USING FUZZY NUMBERS AND OWA OPERATORS IN THE WEIGHTED AVERAGE AND ITS APPLICATION IN DECISION MAKING

José M. Merigó Lindahl, jmerigo@ub.edu, University of Barcelona

Montserrat Casanovas Ramón, mcasanovas@ub.edu, University of Barcelona

RESUMEN

Se presenta un nuevo método para tratar situaciones de incertidumbre en los que se utiliza el operador OWAWA (media ponderada – media ponderada ordenada). A este operador se le denomina operador OWAWA borroso (FOWAWA). Su principal ventaja se encuentra en la posibilidad de representar la información incierta del problema mediante el uso de números borrosos los cuales permiten una mejor representación de la información ya que consideran el mínimo y el máximo resultado posible y la posibilidad de ocurrencia de los valores internos. Se estudian diferentes propiedades y casos particulares de este nuevo modelo. También se analiza la aplicabilidad de este operador y se desarrolla un ejemplo numérico sobre toma de decisiones en la selección de políticas fiscales.

Palabras clave: Toma de decisiones; Operador OWA; Media ponderada; Números borrosos; Selección de políticas fiscales.

ABSTRACT

We present a new approach for dealing with an uncertain environment when using the ordered weighted averaging – weighted averaging (OWAWA) operator. We call it the fuzzy OWAWA (FOWAWA) operator. The main advantage of this new aggregation operator is that it is able to represent the uncertain information with fuzzy numbers. Thus, we are able to give more complete information because we can consider the maximum and the minimum of the problem and the internal information between these two results. We study different properties and different particular cases of this approach. We also analyze the applicability of the new model and we develop a numerical example in a decision making problem about selection of fiscal policies.

Keywords: Decision making; OWA operator; Weighted average; Fuzzy numbers; Selection of fiscal policies.

1. INTRODUCTION

The weighted average (WA) is one of the most common aggregation operators found in the literature. It can be used in a wide range of different problems including statistics, economics, engineering, etc. Another interesting aggregation operator is the ordered weighted averaging (OWA) operator (Yager, 1988). The OWA operator provides a parameterized family of aggregation operators that range from the maximum to the minimum. For further reading on the OWA operator and some of its applications, refer to (Beliakov et al., 2007; Calvo et al., 2002; Casanovas and Merigó, 2007; Merigó, 2008; Merigó and Casanovas, 2007; 2008a; 2008b; Merigó and Gil-Lafuente, 2009; Sadiq and Tesfamariam, 2008; Torra, 1997; Torra and Narukawa, 2007; Xu, 2006; 2007; Xu and Da, 2003; Xu and Yager, 2008; Yager, 1993; 1996; 2007; Yager and Filev, 1994; Yager and Kacprzyk, 1997).

Usually, when using these approaches it is considered that the available information are exact numbers. However, this may not be the real situation found in the specific problem considered. Sometimes, the available information is vague or imprecise and it is not possible to analyze it with exact numbers. Then, it is necessary to use another approach that is able to assess the uncertainty such as the use of fuzzy numbers (FNs). In order to develop the fuzzy approach, we will follow the ideas of (Chang and Zadeh, 1972; Dubois and Prade, 1980; Kaufmann and Gil-Aluja, 1987; Kaufmann and Gupta, 1985). Note that in the literature, there are a lot of studies dealing with uncertain information represented in the form of FNs in different problems such as (Casanovas and Merigó, 2007; Chen and Chen, 2003; 2008; Kaufmann and Gil-Aluja, 1987; Merigó, 2008; Merigó and Casanovas, 2007; 2008a; Sadiq and Tesfamariam, 2008; Xu, 2007; Xu and Yager, 2008).

