\infty<s\leqslant 0} |\phi(s)|^4].$$

$$\begin{aligned}
&\frac{1}{2}E[\sup_{0\leqslant s\leqslant t} |u_n(s)|^2] + 2\nu \int_0^t E\|u_n(s)\|^2 ds \\
&\leqslant E|u_0|^2 + E\int_0^t |f(s)|^2 ds + (4L_{g_1}^2 + 65L_{g_2}^2)E[\sup_{-\infty<s\leqslant 0} |\phi(s)|^4] + (1 + 4L_{g_1}^2 + 65L_{g_2}^2)\int_0^t E[\sup_{0\leqslant\tau\leqslant s} |u_n(\tau)|^2]ds.
\end{aligned}$$

Then the conclusion follows directly from the Gronwall Lemma. □

Lemma 4.2.4. *Assume that $(g_1) - (g_3)$ hold and $E\int_0^t |f(s)|^4 ds < \infty$. Then, for $\phi \in I^2(-\infty, 0; V) \cap L^2(\Omega; BCL_{-\infty}(H))$ such that $E[\sup_{-\infty<s\leqslant 0} |\phi(s)|^4] < \infty$, there exists a $\delta > 0$, which is independent of n and will be specified later, such that $E\int_0^t |u_n(s)|_4^4 ds < \delta$.*

Proof. Applying the Itô formula to $|u_n(t)|^4$,

$$|u_n(t)|^4 = |P_n u_0|^4 + 4\int_0^t |u_n(s)|^2(u_n(s), -\nu A u_n(s) - B(u_n(s)))ds + 4\int_0^t |u_n(s)|^2(f(s) + g_1(s, u_{ns}), u_n(s))ds$$
$$+ 4\int_0^t |u_n(s)|^2(u_n(s), g_2(s, u_{ns}))dW(s) + 6\int_0^t |u_n(s)|^2|g_2(s, u_{ns})|^2_{L_2^0}ds.$$

Taking supremum and expectation,

$$\begin{aligned}
&E\left[\sup_{0\leqslant\tau\leqslant t}|u_n(\tau)|^4\right] + 4\nu E\left[\int_0^t |u_n(s)|^2\|u_n(s)\|^2 ds\right]\\
&\leqslant E|u_0|^4 + 2E\left[\sup_{0\leqslant\tau\leqslant t}\int_0^\tau |u_n(s)|^2(|f|^2 + |u_n(s)|^2)ds\right] + 2E\left[\sup_{0\leqslant\tau\leqslant t}\int_0^\tau |u_n(s)|^2(L_{g_1}^2|u_{ns}|^2 + |u_n(s)|^2)ds\right]\\
&+ 4E\left[\sup_{0\leqslant\tau\leqslant t}\int_0^\tau |u_n(s)|^2(u_n(s), g_2(s, u_{ns}))dW(s)\right] + 6L_{g_2}^2 E\left[\sup_{0\leqslant\tau\leqslant t}\int_0^\tau |u_n(s)|^2|u_{ns}|^2 ds\right]\\
&= E|u_0|^4 + I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Now we estimate $I_i,\ i = 1, 2, 3, 4$, one by one.

$$I_1 = 2E\left[\sup_{0\leqslant\tau\leqslant t}\int_0^\tau |u_n(s)|^2(|f|^2 + |u_n(s)|^2)ds\right] \leqslant 3\int_0^t E[\sup_{0\leqslant\tau\leqslant s}|u_n(\tau)|^4]ds + E\int_0^t |f(s)|^4 ds.$$

$$I_2 = 2E\left[\sup_{0\leqslant\tau\leqslant t}\int_0^\tau |u_n(s)|^2(L_{g_1}^2|u_{ns}|^2 + |u_n(s)|^2)ds\right] \leqslant 2(1 + L_{g_1}^2)\int_0^t E[\sup_{0\leqslant\tau\leqslant s}|u_n(\tau)|^4]ds + L_{g_1}^2 E[\sup_{-\infty<s\leqslant 0}|\phi(s)|^4].$$

Using Burkholder-Davis-Gundy's inequality,

$$I_3 \leqslant \frac{1}{2}E[\sup_{0\leqslant\tau\leqslant t}|u_n(\tau)|^4] + 256L_{g_2}^2\int_0^t E[\sup_{0\leqslant\tau\leqslant s}|u_n(\tau)|^4]ds + 256L_{g_2}^2 E[\sup_{-\infty<s\leqslant 0}|\phi(s)|^4].$$

$$I_4 \leqslant 6L_{g_2}^2\int_0^t E[\sup_{0\leqslant\tau\leqslant s}|u_n(\tau)|^4]ds + 6L_{g_2}^2 E[\sup_{-\infty<s\leqslant 0}|\phi(s)|^4].$$

Consequently,

$$\frac{1}{2}E\left[\sup_{0\leqslant\tau\leqslant t}|u_n(\tau)|^4\right] + 4\nu E\left[\int_0^t |u_n(s)|^2\|u(s)\|^2 ds\right] \leqslant c_f + c_g\int_0^t E[\sup_{0\leqslant\tau\leqslant s}|u_n(\tau)|^4]ds.$$

By the Gronwall Lemma, there exists a $C_0 > 0$ such that

$$E\left[\sup_{0\leqslant\tau\leqslant t}|u_n(\tau)|^4\right] + 8\nu E\left[\int_0^t |u_n(s)|^2\|u(s)\|^2 ds\right] \leqslant C_0.$$

From inequality A.1.1, we obtain that

$$|u_n(s)|_4 \leqslant 2^{-\frac{1}{4}}|u_n(s)|^{1/2}\|u_n(s)\|^{1/2}.$$

Thus, there exists a constant $\delta = \frac{C_0}{16\nu}$ such that

$$E\int_0^t |u_n(s)|_4^4 ds \leqslant \frac{1}{2}E\int_0^t |u_n(s)|^2\|u_n(s)\|^2 ds < \delta.$$

The proof is completed. □

Under more suitable assumption, we can prove the existence and uniqueness of solutions of our problem. There is a positive constant λ (the same as the one in Lemma 4.1.3) such that for all $u, v \in L^2(-\infty, T; V)$ and for all $t \in [0, T]$, it holds

$$\begin{aligned}
&\int_0^t |g_2(s, u_s) - g_2(s, v_s)|^2 ds - \nu\int_0^t \|u(s) - v(s)\|^2 ds \\
&\leqslant \lambda\int_0^t |v(s)|_4^4|u(s) - v(s)|^2 + 2\int_0^t (B(u) - B(v), u(s) - v(s))ds \\
&\quad + \nu\int_0^t (Au(s) - Av(s), u(s) - v(s))ds - 2\int_0^t (g_1(s, u_s) - g_1(s, v_s), u(s) - v(s))ds.
\end{aligned} \tag{4.2.2}$$

We have the next Theorem:

Theorem 4.2.5. *Assume that* $(g1) - (g3)$ *and* (4.2.2) *and* $E\int_0^t |f(s)|^4 ds < \infty$ *hold. Then, for* $\phi \in I^2(-\infty, 0; V) \cap L^2(\Omega; BCL_{-\infty}(H))$ *such that* $E[\sup_{-\infty<s\leqslant 0}|\phi(s)|^4] < \infty$*, there exists a unique solution* u *to*

$$\begin{aligned}
&u(t) = \phi(0) - \nu\int_0^t Au(s)ds - \int_0^t B(u(s))ds + \int_0^t (f(s) + g_1(s, u_s))\,ds + \int_0^t g_2(s, u_s)dW(s), \\
&u(t) = \phi(t), \quad t \in (-\infty, 0],
\end{aligned} \tag{4.2.3}$$

where the equation holds as an identity in V' *almost surely for every* $t \in [0, T]$.

Proof. By Lemma 4.2.3-4.2.4, we obtain that the subsequence $u_n(t)$ (relabeled the same) converges weakly to $u(t) \in L^2(\Omega; L^\infty(0, T; H)) \cap L^2(\Omega \times [0, T]; V) \cap L^4(\Omega \times [0, T]; L^4(\mathcal{O}))$. Moreover,

$$-\nu Au_n - B(u_n) \rightharpoonup \chi \ \text{ weakly in } L^2(\Omega \times [0, T]; V'),$$

$$g_1(t, u_{nt}) \rightharpoonup \zeta \ \text{ weakly in } L^2(\Omega \times [0, T]; H),$$
$$g_2(t, u_{nt}) \rightharpoonup \sigma \ \text{ weakly in } L^2(\Omega \times [0, T]; H).$$

We use the absolutely continuous function φ_k on $[0, T]$ with $\varphi_k' \in L^2(0, T)$ and $\varphi_k(T) = 0$ defined as follows

$$\varphi_k(s) = \begin{cases} 1 & 0 \leqslant s \leqslant t - \frac{1}{2k}, \\ \frac{1}{2} + k(t - s) & t - \frac{1}{2k} < s \leqslant t + \frac{1}{2k}, \\ 0 & t + \frac{1}{2k} < s \leqslant T. \end{cases}$$

Apply Itô formula to function $(u_n(s), \xi)\varphi_k(s)$, $\xi \in H_0^1(\mathcal{O})$, to get

$$0 = (u_{n0}, \xi)\varphi_k(0) - k\int_{t-\frac{1}{2k}}^{t+\frac{1}{2k}} (u_n(s), \xi)ds + \int_0^T (-\nu Au_n(s) - B(u_n(s)), \xi)\varphi_k(s)ds$$
$$+ \int_0^T (g_1(s, u_{ns}), \xi)\varphi_k(s)ds + \int_0^T (g_2(s, u_{ns}), \xi)\varphi_k(s)dW(s) + \int_0^T (f(s), \xi)\varphi_k(s)ds.$$

Let $k \to \infty$ in above inequality, we have

$$(u_n(t), \xi) = (u_{n0}, \xi) + \int_0^T (-\nu Au_n(s) - B(u_n(s)), \xi)ds$$
$$+ \int_0^T (g_1(s, u_{ns}), \xi)ds + \int_0^T (g_2(s, u_{ns}), \xi)dW(s) + \int_0^T (f(s), \xi)ds.$$

Take $n \to \infty$,

$$u(t) = u_0 + \int_0^t (\chi(s) + f(s) + \zeta)ds + \int_0^t \sigma dW(s).$$

Define $\rho(t) = \int_0^t |z(s)|_4^4 ds$, $z \in L^4(\Omega \times [0, T]; L^4(\mathcal{O}))$ and $z(s) = \phi(s), s \leqslant 0$. Using Itô formula to $\exp(-\lambda\rho(t))|u(t)|^2$ and $\exp(-\lambda\rho(t))|u_n(t)|^2$, respectively.

$$Ee^{-\lambda\rho(t)}|u(t)|^2 = E|u_0|^2 - E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u(s)|^2 ds + 2E\int_0^t e^{-\lambda\rho(s)}(\chi(s) + f(s) + \zeta, u(s))ds$$
$$+ E\int_0^t e^{-\lambda\rho(s)}|\sigma|^2 ds,$$

and

$$Ee^{-\lambda\rho(t)}|u_n(t)|^2 = E|u_{n0}|^2 - E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u_n(s)|^2 ds$$
$$+ 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Au_n(s) - B(u_n(s)) + f(s), u_n(s))ds$$
$$+ 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s, u_{ns}), u_n(s))ds$$
$$+ E\int_0^t e^{-\lambda\rho(s)}|g_2(s, u_{ns})|^2 ds.$$

Define α_n, β_n and γ_n as follows

$$\alpha_n = -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u_n(s) - z(s)|^2 ds + 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Au_n(s) - B(u_n(s)), u_n(s) - z(s))ds$$
$$- 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az - B(z), u_n(s) - z(s))ds + 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s, u_{ns}) - g_1(s, z_s), u_n(s) - z(s))ds$$
$$+ E\int_0^t e^{-\lambda\rho(s)}|g_2(s, u_{ns}) - g_2(s, z_s)|^2 ds.$$

$$\beta_n = -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u_n(s)|^2 ds + 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Au_n(s) - B(u_n(s)), u_n(s))ds$$
$$+ 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s,u_{ns}), u_n(s))ds + E\int_0^t e^{-\lambda\rho(s)}|g_2(s,u_{ns})|^2 ds.$$

$$\begin{aligned}\gamma_n = &-E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4(|z(s)|^2 - 2(u_n(s), z(s)))ds \\ &+ 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Au_n(s) - B(u_n(s)), -z(s))ds \\ &- 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az - B(z(s)) + g_1(s,z_s), u_n(s) - z(s))ds \\ &+ 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s,u_{ns}), -z(s))ds + E\int_0^t e^{-\lambda\rho(s)}(g_2(s,z_s) - 2g_2(s,u_{ns}), g_2(s,z_s))ds.\end{aligned}$$

Obviously,

$$\alpha_n = \beta_n + \gamma_n.$$

By Lemma 4.1.3 and (4.2.2), we have $\alpha_n \leqslant 0$.

$$\begin{aligned}0 &\geqslant \liminf_{n\to\infty} \alpha_n \\ &\geqslant -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u(s) - z(s)|^2 ds + 2E\int_0^t e^{-\lambda\rho(s)}(\chi, u(s) - z(s))ds \\ &\quad - 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az - B(z), u(s) - z(s))ds + 2E\int_0^t e^{-\lambda\rho(s)}(\zeta - g_1(s,z_s), u(s) - z(s))ds \\ &\quad + E\int_0^t e^{-\lambda\rho(s)}|\sigma - g_2(s,z_s)|^2 ds.\end{aligned}$$

Take $z(t) = u(t)$ in above inequality, it follows that $\sigma = g_2(t,u_t)$, $t \in [0,T]$, where we use the fact that $e^{-\lambda\rho(t)}$ is bounded for $t \in [0,1]$. On the other hand, notice that

$$\beta_n = Ee^{-\lambda\rho(t)}|u_n(t)|^2 - E|u_n(0)|^2 - 2E\int_0^t e^{-\lambda\rho(s)}(f(s), u_n(s))ds.$$

$$\begin{aligned}\liminf_{n\to\infty}\beta_n &\geqslant Ee^{-\lambda\rho(t)}|u(t)|^2 - E|u(0)|^2 - 2E\int_0^t e^{-\lambda\rho(s)}(f(s), u(s))ds \\ &= -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u(s)|^2 ds + 2E\int_0^t e^{-\lambda\rho(s)}(\chi + \zeta, u(s))ds \\ &\quad + E\int_0^t e^{-\lambda\rho(s)}|g_2(s,u_s)|^2 ds.\end{aligned}$$

$$\liminf_{n\to\infty} \gamma_n \geqslant -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4(|z(s)|^2 - 2(u(s),z(s)))ds + 2E\int_0^t e^{-\lambda\rho(s)}(\chi, -z(s))ds$$
$$- 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az - B(z), u(s)-z(s))ds - 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s,z_s), u(s)-z(s))ds$$
$$+ 2E\int_0^t e^{-\lambda\rho(s)}(\zeta, -z(s))ds + E\int_0^t e^{-\lambda\rho(s)}(g_2(s,z_s) - 2g_2(s,u_s), g_2(s,z_s))ds.$$

$$0 \geqslant \liminf_{n\to\infty} \alpha_n \geqslant \liminf_{n\to\infty} \beta_n + \liminf_{n\to\infty} \gamma_n$$
$$\geqslant -E\int_0^t \lambda e^{-\lambda\rho(s)}|z(s)|_4^4|u(s)-z(s)|^2 ds + 2E\int_0^t e^{-\lambda\rho(s)}(\chi+\zeta, u(s)-z(s))ds$$
$$- 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az - B(z), u(s)-z(s))ds - 2E\int_0^t e^{-\lambda\rho(s)}(g_1(s,z_s), u(s)-z(s))ds$$
$$+ E\int_0^t e^{-\lambda\rho(s)}|g_2(s,z_s) - g_2(s,u_s)|^2 ds.$$

Thus

$$0 \leqslant E\int_0^t e^{-\lambda\rho(s)}|g_2(s,z_s) - g_2(s,u_s)|^2 ds$$
$$\leqslant 2E\int_0^t e^{-\lambda\rho(s)}(-\nu Az - B(z) + g_1(s,z_s), u(s)-z(s))ds - 2E\int_0^t e^{-\lambda\rho(s)}(\chi+\zeta, u(s)-z(s))ds$$
$$+ \lambda E\int_0^t e^{-\lambda\rho(s)}|z(s)|_4^4|u(s)-z(s)|^2 ds.$$

For any fixed $v \in C_0^\infty(O)$. Set $z(t) = u(t) - \theta v$.

$$0 \leqslant 2E\int_0^t e^{-\lambda\rho(s)}(-\nu A(u-\theta v) - B(u-\theta v) + g_1(s, u_s - \theta v), v)ds$$
$$- 2E\int_0^t e^{-\lambda\rho(s)}(\chi+\zeta, v)ds + \theta\lambda E\int_0^t e^{-\lambda\rho(s)}|z(s)|_4^4|v|^2 ds.$$

Let $\theta \to 0$, we obtain

$$E\int_0^t (\chi + \zeta + \nu Au(s) + B(u(s)) - g_1(s,u_s), v)ds = 0.$$

Since v is any, and $\overline{C_0^\infty}(O) = H_0^1(O)$.

$$\int_0^t (-\nu Au(s) - B(u(s)) + g_1(s,u_s))ds = \int_0^t (\chi + \zeta)ds.$$

Hence,

$$u(t) = u_0 - \int_0^t (\nu Au(s) + B(u(s)))ds + \int_0^t f(s)ds + \int_0^t g_1(s,u_s)ds + \int_0^t g_2(s,u_s)dW(s), \quad a.e.\ \omega \in \Omega.$$

Therefore, there exists a unique weak solution to (4.0.1) on $[0,T]$. This completes the proof of the theorem. □

Corollary 4.2.6. *Assume that* $(g1)-(g3)$ *hold,* $E[\sup_{-\infty<s\leqslant 0}|f(s)|^4]<\infty$, *and*

$$\nu\lambda_1 > 2L_{g_1} + L_{g_2}^2.$$

Then for every $\phi \in I^2(-\infty,0;V)\cap L^2(\Omega;BCL_{-\infty}(H))$ *such that* $E[\sup_{-\infty<s\leqslant 0}|\phi(s)|^4]<\infty$, *there exists a unique solution* u *to* (4.2.3).

Proof. It is not difficult to verify that the assumption $\nu\lambda_1 > 2L_{g_1} + L_{g_2}^2$ is sufficient to obtain (4.2.2). Therefore, the proof is finished by Theorem 4.2.5. □

4.3 Asymptotic behavior of solutions

In this section we analyze the long time behavior of solutions in a neighborhood of a stationary solution to (4.0.1). First, we state a general result ensuring the existence and uniqueness of stationary solutions. Then, we show two methods to study the stability properties: the Lyapunov function as well as the construction of Lyapunov functionals. Both cases will be related to the unbounded variable delay case. We also would like to point out that, although we will provide some sufficient condition ensuring the asymptotic stability of stationary solutions, to prove that this stability is indeed exponential remains as an open problem in general in the unbounded variable delay case. Nevertheless we will be able to prove polynomial asymptotic stability in some particular cases which have some relevance in applications.

4.3.1 Existence and uniqueness of stationary solutions

For convenience, we consider our model in an abstract formulation as

$$\frac{du}{dt} + \nu Au + B(u) = f + g_1(t,u_t) + g_2(t,u_t)\frac{dW}{dt}, \tag{4.3.1}$$

with $f(t)\equiv f\in V'$. A stationary solution u^* to (6.2.1) must satisfy

$$(\nu Au^* + B(u^*) - f - g_1(t,u^*))t = \int_0^t g_2(s,u^*)\,dW(s), \quad \forall t>0, \tag{4.3.2}$$

and, according to [37, Remark 3.1], this means that u^* must be a stationary solution of the deterministic equation, in other words

$$\nu Au^* + B(u^*) = f + g_1(t,u^*), \tag{4.3.3}$$

which is an equality in V' and is a deterministic case of equation (4.3.2).

Thus, to discuss the stability of weak solutions to stochastic (4.3.1), we first need to consider the existence of stationary solutions to equation (4.3.3).

To carry out our analysis, for any $u\in H$ we denote by $\hat{u}$ the function defined in $(-\infty,0]$ by $\hat{u}(\theta)=u$, for all $\theta\leqslant 0$, and we assume that the forcing term g_1, g_2 satisfy that

$$g_i(t,\hat{u}) = G_i(u),\ i=1,2, \text{for all } u\in H,$$

where $G_i : \mathbb{R}^2 \to \mathbb{R}^2,\ i = 1, 2$ are functions satisfying

$$G_1(0) = 0, \tag{4.3.4}$$

and that there exists $M_i > 0,\ i = 1, 2$ for which

$$|G_i(u) - G_i(v)|_{\mathbb{R}^2} \leqslant M_i|u - v|_{\mathbb{R}^2},\ \forall u, v \in \mathbb{R}^2,\ i = 1, 2, \tag{4.3.5}$$

and by $G_i(u)$ we denote the element in H defined by $G_i(u)(x) = G_i(u(x))$ for all $x \in \mathcal{O}$. Then equation (4.3.2) and equation (4.3.3) can be rewritten respectively as

$$\nu Au^* + B(u^*) - f - G_1(u^*) = G_2(u^*)\frac{W(t)}{t},\ \ \forall t > 0, \tag{4.3.6}$$

and

$$\nu Au^* + B(u^*) = f + G_1(u^*). \tag{4.3.7}$$

Remark 4.3.1. *As it is pointed out in [37, Remark 3.1], any stationary solution, for instance u^*, to (4.3.6) is also a stationary solution to equation (4.3.7) , but it is possible that equation (4.3.7) possesses more than one stationary solution, for example u_1, and $u_1 \neq u^*$. However, if we assume that equation (4.3.7) has a unique stationary solution u_1, then $u_1 = u^*$, and in this case it must hold $G_2(u^*) = 0$ since (4.3.6) must hold for all $t > 0$.*

From now on, we always suppose that the stochastic equation (4.3.1) has a time-independent solution u_∞, which satisfies equation (4.3.6) . Actually, this can happen when we take g_2 in such a way that g_2 vanishes at a stationary point, for instance, $g_2(t, u) = G_2(u - u_\infty)$. See [37, Remark 4.3] for more details.

Theorem 4.3.2. *Suppose that G_1 satisfies conditions (4.3.4)-(4.3.5) and $\nu > \lambda_1^{-1}M_1$. Then,*

(a) for all $f \in V'$ there exists at least one stationary solution to (4.3.1);

(b) if $f \in (L^2(\mathcal{O}))^2$, the stationary solutions belong to $D(A)$;

(c) if $(\nu - \lambda_1^{-1}M_1)^2 > (2\lambda_1)^{-\frac{1}{2}}\|f\|_*$, then the stationary solution to (4.3.1) is unique.

Proof. By a similar method as that of [37, Theorem 4.1, p. 1087], the theorem can be proved. We omit it here. □

4.3.2 Local stability: A direct approach

In this subsection, we prove the local stability of stationary solution by a straightforward way. Suppose that $\rho \in C^1([0, +\infty))$, $\rho(t) \geqslant 0$ for all $t \geqslant 0$ and $\rho_* = \sup_{t\geqslant 0} \rho'(t) < 1$.