Recently, some authors have tried to unify the WA and the OWA in the same formulation. It is worth noting the work developed by Torra (1997) with the introduction of the weighted OWA (WOWA) operator and the work of Xu and Da (2003) about the hybrid averaging (HA) operator. Both models arrived to a unification between the OWA and the WA because both concepts were included in the formulation as particular cases. However, as it has been studied in (Merigó, 2008), these models seem to be a partial unification but not a real one because they can unify them but they cannot consider how relevant these concepts are in the specific problem considered. For example, in some problems we may prefer to give more importance to the OWA operator because we believe that it is more relevant and vice versa. This problem is solved with the ordered weighted averaging – weighted averaging (OWAWA) operator (Merigó, 2008).

In this paper, we present a new approach to unify the OWA operator with the WA when the available information is uncertain and can be assessed with FNs. We call it the fuzzy ordered weighted averaging – weighted averaging (FOWAWA) operator. The main advantage of this approach is that it unifies the OWA and the WA taking into account the degree of importance of each case in the formulation and considering that the information is given with FNs. Thus, we are able to consider situations where we give more or less importance to the FOWA and the FWA depending on our interests and the problem analysed. We study different properties of the FOWAWA operator and different particular cases. Moreover, we are also able to unify the fuzzy arithmetic mean (or simple fuzzy average) with the FOWA operator when the weights of the FWA are equal. We study other families such as the step-FOWAWA, the median-FOWAWA, the olympic-FOWAWA, the S-FOWAWA, the centered-FOWAWA, etc.

We also analyze the applicability of the new approach and we see that it is possible to develop an astonishingly wide range of applications. For example, we can apply it in a lot of problems about statistics, economics, engineering, decision theory, etc. In this paper, we focus on a decision making problem about selection of fiscal policies. The main advantage of the FOWAWA in these problems is that it is possible to consider the subjective probability (or the degree of importance) and the attitudinal character of the decision maker at the same time.

This paper is organized as follows. In Section 2 we revise the FNs, the FWA and the FOWA operator. In Section 3 we present the new approach. Section 4 analyzes different families of FOWAWA operators. In Section 5 we analyze the applicability of the new approach in a decision making problem. Section 6 presents a numerical example and in Section 7 we summarize the main conclusions of the paper.

2. PRELIMINARIES

In this Section, we briefly describe some basic concepts to be used throughout the paper such as the FNs, the OWAWA operator and the FOWA operator.

2.1. FUZZY NUMBERS

A FN A is defined as a fuzzy subset of a universe of discourse that is both convex (i.e., $\mu_A(\lambda x_1 + (1 - \lambda)x_2 \geq \min(\mu_A(x_1), \mu_A(x_2))$; for $\forall x_1, x_2 \in R$ and $\lambda \in [0, 1]$) and normal (i.e., $\sup_{x \in R} \mu_A(x) = 1$).

Note that the FN may be considered as a generalization of the interval number although it is not strictly the same because the interval numbers may have different meanings. In the literature, we find a wide range of FNs (Casanovas and Merigó, 2007; Chen and Chen, 2003; 2008; Kaufmann and Gil-Aluja, 1987; Merigó, 2008; Merigó and Casanovas, 2007; 2008a; Sadiq and Tesfamariam, 2008; Xu, 2007; Xu and Yager, 2008) such as the TFN, the TpFN, the IVFN, the IFN, the GFN, the IVGFN, etc.

For example, a TpFN A of a universe of discourse R can be characterized by a trapezoidal membership function (α–cut representation) $A = (\underline{a}, \overline{a})$ such that

$$\begin{aligned} \underline{a}(\alpha) &= a_1 + \alpha(a_2 - a_1), \\ \overline{a}(\alpha) &= a_4 - \alpha(a_4 - a_3). \end{aligned} \tag{1}$$

where $\alpha \in [0, 1]$ and parameterized by (a_1, a_2, a_3, a_4) where $a_1 \leq a_2 \leq a_3 \leq a_4$, are real values. Note that if $a_1 = a_2 = a_3 = a_4$, then, the FN is a crisp value and if $a_2 = a_3$, the FN is represented by a TFN. Note that the TFN can be parameterized by (a_1, a_2, a_4).