Theorem 4.3.3. *Let $g_i(t, u_t) = G_i(u(t-\rho(t))), i = 1, 2,$ satisfying (4.3.4)-(4.3.5). Assume that there exists constant $c_1 > 0$, depending only on $\mathcal{O}$, such that if $f \in (L^2(\mathcal{O}))^2$ and $\nu > \lambda_1^{-1}M_1 + (2\lambda_1)^{-\frac{1}{4}}\|f\|_*^{\frac{1}{2}}$ satisfies in addition*

$$2\nu > \frac{(2-\rho_*)M_1 + M_2^2}{\lambda_1(1-\rho_*)} + \frac{c_1|f|}{\lambda_1(\nu - \lambda_1^{-1}M_1)}. \tag{4.3.8}$$

Then there exists a unique stationary solution $u_\infty \in D(A)$ of (4.3.7), and for all $\phi \in BCL_{-\infty}(H) \cap L^2((-\infty, 0); H)$, the corresponding solution u of (4.0.1) with $f(t) \equiv f$ satisfies

$$E|u(t) - u_\infty|^2 \leqslant E|u_0 - u_\infty|^2 + \frac{M_1 + M_2^2}{(1-\rho_*)}\int_{-\infty}^0 E[|\phi(s) - u_\infty|^2]ds. \tag{4.3.9}$$

Proof. Consider u the solution of (4.0.1) for $f(t) \equiv f$, and let $u_\infty \in D(A)$ be a stationary solution to (4.3.1). Apply Itô formula to function $|u(t) - u_\infty|^2$,

$$\begin{aligned}
|u(t) - u_\infty|^2 &= |u_0 - u_\infty|^2 + 2\int_0^t (-\nu A(u-u_\infty) - B(u) + B(u_\infty), u - u_\infty)ds \\
&\quad + 2\int_0^t (G_1(u(s-\rho(s))) - G_1(u_\infty), u - u_\infty)ds \\
&\quad + 2\int_0^t (G_2(u(s-\rho(s))) - G_2(u_\infty), u - u_\infty)dW(s) + \int_0^t |G_2(u(s-\rho(s)))|^2_{L_2^0}ds
\end{aligned} \tag{4.3.10}$$

Take expectation,

$$\begin{aligned}
E|u(t) - u_\infty|^2 &= E|u_0 - u_\infty|^2 - 2\nu\int_0^t E[\|u - u_\infty\|^2]ds - 2\int_0^t E(B(u) - B(u_\infty), u - u_\infty)ds \\
&\quad + 2\int_0^t E(G_1(u(s-\rho(s))) - G_1(u_\infty), u - u_\infty)ds + \int_0^t E|G_2(u(s-\rho(s)))|^2_{L_2^0}ds.
\end{aligned} \tag{4.3.11}$$

$$2(B(u) - B(u_\infty), u - u_\infty) = 2b(u - u_\infty, u_\infty, u - u_\infty) \leqslant \frac{c_1}{\sqrt{\lambda_1}}\|u - u_\infty\|^2\|u_\infty\|.$$

Since,

$$\nu A u_\infty + B(u_\infty) = f + G_1(u_\infty),$$

$$\|u_\infty\| \leqslant \frac{|f|}{\sqrt{\lambda_1}(\nu - \lambda_1^{-1}M_1)}.$$

On the other hand,

$$\begin{aligned}
&2\int_0^t E(G_1(u(s-\rho(s))) - G_1(u_\infty), u - u_\infty)ds \\
&\leqslant \frac{(2-\rho_*)M_1}{\lambda_1(1-\rho_*)}\int_0^t E[\|u - u_\infty\|^2]ds + \frac{M_1}{(1-\rho_*)}\int_{-\infty}^0 E[\sup_{-\infty<s\leqslant 0}|\phi(s) - u_\infty|^2]ds.
\end{aligned}$$

By (4.3.2), we have

$$\int_0^t E|G_2(u(s-\rho(s)))|^2_{L_2^0} ds \leqslant \frac{M_2^2}{\lambda_1(1-\rho_*)} \int_0^t E[\|u-u_\infty\|^2]ds + \frac{M_2^2}{(1-\rho_*)} \int_0^t E[\sup_{-\infty<s\leqslant 0} |\phi(s)-u_\infty|^2]ds.$$

Hence,

$$E|u(t)-u_\infty|^2 \leqslant E|u_0-u_\infty|^2 + \left(-2\nu + \frac{c_1|f|}{\sqrt{\lambda_1}(\nu-\lambda_1^{-1}M_1)} + \frac{(2-\rho_*)M_1}{\lambda_1(1-\rho_*)} + \frac{M_2^2}{\lambda_1(1-\rho_*)}\right)\cdot$$
$$\int_0^t E[\|u-u_\infty\|^2]ds + \frac{M_1+M_2^2}{(1-\rho_*)} \int_{-\infty}^0 E[\sup_{-\infty<s\leqslant 0} |\phi(s)-u_\infty|^2]ds.$$

Therefore, by (4.3.8) we have

$$E|u(t)-u_\infty|^2 \leqslant E|u_0-u_\infty|^2 + \frac{M_1+M_2^2}{(1-\rho_*)} E[|\phi(s)-u_\infty|^2_{L^2(-\infty,0;H)}].$$

The proof is completed. □

Remark 4.3.4. *In order to obtain that the weak solution to Eq.(4.3.1) converges exponentially to u_∞ and thus u_∞ is exponentially stable in the mean square by this technique, we need that $\rho(t)$ be bounded. See [120] for details.*

4.3.3 Stability via the construction of Lyapunov functionals

In this subsection, we first show the asymptotic stability of the trivial solution by constructing suitable Lyapunov functionals of the following class of nonlinear stochastic partial differential equations, and later we will apply these abstract results to our Navier-Stokes model. See [149] for more details.

Let us consider the following problem

$$\begin{aligned} du(t) &= (A(t,u(t)) + f(t,u_t))dt + g(t,u_t)dW(t), \quad t\in[0,T], \\ u(t) &= \phi(t), \qquad t\in(-\infty,0], \end{aligned} \tag{4.3.12}$$

where $A(t,\cdot): V \to V'$ with $\langle A(t,u),u\rangle \leqslant 0$, for all $v\in V$, $f(t,\cdot): BCL_{-\infty}(H) \to H$ and $g(t,\cdot): BCL_{-\infty}(H) \to \mathcal{L}(K,H)$ satisfy the following Lipschitz conditions: there exist L_f, L_g such that for all $t\geqslant 0$ and all $\xi,\eta\in BCL_{-\infty}(H)$,

$$\begin{aligned} |f(t,\xi)-f(t,\eta)| &\leqslant L_f\|\xi-\eta\|_{BCL_{-\infty}(H)}, \\ |g(t,\xi)-g(t,\eta)| &\leqslant L_g\|\xi-\eta\|_{BCL_{-\infty}(H)}. \end{aligned} \tag{4.3.13}$$

The existence and uniqueness of solution to (4.3.12) can be proved by a similar process as we did in Section 3. For a fixed $T>0$, given an initial value $\phi\in I^2(-\infty,0;V)\cap L^2(\Omega;BCL_{-\infty}(H))$, a solution to (4.3.12) is a process $u(\cdot)\in I^2(-\infty,T;V)\cap L^2(\Omega;C(-\infty,T;H))$ such that

$$\begin{aligned} u(t) &= \phi(0) + \int_0^t A(s,u(s))ds + \int_0^t f(s,u_s)ds + \int_0^t g_(s,u_s)dW(s), \quad t\in[0,T], \ P-a.s. \\ u(t) &= \phi(t), \quad t\in(-\infty,0], \end{aligned} \tag{4.3.14}$$

where the first equality is defined in V'.

From now on, we are interested in the longtime behavior of the solutions to (4.3.12). To this end, we need the Itô formula for the solutions of (4.3.14). We define an associate operator L which is usually called the "generator" of equation (4.3.14). To deal with the stochastic differential of the process $\sigma(t) = v(t, u(t))$, where $u(t)$ is a solution of equation (4.3.12), and the function $v(t, u) : [0, \infty) \times V \to \mathbb{R}_+$ has continuous partial derivatives

$$v'_t(t, u) = \frac{\partial v(t, u)}{\partial t}, \quad v'_u(t, u) = \frac{\partial v(t, u)}{\partial u}, \quad v''_{uu}(t, u) = \frac{\partial^2 v(t, u)}{\partial u^2},$$

the Itô formula for $\sigma(t)$ reads

$$d\sigma(t) = Lv(t, u(t))dt + \langle v'_u(t, u(t)), g(t, u_t)dW(t)\rangle,$$

where the generator L is defined in the following way

$$Lv(t, u(t)) = v'_t(t, u(t)) + \langle v'_u(t, u(t)), A(t, u_t) + f(t, u_t)\rangle + \frac{1}{2}Tr[v''_{uu}(t, u(t))g(t, u_t)Qg^*(t, u_t)].$$

The generator L can be applied also for some functionals $V(t, \varphi) : [0, \infty) \times H \to \mathbb{R}_+$. Notice that, although we are using the same letter V to denote the functionals and the Hilbert space, no confusion is possible. Suppose that a functional $V(t, \varphi)$ can be represented in the form $V(t, \varphi) = V(t, \varphi(0), \varphi(\theta))$, $\theta < 0$ and for $\varphi = u_t$, which is defined as $\varphi(\theta) = u(t + \theta)$ put

$$\begin{aligned} V_\varphi(t, u) &= V(t, \varphi) = V(t, u, \varphi(\theta)), \quad \theta \leqslant 0, \\ u &= \varphi(0) = u(t). \end{aligned} \tag{4.3.15}$$

Denote by D be the set of functionals for which the function $V_\varphi(t, u)$, defined by (4.3.15), has a continuous derivative with respect to t and two continuous derivatives with respect to u. For functionals from D, the generator L of the equation (4.3.12) has the from

$$LV(t, u_t) = V'_{\varphi t}(t, u(t)) + \langle V'_{\varphi u}(t, u(t)), A(t, u(t)) + f(t, u_t)\rangle + \frac{1}{2}Tr[v''_{uu}(t, u(t))g(t, u_t)Qg^*(t, u_t)].$$

For functionals from D, the Itô formula implies

$$E[V(t, u_t) - V(s, u_s)] = \int_s^t ELV(r, u_r)dr, \quad t \geqslant s. \tag{4.3.16}$$

Next proposition is a generalization of Theorem 2.1 in [149, p. 34] to an infinite dimensional framework. More precisely, Theorem 2.1 in [149] was stated and proved for stochastic ordinary differential equations with finite delays while we will prove it for stochastic partial differential equations with unbounded delays.

Proposition 4.3.5. *Let $V(t, u_t)$ be a continuous functional such that for any solution $u(t)$ of problem (4.3.12) with $p \geqslant 2$, the following inequalities hold:*

$$\begin{aligned} &EV(t, u_t) \geqslant \gamma_1 E|u(t)|^p, \quad \forall t \geqslant 0, \\ &EV(0, \phi) \leqslant \gamma_2 \|\phi\|_1^p, \\ &E[V(t, u_t) - V(0, \phi)] \leqslant -\gamma_3 \int_0^t E|u(s)|^p ds, \quad t \geqslant 0, \end{aligned}$$

where $\|\phi\|_1^p := \sup_{\theta \leqslant 0} E|\varphi(\theta)|^p$. Then the trivial solution of (4.3.12) is asymptotically p-stable (i.e. asymptotically stable in the pth-moment).

Proof. For simplicity, we only prove the case when $p = 2$, although the proof for $p \neq 2$ can be obtained in a similar way. By the assumption, we know that for any $\phi \in BCL_{-\infty}(H)$,

$$\gamma_1 E|u(t)|^2 \leqslant EV(t, u_t) \leqslant EV(0, \phi) \leqslant \gamma_2 \|\phi\|_1^2 = \gamma_2 \sup_{\theta \leqslant 0} E|\varphi(\theta)|^2, \tag{4.3.17}$$

which implies that the trivial solution is stable.

Notice that, by (4.3.17), we have

$$\sup_{t \geqslant 0} E|u(t)|^2 \leqslant \frac{\gamma_2}{\gamma_1} \|\phi\|_1^2. \tag{4.3.18}$$

On the other hand, it follows from the condition of this proposition, we find

$$\int_0^\infty E|u(s)|^2 ds \leqslant \frac{1}{\gamma_3} EV(0, \phi) \leqslant \frac{\gamma_2}{\gamma 3} \|\phi\|_1^2 < \infty, \tag{4.3.19}$$

Applying the generator L to function $|u(t)|^2$ and using (4.3.13), we obtain

$$\begin{aligned} EL|u(t)|^2 &= 2E(u(t), A(t, u(t)) + f(t, u_t)) + E|g_(t, u_t)|^2 \\ &\leqslant 2E(u(t), f(t, u_t)) + E|g_2(t, u_t)|^2 \\ &\leqslant E|u(t)|^2 + (L_f^2 + L_g^2) E|u_t|^2_{BCL_\infty(H)} \\ &= E|u(t)|^2 + (L_f^2 + L_g^2) E \sup_{\theta \leqslant 0} |u(t + \theta)|^2 \\ &\leqslant E|u(t)|^2 + (L_f^2 + L_g^2) E \sup_{\theta \leqslant -t} |u(t + \theta)|^2 + (L_f^2 + L_g^2) E \sup_{\theta > -t} |u(t + \theta)|^2 \\ &\leqslant E|u(t)|^2 + (L_f^2 + L_g^2) \|\phi\|_1^2 + (L_f^2 + L_g^2) \sup_{t \geqslant 0} E|u(t)|^2 \\ &\leqslant \gamma_4, \end{aligned} \tag{4.3.20}$$

where γ_4 is a positive constant. Since from (4.3.16),

$$E[V(t, u_t) - V(s, u_s)] = \int_s^t ELV(r, u_r) dr.$$

By (4.3.20), we have for any $t_2 \geqslant t_1 \geqslant 0$,

$$|E|u(t_2)|^2 - E|u(t_1)|^2| \leqslant \gamma_4(t_2 - t_1),$$

that is to say the function $E|u(t)|^2$ is Lipschitz, from which and (4.3.18)-(4.3.19), we obtain that $\lim_{t\to+\infty} E|u(t)|^2 = 0$.Therefore, the proof is finished. □

Theorem 4.3.6. *Assume that the forcing terms $g_i(t, u_t)$ are given by $g_i(t, u_t) = G_i(u(t - \rho(t))), i = 1, 2,$ and satisfy (4.3.4)-(4.3.5). Let $f = 0$ and*

$$2\nu > \frac{(2 - \rho_*)M_1 + M_2^2}{\lambda_1(1 - \rho_*)}.$$

Then, there exists a unique stationary solution $u_\infty = 0$ to (4.3.7), and any weak solution $u(t)$ to (4.3.1) converges to zero in mean square. In other words, zero is asymptotically mean-square stable.

Proof. We prove this theorem by constructing a Lyapunov functional following the general method of construction described in [40]. Setting

$$V_1(t, u_t) = |u(t)|^2,$$

then

$$\begin{aligned} LV_1(t, u_t) &= 2(-\nu Au(t) - B(u(t)) + G_1(u(t - \rho(t))), u(t)) + |G_2(u(t - \rho(t)))|^2 \\ &\leqslant -2\nu\|u(t)\|^2 + 2(G_1(u(t - \rho(t))), u(t)) + |G_2(u(t - \rho(t)))|^2 \\ &\leqslant (-2\nu\lambda_1^{-1} + M_1)|u(t)|^2 + (M_1 + M_2^2)|u(t - \rho(t))|^2. \end{aligned}$$

Let

$$V_2(t, u_t) = \frac{M_1 + M_2^2}{1 - \rho_*}\int_{t-\rho(t)}^{t} |u(s)|^2 ds,$$

so we have

$$LV_2(t, u_t) \leqslant \frac{M_1 + M_2^2}{1 - \rho_*}|u(t)|^2 - (M_1 + M_2^2)|u(t - \rho(t))|^2.$$

Thanks to the above inequalities and the fact that $2\nu > \lambda_1^{-1}M_1 + \frac{M_1+M_2^2}{\lambda_1(1-\rho_*)}$, we obtain that there exists a positive constant γ, such that the Lyapunov functional $V(t, u_t)$ defined by $V_1(t, u_t) + V_2(t, u_t)$ fulfills

$$LV(t, u_t) = L(V_1(t, u_t) + V_2(t, u_t)) \leqslant (-2\nu\lambda_1^{-1} + M_1 + \frac{M_1 + M_2^2}{1 - \rho_*})|u(t)|^2 \leqslant -\gamma|u(t)|^2 \leqslant 0.$$

Therefore, the functional $V(t, u_t) = V_1(t, u_t) + V_2(t, u_t) = |u(t)|^2 + \frac{M_1+M_2^2}{1-\rho_*}\int_{t-\rho(t)}^{t} |u(s)|^2 ds$ satisfies the conditions in Proposition 4.3.5, thus the trivial solution of (4.3.7) is asymptotically mean-square stable, which also means that the stationary solution to (4.3.7) is unique. □

4.4 Polynomial stability for special case

In this subsection, we will prove the polynomial stability and convergence of solution when the unbounded variable delay term is specially taken as $\rho(t) = (1-\lambda)t$ with $0 < \lambda < 1$.

To do this, we need to introduce the following stochastic pantograph equation and some technical lemmas that are needed later. But these lemmas have been presented in Chapter 3, we omit them here.

Theorem 4.4.1. *Consider* (4.0.1) *with* $f = 0$, $g_1(t, u_t) = L_{g_1} u(qt)$, $g_2(t, u_t) = L_{g_2} u(qt)$ *with* $0 < q < 1$ *and* $2\lambda_1 \nu > 2|L_{g_1}| + L_{g_2}^2$, *then there exists unique trivial solution* $u = 0$ *of* (4.0.1)*, and all the solutions of* (4.0.1) *converges to zero polynomially, namely, there exist* $C > 0$ *and* $\mu < 0$ *such that*

$$E|u(t)|^2 < CE|u(0)|^2(1+t)^{\mu}, \quad \text{for all } t \geqslant 0 \tag{4.4.1}$$

where μ *satisfies* $|L_{g_1}| - 2\nu\lambda_1 + (|L_{g_1}| + L_{g_2}^2)q^{\mu} = 0$.

Proof. Let $f \equiv 0$. Applying Itô formula to $|u(t)|^2$, using a similar process as it did in Theorem 4.1 in [1], we obtain

$$E[|u(t+h)|^2] - E[|u(t)|^2] \leqslant (-2\nu\lambda_1 + |L_{g_1}|)E\int_t^{t+h} |u(s)|^2 ds + (|L_{g_1}| + L_{g_2}^2)E\int_t^{t+h} |u(qs)|^2 ds.$$

Denote by $v(t) = E|u(t)|^2$,

$$v'(t) \leqslant (-2\lambda_1\nu + |L_{g_1}|)v(t) + (|L_{g_1}| + L_{g_2}^2)w(qt). \tag{4.4.2}$$

By Lemma 3.4.1-3.4.2, there exist $C = C(L_{g_1}, L_{g_2}, \lambda_1, \nu) > 0$ and $\mu \in \mathbb{R}$ such that

$$v(t) \leqslant Cv(0)(1+t)^{\mu},$$

Since $-2\lambda_1\nu + 2|L_{g_1}| + L_{g_2}^2 < 0$, it holds that $\mu < 0$, and

$$E|u(t)|^2 \leqslant CE|u(0)|^2(1+t)^{\mu}.$$

Then the polynomial decay of solutions follows directly. □

Remark 4.4.2. *(i) In this special case,* $g_i(t, u_t) = L_{g_i} u(qt)$, $i = 1, 2$ *with* $0 < q < 1$ *and* $f \equiv 0$. *As long as we have* $2\lambda_1\nu > 2|L_{g_1}| + L_{g_2}^2$, *then we can prove that the solution converges polynomially to zero. In this sense, this result improves the stability results we established previously.*

(ii) In fact, our result can be extended to more general case, namely, if the delay term $g(t, \phi)$ *is defined as* $g(t, \phi) = G(\phi(-(1-\lambda)t))$, *with* G *satisfying a Lipschitz condition with Lipschitz constant* L_g.

As one of the most important Newtonian fluids, both the deterministic and the stochastic Navier-Stokes equations with unbounded delay have been discussed in this Part II. We obtained the polynomial stability of stationary solution in some special case, but the exponential stability, in general, is still an open problem as well as the existence of attractor in this case. However, at the same time, another type of fluids are also worth analyzing, namely, the so-called non-Newtonian fluids. Actually, many fluid materials, such as molten plastics, synthetic fibers, paints and greases, polymer solutions, suspensions, adhesives, dyes, varnishes, and biological fluids like blood etc., their flow behavior cannot be characterized by Newtonian relationships in the real world, these fluids belong to the class of non-Newtonian ones. Therefore Part III will be devoted to studying the asymptotic behavior of some non-Newtonian fluids with finite delay.

Part III

Non-Newtonian models with delay

As it is well known, the Navier-Stokes model of fluid restricts the linear relation between the stress tensor and the velocity gradient (see [143, 160]). Fluids satisfying such constitutive relationship are called Newtonian fluids, e.g. air, gases, water, motor oil, alcohols, and simply hydrocarbon compounds. However, for many fluid materials, such as molten plastics, synthetic fibers, paints and greases, polymer solutions, suspensions, adhesives, dyes, varnishes, and biological fluids like blood etc., their flow behavior cannot be characterized by Newtonian relationships in the real world. By weaken the constraints of the Stokes hypothesis, the mathematical theory of viscous non-Newtonian fluids generalizes the usual Stokes model in three important aspects: nonlinear constitutive relations between the viscous part of the stress tensor and velocity gradients, dependence of the viscous stress tensor on velocity gradients of order two or higher, and constitutive relations for higher order stress tensors which must be present in the balance of energy equations as soon as higher order velocity gradients are considered into the theory [15, 84].

Actually, non-Newtonian fluids models have many application in medicine, chemical engineering and environmental protection. In medicine, for example, human blood belongs to non-Newtonian fluid, understanding non-Newtonian viscous features and hemodynamics of blood is not only beneficial for blood observation and control, but also help for cardiovascular disease diagnose and treatment, and because artherosclerosis arising in the arterial wall shear stress is closely related, considering that the blood is non-Newtonian fluid, therefore studying blood flow in the brain direction helps to determine the location of cerebral aneurysms. In chemical industry, making full use of the viscous characteristics of non-Newtonian fluid can be applied to wastewater treatment, which is very conducive to environmental protection.

Part III focuses on the following incompressible non-Newtonian fluid with finite delay on a bounded domain:

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p = \nabla\cdot\mu(e(u)) + f(t,u_t) + g(x,t), \quad \text{in } (\tau,+\infty)\times\mathcal{O}, \tag{4.4.3}$$

$$\nabla\cdot u = 0, \quad \text{in } (\tau,+\infty)\times\mathcal{O}, \tag{4.4.4}$$

$$u(\tau+\theta,x) = \phi(\theta,x), \quad \theta\in[-h,0],\ x\in\mathcal{O}. \tag{4.4.5}$$

which is supplemented by the boundary conditions ($\nu_{ij}e = 2\mu_1\frac{\partial e_{ij}}{\partial x_l}$, $i,j,l = 1,2$, and $n = (n_1,n_2)$ the exterior unit normal to $\partial\mathcal{O}$)

$$u = 0, \quad \nu_{ij}n_jn_l = 0, \ i,j,k = 1,2, \text{ on } \partial\mathcal{O}\times(\tau,+\infty), \tag{4.4.6}$$

where $\mathcal{O}$ is a smooth bounded domain of $\mathbb{R}^2$, the unknown vector function $u = u(x,t) = (u^{(1)},u^{(2)})$ denotes the velocity of the fluid, $g(x,t) = g(t) = (g^{(1)},g^{(2)})$ is a time-dependent external function, and the scalar function p represents the pressure. The first condition in (4.4.6) represents the usual non-slip condition associated with a viscous fluid, while the second one expresses the fact that the first moments of the traction vanish on $\partial\mathcal{O}$, it is a direct consequence of the principle of virtual work. The time-dependent delay term $f(t,u_t)$ represents, for instance, the influences of an external force with some kind of delay, memory or hereditary characteristics, although we can also model some kind of feedback controls. Here, u_t denotes a segment of the solution, in other words, given $h>0$ and a function $u:[s-h,+\infty)\times\mathcal{O}\to\mathbb{R}^2$, for each $t\geqslant s$ we define the mapping $u_t:[-h,0]\times\mathcal{O}\to\mathbb{R}^2$ by

$$u_t(\theta,x) = u(t+\theta,x), \quad \text{for } \theta\in[-h,0],\ x\in\mathcal{O}.$$

In this way, this abstract formulation includes several types of delay terms in a unified way. For example, terms like

$$F_1(t, u(t-h)), \quad F_2(u(t-\rho(t))), \quad \int_{-h}^{0} F_3(t, \theta, u(t+\theta))d\theta, \tag{4.4.7}$$

where F_i $(i = 1, 2, 3)$ are suitable functions, and $\rho : \mathbb{R} \mapsto [0, h]$, can all be described by the following corresponding f_i defined as

$$f_1(t, \phi) = F_1(t, \phi(-h)), \;\; f_2(t, \phi) = F_2(\phi(-\rho(t))), \;\; f_3(t, \phi) = \int_{-h}^{0} F_3(t, \theta, \phi(\theta))d\theta, \tag{4.4.8}$$

where $\phi : [-h, 0] \to X$ (X denotes certain Banach or Hilbert space concerning the spatial variable). Then, when we replace ϕ by u_t in (4.4.8), we obtain (4.4.7). Readers are referred to [30, 31, 33] for more details.

The structure of Part III is as follows. Since Chapter 5 and Chapter 6 work on exactly the same systems, namely, problem (4.4.3)-(4.4.5), in order to avoid unnecessary repetitions, in Preliminaries below we recall some definitions and abstract spaces, as well as the reformulation of problem (4.4.3)-(4.4.5). Besides, we will state assumptions on the delay term $f(t, u_t)$ in this section. Then, in Chapter 5, we investigate the dynamics of a non-autonomous incompressible non-Newtonian fluids with delay, and prove the existence of pullback attractor. Finally, in Chapter 6, we analyze the exponential stability of problem (4.4.3)-(4.4.5).

Preliminaries

We first recall some notations which are necessary for our analysis although they are similar to those in [7, 15, 103], but we prefer to introduce them here for completeness.

$L^p(\mathcal{O})$ will denote the 2D vector Lebesgue space with norm $\|\cdot\|_{L^p(\mathcal{O})}$; particularly, $\|\cdot\|_{L^2(\mathcal{O})} = \|\cdot\|$,

$H^m(\mathcal{O})$ is the 2D vector Sobolev space $\{\phi : \phi = (\phi_1, \phi_2) \in L^2(\mathcal{O}), \nabla^k\phi \in L^2(\mathcal{O}), k \leqslant m\}$ with norm $\|\cdot\|_{H^m(\mathcal{O})}$,

$H_0^1(\mathcal{O})$ is the closure of $\{\phi : \phi = (\phi_1, \phi_2) \in C^\infty(\mathcal{O}) \times C^\infty(\mathcal{O})\}$ in $H^1(\mathcal{O})$,

$\mathcal{V}$ denotes the $\{\phi \in C^\infty(\mathcal{O}) \times C^\infty(\mathcal{O}) : \phi = (\phi_1, \phi_2),\ \nabla \cdot \phi = 0\}$,

H is the closure of $\mathcal{V}$ in $L^2(\mathcal{O})$ with norm $\|\cdot\|$; H' is the dual space of H,

W denotes the closure of $\mathcal{V}$ in $H^2(\mathcal{O})$ with norm $\|\cdot\|_W$; W'=dual space of W,

$(\cdot,\cdot)$−the inner product in H, $\langle\cdot,\cdot\rangle$-the dual pairing between W and W'.

$\mathrm{dist}_M(X, Y)$−the Hausdorff semi-distance between $X, Y \subset M$, where M is a normed space, defined by

$$\mathrm{dist}_M(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_M.$$

Set

$$a(u, v) = \sum_{i,j,k=1}^{2} \left(\frac{\partial e_{ij}(u)}{\partial x_k}, \frac{\partial e_{ij}(v)}{\partial x_k} \right) = \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \frac{\partial e_{ij}(u)}{\partial x_k} \cdot \frac{\partial e_{ij}(v)}{\partial x_k} dx, \quad u, v \in W. \tag{4.4.9}$$

On the one hand, from the definition of $a(\cdot,\cdot)$ and Lemma A.1.3 in Section 6, we see that $a(\cdot,\cdot)$ defines a positive definite symmetric bilinear form on W. As a consequence of the Lax-Milgram Lemma, we obtain an isometric operator $A \in \mathcal{L}(W, W')$, via

$$\langle Au, v\rangle = a(u, v), \quad u, v \in W.$$

On the other hand, denoting $D(A) = \{u \in W : Au \in H\}$, it turns out that $D(A)$ is a Hilbert space and A is also an isometry from $D(A)$ to H. Actually, $A = P\Delta^2$, where P is the Leray projector from $L^2(\mathcal{O})$ to H and, for any $u \in D(A)$, we have (see [180])

$$c_1\|u\|_W \leqslant \|Au\|. \tag{4.4.10}$$

We also define a continuous trilinear form on $H_0^1(\mathcal{O}) \times H_0^1(\mathcal{O}) \times H_0^1(\mathcal{O})$ by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad u, v, w \in H_0^1(\mathcal{O}).$$

Since $W \subset H_0^1(\mathcal{O})$, $b(\cdot,\cdot,\cdot)$ is continuous on $W \times W \times W$ and it is easy to check that (see [160])

$$b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0, \quad \forall u, v, w \in W. \tag{4.4.11}$$

Now we can define below continuous functional $B(u) := B(u, u)$ from $W \times W$ to W', for any $u \in W$, in the following way,

$$\langle B(u), w\rangle = b(u, u, w), \quad \forall w \in W. \tag{4.4.12}$$

To finish, we set

$$\mu(u) = 2\mu_0(\epsilon + |e(u)|^2)^{-\alpha/2},$$

for $u \in W$, and define $N(u)$ as

$$\langle N(u), v\rangle = \sum_{i,j=1}^{2} \int_{\mathcal{O}} \mu(u) e_{ij}(u) e_{ij}(v) dx, \quad \forall v \in W. \tag{4.4.13}$$

Then the functional $N(u)$ is continuous from W to W'. When $u \in D(A)$, we can extend $N(u)$ to H by setting

$$\langle N(u), v\rangle = -\int_{\mathcal{O}} \{\nabla \cdot [\mu(u)e(u)] \cdot v\}\, dx, \quad \forall v \in H. \tag{4.4.14}$$

From a physical point of view, the initial boundary problem of Eq. (4.4.3) can be formulated as

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot \left(2\mu_0(\epsilon + |e|^2)^{-\frac{\alpha}{2}} - 2\mu_1 \Delta e\right) + f(t, u_t) + g(x, t), \text{ in } (\tau, +\infty) \times \mathcal{O}, \tag{4.4.15}$$

$$\nabla \cdot u = 0, \quad \text{in } (\tau, +\infty) \times \mathcal{O}, \tag{4.4.16}$$

$$u = 0, \quad v_{ij} n_j n_l = 0, \quad \text{on } \partial\mathcal{O} \times (\tau, \infty), \tag{4.4.17}$$

$$u(\tau+\theta,x)=\phi(\theta,x),\quad \theta\in[-h,0],\ x\in\mathcal{O}. \tag{4.4.18}$$

As usual, in the variational set-up, we get rid of the pressure and rewrite our problem (4.4.15)-(4.4.18) in a weak formulation as follows (see [15, 177])

$$\frac{\partial u}{\partial t}+2\mu_1 Au+B(u)+N(u)=f(t,u_t)+g(x,t),\ \text{in } (\tau,+\infty)\times\mathcal{O}, \tag{4.4.19}$$

$$u(\tau+\theta,x)=\phi(\theta,x),\quad \theta\in[-h,0],\ x\in\mathcal{O}. \tag{4.4.20}$$

We now state the assumptions that will be imposed on the function $f:[\tau,T]\times C_H\mapsto(L^2(\mathcal{O}))^2$ containing the delay along our analysis. We will assume that the given delay term satisfies:

(H1) For any $\xi\in C_H$, the mapping $[\tau,T]\ni t\mapsto f(t,\xi)\in(L^2(\mathcal{O}))^2$ is measurable,

(H2) $f(\cdot,0)=0$,

(H3) $\exists\, L_f>0$ such that for any $t\in[\tau,T]$ and all $\xi,\eta\in C_H$,

$$\|f(t,\xi)-f(t,\eta)\|_{L^2(\mathcal{O})}\leqslant L_f\|\xi-\eta\|_{C_H},$$

Remark 4.4.3. *As it is pointed out in [30, 70, 130], (H2) is not really a restriction, and condition (H2) and (H3) imply that*

$$\|f(t,\xi)\|_{L^2(\mathcal{O})}\leqslant L_f\|\xi\|_{C_H},$$

so that $\|f(\cdot,\xi)\|_{L^2(\mathcal{O})}\in L^\infty(\tau,T)$.

Examples of delay terms satisfying (H1) – (H3) can be seen in Chapter 3 but in a finite interval for the delay. Now we are in the position to study the global dynamics of a non-autonomous incompressible non-Newtonian fluids with delay.

Chapter 5

Dynamics of a non-autonomous incompressible non-Newtonian fluids with delay

The objective of this chapter is to study the well-posedness and dynamical behavior of the following non-autonomous incompressible non-Newtonian fluids with delay in a 2D bounded domain, and for the completeness of the chapter, we rewrite the non-Newtonian fluid system as:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot \mu(e(u)) + f(t, u_t) + g(x, t), \quad \text{in } (\tau, +\infty) \times \mathcal{O}, \tag{5.0.1}$$

$$\nabla \cdot u = 0, \quad \text{in } (\tau, +\infty) \times \mathcal{O}, \tag{5.0.2}$$

$$u(\tau + \theta, x) = \phi(\theta, x), \quad \theta \in [-h, 0], \ x \in \mathcal{O}. \tag{5.0.3}$$

System (5.0.1)-(5.0.3) is supplemented with the boundary conditions ($\nu_{ij}e = 2\mu_1 \frac{\partial e_{ij}}{\partial x_l}$, $i, j, l = 1, 2$, and $n = (n_1, n_2)$ the exterior unit normal to $\partial\mathcal{O}$)

$$u = 0, \quad \nu_{ij} n_j n_l = 0, \ i, j, k = 1, 2, \ \text{on } \partial\mathcal{O} \times (\tau, +\infty), \tag{5.0.4}$$

where $\mathcal{O}$ is a smooth bounded domain of $\mathbb{R}^2$.

Problem (5.0.1)-(5.0.4) models the motion of an isothermal incompressible viscous fluid with $\mu(e(u)) = (\mu_{ij}(e(u)))_{2\times 2}$, which is usually called the extra stress tensor of the fluid and is a matrix of order 2×2 in which

$$\begin{aligned} \mu_{ij}(e(u)) &= 2\mu_0(\epsilon_0 + |e|^2)^{-\frac{\alpha}{2}} e_{ij} - 2\mu_1 \Delta e_{ij}, \quad i, j = 1, 2, \\ e_{ij} = e_{ij}(u) &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad |e|^2 = \sum_{i,j=1}^{2} |e_{ij}|^2, \end{aligned} \tag{5.0.5}$$

where μ_0, μ_1, ϵ_0 and α $(0 < \alpha < 1)$ are positive constants which generally depend on the temperature and pressure. In (5.0.5) if $\mu_{ij}(e(u))$ depends linearly on $e_{ij}(u)$, then we say the corresponding fluid is a Newtonian one. If the relation between $\mu_{ij}(e(u))$ and $e_{ij}(u)$ is nonlinear, then the fluid is said to be non-Newtonian. One can refer to [12, 15] and related references therein for more physical explanations.

The existence and uniqueness of solution of non-Newtonian flow is studied in [7, 12, 15]. In [177, 180, 181] the existence of (compact, global, pullback) attractor for a non-Newtonian equation without delay has been analyzed, while [103] focused on pullback attractor of a non-autonomous non-Newtonian equation with variable delays. Caraballo and Real [36] proved the existence and uniqueness of solution for functional Navier-Stokes models with delay, and a non-classical non-autonomous diffusion equation with delay was considered in [31].

Enlightened by [31], in this paper we first aim to show the existence, uniqueness and continuity of solutions to (5.0.1)-(5.0.4) by the energy method (see [31, 70, 71]) and the classical Galerkin approximation (see [160]). Our second goal is to establish the existence of pullback attractor in phase space $C([-h,0];H^2(\mathcal{O}))$ by using pullback $\mathcal{D}-\omega-$limit compactness and a priori estimates.

We would like to mention that we will give a relatively complete proof of the existence, uniqueness and continuity of solutions to Eq.(5.0.1), which will be obtained assuming that g belongs to a more general space than the one in [103], namely, $g \in L^2_{loc}(\mathbb{R};W')$ instead of $g \in L^2_{loc}(\mathbb{R};H)$. And the assumption $g \in L^2_{loc}(\mathbb{R};H)$ is needed only when we show the existence of pullback absorbing set in the space C_W. Moreover, we only need $g \in L^2_{loc}(\mathbb{R};H)$ and satisfying (5.3.14), i.e., $\lim_{m\to+\infty}\sup_{t\geqslant\tau}\int_\tau^t e^{-2\mu_1\lambda_{m+1}(t-s)}\|g(s)\|^2ds = 0$, to establish that the process is pullback $\mathcal{D}-\omega-$limit compact in C_W. However, in some references, the fact that $g \in C(\mathbb{R};H)$ is required to prove the pullback $\mathcal{D}-\omega-$limit compactness in C_W which is a much stronger assumption than ours. Besides, in [103] the authors established the existence of pullback attractor for non-Newtonian fluid with variable delay, and we generalize this result to model more general delay. In other words, our result is true for both variable and distributed delays.

5.1 Definition and Basic Theory

We now recall some definitions and results concerning dynamical systems and pullback attractors. These definitions and results can be found in [30, 100, 180, 184].

Let (X, d_X) be a metric space, and denote $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : \tau \leqslant t\}$. A process U on X is a mapping $\mathbb{R}^2_d \times X \ni (t,\tau,x) \mapsto U(t,\tau)x \in X$ such that $U(\tau,\tau)x = x$ for any $\tau \in \mathbb{R}$, $x \in X$, and $U(t,r)(U(r,\tau)x) = U(t,\tau)x$ for any $\tau \leqslant r \leqslant t$ and all $x \in X$.

Let $P(X)$ denote the family of all nonempty subsets of X, and consider a family of nonempty sets $D_0 = \{D_0(t) : t \in \mathbb{R}\} \subset P(X)$. Let $\mathcal{D}$ be a given nonempty class of sets parameterized in time, $D = \{D(t) : t \in \mathbb{R}\} \subset P(X)$. The class $\mathcal{D}$ will be called a universe in $P(X)$.

Definition 5.1.1. *For any $\sigma > 0$, we will denote by $\mathcal{D}_\sigma(X)$ the class of all families of nonempty subsets $D = \{D(t) : t \in \mathbb{R}\} \subset P(X)$ such that*

$$\lim_{t\to-\infty}\left(e^{\sigma t}\sup_{u\in D(t)}\|u\|^2_X\right) = 0.$$

Definition 5.1.2. *It is said that $D_0 = \{D_0(t) : t \in \mathbb{R}\} \subset P(X)$ is pullback $\mathcal{D}-$absorbing for the process $\{U(t,\tau) : t \geqslant \tau\}$ on X if for any $t \in \mathbb{R}$ and any $D = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$, there exists a $\tau_0(t, D) \leqslant t$ such that*

$$U(t,\tau)D(\tau) \subset D_0(t) \text{ for all } \tau \leqslant \tau_0(t, D).$$

Definition 5.1.3. *Let $\{U(t,\tau)\}$ be a process on X. We say that $\{U(t,\tau)\}$ is pullback $\mathcal{D} - \omega-$limit compact with respect to each $t \in \mathbb{R}$, if for any family $B = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and for any $\epsilon > 0$, there exists $t_1 = t_1(B, t, \epsilon) > 0$, such that*

$$\kappa\left(\bigcup_{s \geqslant t_1} U(t, t-s)B(t-s)\right) \leqslant \epsilon,$$

where κ is the Kuratowski measure of non-compactness (see [125] for more information).

Definition 5.1.4. *The family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\} \subset P(X)$ is a pullback $\mathcal{D}-$attractor for the process $\{U(t,\tau) : t \geqslant \tau\}$ in X if :*

(i) for any $t \in \mathbb{R}$, the set $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X,

(ii) $\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}-$attracting, i.e.,

$$\lim_{\tau \to -\infty} dist_X\left(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)\right) = 0, \text{ for all } D \in \mathcal{D}, \text{ for any } t \in \mathbb{R},$$

(iii) $\mathcal{A}_{\mathcal{D}}$ is invariant, i.e.,

$$U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t), \text{ for all } \tau \leqslant t.$$

To analyze our problem with delay, we need to construct our process in a Banach space of segments of solutions. Namely, the space C_X which we will define below. Let X be a Banach space and let $h > 0$ be a given positive number (the time delay). Denote by C_X the Banach space $C([-h, 0]; X)$ endowed with the norm $\|\phi\|_{C_X} = \sup_{\theta \in [-h,0]} \|\phi(\theta)\|_X$. To study the pullback $\mathcal{D} - \omega-$limit compactness of the process on C_X, we borrow some techniques from [100, 170].

Proposition 5.1.5. *(see [100]) Let $\{U(t,\tau)\}$ be a continuous process on C_X. Suppose that for each $t \in \mathbb{R}$, $B = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and $\epsilon > 0$, there exist $\tau_0 = \tau_0(t, B, \epsilon) > 0$, a finite dimensional subspace X_1 of X and $\delta > 0$ such that*

(i) for each fixed $\theta \in [-h, 0]$

$$\left\| \bigcup_{s \geqslant \tau_0} \bigcup_{u_t(\cdot) \in U(t,t-s)B(t-s)} Pu(t+\theta) \right\|_X \text{ is bounded};$$

(ii) for all $s \geqslant \tau_0$, $u_t(\cdot) \in U(t, t-s)B(t-s)$, $\theta_1, \theta_2 \in [-h, 0]$ with $|\theta_2 - \theta_1| < \delta$,

$$\|P(u(t+\theta_1) - u(t+\theta_2))\|_X < \epsilon;$$

(iii) *for all* $s \geqslant \tau_0$, $u_t(\cdot) \in U(t, t-s)B(t-s)$,

$$\sup_{\theta\in[-h,0]} \|(I-P)u(t+\theta)\|_X < \epsilon,$$

where $P : X \to X_1$ *is the canonical projector. Then* $\{U(t,\tau)\}$ *is pullback* $\mathcal{D}-\omega-$*limit compact in* C_X *with respect to each* $t \in \mathbb{R}$.

The following proposition is similar to that of [28, 125, 184].

Proposition 5.1.6. *Let* $\{U(t,\tau)\}_{t\geqslant\tau}$ *be a process on Banach space* C_X *and* $\mathcal{D}$ *be a universe in* $P(C_X)$. *Then,* $\{U(t,\tau)\}_{t\geqslant\tau}$ *possesses a unique pullback* $\mathcal{D}-$*attractor* $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$, *for any* $t \in \mathbb{R}$ *and* $D \in \mathcal{D}$,

$$\mathcal{A}_{\mathcal{D}}(t) = \omega(D,t) = \bigcap_{\tau_0\leqslant t}\overline{\bigcup_{\tau\leqslant\tau_0} U(t,\tau)D(\tau)}$$

if and only if

(a) $\{U(t,\tau)\}_{t\geqslant\tau}$ *has a pullback* $\mathcal{D}-$*absorbing set in* C_X,

(b) $\{U(t,\tau)\}_{t\geqslant\tau}$ *is pullback* $\mathcal{D}-\omega-$*limit compact in* C_X.

5.2 Existence and continuity of solutions

In this section, by classical Faedo-Galerkin approximation and the energy method, we prove the existence, uniqueness and continuity of solutions to problem (5.0.1)-(5.0.4). But, in order to simplify the process of prove, we reformulate (5.0.1)-(5.0.4) into an abstract way, namely, (4.4.19)-(4.4.20)

Theorem 5.2.1. *(Existence and uniqueness of solution) Assume* $(H1)-(H3)$ *hold. Let* $g \in L^2_{loc}(\mathbb{R}, W')$ *and* $\phi \in C_H$. *Then, for any* $\tau \in \mathbb{R}$,

(a) *there exists a unique weak solution* u *to problem* (4.4.19) *satisfying*

$$u \in C([\tau-h,T];H) \cap L^\infty(\tau,T;H) \cap L^2(\tau,T;W), \quad \forall T > \tau.$$

(b) *If* $\phi(0) \in W$, *and* $g \in L^2_{loc}(\mathbb{R}, H)$, *then there exists a unique strong solution* u *to problem* (4.4.19) *satisfying*

$$u \in C([\tau-h,T];W) \cap L^\infty(\tau,T;W) \cap L^2(\tau,T;D(A)), \quad \forall T > \tau.$$

Proof. We split the proof into several steps.

Step 1. A Galerkin Scheme.

By the definition of A and the classical spectral theory of elliptic operators (see [126]), we see that operator A possesses a sequence of eigenvalues $\{\lambda_n\}_{n=1}^\infty$ and a corresponding family of eigenfunctions

$\{w^n\}_{n=1}^{\infty} \subset W \cap D(A)$, which form a basis of W and are orthonormal in H, we consider the subspace $W_m = \mathrm{span}\{w^1, w^2, \cdots, w^m\}$, and the projector $P_m : H \to W_m$ defined as

$$P_m u = \sum_{n=1}^{m} (u, w^n) w^n, \quad u \in H.$$

Define

$$u^m(t) = \sum_{n=1}^{m} \gamma_{mn} w^n,$$

where the upper script m will be used instead of (m) for short, since no confusion is possible with powers of u, and the coefficients γ_{mn} are required to satisfy the following system:

$$\begin{aligned} &\left(\frac{\partial}{\partial t} u^m(t), w^n\right) + 2\mu_1 (Au^m, w^n) + \langle B(u^m(t)), w^n\rangle + \langle N(u^m(t)), w^n\rangle \\ &\quad = (f(t, u_t^m), w^n) + \langle g(t,x), w^n\rangle, \quad a.e.\ t > \tau, \quad 1 \leqslant n \leqslant m, \end{aligned} \tag{5.2.1}$$

and where the equations are understood in the sense of $\mathcal{D}'(\tau, T)$, and the initial conditions are

$$u^m(\tau + \theta) = P_m \phi(\theta), \quad \text{for } \theta \in [-h, 0].$$

The above system of ordinary differential equations with finite delay fulfills the conditions for existence and uniqueness of local solution in [30, Theorem A1, p. 2450]. Hence, we can ensure that problem (5.2.1) has a unique local solution defined in $[\tau, t_m]$ with $\tau < t_m \leqslant +\infty$ (see [91] for a similar result).

Next, by a priori estimates, we verify that solutions u^m do exist for all time $t \in [\tau, +\infty)$.