In the following, we are going to review some basic FN arithmetic operations as follows. Let A and B be two TFNs, where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Then:

1) $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

2) $A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

3) $A \times k = (k \times a_1, k \times a_2, k \times a_3)$; for $k > 0$.

Note that other operations could be studied but in this paper we will focus on these ones. For more complete overviews about FNs, see for example (Chang and Zadeh, 1972; Dubois and Prade, 1980; Kaufmann and Gil-Aluja, 1987; Kaufmann and Gupta, 1985).

2.2. THE OWAWA OPERATOR

The ordered weighted averaging – weighted averaging (OWAWA) operator is an aggregation operator that unifies the WA and the OWA operator in the same formulation (Merigó, 2008). It can be defined as follows.

Definition 1. An OWAWA operator of dimension n is a mapping OWAWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWAWA\ (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{2}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1 - \beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

Note that it is possible to distinguish between descending (DOWAWA) and ascending (AOWAWA) orders. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DOWAWA and w^*_{n-j+1} the jth weight of the AOWAWA operator. By choosing a different manifestation in the weighting vector, we are able to obtain a wide range of particular types of OWAWA operators (Merigó, 2008). Especially, when $\beta = 0$, we get the WA, and if $\beta = 1$, we get the OWA operator.

2.3. FUZZY OWA OPERATOR

The FOWA operator is an extension of the OWA operator that uses uncertain information in the arguments represented in the form of FNs. The reason for using this aggregation operator is that sometimes the available information cannot be assessed with exact numbers and it is necessary to use other techniques such as FNs. The FOWA operator provides a parameterized family of aggregation operators that include the fuzzy maximum, the fuzzy minimum and the fuzzy average criteria, among others.

Definition 2. Let Ψ be the set of FNs. A FOWA operator of dimension n is a mapping $FOWA$: $\Psi^n \rightarrow \Psi$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$FOWA\ (\tilde{a}_1, \tilde{a}_2\ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j b_j \tag{3}$$

where b_j is the jth largest of the $\tilde{a}_i$, and the $\tilde{a}_i$ are FNs.

Note that sometimes, it is not clear how to reorder the arguments. Then, it is necessary to establish a criterion for comparing FNs. For simplicity, we recommend the following method. Select the FN with the highest value in its highest membership level, usually, when $\alpha = 1$. Note that if the membership level $\alpha = 1$ is an interval, then, we will calculate the average of the interval. If there is still a tie, then, we recommend to use an average or a weighted average of the FN according to the interests of the decision maker.

From a generalized perspective of the reordering step, it is possible to distinguish between the descending FOWA (DFOWA) operator and the ascending FOWA (AFOWA) operator. By using a different manifestation in the weighting vector, we can obtain different families of FOWA operators such as the step-FOWA operator, the olympic-FOWA operator, the S-FOWA, the nonmonotonic-FOWA, the centered-FOWA operator, etc (Beliakov et al., 2007; Calvo et al., 2002; Casanovas and Merigó, 2007; Merigó, 2008; Merigó and Casanovas, 2007; 2008a; 2008b; Merigó and Gil-Lafuente, 2009; Xu and Da, 2003; Yager, 1993; 1996; 2007; Yager and Filev, 1994; Yager and Kacprzyk, 1997).

3. FUZZY OWAWA OPERATOR

The fuzzy ordered weighted averaging – weighted averaging (FOWAWA) operator is a new model that unifies the FOWA operator and the FWA in the same formulation. Therefore, both concepts can be seen as a particular case of a more general one that includes the WA, the OWA operator and uncertain information represented in the form of FNs. This approach seems to be complete, at least as an initial real unification between FOWA operators and FWAs. It can also be seen as a unification between decision making problems under uncertainty (with FOWA operators) and under risk (with probabilities).