Step 2: A priori estimates

Multiplying (5.2.1) by γ_{mn}, summing from $n = 1$ to $n = m$, and using Lemma A.1.3 in Appendix, we obtain, for all $t \in [\tau, t_m]$, that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^m(t)\|^2 + 2c_1\mu_1\|u^m(t)\|_W^2 + \langle B(u^m(t)), u^m(t)\rangle + \langle N(u^m(t)), u^m(t)\rangle \\ &\quad \leqslant (f(t, u_t^m), u^m(t)) + \langle g, u^m(t)\rangle. \end{aligned} \tag{5.2.2}$$

Integrating over $[\tau, t]$,

$$\begin{aligned} &\frac{1}{2}\|u^m(t)\|^2 + 2c_1\mu_1 \int_\tau^t \|u^m(s)\|_W^2 ds + \int_\tau^t \langle B(u^m(s)), u^m(s)\rangle ds + \int_\tau^t \langle N(u^m(s)), u^m(s)\rangle ds \\ &\quad \leqslant \frac{1}{2}\|u^m(\tau)\|^2 + \int_\tau^t (f(s, u_s^m), u^m(s)) ds + \int_\tau^t \langle g, u^m(s)\rangle ds. \end{aligned} \tag{5.2.3}$$

First, by (4.4.11) and (4.4.13),

$$\int_\tau^t \langle B(u^m(s)), u^m(s)\rangle ds = 0, \tag{5.2.4}$$

and

$$\int_\tau^t \langle N(u^m(s)), u^m(s)\rangle ds \geqslant 0. \tag{5.2.5}$$

By the fact that $\|v\|_W \geqslant \|v\|$ for all $v \in W$,

$$\int_\tau^t \langle g, u^m(s)\rangle ds \leqslant \frac{c_1\mu_1}{2}\int_\tau^t \|u^m(s)\|_W^2 ds + \frac{1}{2c_1\mu_1}\int_\tau^t \|g(s)\|_{W'}^2 ds. \tag{5.2.6}$$

From (H3) and Young's inequality,

$$\begin{aligned}\int_\tau^t (f(s,u_s^m), u^m(s))ds &\leqslant \int_\tau^t \|f(s,u_s^m)\|\cdot\|u^m(s)\|ds\\ &\leqslant L_f\int_\tau^t \|u_s^m\|_{C_H}\|u^m(s)\|ds\\ &\leqslant \frac{c_1\mu_1}{2}\int_\tau^t \|u^m(s)\|_W^2 ds + \frac{L_f^2}{2c_1\mu_1}\int_\tau^t \|u_s^m\|_{C_H}^2 ds.\end{aligned} \tag{5.2.7}$$

It follows from (5.2.3)-(5.2.7) that

$$\|u^m(t)\|^2 + 2c_1\mu_1\int_\tau^t \|u^m(s)\|_W^2 ds \leqslant \|\phi\|_{C_H}^2 + \frac{L_f^2}{c_1\mu_1}\int_\tau^t \|u_s^m\|_{C_H}^2 ds + \frac{1}{c_1\mu_1}\int_\tau^t \|g(s)\|_{W'}^2 ds, \quad \forall\, t \geqslant \tau. \tag{5.2.8}$$

Replacing t by $t+\theta$ in (5.2.8) we obtain

$$\|u_t^m\|_{C_H}^2 \leqslant \|\phi\|_{C_H}^2 + \frac{L_f^2}{c_1\mu_1}\int_\tau^t \|u_s^m\|_{C_H}^2 ds + \frac{1}{c_1\mu_1}\int_\tau^t \|g(s)\|_{W'}^2 ds, \quad \forall\, t \geqslant \tau,$$

and therefore, the Gronwall Lemma implies

$$\|u_t^m\|_{C_H}^2 \leqslant e^{\frac{L_f^2}{c_1\mu_1}(t-\tau)}\left(\|\phi\|_{C_H}^2 + \frac{1}{c_1\mu_1}\int_\tau^t \|g(s)\|_{W'}^2 ds\right), \quad \forall\, t \geqslant \tau,\ \forall\, m \geqslant 1. \tag{5.2.9}$$

Then, by (5.2.9), we can check that for each $T > \tau$ and $R > 0$, there exists a positive constant $C(\tau, T, R, L_f)$, depending on the constants of the problem c_1, μ_1, L_f, g, and on τ, T, R, such that for all $m \geqslant 1$,

$$\|u_t^m\|_{C_H}^2 + \|u^m\|_{L^2(\tau,T;W)}^2 \leqslant C(\tau, T, R, L_f),\ \|\phi\|_{C_H} \leqslant R.$$

In particular, thanks to inequalities (5.2.8) and (5.2.9), and the fact that $g \in L^2_{loc}(\mathbb{R}; W')$, we deduce

$$\{u^m\} \text{ is bounded in } L^\infty(\tau - h, T; H) \cap L^2(\tau, T; W), \quad \forall T > \tau. \tag{5.2.10}$$

On the other hand, for almost all t, $B(u(t))$ and $N(u(t))$ are elements of W', and the measurability of the mappings

$$t \in [0,T] \to B(u(t)) \in W',$$

and

$$t \in [0,T] \to N(u(t)) \in W'$$

are straightforward. Moreover, thanks to (4.4.11), the Hölder inequality, embedding theorems, and Lemma A.1.1 in Appendix, we have that for all $\varphi \in W$,

$$\begin{aligned} |\langle B(u),\varphi\rangle| = |b(u,u,\varphi)| = |-b(u,\varphi,u)| = \left|\sum_{i,j=1}^{2}\int_O u_i\frac{\partial\varphi_j}{\partial x_i}u_j dx\right| \\ \leqslant c\|u\|_{L^4}^2\|\varphi\|_{H_0^1(O)} \leqslant c\|u\|\cdot\|\nabla u\|\cdot\|\varphi\|_{H_0^1(O)} \leqslant c\|u\|\cdot\|\Delta u\|\cdot\|\Delta\varphi\|. \end{aligned} \tag{5.2.11}$$

Using the fact that $\mu(u) = \mu_0(\epsilon_0 + |e|^2)^{-\frac{\alpha}{2}} \leqslant \mu_0\epsilon_0^{-\frac{\alpha}{2}}$, we can also obtain

$$\begin{aligned} |\langle N(u),\varphi\rangle| &= \left|2\sum_{i,j=1}^{2}\int_O \mu(e(u))e_{ij}(u)e_{ij}(\varphi)dx\right| \\ &\leqslant 2\mu_0\epsilon_0^{-\frac{\alpha}{2}}\int_O\sum_{i,j=1}^{2}|e_{ij}(u)e_{ij}(\varphi)|dx \\ &\leqslant c\|\nabla u\|\cdot\|\nabla\varphi\| \\ &\leqslant c\|\Delta u\|\cdot\|\Delta\varphi\|. \end{aligned} \tag{5.2.12}$$

As a consequence of (5.2.11) and (5.2.12), the estimates hold true,

$$\|B(u)\|_{W'} \leqslant c\|u\|\cdot\|\Delta u\| \tag{5.2.13}$$

and

$$\|N(u)\|_{W'} \leqslant c\|\Delta u\|. \tag{5.2.14}$$

Hence,

$$\int_\tau^T \|B(u(s))\|_{W'}^2 ds \leqslant c\int_\tau^T \|u(s)\|^2\|\Delta u(s)\|^2 ds \leqslant c\int_\tau^T \|u_s\|_{C_H}^2\|\Delta u(s)\|^2 ds < \infty \tag{5.2.15}$$

and

$$\int_\tau^T \|N(u(s))\|_{W'}^2 ds \leqslant c\int_\tau^T \|\Delta u(s)\|^2 ds < \infty. \tag{5.2.16}$$

To this end, we need to show that $\{(u^m)'\}$ is bounded in $L^2(\tau,T;W')$. Let $\varphi \in C^1([0,T],W)$ and φ^m be the projection of φ in W, onto the space $W_m = \text{span}\{w^1,w^2,\cdots,w^m\}$. By (5.2.1), we have

$$\begin{aligned} &\int_O \frac{\partial u^m}{\partial t}w^n dx \\ &= -2\mu_1\sum_{i,j,k=1}^{2}\int_O \frac{\partial e_{ij}(u^m)}{\partial x_k}\frac{\partial e_{ij}(w^n)}{\partial x_k}dx - \sum_{i,j=1}^{2}\int_O u_i^m\frac{\partial u_j^m}{\partial x_i}w_j^n dx \\ &\quad - \sum_{i,j=1}^{2}\int_O \mu(u^m)e_{ij}(u^m)e_{ij}(w^n)dx + \int_O f(t,u_t^m)w^n dx + \int_O g(x,t)w^n dx,\ n = 1,2,\cdots,m. \end{aligned} \tag{5.2.17}$$

Using (5.2.17) and the definition of φ^m,

$$\begin{aligned}\int_\tau^T\int_O \frac{\partial u^m}{\partial t}\varphi dxdt &= \int_\tau^T\int_O \frac{\partial u^m}{\partial t}\varphi^m dxdt\\ &= -2\mu_1\sum_{i,j,k=1}^{2}\int_\tau^T\int_O \frac{\partial e_{ij}(u^m)}{\partial x_k}\frac{\partial e_{ij}(\varphi^m)}{\partial x_k}dxdt - \sum_{i,j=1}^{2}\int_\tau^T\int_O u_i^m\frac{\partial u_j^m}{\partial x_i}\varphi_j^m dxdt\\ &\quad -\sum_{i,j=1}^{2}\int_\tau^T\int_O \mu(u^m)e_{ij}(u^m)e_{ij}(\varphi^m)dxdt + \int_\tau^T\int_O f(t,u_t^m)\varphi^m dxdt\\ &\quad + \int_\tau^T\int_O g(x,t)\varphi^m dxdt\\ &= I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{5.2.18}$$

From (5.2.10),

$$|I_1 + I_4 + I_5| \leqslant C_1\|\varphi^m\|_{L^2(\tau,T;W)}. \tag{5.2.19}$$

By a similar argument to that one in (5.2.11) and (5.2.12), we can check that

$$|I_2| \leqslant C_2\int_\tau^T \|u^m\|\cdot\|\nabla u^m\|\cdot\|\nabla\varphi^m\|dt \leqslant C_3\|\varphi^m\|_{L^2(\tau,T;W)} \tag{5.2.20}$$

as well as

$$|I_3| \leqslant C_4\|\varphi^m\|_{L^2(\tau,T;W)}. \tag{5.2.21}$$

Hence, from (5.2.17)-(5.2.21), we can conclude that

$$\left|\int_\tau^T\int_O \frac{\partial u^m}{\partial t}\varphi dxdt\right| \leqslant C_5\|\varphi^m\|_{L^2(\tau,T;W)} \leqslant C_5\|\varphi\|_{L^2(\tau,T;W)}, \tag{5.2.22}$$

and

$$\left\|\frac{\partial u^m}{\partial t}\right\|_{L^2(\tau,T;W')} \leqslant C_6, \tag{5.2.23}$$

where C_i $(i = 1, 2, \cdots, 6)$ are positive constants. Thus,

$$\{(u^m)'\} \text{ is bounded in } L^2(\tau,T;W'), \quad \forall T \geqslant \tau. \tag{5.2.24}$$

Step 3: The energy method and compactness results

Now, we combine some well-known compactness results with the energy method to pass to the limit in a subsequence of $\{u^m\}$ to obtain a solution of (5.0.1). Observe that

$$u^m|_{[\tau-h,\tau]} = P_m\phi \to \phi \text{ in } C_H. \tag{5.2.25}$$

By Step 1, Step 2 and compactness theorem, we deduce that there exist a subsequence (which we relabel the same) $\{u^m\}$, a function $u \in C([\tau-h,\infty);H)$, with $u|_{[\tau-h,\tau]} = \phi$, $u \in L^2(\tau,T;W)$, $\chi \in$

$L^2(\tau, T; W')$ for all $T > \tau$, and an element $\xi \in L^\infty(\tau, T, H)$ for all $T > \tau$, such that

$$\begin{aligned} &u^m \overset{*}{\rightharpoonup} u \;\text{ weakly-star in }\; L^\infty(\tau, T; H),\\ &u^m \rightharpoonup u \;\text{ weakly in }\; L^2(\tau, T; W),\\ &(u^m)' \rightharpoonup \chi \;\text{ weakly in }\; L^2(\tau, T; W'),\\ &u^m \to u \;\text{ strongly in }\; L^2(\tau, T; H),\\ &f(\cdot, u^m_\cdot) \overset{*}{\rightharpoonup} \xi \;\text{ weakly-star in }\; L^\infty(\tau, T; H). \end{aligned} \tag{5.2.26}$$

We first prove that $\chi = u' = \frac{du}{dt}$. Indeed, since the approximate solutions $\{u^m\}$ satisfy

$$u^m(s) = P_m\phi(\tau) + \int_\tau^s \frac{du^m}{dt}dt, \quad s \in [\tau, T], \; m = 1, 2, \cdots .$$

From (5.2.25), we know

$$P_m\phi(\tau) \to \phi(\tau).$$

Then

$$u(s) = \phi(\tau) + \int_\tau^s \chi dt,$$

by [160, Lemma 3.1, Chapter II], we immediately deduce that $\chi = u' = \frac{du}{dt}$.

Using $(5.2.26)_4$, we can also assume that

$$u^m(t) \to u(t) \;\text{ in }\; H \;\; a.e.\; t \in [\tau, T], \tag{5.2.27}$$

which is not enough to deduce that $\xi(\cdot) = f(\cdot, u_\cdot)$.

However, we can obtain convergence for all $t > \tau$ with a little more effort and in a more general case. Notice that,

$$u^m(t) - u^m(s) = \int_s^t (u^m)'(r)dr \;\text{ in } W', \; \forall\; s, t \in [\tau, T],$$

and by (5.2.24) we have that $\{u^m\}$ is equi-continuous on $[\tau, T]$ with values in W', for all $T > \tau$.

Since the injection of W in H is compact, the injection of H into W' is compact as well. Thus, from (5.2.10) and the equi-continuity of $\{u^m\}$ in W', using Arzelà-Ascoli theorem, we have (again, up to a subsequence)

$$u^m \to u \;\text{ in } C([\tau, T]; W'), \;\; \forall\; T > \tau. \tag{5.2.28}$$

This, jointly with the fact $H \subset W'$, (5.2.10) and [160, Lemma 3.3, Chapter II], allows us to claim that for any sequence $\{t_m\} \subset [\tau, \infty)$, with $t_m \to t$,

$$u^m(t_m) \rightharpoonup u(t) \;\text{ weakly in } H, \tag{5.2.29}$$

where we have used (5.2.28) in order to identify which is the weak limit.

Now we prove that in fact

$$u^m(t_m) \to u(t) \ \text{ in } \ C([\tau, T]; H) \ \ \forall\, T > \tau. \tag{5.2.30}$$

If not, then, taking into account that $u \in C([\tau, \infty); H)$, there would exist $T > \tau$, $\epsilon_1 > 0$, a value $t_0 \in [\tau, T]$, and subsequences (relabelled the same) $\{u^m\}$ and $\{t_m\} \subset [\tau, T]$, with $\lim_{m\to+\infty} t_m = t_0$, such that

$$\|u^m(t_m) - u(t_0)\| \geqslant \epsilon_1, \ \forall\, m \geqslant 1.$$

To conclude that this is false, we use an energy method. Note that the following energy equality holds for all u^m:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^m(t)\|^2 + 2\mu_1 a(u^m(t), u^m(t)) + \langle B(u^m(t)), u^m(t)\rangle + \langle N(u^m(t)), u^m(t)\rangle \\ &\quad = (f(t, u^m_t), u^m(t)) + \langle g, u^m(t)\rangle. \end{aligned} \tag{5.2.31}$$

By Lemma A.1.3, we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^m(t)\|^2 + 2c_1\mu_1\|u^m(t)\|_W^2 + \langle B(u^m(t)), u^m(t)\rangle + \langle N(u^m(t)), u^m(t)\rangle \\ &\quad \leqslant (f(t, u^m_t), u^m(t)) + \langle g, u^m(t)\rangle. \end{aligned} \tag{5.2.32}$$

Integrating (5.2.32) over $[s, t]$ with respect to t,

$$\begin{aligned} &\frac{1}{2}\|u^m(t)\|^2 + 2c_1\mu_1\int_s^t \|u^m(r)\|_W^2 dr + \int_s^t \langle B(u^m(r)), u^m(r)\rangle dr + \int_s^t \langle N(u^m(r)), u^m(r)\rangle dr \\ &\leqslant \frac{1}{2}\|u^m(s)\|^2 + \int_s^t (f(r, u^m_r), u^m(r)) dr + \int_s^t \langle g(r), u^m(r)\rangle dr. \end{aligned} \tag{5.2.33}$$

Since $\langle B(u^m(r)), u^m(r)\rangle = 0$ and $\langle N(u^m(r)), u^m(r)\rangle \geqslant 0$, and

$$\begin{aligned} \int_s^t (f(r, u^m_r), u^m(r)) dr &\leqslant \frac{c_1\mu_1}{2}\int_s^t \|u^m(r)\|_W^2 dr + \frac{1}{2c_1\mu_1}\int_s^t \|f(r, u^m_r)\|^2 dr \\ &\leqslant \frac{c_1\mu_1}{2}\int_s^t \|u^m(r)\|_W^2 dr + \frac{L_f^2}{2c_1\mu_1}\int_s^t \|u^m_r\|_{C_H} dr \\ &\leqslant \frac{c_1\mu_1}{2}\int_s^t \|u^m(r)\|_W^2 dr + C(t-s), \ \forall \tau \leqslant s \leqslant t \leqslant T, \end{aligned}$$

where $C = \frac{L_f^2 D}{2c_1\mu_1}$, and D corresponds to the upper bound of $\|u_t\|_{C_H}$, it follows

$$\|u^m(t)\|^2 \leqslant \|u^m(s)\|^2 + 2\int_s^t \langle g(r), u^m(r)\rangle dr + 2C(t-s), \quad \forall \tau \leqslant s \leqslant t \leqslant T. \tag{5.2.34}$$

On the one hand, observe that by (5.2.26), passing to the limit in (5.2.1), we have that $u \in C([\tau, T]; H)$ is a solution of a similar problem to (5.0.1), namely,

$$\left(\frac{\partial}{\partial t}u(t), w\right) + 2\mu_1 a(u(t), w) + \langle B(u(t)), w\rangle + \langle N(u(t)), w\rangle = (\xi, w) + \langle g, w\rangle, \quad \forall w \in W,$$

fulfilled with the initial datum $u(\tau) = \phi(0)$. Therefore, it satisfies the energy equality

$$\begin{aligned} &\frac{1}{2}\|u(t)\|^2 + 2\mu_1 \int_s^t a(u(r), u(r))dr + \int_s^t \langle B(u(r)), u(r)\rangle dr + \int_s^t \langle N(u(r)), u(r)\rangle dr \\ &= \frac{1}{2}\|u(s)\|^2 + \int_s^t (\xi(r), u(r))dr + \int_s^t \langle g(r), u(r)\rangle dr, \quad \forall \tau \leqslant s \leqslant t \leqslant T. \end{aligned}$$

On the other hand, from $(5.2.26)_5$ we deduce that

$$\int_s^t \|\xi(r)\|^2 dr \leqslant \liminf_{m\to+\infty} \int_s^t \|f(r, u_r^m)\|^2 dr \leqslant D(t-s), \quad \forall \tau \leqslant s \leqslant t \leqslant T,$$

which implies that u also satisfies inequality (5.2.34) (here we applied Lemma A.1.3) with the same constant c_1.

Now, consider the functions $J_m, J : [\tau, T] \mapsto \mathbb{R}$ defined by

$$\begin{aligned} J_m(t) &= \frac{1}{2}\|u^m(t)\|^2 - \int_\tau^t \langle g(r), u^m(r)\rangle dr - C(t-\tau), \\ J(t) &= \frac{1}{2}\|u(t)\|^2 - \int_\tau^t \langle g(r), u(r)\rangle dr - C(t-\tau), \end{aligned}$$

with C defined in (5.2.34). By (5.2.34) and the analogous inequality for u, it is clear that J_m and J are non-increasing continuous functions. Moreover, by (5.2.26) and (5.2.27),

$$J_m(t) \to J(t) \quad a.e.\ t \in [\tau, T]. \tag{5.2.35}$$

Now we are ready to prove that

$$u^m(t_m) \to u(t_0) \text{ in } H. \tag{5.2.36}$$

Recall that from (5.2.29) we have

$$\|u(t_0)\| \leqslant \liminf_{m\to+\infty} \|u^m(t_m)\|. \tag{5.2.37}$$

Therefore, if we show that

$$\limsup_{m\to+\infty} \|u^m(t_m)\| \leqslant \|u(t_0)\|, \tag{5.2.38}$$

then combining with (5.2.37), we can obtain $\lim_{m\to+\infty} \|u^m(t_m)\| = \|u(t_0)\|$, which means (5.2.36) holds true.

Note that the case $t_0 = \tau$ follows directly from (5.2.25) and (5.2.34) with $s = \tau$. Hence, we can assume that $t_0 > \tau$. Owing to this result, we approach t_0 from the left by a sequence $\{\tilde{t}_k\}$, namely,

$\lim_{k\to+\infty} \tilde{t}_k \nearrow t_0$, being $\{\tilde{t}_k\}$ values where (5.2.35) holds. Since $J(\cdot)$ is continuous at t_0, for any $\epsilon > 0$ there is k_ϵ such that $|J(\tilde{t}_k) - J(t_0)| < \frac{\epsilon}{2}$ for all $k \geqslant k_\epsilon$. On the other hand, taking $m \geqslant m(k_\epsilon)$ such that $t_m > \tilde{t}_{k_\epsilon}$, as J_m is non-increasing and for all $\tilde{t}_k$ the convergence (5.2.35) holds, one has

$$J_m(t_m) - J(t_0) \leqslant |J_m(\tilde{t}_{k_\epsilon}) - J(\tilde{t}_{k_\epsilon})| + |J(\tilde{t}_{k_\epsilon}) - J(t_0)|,$$

and taking $m \geqslant m'(k_\epsilon) \geqslant m(k_\epsilon)$, such that $|J_m(\tilde{t}_{k_\epsilon}) - J(\tilde{t}_{k_\epsilon})| < \frac{\epsilon}{2}$. It can also be deduced from (5.2.26) that

$$\int_\tau^{t_m} \langle g(r), u^m(r)\rangle dr \to \int_\tau^{t_0} \langle g(r), u(r)\rangle dr,$$

We conclude that (5.2.38) holds. Thus, (5.2.36) and finally (5.2.30) are also true, as claimed. This also implies, thanks to (5.2.25), that $u_t^m \to u_t$ in C_H for all $t \geqslant \tau$. Therefore, we identify the weak limit ξ from (5.2.26). Indeed, from the above convergence and since f satisfies (H3), we have that $f(\cdot, u_\cdot^m) \to f(\cdot, u_\cdot)$ in $L^2(\tau, T; (L^2(O))^2)$ for all $T > \tau$. Thus, we can pass to the limit finally in (5.2.1) concluding that u solves (5.0.1).

Step 4: The uniqueness of solution

This can be obtained in the following way. Consider two weak solutions of (5.0.1), u and v, with the same initial data, and denote $w = u - v$. We notice that

$$|b(u, v, w)| \leqslant 2^{-\frac{1}{2}} \|u\|^{\frac{1}{2}} \|\nabla u\|^{\frac{1}{2}} \|\nabla v\| \|w\|^{\frac{1}{2}} \|\nabla w\|^{\frac{1}{2}}, \quad u, v, w \in W.$$

$$\frac{\partial w}{\partial t} + 2\mu_1 Aw + B(u) - B(v) + N(u) - N(v) = f(t, u_t) - f(t, v_t), \tag{5.2.39}$$

with initial value

$$w(\tau) = 0. \tag{5.2.40}$$

Take the inner product of (5.2.39) with w to yield that

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + 2\mu_1 a(w, w) + \langle B(u) - B(v), w\rangle + \langle N(u) - N(v), w\rangle = (f(t, u_t) - f(t, v_t), w).$$

From the monotonicity of $\mu(u)$, it follows that

$$\langle N(u) - N(v), w\rangle = 2\int_O [\mu(u)e_{ij}(u) - \mu(v)e_{ij}(v)]e_{ij}(w)dx \geqslant 0, \tag{5.2.41}$$

and we also have

$$\langle B(u) - B(v), w\rangle = b(u, u, w) - b(v, v, w) = b(w, v, w). \tag{5.2.42}$$

Then, for some $\alpha_i > 0$, $i = 1, 2, 3$,

$$\begin{aligned} |b(u, u, w) - b(v, v, w)| &\leqslant \left| \int_O w_i \frac{\partial u_j}{\partial x_i} w_j dx \right| \leqslant \alpha_1 \|w\|_{L^4}^2 \|\nabla u\| \\ &\leqslant \alpha_2 \|w\|_{L^2} \|\nabla w\| \cdot \|\nabla u\| \leqslant \alpha_3 \|w\|_{L^2} \|\Delta w\| \cdot \|\Delta u\|. \end{aligned} \tag{5.2.43}$$

Using Lemma A.1.3, together with (5.2.42)-(5.2.43), we find that, for some $c > 0$,

$$\begin{aligned}\frac{d}{dt}\|w\|^2 &\leqslant c\|w\|\cdot\|\Delta w\|\cdot\|\Delta u\| + 2(f(t,u_t) - f(t,v_t), w)\\ &\leqslant c\|w\|\cdot\|\Delta w\|\cdot\|\Delta u\| + 2L_f\|u_t - v_t\|\cdot\|w\|\\ &\leqslant c\|w\|\cdot\|\Delta w\|\cdot\|\Delta u\| + 2L_f\|w_t\|\cdot\|w\|\\ &\leqslant c_1\mu_1\|w\|_W^2 + c\|w\|^2\|\Delta u\|^2 + 2L_f\|w_t\|_{C_H}^2, \quad \text{for all } t > \tau,\end{aligned}$$

and therefore,

$$\frac{d}{dt}\|w\|^2 \leqslant c\|w\|^2\|\Delta u\|^2 + 2L_f\|w_t\|_{C_H}^2, \quad \text{for all } t > \tau.$$

Integrating the above inequality over $[\tau, t]$ with respect to t,

$$\|w(t)\|^2 \leqslant c\int_\tau^t \|w(r)\|^2\|\Delta u(r)\|^2 dr + 2L_f\int_\tau^t \|w_r\|_{C_H}^2 dr.$$

Hence,

$$\|w_t\|_{C_H}^2 \leqslant \int_\tau^t (c\|\Delta u(r)\|^2 + 2L_f)\|w_r\|_{C_H}^2 dr.$$

By the Gronwall lemma, we have

$$\|w_t\|_{C_H}^2 \equiv 0.$$

Finally, the regularity in (b) is a consequence of well-known regularity results and the fact that, if $g \in L^2_{loc}(\mathbb{R}; (L^2(O))^2)$, then the function $\hat{g}(t) = g(t) + f(t, u_t)$, $t > \tau$, belongs to $L^2_{loc}(\tau, \infty; (L^2(O))^2)$.

The proof is finished. □

Theorem 5.2.2. *(Continuous dependence of solutions on initial values) Let $g \in L^2_{loc}(\mathbb{R}; W')$, $f : \mathbb{R}\times C_H \mapsto (L^2(O))^2$ satisfying (H1) – (H3), and ϕ, $\psi \in C_H$ be given. Let us denote $u = u(\cdot;\tau,\phi)$ and $v = v(\cdot;\tau,\psi)$ the corresponding weak solutions to problem* (5.0.1)*. Then, the following estimate holds:*

$$\|u_t - v_t\|_{C_H}^2 \leqslant \|\phi - \psi\|_{C_H}^2 \exp\left\{\int_\tau^t \left(c\|\Delta u(s)\|^2 + 2L_f\right) ds\right\}.$$

Proof. Denote $w = u - v$. Analogously to the arguments in Theorem 5.2.1 for the proof of uniqueness of weak solution to problem (5.0.1) we obtain

$$\frac{\partial w}{\partial t} + 2\mu_1 Aw + (B(u) - B(v)) + (N(u) - N(v)) = f(t,u_t) - f(t,v_t). \tag{5.2.44}$$

Multiplying (5.2.44) by w,

$$\frac{d}{dt}\|w\|^2 + 3c_1\mu_1\|w\|_W^2 \leqslant c\|w\|^2\|\Delta u\|^2 + 2L_f\|w_t\|_{C_H}^2.$$

Integrate the above inequality over $[\tau, t]$ with respect to t to get

$$\|w(t)\|^2 \leqslant \|w(\tau)\|^2 + c\int_\tau^t \|w(s)\|^2\|\Delta u(s)\|^2 ds + 2L_f\int_\tau^t \|w_s\|_{C_H}^2 ds,$$

particularly,

$$\|w_t\|_{C_H}^2 \leqslant \|w(\tau)\|^2 + \int_\tau^t (c\|\Delta u(s)\|^2 + 2L_f)\|w_s\|_{C_H}^2 ds.$$

Again by the Gronwall lemma, we have

$$\|w_t\|_{C_H}^2 \leqslant \|w(\tau)\|^2 \exp\left\{\int_\tau^t \left(c\|\Delta u(s)\|^2 + 2L_f\right) ds\right\},$$

namely,

$$\|u_t - v_t\|_{C_H}^2 \leqslant \|\phi - \psi\|_{C_H}^2 \exp\left\{\int_\tau^t \left(c\|\Delta u(s)\|^2 + 2L_f\right) ds\right\}.$$

The proof is completed immediately. □

5.3 Uniform Estimates

In this section, we analyze the existence of pullback $\mathcal{D}-$attractor in C_W.

5.3.1 Existence of pullback absorbing sets

Now, by the previous results, we are able to define correctly a process U on C_H and C_W associated to (4.4.19), and then to obtain the existence of pullback attractors.

Theorem 5.3.1. *Let $g \in L^2_{loc}(\mathbb{R}; W')$ and $f : \mathbb{R} \times C_H \to (L^2(\mathcal{O}))^2$ satisfying (H1) − (H3). Then, the process $U(t, \tau) : C_H \to C_H$, with $\tau \leqslant t$, given by*

$$U(t, \tau)\phi = u_t(\cdot; \tau, \phi),$$

where $u = u(\cdot; \tau, \phi)$ is the unique weak solution to (4.4.19), defines a continuous process on C_H.

Proof. It is a consequence of theorems 5.2.1 and 5.2.2. □

Remark 5.3.2. *By a reasoning similar to the one in Theorem 5.2.2, we can conclude that U depends continuously on the initial values in C_W, which jointly with Theorem 5.2.1, allow us to show that U is also a well-defined process on C_W with $g \in L^2_{loc}(\mathbb{R}; H)$ and initial datum $\phi \in C_W$.*

Next, we show the existence of pullback $\mathcal{D}-$absorbing sets of U in C_H and C_W, and then verify the pullback $\mathcal{D}-\omega-$limit compactness of U in C_W. Hereafter, we suppose that

$$\text{there exists } 0 < \beta < c_1\mu_1 \text{ such that } \sigma := \beta - \frac{L_f^2 e^{\beta h}}{c_1\mu_1} > 0, \tag{5.3.1}$$

and

$$\int_{-\infty}^{t} e^{\sigma s}\left(\|g(s)\|_{W'}^2 + \|g(s)\|^2\right) ds < \infty, \ \forall\, t \in \mathbb{R},$$

and denote by $\mathcal{D}$ the class of all families of nonempty subsets $D = \{D(t)\}_{t\in\mathbb{R}} \subset P(C_X)$ such that

$$\lim_{t\to-\infty}\left(e^{\sigma t}\sup_{u\in D(t)}\|u\|_{C_X}^2\right) = 0,$$

where σ is defined in (5.3.1), $C_X = C_H$ or $C_X = C_W$.

Remark 5.3.3. *If $c_1\mu_1 h > 1$, there exists β which satisfies (5.3.1) when $ehL_f^2 < c_1\mu_1$; if $c_1\mu_1 h \leqslant 1$, then we can choose β which satisfies (5.3.1) as long as $e^{c_1\mu_1 h}L_f^2 < (c_1\mu_1)^2$ holds.*

We now prove the existence of a pullback absorbing set in C_H.

Lemma 5.3.4. *(Pullback absorbing set in C_H) Assume that (H1)-(H3) hold and $g \in L^2_{loc}(\mathbb{R}; W')$. Let $B = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$. Then, there exists $T_B > 0$, such that for any $t \in \mathbb{R}$, all $r > T_B$ and $\phi \in B(t-r) \subset C_H$, the weak solution $u(\cdot; t-r, \phi)$ of Eq. (4.4.19) satisfies*

$$\|u_t\|_{C_H}^2 = \|U(t, t-r)\phi\|_{C_H}^2 \leqslant \rho_1^2(t),$$

where $\rho_1^2(t) := 1 + e^{\beta h - \sigma t}\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|_{W'}^2 ds$.

Proof. The uniform estimates that we require for the solutions which define the process U are analogous to those provided in the proof of theorems 5.2.1-5.2.2, but there with Galerkin approximations.

For the sake of brevity, we only sketch the main ideas:

Multiplying (4.4.19) by u, by Lemma A.1.3, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + 2c_1\mu_1\|u\|_W^2 + \langle B(u), u\rangle + \langle N(u), u\rangle \leqslant (f(t, u_t), u) + \langle g, u\rangle. \tag{5.3.2}$$

Observe that

$$\langle B(u), u\rangle = 0, \ \ \langle N(u), u\rangle \geqslant 0.$$

By (H3) and the Young inequality,

$$(f(t, u_t), u) \leqslant \|f(t, u_t)\| \cdot \|u\| \leqslant L_f\|u_t\| \cdot \|u\| \leqslant \frac{L_f^2}{2c_1\mu_1}\|u_t\|_{C_H}^2 + \frac{c_1\mu_1}{2}\|u\|_W^2,$$

and

$$\langle g, u\rangle \leqslant \frac{c_1\mu_1}{2}\|u\|_W^2 + \frac{1}{2c_1\mu_1}\|g\|_{W'}^2.$$

From the above inequalities we obtain

$$\frac{d}{dt}\|u\|^2 + 2c_1\mu_1\|u\|_W^2 \leqslant \frac{L_f^2}{c_1\mu_1}\|u_t\|_{C_H}^2 + \frac{1}{c_1\mu_1}\|g\|_{W'}^2. \tag{5.3.3}$$

Multiplying (5.3.3) by $e^{\beta t}$ with $0 < \beta < c_1\mu_1$, and integrating the resulting over $[\tau, t]$ yield

$$e^{\beta t}\|u(t)\|^2 \leqslant e^{\beta\tau}\|u(\tau)\|^2 + \frac{L_f^2}{c_1\mu_1}\int_\tau^t e^{\beta s}\|u_s\|_{C_H}^2 ds + \frac{1}{c_1\mu_1}\int_\tau^t e^{\beta s}\|g(s)\|_{W'}^2 ds.$$

In particular we have

$$e^{\beta t}\|u_t\|_{C_H}^2 \leqslant e^{\beta(\tau+h)}\|u(\tau)\|^2 + \frac{L_f^2 e^{\beta h}}{c_1\mu_1}\int_\tau^t e^{\beta s}\|u_s\|_{C_H}^2 ds + \frac{e^{\beta h}}{c_1\mu_1}\int_\tau^t e^{\beta s}\|g(s)\|_{W'}^2 ds. \tag{5.3.4}$$

By Lemma A.1.4 in Appendix, we obtain that

$$e^{\beta t}\|u_t\|_{C_H}^2 \leqslant e^{\beta(h+\tau)+\frac{L_f^2 e^{\beta h}}{c_1\mu_1}(t-\tau)}\|\phi\|_{C_H}^2 + \frac{e^{\beta h}}{c_1\mu_1}\int_\tau^t e^{\beta s+\frac{L_f^2 e^{\beta h}}{c_1\mu_1}(t-s)}\|g(s)\|_{W'}^2 ds,\ \forall\, t \geqslant \tau,$$

which means that

$$\|u_t\|_{C_H}^2 \leqslant e^{\beta h-\sigma(t-\tau)}\|\phi\|_{C_H}^2 + \frac{e^{\beta h}}{c_1\mu_1}\int_\tau^t e^{-\sigma(t-s)}\|g(s)\|_{W'}^2 ds,\ \forall\, t \geqslant \tau.$$

We now consider the initial time $t - r$ instead of τ, and then

$$\begin{aligned}\|u_t\|_{C_H}^2 = \|U(t, t-r)\phi\|_{C_H}^2 &\leqslant e^{\beta h-\sigma r}\|\phi\|_{C_H}^2 + e^{\beta h-\sigma t}\int_{t-r}^t e^{\sigma s}\|g(s)\|_{W'}^2 ds \\ &\leqslant e^{\beta h-\sigma r}\|\phi\|_{C_H}^2 + e^{\beta h-\sigma t}\int_{-\infty}^t e^{\sigma s}\|g(s)\|_{W'}^2 ds.\end{aligned} \tag{5.3.5}$$

We deduce from (5.3.5) that there exists $T_B > 0$, such that for all $r > T_B$ and all $t \in \mathbb{R}$, it holds

$$\|u_t\|_{C_H}^2 \leqslant 1 + e^{\beta h-\sigma t}\int_{-\infty}^t e^{\sigma s}\|g(s)\|_{W'}^2 ds := \rho_1^2(t). \tag{5.3.6}$$

The proof is finished. □

Denoting by $B_{C_H}(0, \rho_1(t))$ the closed ball in C_H of center zero and radius $\rho_1(t)$, it is easy to check that $\lim_{t\to-\infty} e^{\sigma t}\rho_1^2(t) = 0$. Hence, $B_{C_H}(0, \rho_1(t))$ is a pullback $\mathcal{D}-$absorbing set for the process U in C_H.

To our purpose, the following lemma is needed.

Lemma 5.3.5. *Assume that (H1)-(H3) hold and $g \in L^2_{loc}(\mathbb{R}; W')$. Then for T_B the absorbing time corresponding to the set $B_{C_H}(0, \rho_1(t))$ in Lemma 5.3.4, there holds*

$$\int_{t-1}^{t} a(u(s; t-r, \phi), u(s; t-r, \phi))ds \leqslant \rho_2^2(t),$$

for all $r \geqslant T_B, t \in \mathbb{R}$, where $\rho_2^2(t) := c\rho_1^2(t) + ce^{-\sigma(t-1)} \int_{-\infty}^{t} e^{\sigma s} \|g(s)\|^2_{W'} ds$.

Proof. Denote $u(\cdot) = u(\cdot; t_0 - r, \phi)$ for $\phi \in B(t_0 - r) \subset C_H$, where $t_0 \in \mathbb{R}$ is a fixed, but arbitrary, number, and let us take $r \geqslant T_B$, where we have chosen the same σ than in that proof. We can then integrate (5.3.3) over $[t-1, t]$ for $t \geqslant t_0$ and $r \geqslant T_B$,

$$2c_1\mu_1 \int_{t-1}^{t} \|u(s)\|^2_W ds \leqslant \|u(t-1)\|^2 + \frac{L_f^2}{c_1\mu_1} \int_{t-1}^{t} \|u_s\|^2_{C_H} ds + \frac{1}{c_1\mu_1} \int_{t-1}^{t} \|g(s)\|^2_{W'} ds.$$

Therefore,

$$\begin{aligned} \frac{c_1\mu_1}{c_2} \int_{t-1}^{t} a(u(s), u(s))ds &\leqslant c_1\mu_1 \int_{t-1}^{t} \|u(s)\|^2_W ds \\ &\leqslant \frac{1}{2}\|u(t-1)\|^2 + \frac{L_f^2}{2c_1\mu_1} \int_{t-1}^{t} \|u_s\|^2_{C_H} ds + \frac{1}{2c_1\mu_1} \int_{t-1}^{t} \|g(s)\|^2_{W'} ds. \end{aligned}$$

Notice that by Lemma 5.3.4, for all $r \geqslant T_B$, it follows

$$\frac{1}{2}\|u(t-1)\|^2 + \frac{L_f^2}{2c_1\mu_1} \int_{t-1}^{t} \|u_s\|^2_{C_H} ds \leqslant c\rho_1^2(t),$$

and

$$\int_{t-1}^{t} \|g(s)\|^2_{W'} ds \leqslant \int_{t-1}^{t} e^{\sigma(s-t+1)} \|g(s)\|^2_{W'} ds \leqslant e^{-\sigma(t-1)} \int_{-\infty}^{t} e^{\sigma s} \|g(s)\|^2_{W'} ds.$$

Hence, we can deduce for all $r \geqslant T_B$,

$$\int_{t-1}^{t} a(u(s), u(s))ds \leqslant c\rho_1^2(t) + ce^{-\sigma(t-1)} \int_{-\infty}^{t} e^{\sigma s} \|g(s)\|^2_{W'} ds := \rho_2^2(t).$$

The proof is completed immediately. □

Lemma 5.3.6. *(Pullback absorbing set in C_W) Assume that (H1)-(H3) hold and $g \in L^2_{loc}(\mathbb{R}; H)$. Then the weak solution u of* (4.4.19) *satisfies*

$$\|u_t\|^2_{C_W} = \|U(t, t-r)\phi\|^2_{C_W} \leqslant \rho_3^2(t),$$

for all $t \geqslant T_B + 1 + h$ and $t \in \mathbb{R}$, where $\rho_3^2(t) := \frac{1}{c_1}(a_2 + a_3)e^{a_1}$, $a_1 = c\left(1 + \rho_1^2(t)\rho_2^2(t)\right)$, $a_2 = c\left(\rho_1^2(t) + e^{-\sigma(t-1)} \int_{-\infty}^{t} e^{\sigma s} \|g(s)\|^2 ds\right)$, and $a_3 = \rho_2^2(t)$.

Proof. Denote $u(\cdot) = u(\cdot; t_0 - r, \phi)$ for $\phi(0) \in W$, where $t_0 \in \mathbb{R}$ is a fixed number, and let us take $r \geqslant T_B$. We multiply (4.4.19) by Au and obtain that for $s \geqslant t_0$,

$$\frac{1}{2}\frac{d}{ds}a(u(s), u(s)) + 2\mu_1\|Au\|^2 + \langle B(u), Au\rangle + \langle N(u), Au\rangle = (f(s, u_s), Au) + (g, Au). \tag{5.3.7}$$

On the one hand,

$$(f(s, u_s), Au) \leqslant \|f(s, u_s)\| \cdot \|Au\| \leqslant L_f\|u_s\|_{C_H} \cdot \|Au\| \leqslant \frac{\mu_1}{4}\|Au\|^2 + \frac{L_f^2}{\mu_1}\|u_s\|_{C_H}^2, \tag{5.3.8}$$

$$(g, Au) \leqslant \frac{\mu_1}{4}\|Au\|^2 + \frac{1}{\mu_1}\|g\|^2. \tag{5.3.9}$$

By Hölder's inequality and the Gagliardo-Nirenberg inequality,

$$\begin{aligned} |\langle B(u), Au\rangle| &\leqslant \|Bu\| \cdot \|Au\| \leqslant \|u\|_{L^4}\|\nabla u\|_{L^4} \cdot \|Au\| \leqslant c\|u\|_W^{\frac{1}{2}}\|u\|_{H^1}\|u\|^{\frac{1}{2}}\|Au\| \\ &\leqslant c\|Au\|^{\frac{3}{2}}\|u\|_{H^1}\|u\|^{\frac{1}{2}} \leqslant \frac{\mu_1}{4}\|Au\|^2 + \frac{1}{\mu_1}\|u\|^2\|u\|_{H^1}^4. \end{aligned} \tag{5.3.10}$$

Moreover, from the definition of $N(u)$, one can check that

$$\begin{aligned} \langle N(u), Au\rangle &= -\int_O \left\{\nabla \cdot [\mu(u) \cdot e(u)]\right\} \cdot Au\, dx \\ &\leqslant c(\|\nabla u\| + \|\Delta u\|) \cdot \|Au\| \leqslant \frac{\mu_1}{4}\|Au\|^2 + c\|\Delta u\|^2. \end{aligned} \tag{5.3.11}$$

It follows from (5.3.7)-(5.3.11) that

$$\frac{d}{ds}a(u, u) + 2\mu_1\|Au\|^2 \leqslant \frac{2L_f^2}{\mu_1}\|u_s\|_{C_H}^2 + \frac{2}{\mu_1}\|g\|^2 + c\left(1 + a(u, u)\|u\|^2\right)a(u, u). \tag{5.3.12}$$

On the other hand, from Lemma 5.3.4 we have for all $r \geqslant T_B$,

$$\int_{t-1}^{t}\left[\frac{2L_f^2}{\mu_1}\|u_s\|_{C_H}^2 + \frac{2}{\mu_1}\|g\|^2\right]ds \leqslant c\left(\rho_1^2(t) + e^{-\sigma(t-1)}\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2 ds\right).$$

In view of Lemma 5.3.5, for all $r \geqslant T_B$,

$$\int_{t-1}^{t} a(u(s), u(s))\|u(s)\|^2 ds \leqslant \int_{t-1}^{t} a(u(s), u(s))e^{-\sigma s}e^{\sigma s}\|u_s\|_{C_H}^2 ds \leqslant e^{\sigma}\rho_1^2(t)\rho_2^2(t).$$

Now, by Lemma A.1.3 and A.1.5 in Appendix, we can conclude that

$$\|u(s)\|_W^2 \leqslant \frac{1}{c_1}a(u, u) \leqslant \frac{1}{c_1}(a_2 + a_3)e^{a_1}, \text{ for all } s \geqslant t_0 + 1, \text{ provided } r \geqslant T_B,$$

where $a_1 = c\left(1+\rho_1^2(t)\rho_2^2(t)\right)$, $a_2 = c\left(\rho_1^2(t)+e^{-\sigma(t-1)}\int_{-\infty}^t e^{\sigma s}\|g(s)\|^2 ds\right)$, $a_3 = \rho_2^2(t)$, and consequently, if we take $r \geqslant T_B + h + 1$,

$$\sup_{\theta\in[-h,0]} \|u(t_0+\theta)\|_W^2 \leqslant \frac{1}{c_1}(a_2+a_3)e^{a_1} := \rho_3^2(t), \tag{5.3.13}$$

where the constants a_1, a_2, a_3 and c_1 in (5.3.13) are independent of the fixed time $t_0 \in \mathbb{R}$. Thus (5.3.13) holds true for all $t_0 \in \mathbb{R}$. Denoting from now on

$$u(\cdot) = u(\cdot; t-r, \phi),$$

we have for all $t \in \mathbb{R}$, $r \geqslant T_B + h + 1$,

$$\|u_t\|_{C_W}^2 = \|U(t,t-r)\phi\|_{C_W}^2 \leqslant \frac{1}{c_1}(a_2+a_3)e^{a_1} := \rho_3^2(t),$$

as claimed. □

Obviously, it is easy to check that $\lim_{t\to-\infty} e^{\sigma t}\rho_3^2(t) = 0$. Denote by $B_{C_W}(0, \rho_3(t))$ the closed ball in C_W of center zero and radius $\rho_3(t)$. Thus, $B_{C_W}(0, \rho_3(t))$ is a pullback $\mathcal{D}-$absorbing set for the process U in C_W.

5.3.2 pullback $\mathcal{D}-\omega-$limit compactness

From now on, we assume that

$$\lim_{m\to+\infty} \sup_{t\geqslant\tau} \int_\tau^t e^{-2\mu_1\lambda_{m+1}(t-s)}\|g(s)\|^2 ds = 0. \tag{5.3.14}$$

Remark 5.3.7. *An example for g satisfying (5.3.14) is given in [170], i.e., if g is normal in $L^2_{loc}(\mathbb{R}; H)$, then (5.3.14) holds, which is proved in Lemma 3.1 of [123].*

Now, we are in a position to prove pullback $\mathcal{D}-\omega-$limit compactness of the process U in C_W.

Lemma 5.3.8. *Suppose that (H1)-(H3) and (5.3.14) hold. Then the process $\{U(t,\tau)\}$ corresponding to problem (4.4.19)-(4.4.20) is pullback $\mathcal{D}-\omega-$limit compact on C_W.*

Proof. By the classical spectral theory of elliptic operators, there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ satisfying

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots, \quad \lambda_n \to +\infty \text{ as } n \to +\infty, \tag{5.3.15}$$

and a family of elements $\{w_n\}_{n=1}^\infty \subset D(A)$, which forms a basis of W and is orthonormal in H, such that

$$Aw_n = \lambda_n w_n, \quad \forall n \in \mathbb{N}. \tag{5.3.16}$$

Let $W_m = \text{span}\{w_1, w_2, \cdots, w_m\}$, where $m \in \mathbb{N}$ will be specified later. Then W_m is a finite-dimensional subspace of W. Denote by P_m the orthogonal projector from W into W_m and we obviously have $\|P_m\| \leqslant 1$ for each $m \in \mathbb{N}$.

Set $u = u_1 + u_2$, where $u_1 = P_m u$ and $u_2 = (I - P_m)u$. We decompose Eq. (4.4.19) as follows:

$$\frac{\partial u_2(t)}{\partial t} + 2\mu_1 A u_2 + B(u) - P_m B(u_1) + N(u) - P_m N(u_1) = f(t, u_t) - P_m f(t, u_{1t}) + (I - P_m)g \quad (5.3.17)$$

with initial data

$$u_2(\tau + t) = (I - P_m)\phi(t), \quad t \in [-h, 0], \quad (5.3.18)$$

and

$$\frac{\partial u_1(t)}{\partial t} + 2\mu_1 A u_1 + P_m B(u_1) + P_m N(u_1) = P_m f(t, u_{1t}) + P_m g \quad (5.3.19)$$

with initial data

$$u_1(\tau + t) = P_m \phi(t), \quad t \in [-h, 0]. \quad (5.3.20)$$

We divide the proof into two steps: Step 1: For every fixed $t \in \mathbb{R}$, any $B = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and any $\epsilon > 0$ we observe that for any $T \geqslant t - s$ with $s \geqslant 0$, $U(T, t - s)(\phi) = \{u_T(\cdot; t - s, \phi) : u \text{ is a strong solution to the problem (4.4.19) with } \phi \in B(t - s)\}$. We now show that condition (iii) of Proposition 5.1.5 holds.