Note that some previous models already considered the possibility of using OWA operators and WAs in the same formulation. The main models are the weighted OWA (WOWA) operator (Torra, 1997; Torra and Narukawa, 2007) and the hybrid averaging (HA) operator (Xu and Da, 2003). For the case with FNs, we would get the fuzzy WOWA (FWOWA) and the fuzzy HA (FHA) operator. Although they seem to be a good approach, they are not so complete than the FOWAWA because it can unify FOWAs and FWAs in the same model but they can not take in consideration the degree of importance of each case in the aggregation process. Moreover, in some particular cases we also find inconsistencies (Merigó, 2008). In the following, we are going to analyze the FOWAWA operator. It can be defined as follows.

Definition 3. A FOWAWA operator of dimension n is a mapping FOWAWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\text{FOWAWA } (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{4}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (FWA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (FWA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

Note that it is also possible to formulate the FOWAWA operator separating the part that strictly affects the FOWA operator and the part that affects the FWAs. This representation is useful to see both models in the same formulation but it does not seem to be as an unique equation that unifies both models.

Definition 4. A FOWAWA operator is a mapping FOWAWA: $R^n \rightarrow R$ of dimension n, if it has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$ and a weighting vector V that affects the FWA, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$\text{FOWAWA } (a_1, \ldots, a_n) = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_i a_i \tag{5}$$

where b_j is the jth largest of the arguments a_i and $\beta \in [0, 1]$.

Note that sometimes, it is not clear how to reorder the arguments. Then, it is necessary to establish a criterion for comparing FNs. For simplicity, we recommend the following method. Select the FN with the highest value in its highest membership level, usually, when $\alpha = 1$. Note that if the membership level $\alpha = 1$ is an interval, then, we will calculate the average of the interval. If there is still a tie, then, we recommend to use an average or a weighted average of the FN according to the interests of the decision maker.

From a generalized perspective of the reordering step, it is possible to distinguish between the descending FOWAWA (DFOWAWA) and the ascending FOWAWA (AFOWAWA) operator by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DFOWAWA and w^*_{n-j+1} the jth weight of the AFOWAWA operator.

Note that different types of FNs can be used in the aggregation process according to the interests or necessities of the decision maker. For example, we could mention the following ones (Merigo, 2008):

- Triangular FNs (Interval-valued, generalized, etc.).
- Trapezoidal FNs (Interval-valued, generalized, etc.).
- L-R FNs (Interval-valued, generalized, intuitionistic, Type 2 and n, etc.).
- Interval-valued FNs (triplets, quadruplets, etc.).
- Generalized FNs (simple, interval-valued, intuitionistic, Type 2 and n, etc.).
- Intuitionistic FNs (simple, interval-valued, generalized, etc.).

- Type 2 and n FNs (simple, generalized, etc.).
- Etc.

If B is a vector corresponding to the ordered arguments b_j, we shall call this the ordered argument vector and W^T is the transpose of the weighting vector, then, the FOWAWA operator can be expressed as:

$$\text{FOWAWA}(a_1, \ldots, a_n) = W^T B \tag{6}$$

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the FOWAWA operator can be expressed as:

$$\text{FOWAWA}(a_1, \ldots, a_n) = \frac{1}{W} \sum_{j=1}^{n} \hat{v}_j b_j \tag{7}$$

The FOWAWA is monotonic, commutative, bounded and idempotent. It is monotonic because if $a_i \geq u_i$, for all a_i, then, FOWAWA($a_1, a_2, \ldots, a_n$) $\geq$ FOWAWA($u_1, u_2 \ldots, u_n$). It is commutative because any permutation of the arguments has the same evaluation. That is, FOWAWA($a_1, a_2, \ldots, a_n$) = FOWAWA($u_1, u_2, \ldots, u_n$), where ($u_1, u_2, \ldots, u_n$) is any permutation of the arguments ($a_1, a_2, \ldots, a_n$). It is bounded because the FOWAWA aggregation is delimitated by the fuzzy minimum and the fuzzy maximum. That is, Min$\{a_i\}$ $\leq$ FOWAWA($a_1, a_2, \ldots, a_n$) $\leq$ Max$\{a_i\}$. It is idempotent because if $a_i = a$, for all a_i, then, FOWAWA($a_1, a_2, \ldots, a_n$) = a.

Another interesting issue to analyze are the measures for characterizing the weighting vector W. Following a similar methodology as it has been developed for the FOWA operator (Merigó, 2008; Yager, 1988) we can formulate the attitudinal character, the entropy of dispersion, the divergence of W and the balance operator. Note that these measures affect the weighting vector W but not the WAs because they are given as some kind of objective information.

4. FAMILIES OF FOWAWA OPERATORS

First of all we are going to consider the two main cases of the FOWAWA operator that are found by analyzing the coefficient β. Basically, if $\beta = 0$, then, we get the FWA and if $\beta = 1$, the FOWA operator. Note that if $v_i = 1/n$, for all i, then, we get the unification between the fuzzy arithmetic mean (or simple fuzzy average) and the FOWA operator. Another interesting result is when the FNs are reduced to the usual exact numbers and to the interval numbers (Moore, 1966).

Theorem 1. If the FNs are reduced to the usual exact numbers, then, the FOWAWA operator becomes the OWAWA operator (Merigó, 2008).

Proof. Assume a TpFN = (a_1, a_2, a_3, a_4). If $a_1 = a_2 = a_3 = a_4$, then (a_1, a_2, a_3, a_4) = a, thus, we get the OWAWA operator.

Remark 1. In a similar way, we could develop the same proof for all the other types of FNs available in the literature.

Theorem 2. If the FNs are reduced to the usual exact numbers, then, the FOWAWA operator becomes the uncertain OWAWA (UOWAWA) operator (Merigó, 2008).

Proof. Assume a TpFN = (a_1, a_2, a_3, a_4). If we only consider the points (a_1, a_2, a_3, a_4), then, the FN becomes an interval number (a quadruplet). Therefore, the FOWAWA operator becomes the UOWAWA operator.

Remark 2. In a similar way, we could develop the same proof for all the other types of FNs.

Remark 3. Note that similar analysis could be developed for considering situations when the FNs are representing linguistic variables, etc.

By choosing a different manifestation of the weighting vector in the FOWAWA operator, we are able to obtain different types of aggregation operators. For example, we can obtain the partial fuzzy maximum, the partial fuzzy minimum, the partial fuzzy average and the partial fuzzy weighted average.

Remark 4. The partial fuzzy maximum is found when $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$. The partial fuzzy minimum is formed when $w_n = 1$ and $w_j = 0$ for all $j \neq n$. More generally, the step-FOWAWA is formed when $w_k = 1$ and $w_j = 0$ for all $j \neq k$. Note that if $k = 1$, the step-FOWAWA is transformed to the partial fuzzy maximum, and if $k = n$, the step-FOWAWA becomes the partial fuzzy minimum operator.

Remark 5. The partial fuzzy average is obtained when $w_j = 1/n$ for all j, and the partial fuzzy weighted average is obtained when the ordered position of i is the same as the ordered position of j.

Remark 6. For the median-FOWAWA, if n is odd we assign $w_{(n+1)/2} = 1$ and $w_{j*} = 0$ for all others. If n is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_{j*} = 0$ for all others. For the weighted median-FOWAWA, we select the argument b_k that has the kth largest argument such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.

Remark 7. Another interesting family is the S-FOWAWA operator. It can be subdivided into three classes: the “or-like,” the “and-like” and the generalized S-FOWAWA operators. The generalized S-FOWAWA operator is obtained if $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$, where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-FOWAWA operator becomes the “and-like” S-FOWAWA operator, and if $\beta = 0$, it becomes the “or-like” S-FOWAWA operator.