Taking the inner product of (5.3.17) with $Au_2 = A(I - P_m)u$ in H, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}a(u_2, u_2) + 2\mu_1(Au_2, Au_2) + \langle B(u) - P_m B(u_1), Au_2\rangle + \langle N(u) - P_m N(u_1), Au_2\rangle \\ &\quad = (f(t, u_t) - P_m f(t, u_{1t}), Au_2) + ((I - P_m)g, Au_2). \end{aligned} \quad (5.3.21)$$

Since $(u_1, u_2) = 0$, from Hölder's inequality and Gagliardo-Nirenberg's inequality,

$$\frac{1}{2}\frac{d}{dt}a(u_2, u_2) + 2\mu_1\|Au_2\|^2 \leqslant |\langle B(u), Au_2\rangle| + |\langle N(u), Au_2\rangle| + |(f(t, u_t), Au_2)| + |((I - P_m)g, Au_2)|, \quad (5.3.22)$$

$$|(f(t, u_t), Au_2)| \leqslant \frac{\mu_1}{4}\|Au_2\|^2 + \frac{L_f^2}{\mu_1}\|u_t\|_{C_H}^2, \quad (5.3.23)$$

$$|((I - P_m)g, Au_2)| \leqslant \frac{\mu_1}{4}\|Au_2\|^2 + \frac{1}{\mu_1}\|g\|^2, \quad (5.3.24)$$

$$\begin{aligned} |\langle B(u), Au_2\rangle| &\leqslant \|B(u)\| \cdot \|Au_2\| \leqslant \|u\|_{L^4}\|\nabla u\|_{L^4}\|Au_2\| \leqslant c\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|u\|^{\frac{1}{4}}\|\Delta u\|^{\frac{3}{4}}\|Au_2\| \\ &\leqslant c\|u\|^{\frac{3}{4}}\|\nabla u\|^{\frac{1}{2}}\|u\|_W^{\frac{3}{4}}\|Au_2\| \leqslant \frac{\mu_1}{4}\|Au_2\|^2 + c\|u\|^{\frac{3}{2}}\|\nabla u\| \cdot \|u\|_W^{\frac{3}{2}} \\ &\leqslant \frac{\mu_1}{4}\|Au_2\|^2 + c\|u\|^{\frac{3}{2}}\|u\|_W^{\frac{5}{2}}, \end{aligned} \quad (5.3.25)$$

$$\begin{aligned} |\langle N(u), Au_2\rangle| &= \left| -\int_O \{\nabla \cdot [\mu(u)e(u)]\} \cdot Au_2 dx \right| \\ &\leqslant c(\|\nabla u\| + \|\Delta u\|) \cdot \|Au_2\| \leqslant \frac{\mu_1}{4}\|Au_2\|^2 + c\|\Delta u\|^2. \end{aligned} \quad (5.3.26)$$

From (5.3.22)-(5.3.26) it follows

$$\frac{d}{dt}a(u_2, u_2) + 2\mu_1\|Au_2\|^2 \leqslant \frac{2L_f^2}{\mu_1}\|u_t\|_{C_H}^2 + \frac{2}{\mu_1}\|g\|^2 + c\|u\|^{\frac{3}{2}}\|u\|_W^{\frac{5}{2}} + c\|u\|_W^2. \quad (5.3.27)$$

On the other hand, from (5.3.15)-(5.3.16), we infer

$$\|Au_2\|^2 \geqslant \lambda_{m+1}(Au_2, u_2) = \lambda_{m+1}a(u_2, u_2),$$

which along with (5.3.27) give

$$\frac{d}{dt}a(u_2, u_2) + 2\mu_1\lambda_{m+1}a(u_2, u_2) \leqslant \frac{2L_f^2}{\mu_1}\|u_t\|_{C_H}^2 + \frac{2}{\mu_1}\|g\|^2 + c\|u\|^{\frac{3}{2}}\|u\|_W^{\frac{5}{2}} + c\|u\|_W^2. \tag{5.3.28}$$

Applying the Gronwall lemma to (5.3.28) in the interval $[\tau, t+\theta]$,

$$\begin{aligned} a(u_2(t+\theta), u_2(t+\theta)) &\leqslant a(u_2(\tau), u_2(\tau))e^{-2\mu_1\lambda_{m+1}(t+\theta-\tau)} \\ &+ c\int_\tau^{t+\theta} e^{-2\mu_1\lambda_{m+1}(t+\theta-s)}\left(\|u_s\|_{C_H}^2 + \|g(s)\|^2 + \|u(s)\|^{\frac{3}{2}}\|u(s)\|_W^{\frac{5}{2}} + c\|u(s)\|_W^2\right)ds. \end{aligned}$$

From (5.3.14) and Lemma 3.1 in [123], we can select $m+1$ large enough such that for all $\epsilon > 0$ and $t \geqslant \tau + h$, we have $2\mu_1\lambda_{m+1} - \sigma > 0$, and

$$\sup_{\theta\in[-h,0]}\int_\tau^{t+\theta} e^{-2\mu_1\lambda_{m+1}(t+\theta-s)}\|g(s)\|^2 ds < \frac{c_1\epsilon}{2}. \tag{5.3.29}$$

Thanks to Lemma 5.3.4 and 5.3.6, we can deduce that for large enough $m+1$,

$$\begin{aligned} &\sup_{\theta\in[-h,0]} a(u_2(\tau), u_2(\tau))e^{-2\mu_1\lambda_{m+1}(t+\theta-\tau)} \\ &\leqslant c_2\sup_{\theta\in[-h,0]}\|u_{2\tau}\|_{C_W}^2 e^{-2\mu_1\lambda_{m+1}(t+\theta-\tau)} \leqslant \rho_3^2(\tau)e^{-2\mu_1\lambda_{m+1}(t+\theta-\tau)} \leqslant \frac{c_1\epsilon}{4}, \end{aligned} \tag{5.3.30}$$

and

$$c\sup_{\theta\in[-h,0]}\int_\tau^{t+\theta} e^{-2\mu_1\lambda_{m+1}(t+\theta-s)}\left(\|u_s\|_{C_H}^2 + \|u(s)\|^{\frac{3}{2}}\|u(s)\|_W^{\frac{5}{2}} + \|u(s)\|_W^2\right)ds < \frac{c_1\epsilon}{4}. \tag{5.3.31}$$

Therefore, from (5.3.29)-(5.3.31) we have

$$\|u_{2t}\|_{C_W}^2 \leqslant \frac{1}{c_1}a(u_2(t+\theta), u_2(t+\theta)) < \epsilon,$$

as claimed.

Step 2: We consider problem (5.3.19) and check condition (ii) in Proposition 5.1.5. Notice that

$$\|Au_1\|_W^2 \leqslant \lambda_m\|u_1\|_W^2 \leqslant \lambda_m^2\|u_1\|^2.$$

Without loss of generality, we assume that $\theta_1, \theta_2 \in [-h, 0]$ with $0 < \theta_1 - \theta_2 < 1$. Hence,

$$\begin{aligned} \|u_1(t+\theta_1) - u_1(t+\theta_2)\|_W &\leqslant \sqrt{\lambda_m}\|u_1(t+\theta_1) - u_1(t+\theta_2)\| = \sqrt{\lambda_m}\int_{t+\theta_1}^{t+\theta_2}\|\frac{du_1}{dt}\|dt \\ &\leqslant \sqrt{\lambda_m}\int_{t+\theta_1}^{t+\theta_2}\left(2\mu_1\sqrt{\lambda_m}\|u_1\|_W + \|B(u_1)\| + \|N(u_1)\| + \|f(s, u_{1s})\| + \|P_m g\|\right)ds. \end{aligned} \tag{5.3.32}$$

Since

$$\|B(u_1)\| \leqslant \|u_1\|_{L^4}\|\nabla u_1\|_{L^4} \leqslant c\|u_1\|^{\frac{1}{2}}\|\nabla u_1\|^{\frac{1}{2}}\|u_1\|^{\frac{1}{4}}\|\Delta u_1\|^{\frac{3}{4}} \leqslant c\|u_1\|^{\frac{3}{4}}\|\nabla u_1\|^{\frac{1}{2}}\|\Delta u_1\|^{\frac{3}{4}} \leqslant c\|\Delta u_1\|^2, \quad (5.3.33)$$

$$\|N(u_1)\| \leqslant c(\|\nabla u_1\| + \|\Delta u_1\|) \leqslant c\|\Delta u_1\|, \quad (5.3.34)$$

and

$$\|f(s, u_{1s})\| \leqslant L_f\|u_{1s}\|_{C_H}. \quad (5.3.35)$$

Thus, it follows from (5.3.32)-(5.3.35) that

$$\|u_1(t+\theta_1) - u_1(t+\theta_2)\|_W \leqslant c\int_{t+\theta_1}^{t+\theta_2}\left(\|u_1(s)\|_W + \|u_1(s)\|_W^2 + \|u_s\|_{C_H} + \|P_m g(s)\|\right)ds. \quad (5.3.36)$$

Using Lemma 5.3.5 and Young's inequality,

$$\begin{aligned} c\int_{t+\theta_1}^{t+\theta_2}\left(\|u_1(s)\|_W + \|u_1(s)\|_W^2\right)ds &\leqslant c\int_{t+\theta_1}^{t+\theta_2}\|u_1(s)\|_{C_W}^2 ds + c|\theta_2 - \theta_1| \\ &\leqslant c\rho_3^2(t)|e^{-\sigma\theta_1} - e^{-\sigma\theta_2}| + c|\theta_2 - \theta_1|. \end{aligned} \quad (5.3.37)$$

and

$$c\int_{t+\theta_1}^{t+\theta_2}\|u_s\|_{C_H}ds \leqslant c\int_{t+\theta_1}^{t+\theta_2}\|u_s\|_{C_H}^2 ds + c|\theta_2 - \theta_1| \leqslant c\rho_1^2(t)|e^{-\sigma\theta_1} - e^{-\sigma\theta_2}| + c|\theta_2 - \theta_1|. \quad (5.3.38)$$

Noting that $g \in L^2_{loc}(\mathbb{R}; H)$,

$$\begin{aligned} \int_{t+\theta_1}^{t+\theta_2}\|P_m g(s)\|ds &\leqslant c\int_{t+\theta_1}^{t+\theta_2}\left(|\theta_1 - \theta_2|^{\frac{1}{2}}\|g(s)\|^2 + \frac{1}{4|\theta_1 - \theta_2|^{\frac{1}{2}}}\right)ds \\ &\leqslant c|\theta_1 - \theta_2|^{\frac{1}{2}}\int_t^{t+\theta_1-\theta_2}\|g(s)\|^2 ds + \frac{1}{4}|\theta_1 - \theta_2|^{\frac{1}{2}} \end{aligned} \quad (5.3.39)$$

From (5.3.36)-(5.3.39), we obtain

$$\|u_1(t+\theta_1) - u_1(t+\theta_2)\|_W = \|P_m(u(t+\theta_1) - u(t+\theta_2))\|_W < \epsilon,$$

for any $\theta_1, \theta_2 \in [-h, 0]$ with $|\theta_1 - \theta_2| < \delta$, so condition (ii) in Proposition 5.1.5 is proved. By Lemma 5.3.6, we know that condition (i) in Proposition 5.1.5 holds true. Hence, we can conclude by Proposition 5.1.5 that the process $\{U(t, \tau)\}$ is pullback $\mathcal{D} - \omega$–limit compact in C_W.

This completes the proof. □

5.4 Existence of Pullback $\mathcal{D}$–attractor

We now state and prove the second main result of this Chapter.

Theorem 5.4.1. *Suppose that (H1) – (H3) and (5.3.14) hold. Then the process $\{U(t,\tau)\}$ associated to problem (5.0.1)-(5.0.4) has a unique pullback $\mathcal{D}-$attractor $\{\mathcal{A}_{\mathcal{D}}(t)\}_{t\in\mathbb{R}}$ in C_W.*

Proof. By Lemma 5.3.6, we know that $\{U(t,\tau)\}$ has a pullback $\mathcal{D}-$absorbing set in C_W, while Lemma 5.3.8 shows that $\{U(t,\tau)\}$ is pullback $\mathcal{D}-\omega-$limit compact in C_W. Consequently, the proof can be completed immediately by Proposition 5.1.6. □

Remark 5.4.2. *We have obtained the existence of pullback attractor to a 2D-dimensional incompressible non-Newtonian fluid with finite delay. But, in our opinion, there is still much work to be done in this field. For example, it will be very meaningful to obtain some results on the finite (fractal or Hausdorff) dimensionality of the pullback attractor. Also, we could consider the regularity of the attractor as well as its internal structure for which it is important to study the existence of steady-state solutions and their stability properties. Also the interesting and important 3D-dimensional case is worth being considered. We plan to analyze all these topics in some forthcoming papers.*

Chapter 6

Exponential stability of an incompressible non-Newtonian fluids with delay

Enlightened by the analysis carried out in [21], in this chapter we study the exponential stability of our incompressible non-Newtonian fluids with delay, i.e., problem (4.4.3)-(4.4.6) analyzed in Chapter 5 and which we recall here again

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot \mu(e(u)) + f(t, u_t) + g(x, t), \quad \text{in } (\tau, +\infty) \times \mathcal{O}, \tag{6.0.1}$$

$$\nabla \cdot u = 0, \quad \text{in } (\tau, +\infty) \times \mathcal{O}, \tag{6.0.2}$$

$$u(\tau + \theta, x) = \phi(\theta, x), \quad \theta \in [-h, 0], \ x \in \mathcal{O}. \tag{6.0.3}$$

System (6.0.1)-(6.0.3) is supplemented by the boundary conditions ($\tau_{ij}e = 2\mu_1 \frac{\partial e_{ij}}{\partial x_l}$, $i, j, l = 1, 2$, and $\nu = (\nu_1, \nu_2)$ denotes the exterior unit normal to $\partial\mathcal{O}$)

$$u = 0, \quad \tau_{ij}\nu_j\nu_l = 0, \ i, j, l = 1, 2, \ \text{on } \partial\mathcal{O} \times (\tau, +\infty), \tag{6.0.4}$$

Our main goal is to study the exponential stability of steady-state solutions by using the several methods developed in [21, 45, 108, 109]. More precisely, the classical Lyapunov theory is used to prove the exponential stability of solutions in the cases in which the delay terms are continuously differentiable. Fortunately, this assumption, which somehow may be restrictive, can be weaken by an appropriate application of the Razumikhin technique, where only the continuity on the operators of the model is needed but more general types of delay are allowed, since continuity is the only requirement for delay terms. A third way to study the asymptotic behavior of our problem is by constructing Lyapunov functionals. In this way, a better stability result is achieved as long as a suitable Lyapunov functionals can be constructed. The fourth alternative is based on a Gronwall-like lemma, which only needs measurability for the delay functions but still ensures exponential stability.

Nevertheless, to establish our main stability results, we first need to prove the existence and eventual uniqueness of stationary solutions, which is not a trivial task due to the difficulties in handling the nonlinear term $N(u)$. Indeed, the proof of the existence of stationary solutions is much more complicated and involved when we compare with other models, for example, Navier-Stokes. In other words, many more technicalities are required to deal with the nonlinear term $N(u)$ and to obtain the existence of stationary solution, what represents one of the main difficulties of this work. In this respect, it is worth mentioning that Guo and Lin studied in [84] the existence and uniqueness of stationary solutions of non-Newtonian viscous incompressible fluids without delay, but this reference does not contain a completed proof for the existence of such stationary solution, a gap which is solved in our current paper since it can be obtained as a particular case of the analysis we are doing in this paper by just taking $h = 0$. We would also like to point out that the existence and uniqueness of solutions, and the existence of pullback attractors of our delay model have been investigated in our previous work [118].

For better reading of this chapter, we rewrite problem (4.4.3)-(4.4.6), which is an abstract version of system (6.0.1)-(6.0.4), as :

$$\frac{\partial u}{\partial t} + 2\mu_1 Au + B(u) + N(u) = f(t, u_t) + g(x,t), \text{ in } (\tau, +\infty) \times \mathcal{O}, \tag{6.0.5}$$

$$u(\tau + \theta, x) = \phi(\theta, x), \quad \theta \in [-h, 0], \ x \in \mathcal{O}. \tag{6.0.6}$$

Here, we would like to emphasize that the delay term $f(t, u_t)$ satisfies assumption $(H1) - (H3)$, which is given in Preliminaries of Part III and $g \in L^2_{loc}(0, T; L^2(\mathcal{O}))$.

6.1 Existence and uniqueness of stationary solutions

In this section, we first recall an existence and uniqueness result concerning our model, completed with a statement about the regularity of solutions. Next we will prove a result ensuring the existence and uniqueness of stationary solutions to our problem by exploiting the techniques of Galerkin's approximations, Lax-Milgram theorem as well as Schauder fixed pointed theorem. The presence of the nonlinear term $N(\cdot)$ requires of a more involved and technical analysis compared with the Newtonian case, which implies the nontrivial character of this proof.

In the sequel, we will use the following inequalities.

$$\|Au\|^2 \geqslant \lambda_1 \|u\|_W^2, \quad \|u\|_W^2 \geqslant \|u\|^2, \tag{6.1.1}$$

where λ_1 is a positive constant.

To make this chapter as much self-contained as possible, let us recall a result ensuring existence and uniqueness of solution to our problem which was stated and proved in Chapter 5, namely, Theorem 5.2.1 (see also [118]).

Theorem 6.1.1. *(see Theorem 5.2.1 and [118]) Assume that $(H1) - (H3)$ hold. Let $g \in L^2_{loc}(\mathbb{R}, W')$ and $\phi \in C_H$. Then, for any $\tau \in \mathbb{R}$,*

(a) there exists a unique weak solution u to problem (6.0.5) satisfying

$$u \in C([\tau - h, T]; H) \cap L^\infty(\tau, T; H) \cap L^2(\tau, T; W), \quad \forall T > \tau.$$

(b) If $\phi(0) \in W$, and $g \in L^2_{loc}(\mathbb{R}, H)$, then there exists a unique strong solution u to problem (6.0.5) *satisfying*

$$u \in C([\tau - h, T]; W) \cap L^\infty(\tau, T; W) \cap L^2(\tau, T; D(A)), \quad \forall T > \tau.$$

Although our interest in this paper is to analyze the stability properties of solutions in the case of variable delays, we can consider the existence of steady-state solutions in a much more general case which is described below. Indeed, to carry out our analysis, we will assume that there exists a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that for any constant function $\xi(\cdot) : [-h, 0] \to W$, i.e. $\xi(\theta) = \xi^* \in W$ for all $\theta \in [-h, 0]$, it holds

$$f(t, \xi^*)(x) = F(\xi^*(x)), \text{ for all } t \in \mathbb{R}, x \in \mathcal{O}, \tag{6.1.2}$$

where F satisfies

$$F(0) = 0 \tag{6.1.3}$$

and that there exits $L_F > 0$ for which

$$|F(u) - F(v)|_{\mathbb{R}^2} \leqslant L_F |u - v|_{\mathbb{R}^2}, \quad \forall\, u, v \in \mathbb{R}^2. \tag{6.1.4}$$

Now, we can study existence and uniqueness of steady-state solutions to the equation

$$\frac{du}{dt} + 2\mu_1 Au + B(u) + N(u) = f(t, u_t) + g, \tag{6.1.5}$$

with $g \in W'$ independent of t. Recall that such a stationary (or steady-state) solution to (6.1.5) is a $u^* \in W$ such that

$$2\mu_1 Au^* + B(u^*) + N(u^*) = f(t, u^*) + g$$

for all $t \geqslant 0$, which can be written, according to our assumption, as

$$2\mu_1 Au^* + B(u^*) + N(u^*) = g + F(u^*). \tag{6.1.6}$$

Theorem 6.1.2. *Suppose that F satisfies* (6.1.3)-(6.1.4) *and $2\lambda_1\mu_1 > L_G$. Then,*

(a) for all $g \in W'$, there exists a stationary solution to (6.1.5)*;*

(b) if $g \in (L^2(\mathcal{O}))^2$, the stationary solutions belong to $D(A)$;

(c) there exists a constant $C_0(\mathcal{O}) > 0$, such that if $(2\lambda_1\mu_1 - L_G)^2 > C_0(\mathcal{O})\|g\|_*$, then the stationary solution to (6.1.5) *is unique.*

Proof. (a) Denote $W_m = \text{span}\{w_1, w_2, \cdots, w_m\}$, where $\{w_n\}_{n=1}^\infty \subset W \cap D(A)$ form a basis of W and are orthonormal in H. Now consider that for fixed $z^m \in W_m$, there exists u^m satisfying

$$2\mu_1(Au^m, v^m) + b(z^m, u^m, v^m) + n(z^m, u^m, v^m) = (F(z^m), v^m) + \langle g, v^m \rangle, \ \forall v^m \in W_m. \tag{6.1.7}$$

Notice that for each $z^m \in W_m$, the functional $(u, v) \mapsto 2\mu_1(Au, v) + b(z^m, u, v) + n(z^m, u, v)$ is bilinear, continuous and coercive in $W_m \times W_m$. On the other hand, the functional $v \mapsto (F(z^m), v) + \langle g, v \rangle$ is obviously linear and continuous. Thanks to Lax-Milgram theorem, for each $z^m \in W_m$, there exists a unique $u^m \in W_m$ such that (6.1.7) holds true.

Define the mapping $T_m : W_m \mapsto W_m$ given by

$$T(z^m) = u^m.$$

We will see that for each m we can apply Schauder's fixed point theorem to the map T_m (restricted to a suitable subset $K_m \subset W_m$) and ensure that we obtain $u^m \in W_m$ such that

$$2\mu_1(Au^m, v^m) + b(u^m, u^m, v^m) + n(u^m, u^m, v^m) = (F(u^m), v^m) + \langle g, v^m \rangle, \ \forall v^m \in W_m. \tag{6.1.8}$$

Indeed, setting $v^m = u^m$ in (6.1.7) yields that

$$2\mu_1(Au^m, u^m) + n(z^m, u^m, u^m) = (F(z^m), u^m) + \langle g, u^m \rangle. \tag{6.1.9}$$

By (6.1.1),

$$2\mu_1(Au^m, u^m) \geqslant 2\lambda_1\mu_1\|u^m\|_W^2,$$

and

$$\begin{aligned}(F(z^m), u^m) + \langle g, u^m \rangle &\leqslant L_F\|z^m\|\|u^m\| + \|g\|_*\|u^m\|_W \\ &\leqslant L_F\|z^m\|\|u^m\|_W + \|g\|_*\|u^m\|_W.\end{aligned}$$

Since $n(z^m, u^m, u^m) \geqslant 0$, the previous inequalities imply

$$2\lambda_1\mu_1\|u^m\|_W \leqslant L_F\|z^m\| + \|g\|_*.$$

Because $2\lambda_1\mu_1 > L_F$, one may take $k > 0$ such that $k(2\lambda_1\mu_1 - L_F) \geqslant \|g\|_*$ and, consequently, $2\lambda_1\mu_1\|u^m\|_W \leqslant L_F\|z^m\| + k(2\lambda_1\mu_1 - L_F)$.