Remark 8. The olympic-FOWAWA is generated when $w_1 = w_n = 0$, and for all others $w_{j*} = 1/(n - 2)$. Note that it is possible to develop a general form of the olympic-FOWAWA by considering that $w_j = 0$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$, and for all others $w_{j*} = 1/(n - 2k)$, where $k < n/2$. Note that if $k = 1$, then this general form becomes the usual olympic-FOWAWA. If $k = (n - 1)/2$, then this general form becomes the median-FOWAWA aggregation. That is, if n is odd, we assign $w_{(n + 1) / 2} = 1$, and $w_{j*} = 0$ for all other values. If n is even, we assign, for example, $w_{n/2} = w_{(n / 2) + 1} = 0.5$ and $w_{j*} = 0$ for all other values.

Remark 9. Note that it is also possible to develop the contrary case, that is, the general olympic-FOWAWA operator. In this case, $w_j = (1/2k)$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$, and $w_j = 0$, for all other values, where $k < n/2$. Note that if $k = 1$, then we obtain the contrary case for the median-FOWAWA.

Remark 10. Another family of aggregation operator that could be used is the centered-FOWAWA operator. We can define a FOWAWA operator as a centered aggregation operator if it is symmetric, strongly decaying and inclusive. Note that these properties have to be accomplished for the weighting vector W of the FOWAWA operator but not necessarily for the weighting vector V of the WA. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$, and it is also possible to remove the third condition. We shall refer to it as a non-inclusive centered-FOWAWA operator.

Remark 11. Another type of aggregation that could be used is the E-Z FOWAWA weights. In this case, we should distinguish between two classes. In the first class, we assign $w_{j*} = (1/q)$ for $j* = 1$ to q and $w_{j*} = 0$ for $j* > q$, and in the second class, we assign $w_{j*} = 0$ for $j* = 1$ to $n - q$ and $w_{j*} = (1/q)$ for $j* = n - q + 1$ to n.

Remark 12. A further interesting type is the non-monotonic-FOWAWA operator. It is obtained when at least one of the weights w_j is lower than 0 and $\sum_{j=1}^{n} w_j = 1$. Note that a key aspect of this operator is that it does not always achieve monotonicity. Therefore, strictly speaking, this particular case is not an FOWAWA operator. However, we can see it as a particular family of operators that is not monotonic but nevertheless resembles an FOWAWA operator.

Remark 13. Note that other families of FOWAWA operators could be used following the recent literature about different methods for obtaining the FOWAWA weights such as (Beliakov et al., 2007; Calvo et al., 2002; Casanovas and Merigó, 2007; Merigó, 2008; Merigó and Casanovas, 2007; 2008a; 2008b; Merigó and Gil-Lafuente, 2009; Xu and Da, 2003; Yager, 1993; 1996; 2007; Yager and Filev, 1994; Yager and Kacprzyk, 1997).

5. DECISION MAKING WITH THE FOWAWA OPERATOR

The FOWAWA operator is applicable in a wide range of situations where it is possible to use the WA (or FWA) and the OWA (or FOWA) operator. Therefore, we see that the applicability is incredibly broad because all the previous models, theories, etc., that uses the FWA can be extended by using the FOWAWA operator. The reason is that most of the problems with WAs deal with uncertainty. Usually, in most of the problems it is assumed a neutral attitudinal character against the FWA but we are still under uncertainty. Thus, sometimes we may prefer to be more or less optimistic against this information. Moreover, by using the FOWA in the FWA, we can under or overestimate the results of a specific problem. Note also that the FWA can be seen as a subjective probability.

Summarizing some of the main fields where it is possible to develop a lot of applications with the FOWAWA operator, we can mention:

- Statistics.
- Mathematics
- Economics
- Business
- Decision theory
- Engineering
- Physics
- Etc.

Note that we can use the FOWAWA operator in practically all the previous studies that have used the FWA or the FOWA in the analysis. In this paper, we will consider a decision making application in the selection of fiscal policies. The use of the FOWAWA operator can be useful in a lot of situations, but the main reason for use it is when we want to consider the subjective probability (or degree of importance) of each state of nature (or characteristic) and the attitudinal character of the decision maker in the same problem.