Define $K_m = \{z \in W_m : \|z\|_W \leqslant k\}$, which is a convex set of W, and also compact since the inclusion $W \subset H_0^1(\mathcal{O})$ is compact as well. Obviously, $T_m : K_m \to K_m$ is well defined due to the choice of the constant k. Now we will use Schauder's fixed point theorem to establish the existence of stationary solutions. To do this, we still need to verify the continuity of T_m. Actually, take $z_1^m, z_2^m \in W_m$, and denote $u_i^m = T(z_i^m)$, $i = 1, 2$, the respective solutions of (6.1.7). For any $v^m \in W_m$ we deduce

$$2\mu_1(A(u_1^m - u_2^m), v^m) + b(z_1^m, u_1^m, v^m) - b(z_2^m, u_2^m, v^m) + n(z_1^m, u_1^m, v^m) - n(z_2^m, u_2^m, v^m) = (F(z_1^m) - F(z_2^m), v^m). \tag{6.1.10}$$

Particulary, put $v^m = u_1^m - u_2^m$ in (6.1.10), then by (6.1.1) once more,

$$2\lambda_1\mu_1\|u_1^m - u_2^m\|_W^2 \leqslant b(z_2^m, u_2^m, v^m) - b(z_1^m, u_1^m, v^m) + n(z_2^m, u_2^m, v^m) - n(z_1^m, u_1^m, v^m) + (F(z_1^m) - F(z_2^m), v^m). \tag{6.1.11}$$

As for the trilinear term,

$$\begin{aligned}b(z_2^m, u_2^m, v^m) - b(z_1^m, u_1^m, v^m) &= b(z_2^m - z_1^m, u_1^m, u_1^m - u_2^m) \\ &\leqslant \|z_2^m - z_1^m\|_{(L^4(\mathcal{O}))^2}\|\nabla u_1^m\|_{(L^2(\mathcal{O}))^2}\|u_2^m - u_1^m\|_{(L^4(\mathcal{O}))^2} \\ &\leqslant c_0\|z_2^m - z_1^m\|_W\|u_1^m\|_W\|u_2^m - u_1^m\|_W \\ &\leqslant c_1\|z_2^m - z_1^m\|_W\|u_2^m - u_1^m\|_W.\end{aligned} \tag{6.1.12}$$

Then we estimate the nonlinear term,

$$
\begin{aligned}
& n(z_2^m, u_2^m, v^m) - n(z_1^m, u_1^m, v^m) \\
&= \sum_{i,j=1}^{2} \int_O [\mu(z_2^m) e_{ij}(u_2^m) - \mu(z_1^m) e_{ij}(u_1^m)] e_{ij}(v^m) dx \\
&= \sum_{i,j=1}^{2} \int_O [\mu(z_2^m) - \mu(z_1^m)] e_{ij}(u_2^m) e_{ij}(v^m) dx - \sum_{i,j=1}^{2} \int_O \mu(z_1^m) |e_{ij}(u_2^m - u_1^m)|^2 dx \\
&\leqslant \sum_{i,j=1}^{2} \int_O [\mu(z_2^m) - \mu(z_1^m)] e_{ij}(u_2^m) e_{ij}(v^m) dx.
\end{aligned}
\tag{6.1.13}
$$

Using the mean value theorem to $\mu(z_2^m) - \mu(z_1^m)$, there exists a constant s with $|e(z_1^m)| < s < |e(z_2^m)|$, such that

$$
\begin{aligned}
\mu(z_2^m) - \mu(z_1^m) &= 2\mu_0(\epsilon + |e(z_2^m)|^2)^{-\frac{\alpha}{2}} - 2\mu_0(\epsilon + |e(z_1^m)|^2)^{-\frac{\alpha}{2}} \\
&= 2\mu_0(-\frac{\alpha}{2})(\epsilon + s^2)^{-\frac{\alpha+2}{2}} (|e(z_2^m)|^2 - |e(z_1^m)|^2) \\
&= -\alpha\mu_0(\epsilon + s^2)^{-\frac{\alpha+2}{2}} (|e(z_2^m)| + |e(z_1^m)|)(|e(z_2^m)| - |e(z_1^m)|).
\end{aligned}
\tag{6.1.14}
$$

Hence,

$$
\begin{aligned}
& n(z_2^m, u_2^m, v^m) - n(z_1^m, u_1^m, v^m) \\
&\leqslant 2\alpha\mu_0 \sum_{i,j=1}^{2} \int_O (\epsilon + |e(z_1^m)|^2)^{-\frac{\alpha+2}{2}} |e(z_2^m)| |e(z_2^m - z_1^m)| |e_{ij}(u_2^m)| |e_{ij}(v^m)| dx \\
&\leqslant 2\alpha\mu_0 \epsilon^{-\frac{\alpha+2}{2}} \|e(z_2^m)\|_{(L^4(O))^2} \|e(z_2^m - z_1^m)\|_{(L^4(O))^2} \|e_{ij}(u_2^m)\|_{(L^4(O))^2} \|e_{ij}(u_1^m - u_2^m)\|_{(L^4(O))^2} \\
&\leqslant 2\alpha\mu_0 \epsilon^{-\frac{\alpha+2}{2}} c_2 \|z_2^m\|_W \|z_1^m - z_2^m\|_W \|u_2^m\|_W \|u_1^m - u_2^m\|_W \\
&\leqslant 2\alpha\mu_0 \epsilon^{-\frac{\alpha+2}{2}} c_3 \|z_1^m - z_2^m\|_W \|u_1^m - u_2^m\|_W .
\end{aligned}
\tag{6.1.15}
$$

On the other hand,

$$
\begin{aligned}
(F(z_1^m) - F(z_2^m), u_1^m - u_2^m) &\leqslant L_F \|z_1^m - z_2^m\| \|u_1^m - u_2^m\| \\
&\leqslant L_F \|z_1^m - z_2^m\|_W \|u_1^m - u_2^m\|_W .
\end{aligned}
$$

By all above inequalities, we obtain

$$
2\lambda_1\mu_1 \|u_1^m - u_2^m\|_W^2 \leqslant (c_1 + 2\alpha\mu_0\epsilon^{-\frac{\alpha+2}{2}} c_3 + L_F) \|z_1^m - z_2^m\|_W \|u_1^m - u_2^m\|_W . \tag{6.1.16}
$$

The continuity of the mapping $T : z \mapsto u$ in K_m follows from (6.1.16). Therefore, by Schauder's fixed point theorem, there exists $z^m \in K_m$ such that $T(z^m) = z^m$, which means that (6.1.8) holds true for every m. Next, we pass to the limit on the solutions and conclude the existence of a stationary solution u to (6.1.5). Put $v^m = u^m$ in (6.1.8), then

$$
2\mu_1(Au^m, u^m) + n(u^m, u^m, u^m) = (F(u^m), u^m) + \langle g, u^m \rangle .
$$

Thanks to some standard computations, we find that

$$(2\lambda_1\mu_1 - L_F)\|u^m\|_W \leqslant \|g\|_*,$$

which gives a uniform bound of u^m in W (namely, $\|u^m\|_W \leqslant (2\lambda_1\mu_1 - L_F)^{-1}\|g\|_*$). We can extract a weakly convergent subsequence (relabeled the same) $u^m \rightharpoonup u$ in W, by the compact injections $((H^2(O))^2 \subset (H_0^1(O))^2 \subset (L^2(O))^2)$, we have $\|u^m - u\|_{(H_0^1(O))^2} \to 0$ and $\|u^m - u\|_{(L^2(O))^2} \to 0$.

To proceed, we fix any $w_j \in W_m$. Since we have a subsequence of equations (6.1.8) for every m greater than j, it is clear that we can pass to the limit on every term to obtain

$$2\mu_1(Au, w_j) + b(u, u, w_j) + n(u, u, w_j) = (F(u), w_j) + \langle g, w_j\rangle. \tag{6.1.17}$$

The first term is obtained by the weak convergence $u^m \rightharpoonup u$ in W. In fact,

$$2\mu_1(Au^m, w_j) = 2\mu_1(\frac{\partial e_{ij}(u^m)}{\partial x_k}, \frac{\partial e_{ij}(w_j)}{\partial x_k}) \rightharpoonup 2\mu_1(\frac{\partial e_{ij}(u)}{\partial x_k}, \frac{\partial e_{ij}(w_j)}{\partial x_k}) = 2\mu_1(Au, w_j) \text{ as } m \to \infty.$$

The trilinear term

$$\begin{aligned} b(u^m, u^m, w_j) - b(u, u, w_j) &= -b(u^m - u, w_j, u^m) - b(u, w_j, u^m - u) \\ &\leqslant c_4\|u^m - u\|_{(L^4(O))^2}\|w_j\|_{(H_0^1(O))^2}\|u^m\|_{(L^4(O))^2} + c_5\|u\|_{(L^4(O))^2}\|w_j\|_{(H_0^1(O))^2}\|u^m - u\|_{(L^4(O)^2} \\ &\leqslant c_6\|u^m - u\|^{1/2}_{(L^2(O))^2}\|u^m - u\|^{1/2}_{(H_0^1(O))^2}\|w_j\|_{(H_0^1(O))^2}\|u^m\|_{(L^2(O))^2}\|u^m\|^{1/2}_{(H_0^1(O))^2} \\ &\quad + c_7\|u\|_{(L^2(O))^2}\|u\|^{1/2}_{(H_0^1(O))^2}\|w_j\|_{(H_0^1(O))^2}\|u^m - u\|^{1/2}_{(L^2(O))^2}\|u^m - u\|^{1/2}_{(H_0^1(O))^2} \to 0. \end{aligned}$$

The nonlinear term

$$\begin{aligned} n(u^m, u^m, w_j) - n(u, u, w_j) &= \langle N(u^m) - N(u), w_j\rangle \\ &\leqslant |\langle N(u^m) - N(u), w_j\rangle| \\ &\leqslant c_8\|u^m - u\|_{(H_0^1(O))^2}\|w_j\|_{(H_0^1(O))^2} \to 0. \end{aligned}$$

And the delay term

$$(F(u^m) - F(u), w_j) \leqslant L_F\|u^m - u\|_{(L^2(O))^2}\|w_j\|_{(L^2(O))^2} \to 0.$$

Thus, (6.1.17) holds true for every $w_j \in W_m$. Since the set of linear combinations of $w_1, w_2, \cdots, w_m, \cdots$ is dense in W, we deduce that (6.1.5) is satisfied at least by $u^* = u$.

(b) Regularity. From (a) we know that

$$2\mu_1 Au + B(u) + N(u) = F(u) + g, \tag{6.1.18}$$

which must be understood in the sense of $\mathcal{D}'$. Now taking the inner product of (6.1.18) with u gives

$$2\mu_1(Au, u) + (N(u), u) = (F(u), u) + (g, u).$$

By standard calculations,

$$\|u\|_W \leqslant (2\lambda_1\mu_1 - L_F)^{-1}\|g\|. \tag{6.1.19}$$

From (6.1.18), we have

$$2\mu_1\|Au\| \leqslant \|B(u,u)\| + \|N(u)\| + \|F(u)\| + \|g\|.$$

Notice that

$$\|B(u,u)\| \leqslant c_9\|u\|\|u\|_{(H_0^1(O))^2} \leqslant c_{10}\|u\|_W^2,$$

and

$$\begin{aligned}\|N(u)\| &= 2\mu_0\Big(\int_O (\epsilon + |\nabla u|^2)^{-\alpha}|\Delta u|^2 dx\Big)^{1/2}\\ &\leqslant 2\mu_0\epsilon^{-\alpha/2}\|\Delta u\|\\ &\leqslant 2\mu_0\epsilon^{-\alpha/2}c_{11}\|u\|_W.\end{aligned}$$

Hence,

$$\begin{aligned}2\mu_1\|Au\| &\leqslant c_{10}\|u\|_W^2 + 2\mu_0\epsilon^{-\alpha/2}c_{11}\|u\|_W + L_F\|u\|_W + \|f\|\\ &\leqslant \Big(c_{10}(2\lambda_1\mu_1 - L_F)^{-1}\|f\| + (2\mu_0\epsilon^{-\alpha/2}c_{11} + 2\lambda_1\mu_1)\Big)(2\lambda_1\mu_1 - L_F)^{-1}\|f\|,\end{aligned}$$

which implies $u \in D(A)$.

(c) Uniqueness. Let u_1, u_2 be two stationary solutions of (6.1.5), and $v = u_1 - u_2$, then

$$2\mu_1(A(u_1-u_2), u_1-u_2) + b(u_1,u_1,v) - b(u_2,u_2,v) + n(u_1,u_1,v) - n(u_2,u_2,v) = (F(u_1) - F(u_2), v).$$

Note that $n(u_1,u_1,v) - n(u_2,u_2,v) \geqslant 0$, and

$$\begin{aligned}|b(u_1,u_1,v) - b(u_2,u_2,v)| &= |b(v,u_2,v)|\\ &\leqslant C_0(O)\|v\|_W^2\|u_2\|_W\\ &\leqslant C_0(O)(2\lambda_1\mu_1 - L_F)^{-1}\|g\|_*\|u_1-u_2\|_W^2,\end{aligned}$$

$$(F(u_1) - F(u_2), v) \leqslant L_F\|u_1 - u_2\|_W^2,$$

whence

$$2\lambda_1\mu_1\|u_1-u_2\|_W^2 \leqslant \Big(L_F + C_0(O)(2\lambda_1\mu_1 - L_F)^{-1}\|g\|_*\Big)\|u_1-u_2\|_W^2,$$

and therefore

$$\Big[(2\lambda_1\mu_1 - L_F) - C_0(O)(2\lambda_1\mu_1 - L_F)^{-1}\|g\|_*\Big]\|u_1-u_2\|_W^2 \leqslant 0.$$

Since $(2\lambda_1\mu_1 - L_F)^2 - C_0(O)\|g\|_* > 0$,

$$\|u_1 - u_2\|_W^2 = 0.$$

This completes the proof. □

6.2 Local asymptotic behavior

In this section, as it was mentioned previously, we will use four approaches to analyze the long time behavior of solutions. They are: the classical Lyapunov function method, the Lyapunov-Razumikhin type argument, the construction of Lyapunov functionals approach, and a Gronwall-like lemma technique.

It is worth pointing out that the first method requires a differentiability assumption on the delay term, which can be relaxed by a Razumikhin method approach but at the price of more continuity with respect to time t for the operators in the problem, in addition to the fact that we have to work with strong solutions instead of weak ones. However, a better stability result can be obtained by constructing appropriate Lyapunov functionals as long as one can find the appropriate ones, which is not a straightforward task. In the end, a Gronwall-like lemma technique is exploited for the stability analysis by only assuming measurability on the delay term. This scheme has already been used in the analysis of stability properties for the stationary solutions of 2D Navier-Stokes equations with delay (see [21] for more details).

6.2.1 Exponential stability: Lyapunov function

Now we will show that under appropriate conditions, our model has a unique stationary solution, u_∞, and every weak solution of (6.0.1) converges to u_∞ exponentially fast as $t \to +\infty$.

Theorem 6.2.1. *Suppose that $f(t, u_t) = F(u(t-\rho(t)))$ with $\rho \in C^1(\mathbb{R}^+; [0, h])$ such that $\rho'(t) \leqslant \rho_* < 1$ for all $t \geqslant 0$. Assume that there exists $l_1 = l_1(\mathcal{O}) > 0$, such that if $g \in (L^2(\mathcal{O}))^2$ and $2\lambda_1\mu_1 > L_F$ and, in addition,*

$$4\lambda_1\mu_1 > \frac{(2-\rho_*)L_F}{1-\rho_*} + \frac{l_1}{2\lambda_1\mu_1 - L_F}\|g\|. \tag{6.2.1}$$

Then, there is a unique stationary solution u_∞ of (6.1.5) and every solution of (6.0.1) converges to u_∞ exponentially as $t \to +\infty$. More precisely, there exist two positive constant C and λ, such that for all $u_0 \in H$ and $\phi \in L^2(-h, 0; W)$, the solution u of (6.0.1) with $g(t) \equiv g$ satisfies

$$\|u(t) - u_\infty\|^2 \leqslant Ce^{-\lambda t}(\|u_0 - u_\infty\|^2 + \|\phi - u_\infty\|^2_{L^2(-h,0;W)}), \tag{6.2.2}$$

for all $t \geqslant 0$.

Proof. Let u be solution of (6.0.1) for $g(t) \equiv g$, and $u_\infty \in D(A)$ be a stationary solution to (6.0.1). Denote $w(t) = u(t) - u_\infty$, since that

$$\frac{dw(t)}{dt} + 2\mu_1 Aw + B(u(t)) - B(u_\infty) + N(u(t)) - N(u_\infty) = F(u(t-\rho(t))) - F(u_\infty).$$

Fix $\lambda > 0$, by standard computations

$$\frac{d}{dt}e^{\lambda t}\|w(t)\|^2 \leqslant (\lambda - 4\lambda_1\mu_1 + L_F)e^{\lambda t}\|w(t)\|^2_W + 2e^{\lambda t}|b(w, w, u_\infty)| + L_F e^{\lambda t}\|w(t-\rho(t))\|^2. \tag{6.2.3}$$

Notice that

$$|b(w, w, u_\infty)| \leqslant l_0 \|w\|_{(L^4(O))^2} \|\nabla w\|_{(L^4(O))^2} \|u_\infty\| \leqslant l_1 \|w\|_W^2 \|u_\infty\|_W.$$

On the other hand,

$$2\mu_1 (Au_\infty, u_\infty) + (N(u_\infty), u_\infty) = (F(u_\infty), u_\infty) + (g, u_\infty),$$

which implies, arguing as in (6.1.18)-(6.1.19)

$$\|u_\infty\|_W \leqslant (2\lambda_1\mu_1 - L_F)^{-1} \|g\|,$$

and

$$\frac{d}{dt} e^{\lambda t} \|w(t)\|^2 \leqslant \left(\lambda - 4\lambda_1\mu_1 + L_F + l_1 (2\lambda_1\mu_1 - L_F)^{-1}\|g\|\right) e^{\lambda t} \|w(t)\|_W^2 + L_F e^{\lambda t} \|w(t - \rho(t))\|^2.$$

Denote by $r(t) = t - \rho(t)$. Then the function $r(\cdot)$ is strictly increasing in $[0, +\infty)$, and there exists a $\mu > 0$ such that $r^{-1}(t) \leqslant t + \mu$ for all $t \geqslant -\rho(0)$. Thus, by performing the change of variable $\eta = s - \rho(s) = r(s)$ in the integral containing the delay, we obtain

$$\begin{aligned}
e^{\lambda t}\|w(t)\|^2 &\leqslant \|w(0)\|^2 + \left(\lambda - 4\lambda_1\mu_1 + L_F + l_1(2\lambda_1\mu_1 - L_F)^{-1}\|g\|\right) \int_0^t e^{\lambda s} \|w(s)\|_W^2 ds \\
&\quad + \int_{-\rho(0)}^{t-\rho(t)} e^{\lambda r^{-1}(\eta)} \|w(\eta)\|^2 \frac{1}{r'(r^{-1}(\eta))} d\eta \\
&\leqslant \|w(0)\|^2 + \left(\lambda - 4\lambda_1\mu_1 + L_F + l_1(2\lambda_1\mu_1 - L_F)^{-1}\|g\|\right) \int_0^t e^{\lambda s} \|w(s)\|_W^2 ds \\
&\quad + \frac{e^{\lambda\mu}}{1-\rho_*} \int_0^t e^{\lambda\eta} \|w(\eta)\|^2 d\eta + \frac{e^{\lambda\mu}}{1-\rho_*} \int_{-h}^0 e^{\lambda\eta} \|w(\eta)\|^2 d\eta \\
&\leqslant \|w(0)\|^2 + \left(\lambda - 4\lambda_1\mu_1 + L_F + l_1(2\lambda_1\mu_1 - L_F)^{-1}\|f\| + \frac{e^{\lambda\mu}}{1-\rho_*}\right) \int_0^t e^{\lambda s} \|w(s)\|_W^2 ds \\
&\quad + \frac{e^{\lambda\mu}}{1-\rho_*} \int_{-h}^0 e^{\lambda\eta} \|w(\eta)\|^2 d\eta.
\end{aligned} \tag{6.2.4}$$

Since (6.2.1) is satisfied, then there exists $\lambda > 0$, small enough, such that

$$\lambda - 4\lambda_1\mu_1 + L_F + l_1(2\lambda_1\mu_1 - L_F)^{-1}\|g\| + \frac{e^{\lambda\mu}}{1-\rho_*} \geqslant 0,$$

which combines with (6.2.4), we conclude that for this $\lambda > 0$,

$$e^{\lambda t}\|w(t)\|^2 \leqslant \|w(0)\|^2 + \frac{e^{\lambda\mu}}{1-\rho_*} \int_{-h}^0 e^{\lambda\eta} \|w(\eta)\|^2 d\eta,$$

which implies (6.2.2).

The uniqueness of u_∞ follows from the fact that if $\hat{u}_\infty$ is another stationary solution of (6.1.5), then $u \equiv \hat{u}_\infty$ is a solution of (6.0.1) with $u_0 = \hat{u}_\infty$ and $\phi = \hat{u}_\infty$, then applying (6.2.2) and letting $t \to +\infty$, one deduces $\|\hat{u}_\infty - u_\infty\|^2 \leqslant 0$. The proof is therefore completed.

□

6.2.2 Exponential stability: A Lyapunov-Razumikhin approach

In the previous part we established the exponential convergence of weak solutions of problem (6.0.1) to the unique stationary solution when the variable delay term is continuously differentiable. And we will relax this condition by a Razumihkin method. Only the continuity with respect to time t of operators in this model and the solutions is required, but we need to work with strong solution rather than the weak ones.

Theorem 6.2.2. *Suppose that g satisfies $(H1)-(H3)$, and for each $\xi \in C([-h,0];W)$, the mapping $t \in [0,+\infty) \mapsto g(t,\xi) \in (L^2(\mathcal{O}))^2$ is continuous. Assume that $2\lambda_1\mu_1 > L_F$ and $g \in (L^2(\mathcal{O}))^2$, and there exists a unique stationary solution u_∞ of (6.1.5) such that for some $\lambda > 0$ it holds*

$$\begin{aligned}&-2\mu_1(A(\phi(0)-u_\infty),\phi(0)-u_\infty)-(B(\phi(0))-B(u_\infty),\phi(0)-u_\infty)-(N(\phi(0))-N(u_\infty),\phi(0)-u_\infty)\\&\quad+(f(t,\phi)-f(t,u_\infty),\phi(0)-u_\infty)<-\lambda\|\phi(0)-u_\infty\|^2,\end{aligned} \tag{6.2.5}$$

whenever $\phi \in C([-h,0];H)$ with $\phi(0) \in W$ satisfies

$$\|\phi-u_\infty\|^2_{C([-h,0];H)} \leqslant e^{\lambda h}\|\phi(0)-u_\infty\|^2.$$

Then, the strong solution $u(t;\phi)$ of (6.0.1) converges exponentially to the unique stationary solution u_∞ as follows

$$\|u(t;\phi)-u_\infty\|^2 \leqslant e^{-\lambda t}\|\phi-u_\infty\|^2_{C([-h,0];H)}. \tag{6.2.6}$$

Proof. If (6.2.6) does not hold true, then there exists an initial datum $\phi \in C([-h,0];H)$ with $\phi(0) \in W$ such that

$$\|u(t;\phi)-u_\infty\|^2 > e^{-\lambda t}\|\phi-u_\infty\|^2_{C([-h,0];H)},$$

for some values of t.

Denote

$$\sigma = \inf\left\{t : \|u(t;\phi)-u_\infty\|^2 > e^{-\lambda t}\|\phi-u_\infty\|^2_{C([-h,0];H)}\right\}.$$

Thus for $0 \leqslant t \leqslant \sigma$,

$$e^{\lambda t}\|u(t;\phi)-u_\infty\|^2 \leqslant e^{\lambda\sigma}\|u(\sigma;\phi)-u_\infty\|^2 = \|\phi-u_\infty\|^2_{C([-h,0];H)}.$$

On the other hand, for any $t \in [\sigma,\sigma+\varepsilon]$, there exists $t_k \searrow \sigma$ such that

$$e^{\lambda t_k}\|u(t_k;\phi)-u_\infty\|^2 > e^{\lambda\sigma}\|u(\sigma;\phi)-u_\infty\|^2. \tag{6.2.7}$$

However,

$$e^{\lambda(\sigma+\theta)}\|u(\sigma+\theta;\phi)-u_\infty\|^2 \leqslant e^{\lambda\sigma}\|u(\sigma;\phi)-u_\infty\|^2,\ \theta \in [-h,0],$$

from which we deduce that

$$\|u_\sigma-u_\infty\|^2_{C([-h,0];H)} \leqslant e^{\lambda h}\|u(\sigma;\phi)-u_\infty\|^2 = e^{\lambda h}\|u_\sigma(0)-u_\infty\|^2,$$

which means that

$$
\begin{aligned}
&-2\mu_1(A(u_\sigma(0)-u_\infty),u_\sigma(0)-u_\infty)-(B(u_\sigma(0))-B(u_\infty),u_\sigma(0)-u_\infty)-(N(u_\sigma(0))-N(u_\infty),u_\sigma(0)-u_\infty)\\
&\quad+(f(t,u_\sigma)-f(t,u_\infty),u_\sigma(0)-u_\infty)<-\lambda\|u_\sigma(0)-u_\infty\|^2.
\end{aligned}
$$

As $u(\cdot;\phi)\in C([-h,+\infty);W)$, by the continuity concerning the operators of the problem, there exists $\epsilon_*>0$ such that for all $\varepsilon\in(0,\epsilon_*]$ and $t\in[\sigma,\sigma+\varepsilon]$,

$$
\begin{aligned}
&-2\mu_1(A(u(t;\phi)-u_\infty),u(t;\phi)-u_\infty)-(B(u(t;\phi))-B(u_\infty),u(t;\phi)-u_\infty)-(N(u(t;\phi))-N(u_\infty),u(t;\phi)-u_\infty)\\
&\quad+(f(t,u_t(\cdot;\phi))-f(t,u_\infty),u(t;\phi)-u_\infty)\leqslant-\lambda\|u(t;\phi)-u_\infty\|^2.
\end{aligned}
$$

Thus, denoting by $w(t)=u(t;\phi)-u_\infty$,

$$
\frac{dw(t)}{dt}+2\mu_1Aw+B(u)-B(u_\infty)+N(u)-N(u_\infty)=f(t,u_t)-f(t,u_\infty).
$$

Take inner product of above equation with w,

$$
\frac{1}{2}\frac{d}{dt}\|w(t)\|^2+2\mu_1(Aw,w)+(B(u)-B(u_\infty),w)+(N(u)-N(u_\infty),w)=(f(t,u_t)-f(t,u_\infty),w)
$$

for all $t\in[\sigma,\sigma+\varepsilon]$, and integrate over $[\sigma,\sigma+\varepsilon]$,

$$
\begin{aligned}
&e^{\lambda(\sigma+\varepsilon)}\|w(\sigma+\varepsilon;\phi)\|^2-e^{\lambda\sigma}\|u(\sigma;\phi)-u_\infty\|^2\\
&=\lambda\int_\sigma^{\sigma+\varepsilon}e^{\lambda t}\|w(t;\phi)\|^2dt-4\mu_1\int_\sigma^{\sigma+\varepsilon}e^{\lambda t}(Aw,w)dt-2\int_\sigma^{\sigma+\varepsilon}e^{\lambda t}(B(u)-B(u_\infty),w)dt\\
&\quad-2\int_\sigma^{\sigma+\varepsilon}e^{\lambda t}(N(u)-N(u_\infty),w)dt+2\int_\sigma^{\sigma+\varepsilon}e^{\lambda t}(f(t,u_t)-f(t,u_\infty),w)dt\leqslant 0,
\end{aligned}
$$

which contradicts (6.2.7). □

The following corollary provides a sufficient condition which implies (6.2.5) but easier to verify.

Corollary 6.2.3. *Suppose that f satisfies $(H1)-(H3)$, and for all $\xi\in C([-h,0];W)$ the mapping $t\in[0,+\infty)\mapsto f(t,\xi)\in(L^2(\mathcal{O}))^2$ is continuous. Assume that $2\lambda_1\mu_1>L_F$ and $g\in(L^2(\mathcal{O}))^2$ so that there exists stationary solution u_∞ of (6.1.5). Assume also that there exists a constant $l_1>0$ such that if*

$$
2\lambda_1\mu_1>L_F+l_1(2\lambda_1\mu_1-L_F)^{-1}\|f\|, \tag{6.2.8}
$$

then the stationary solution u_∞ of (6.1.5) is unique, and for all $\phi\in C([-h,0];H)$ with $\phi(0)\in W$, the stationary solution to (6.0.1) corresponding to this datum, $u(t;\phi)$, satisfies

$$
\|u(t;\phi)-u_\infty\|^2\leqslant e^{-\lambda t}\|\phi-u_\infty\|^2_{C([-h,0];H)},\ \textit{for all } t\geqslant 0.
$$

Proof. Let $\phi \in C([-h,0];H)$ with $\phi(0) \in W$ be such that

$$\|\phi - u_\infty\|^2_{C([-h,0];H)} \leqslant e^{\lambda h}\|\phi(0) - u_\infty\|^2,$$

where $\lambda > 0$ is a constant to be chosen later on. Then

$$\begin{aligned}
&-2\mu_1(A(\phi(0)-u_\infty), \phi(0)-u_\infty) - \langle B(\phi(0)) - B(u_\infty), \phi(0)-u_\infty\rangle \\
&\quad - \langle N(\phi(0)) - N(u_\infty), \phi(0)-u_\infty\rangle + (f(t,\phi) - f(t,u_\infty), \phi(0)-u_\infty) \\
&\leqslant -2\lambda_1\mu_1\|\phi(0)-u_\infty\|^2_W - b(\phi(0)-u_\infty, u_\infty, \phi(0)-u_\infty) + L_F\|\phi - u_\infty\|\|\phi(0)-u_\infty\| \\
&\leqslant -2\lambda_1\mu_1\|\phi(0)-u_\infty\|^2_W + L_F e^{\lambda h}\|\phi(0)-u_\infty\|^2_W - b(\phi(0)-u_\infty, u_\infty, \phi(0)-u_\infty) \\
&\leqslant -2\lambda_1\mu_1\|\phi(0)-u_\infty\|^2_W + L_F e^{\lambda h}\|\phi(0)-u_\infty\|^2_W + l_1(2\lambda_1\mu_1 - L_F)^{-1}\|f\|\|\phi(0)-u_\infty\|^2_W \\
&= \left(-2\lambda_1\mu_1 + L_F e^{\lambda h} + l_1(2\lambda_1\mu_1 - L_F)^{-1}\|f\|\right)\|\phi(0)-u_\infty\|^2_W.
\end{aligned} \tag{6.2.9}$$

If (6.2.8) is satisfied, there exists $\lambda > 0$ such that

$$\lambda - 2\lambda_1\mu_1 + L_F e^{\lambda h} + l_1(2\lambda_1\mu_1 - L_F)^{-1}\|f\| \leqslant 0,$$

and for this fixed λ, we can obtain from (6.2.9) that

$$\begin{aligned}
&-2\mu_1(A(\phi(0)-u_\infty), \phi(0)-u_\infty) - \langle B(\phi(0)) - B(u_\infty), \phi(0)-u_\infty\rangle \\
&\quad - \langle N(\phi(0)) - N(u_\infty), \phi(0)-u_\infty\rangle + (f(t,\phi) - f(t,u_\infty), \phi(0)-u_\infty) \\
&\leqslant -\lambda\|\phi(0)-u_\infty\|^2_W \leqslant -\lambda\|\phi(0)-u_\infty\|^2.
\end{aligned}$$

□

6.2.3 Exponential stability: Constructing of Lyapunov functionals

Our interest in this subsection is to analyze the exponential stability of solutions to problem (6.0.1) by constructing some Lyapunov functionals, a method which was proposed by V. Kolmanovskii and L. Shaikhet and has been extensively used in functional differential equations, in difference equations with discrete time or with continuous time (see [109, 108] for more details and references).

Let $\tilde{A} : W \to W'$; $f_1(t,\cdot) : C([-h,0];H) \to W'$; $f_2(t,\cdot) : C([-h,0];W) \to W'$ be three families of nonlinear operators defined for $t > 0$ satisfying $\tilde{A}(t,0) = 0$, $f_1(t,0) = 0$, $f_2(t,0) = 0$.

Consider the equation

$$\begin{cases} \dfrac{du}{dt} = \tilde{A}(t,u(t)) + f_1(t,u_t) + f_2(t,u_t), & t > 0, \\ u(s) = \psi(s), & s \in [-h,0]. \end{cases} \tag{6.2.10}$$

Denote by $u(\cdot;\psi)$ the solution to (6.2.10) corresponding to the initial value ψ. Now we recall a theorem which will be crucial in our stability investigation.

Theorem 6.2.4. *(See [40]) Assume that there exists a functional $V(\cdot,\cdot) : \mathbb{R} \times C_H \longmapsto [0,+\infty)$ such that the following conditions hold for some positive numbers δ_1, δ_2 and λ:*

$$\begin{aligned}
&V(t,u_t) \geqslant \delta_1 e^{\lambda t}\|u(t)\|^2, \quad t > 0\\
&V(0,u_0) \leqslant \delta_2\|\psi\|^2_{C_H},\\
&\frac{d}{dt}V(t,u_t) \leqslant 0, \quad t \geqslant 0,
\end{aligned}$$

for any $\psi \in C_H$ such that $u(\cdot;\psi) \in C([-h,+\infty); H)$. Then the trivial solution of (6.2.10) is exponentially stable.

Notice that this theorem implies that the stability analysis of Eq. (6.2.10) can be reduced to the construction of appropriate Lyapunov functionals.

To this end, consider the following evolution equation

$$\frac{du}{dt} = \tilde{A}(t,u(t)) + F(u(t-\rho(t))), \tag{6.2.11}$$

where $\tilde{A}(t,\cdot)$, $F : W \to W'$ are proper partial differential operators (see conditions below), which is a particular case of Eq.(6.2.10). And we are going to study exponential stability to problem (6.2.11).

Theorem 6.2.5. *(See [21]) Suppose that the operators in (6.2.11) satisfy*

$$\begin{aligned}
&\langle \tilde{A}(t,u), u\rangle \leqslant -\gamma\|u\|^2_W, \gamma > 0\\
&F : W \to W', \quad \|F(u)\|_* \leqslant \beta\|u\|_W, \quad u \in W,\\
&\rho(t) \in [0,h], \ \rho'(t) \leqslant \rho_* < 1.
\end{aligned}$$

Then the trivial solution of Eq.(6.2.11) is exponentially stable provided

$$\gamma > \frac{\beta}{\sqrt{1-\rho_*}}.$$

Here we apply this method directly to our case, but only give a sketchy proof. We construct a Lyapunov functional V for our model Eq.(6.1.5) with $g(t) \equiv 0$ in the form $V = V_1 + V_2$, where $V_1(t,u_t) = e^{\lambda t}\|u(t)\|^2$, and we obtain

$$\begin{aligned}
\frac{d}{dt}V_1(t,u_t) &= \lambda e^{\lambda t}\|u(t)\|^2 + 2e^{\lambda t}(-2\mu_1 Au(t) - B(u(t)) - N(u(t)), u(t)) + 2e^{\lambda t}(F(u(t-\rho(t))), u(t))\\
&\leqslant \lambda e^{\lambda t}\|u(t)\|^2 - 4\lambda_1\mu_1 e^{\lambda t}\|u(t)\|^2_W + 2L_F e^{\lambda t}\|u(t-\rho(t))\|_W \cdot \|u(t)\|_W\\
&\leqslant (\lambda - 4\lambda_1\mu_1 + \frac{L_F}{\varepsilon})e^{\lambda t}\|u(t)\|^2_W + \varepsilon L_F e^{\lambda t}\|u(t-\rho(t))\|^2_W.
\end{aligned}$$

Set

$$V_2(t,u_t) = \frac{\varepsilon L_F}{1-\rho_*}\int_{t-\rho(t)}^{t} e^{\lambda(s+h)}\|u(s)\|^2_W ds.$$

Then

$$\begin{aligned}\frac{d}{dt}V_2(t,u_t) &= \frac{\varepsilon L_F}{1-\rho_*}e^{\lambda(t+h)}\|u(t)\|_W^2 - \frac{\varepsilon L_F}{1-\rho_*}(1-\rho')e^{\lambda(t-\rho(t)+h)}\|u(t-\rho(t))\|_W^2 \\ &\leqslant \frac{\varepsilon L_F e^{\lambda h}}{1-\rho_*}e^{\lambda t}\|u(t)\|_W^2 - \varepsilon L_F e^{\lambda t}\|u(t-\rho(t))\|_W^2.\end{aligned}$$

Hence, differentiating $V = V_1 + V_2$

$$\frac{d}{dt}V(t,u_t) = \frac{d}{dt}V_1(t,u_t) + \frac{d}{dt}V_2(t,u_t) \leqslant -(4\lambda_1\mu_1 - \frac{L_F}{\varepsilon} - \lambda - \frac{\varepsilon L_F e^{\lambda h}}{1-\rho_*})e^{\lambda t}\|u(t)\|_W^2.$$

Choosing $\varepsilon = \sqrt{1-\rho_*}$, we have

$$\begin{aligned}\frac{d}{dt}V(t,u_t) &\leqslant -(4\lambda_1\mu_1 - \frac{L_F}{\sqrt{1-\rho_*}} - \lambda - \frac{L_F e^{\lambda h}}{\sqrt{1-\rho_*}})e^{\lambda t}\|u(t)\|_W^2 \\ &= -(4\lambda_1\mu_1 - \frac{2L_F}{\sqrt{1-\rho_*}} - \lambda - \frac{L_F(e^{\lambda h}-1)}{\sqrt{1-\rho_*}})e^{\lambda t}\|u(t)\|_W^2.\end{aligned} \tag{6.2.12}$$

Denoting now

$$h(\lambda) = \lambda + \frac{L_F(e^{\lambda h}-1)}{\sqrt{1-\rho_*}}, \quad h(0) = 0,$$

there exists $\lambda > 0$ small enough such that

$$2(2\lambda_1\mu_1 - \frac{L_F}{\sqrt{1-\rho_*}}) \geqslant h(\lambda).$$

Then it follows directly from (6.2.12) that $\frac{d}{dt}V(t,u_t) \leqslant 0$, and the Lyapunov functional $V(t,u_t) = e^{\lambda t}\|u(t)\|^2 + \frac{\varepsilon L_F}{1-\rho_*}\int_{t-\rho(t)}^t e^{\lambda(s+h)}\|u(s)\|_W^2 ds$ satisfies the conditions in Theorem 6.2.4, which implies that the trivial solution of Eq.(6.1.5) is exponentially stable.

Remark 6.2.6. *(a) Here $F : W \to W'$ is a Lipschitz continuous operator with Lipschitz constant $L_F > 0$ and $F(0) = 0$. If $G : H \to H$ with Lipschitz constant L_f with $L_f \geqslant L_F$, then $F : W \to W'$ is Lipschitz , and from $2\lambda_1\mu_1 > \frac{L_f}{\sqrt{1-\rho_*}}$, we obtain that $2\lambda_1\mu_1 > \frac{L_F}{\sqrt{1-\rho_*}}$.*
(b) Although applying this method, we also need the differentiability of variable delay function, the stability result that we obtained is better than the first case, in which $2\lambda_1\mu_1 > \frac{(2-\rho_*)L_F}{2(1-\rho_*)}$ is required, but here we only need $2\lambda_1\mu_1 > \frac{L_F}{\sqrt{1-\rho_*}}$, which means we have more choice for μ_1.

6.2.4 Exponential stability: A Gronwall-like Lemma

Now we study the stability of stationary solutions to Eq.(6.1.5) via a Gronwall-like lemma. For convenience, we will consider Eq.(6.1.5) with $g(t) \equiv 0$ and $f(t,\phi) = F(\phi(-\rho(t)))$, for $\phi \in C_H$, where

$G : H \to H$ is Lipschitz continuous with Lipschitz constant $L_f > 0$ and $F(0) = 0$. For the delay term ρ we only assume that it is measurable and bounded, i.e., $\rho : [0, +\infty) \to [0, h]$. Compared with the ones required in the three previous approaches, this is the weakest assumption. But we still can prove the exponential stability of steady-state solutions.

Lemma 6.2.7. *([45]) Let $y(\cdot) : [-h, +\infty) \to [0, +\infty)$ be a function. Assume that there exist positive numbers $\gamma, \alpha_1, \alpha_2$ such that the following inequality holds:*

$$y(t) \leqslant \begin{cases} \alpha_1 e^{-\gamma t} + \alpha_2 \int_0^t e^{-\gamma(t-s)} \sup\limits_{\theta\in[-h,0]} y(s+\theta)ds, & t \geqslant 0, \\ \alpha_1 e^{-\gamma t}, & t \in [-h, 0]. \end{cases}$$

Then,

$$y(t) \leqslant \alpha_1 e^{-\mu t}, \ \textit{for all } t \geqslant -h,$$

where $\mu \in (0, \gamma)$ is given by the unique root of the equation $\frac{\alpha_2}{\gamma-\mu} e^{\mu h} = 1$ in this interval.

Theorem 6.2.8. *Suppose that $g(t) \equiv 0$ and $f(t, u_t) = F(u(t - \rho(t)))$, where $F : H \to H$ is Lipschitz constant $L_f > 0$ and satisfies $F(0) = 0$. Assume that $\rho : [0, +\infty) \mapsto [0, h]$ is a measurable function. Then the zero solution of (6.0.1) is exponentially stable provided*

$$4\lambda_1\mu_1 > L_f.$$

Proof. Let us first choose a positive constant $\lambda > 0$ such that

$$\lambda - 4\lambda_1\mu_1 + L_f > 0.$$

Notice that the weak solution $u(\cdot)$ to model (6.0.1) corresponding to the initial datum ϕ satisfies

$$\begin{aligned} e^{\lambda t}\|u(t)\|^2 &= \|\phi(0)\|^2 + \lambda \int_0^t e^{\lambda s}\|u(s)\|^2 ds - 4\mu_1 \int_0^t e^{\lambda s}(Au(s), u(s))ds \\ &\quad - 2\int_0^t e^{\lambda s}\langle N(u(s)), u(s)\rangle ds + 2\int_0^t e^{\lambda s}(F(u(s-\rho(s))), u(s))ds \\ &\leqslant \|\phi(0)\|^2 + \lambda \int_0^t e^{\lambda s}\|u(s)\|^2 ds - 4\lambda_1\mu_1 \int_0^t e^{\lambda s}\|u(s)\|^2 ds \\ &\quad + 2L_f \int_0^t e^{\lambda s}\|u(s-\rho(s))\|\|u(s)\|ds \\ &\leqslant \|\phi(0)\|^2 + \lambda \int_0^t e^{\lambda s}\|u(s)\|^2 ds - 4\lambda_1\mu_1 \int_0^t e^{\lambda s}\|u(s)\|^2 ds \\ &\quad + L_f \int_0^t e^{\lambda s}\|u(s)\|^2 ds + L_f \int_0^t e^{\lambda s}\|u(s-\rho(s))\|^2 ds \\ &\leqslant \|\phi(0)\|^2 + (\lambda - 4\lambda_1\mu_1 + L_f) \int_0^t e^{\lambda s}\|u(s)\|^2 ds + L_f \int_0^t e^{\lambda s}\|u(s-\rho(s))\|^2 ds \\ &\leqslant \|\phi\|^2_{C([-h,0];H)} + (\lambda - 4\lambda_1\mu_1 + 2L_f) \int_0^t e^{\lambda s} \sup_{\theta\in[-h,0]} \|u(s+\theta)\|^2 ds. \end{aligned}$$

Hence, from the Lemma 6.2.7, we know that the unique zero solution to Eq.(6.0.1) is exponentially stable.

□

Remark 6.2.9. *In this Chapter we have exhibited several methods to analyze the exponential stability of incompressible non-Newtonian fluids when some hereditary properties are taken into account in the forcing term of the model, and our analysis has been carried out when the delays are bounded.*

In the case of constant delays, the autonomous theory of global attractor may provide an appropriate framework to study the problem. But for more general delay terms, such as variable or distributed delays, the problem becomes non-autonomous and it is necessary to consider a non-autonomous framework for the global asymptotic behavior of the model. Several options, for instance, the theories of skew-product and uniform attractor are available, but we would like to emphasize that the theory of pullback attractors may allow more general non-autonomous terms in the models. In this respect, the existence of pullback attractor of an incompressible non-Newtonian fluid with bounded delay has been established in Chapter 5 (see also [118]).

Although many other aspects on this model have already been investigated (see [7, 12, 15, 177, 180, 181] and the references therein), there are still many interesting problems related to incompressible non-Newtonian fluids that need to be studied in future. For instance, what are the effects that some environmental noise may produce in the phenomenon, which will then become a stochastic non-Newtonian fluid. Amongst the many topics that we could analyze within the field of stochastic non-Newtonian with delay (bounded or unbounded), we could wonder about the existence and uniqueness of solutions, in particular the stationary one, their stability properties, and the existence and structure of random attractors as well. We plan to work on this problems in future.

Appendix A

Some useful lemmas

In this Appendix we recall and prove some useful results from functional analysis.

The following key lemmas have been cited in Section 2 of [15] with appropriate references:

Lemma A.1.1. *If $u \in H_0^1(\mathcal{O})$, then*

$$\|u\|_{L^4(\mathcal{O})} \leqslant 2^{1/4}\|u\|_{L^2(\mathcal{O})}^{1/2}\|\nabla u\|_{L^2(\mathcal{O})}^{1/2}.$$

Lemma A.1.2. *If $u \in W$, then there exists a positive constants c_0, depending only on $\mathcal{O}$, such that*

$$\|\nabla u\|_{L^4(\mathcal{O})} \leqslant c_0\|u\|_{H^1(\mathcal{O})}^{1/2}\|u\|_{H^2(\mathcal{O})}^{1/2}.$$

Lemma A.1.3. *There exist two positive constants c_1 and c_2 which depend only on $\mathcal{O}$ such that*

$$c_1\|u\|_W^2 \leqslant a(u,u) \leqslant c_2\|u\|_W^2, \quad \forall u \in W.$$

Lemma A.1.4. *(Gronwall's Lemma, see [[88], p. 9]) Let x, y, Ψ be real continuous functions defined in $[a,b]$, $y(t) \geqslant 0$ for $t \in [a,b]$. We suppose that on $[a,b]$ we have the inequality*

$$x(t) \leqslant \Psi(t) + \int_a^t y(s)x(s)ds. \tag{A.1.1}$$

Then

$$x(t) \leqslant \Psi(t) + \int_a^t y(s)\Psi(s)\exp\left[\int_s^t y(u)du\right]ds. \tag{A.1.2}$$

in $[a,b]$. Particularly, if Ψ is differentiable, then from (A.1.1) if follows that

$$x(t) \leqslant \Psi(a)\exp\left(\int_a^t y(u)du\right) + \int_a^t \exp\left(\int_s^t y(u)du\right)\Psi'(s)ds, \tag{A.1.3}$$

for all $t \in [a,b]$.

Lemma A.1.5. *(Uniform Gronwall's Lemma [103]) Let $t \in \mathbb{R}$ be given arbitrarily. Let g, h and y be three positive locally integrable functions on $(-\infty, t]$ such that y' is locally integrable on $(-\infty, t]$, which satisfy that*

$$\frac{dy}{dt} \leqslant gy + h \ \text{for} \ \ s \leqslant t,$$

and

$$\int_{t-1}^{t} g(s)ds \leqslant a_1, \ \int_{t-1}^{t} h(s)ds \leqslant a_2, \ \int_{t-1}^{t} y(s)ds \leqslant a_3, \ \ \forall s \leqslant t,$$

where a_1, a_2 and a_3 are positive constants. Then

$$y(t) \leqslant (a_2 + a_3)e^{a_1}, \ \ \forall s \leqslant t.$$

Lemma A.1.6. *(Young's inequality.) Let $a,\ b > 0$. Then for every p, q satisfying $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, it holds*

$$a \cdot b \leqslant \varepsilon \frac{a^p}{p} + \varepsilon^{-\frac{q}{p}} \frac{b^q}{q} \quad \text{for all } \varepsilon > 0.$$

Lemma A.1.7. *(Gagliado-Nirenberg)(see [136]) Suppose that $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Let $u \in L^q(\mathcal{O})$ and its derivatives of order m, $D^m u$ belong to $L^r(\mathcal{O})$, where $1 \leqslant q, r \leqslant \infty$. Then for the derivatives $D^j u, 0 \leqslant j < m$, there holds*

$$\|D^j u\|_{L^p} \leqslant c\|u\|^{\sigma}_{W^{m,r}} \|u\|^{1-\sigma}_{L^q}, \tag{A.1.4}$$

where

$$\frac{1}{p} = \frac{j}{n} + \sigma(\frac{1}{r} - \frac{m}{n}) + (1-\sigma)\frac{1}{q},$$

for all σ in the interval

$$\frac{j}{m} \leqslant \sigma < 1.$$

Here the constant c depends only on n, m, j, q, r and σ.

The following lemmas will be also useful in this thesis, and readers are referred to [77] for details.

Lemma A.1.8. *Let ϕ be a non-negative, absolutely continuous function on $\mathbb{R}_\tau$, $\tau \in \mathbb{R}$, which satisfies for some $\epsilon > 0$ and $0 \leqslant \sigma < 1$ the differential inequality*

$$\frac{d}{dt}\phi + \epsilon\phi \leqslant \Lambda + m_1(t)\phi(t)^{\sigma} + m_2(t) \quad t \in \mathbb{R}_\tau,$$

where $\Lambda \geqslant 0$, and m_1 and m_2 are non-negative locally summable functions on $\mathbb{R}_\tau$. Then

$$\phi(t) \leqslant \frac{1}{1-\sigma}[\phi(\tau)e^{-\epsilon(t-\tau)} + \frac{\Lambda}{\epsilon}] + [\int_\tau^t m_1(y)e^{-\epsilon(1-\sigma)(t-y)}dy]^{\frac{1}{1-\sigma}} + \frac{1}{1-\sigma}\int_\tau^t m_2(y)e^{-\epsilon(t-y)}dy,$$

for any $t \in \mathbb{R}_\tau$.

Lemma A.1.9. *let $m \in \mathcal{T}_b^p(\mathbb{R}, X)$ for some $\tau \in \mathbb{R}$. Then, for every $\epsilon > 0$,*

$$\int_\tau^t m(y) e^{-\epsilon(t-\tau)} dy \leqslant c(\epsilon) \|m\|_{\mathcal{T}_b^p(\mathbb{R},X)},$$

Lemma A.1.10. *(See [138]). Let $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ be a non-negative function, such that if $\mu(s_0) = 0$ for some $s_0 \in \mathbb{R}^+$, then $\mu(s) = 0$ for every $s > s_0$. Let B_0, B, B_1 be three Banach spaces, where B_0, B_1 are reflexive, such that*

$$B_0 \hookrightarrow B \hookrightarrow B_1,$$

where the first injection is compact. Let $C \subset L^2_\mu(\mathbb{R}^+; B)$ satisfy

(i) *C is bounded in $L^2_\mu(\mathbb{R}^+; B_0) \cap H^1_\mu(\mathbb{R}^+; B_1)$,*

(ii) $\sup_{\eta \in C} \|\eta(s)\|_B^2 \leqslant h(s), \ \forall s \in \mathbb{R}^+$, *for some $h(s) \in L^1_\mu(\mathbb{R}^+)$.*

Then C is relatively compact in $L^2_\mu(\mathbb{R}^+; B)$.

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