The process to follow in the selection of fiscal policies with the FOWAWA operator is similar to the process developed in (Gil-Aluja, 1998; Kaufmann and Gil-Aluja, 1987; Merigó, 2008), with the difference that now we are considering a political problem. The 5 steps of the decision process can be summarized as follows:

Step 1: Analysis and determination of the significant characteristics of the available policy for the company. Theoretically, it is represented as: $C = \{C_1, C_2,\ldots, C_i,\ldots, C_n\}$, where C_i is the ith characteristic of the fiscal policy and we suppose a limited number n of characteristics.

Step 2: Fixation of the ideal levels of each characteristic in order to form the ideal policy.

Table 1: Ideal fiscal policy

	C_1	C_2	…	C_i	…	C_n
$P =$	μ_1	μ_2	…	μ_i	…	μ_n

where P is the ideal policy expressed by a fuzzy subset, C_i is the ith characteristic to consider and $\mu_i \in [0, 1]$; $i = 1, 2, \ldots, n$, is a number between 0 and 1 for the ith characteristic.

Step 3: Fixation of the real level of each characteristic for all the fiscal policies considered.

Table 2: Available alternatives

	C_1	C_2	…	C_i	…	C_n
$P_k =$	$\mu_1^{(k)}$	$\mu_2^{(k)}$	…	$\mu_i^{(k)}$	…	$\mu_n^{(k)}$

with $k = 1, 2, \ldots, m$; where P_k is the kth policy expressed by a fuzzy subset, C_i is the ith characteristic to consider and $\mu_i^{(k)} \in [0, 1]$; $i = 1, \ldots, n$, is a number between 0 and 1 for the ith characteristic of the kth policy.

Step 4: Comparison between the ideal policy and the different alternatives considered using the FOWAWA operator. In this step, the objective is to express numerically the removal between the ideal fiscal policy and the different alternatives considered. Note that it is possible to consider a wide range of FOWAWA operators such as those described in Section 3 and 4.

Step 5: Adoption of decisions according to the results found in the previous steps. Finally, we should take the decision about which policy select. Obviously, our decision is to select the policy with the best results according to the type of FOWAWA operator used in the analysis.

6. NUMERICAL EXAMPLE

In the following, we present a numerical example of the new approach in a decision making problem about selection of fiscal policies. Note that similar problems could be developed in the selection of other policies such as monetary policy, commercial policy, etc. We analyze an economic problem about the fiscal policy of a country. Assume the government of a country has to decide on the type of fiscal policy to use the next year. They consider five alternatives:

- A_1 = Develop a strong expansive fiscal policy.
- A_2 = Develop an expansive fiscal policy.
- A_3 = Do not develop any change in the fiscal policy.
- A_4 = Develop a contractive fiscal policy.
- A_5 = Develop a strong contractive fiscal policy.

In order to evaluate these policies, the government has brought together a group of experts. This group considers that the key factor is the economic situation of the world economy for the next period. They consider 5 possible states of nature that could happen in the future:

- S_1 = Very bad economic situation.
- S_2 = Bad economic situation.

- S_3 = Regular economic situation.
- S_4 = Good economic situation.
- S_5 = Very good economic situation.

The results of the available policies, depending on the state of nature S_i and the alternative A_k that the decision maker chooses, are shown in Table 1.

Table 1: Payoff matrix.

	S_1	S_2	S_3	S_4	S_5	S_6
A_1	(30,40,50)	(70,80,90)	(40,50,60)	(20,30,40)	(70,80,90)	(60,70,80)
A_2	(50,60,70)	(40,50,60)	(60,70,80)	(50,60,70)	(60,70,80)	(60,70,80)
A_3	(50,60,70)	(40,50,60)	(20,40,60)	(50,60,70)	(70,80,90)	(70,80,90)
A_4	(60,70,80)	(70,80,90)	(10,20,30)	(70,80,90)	(60,70,80)	(10,20,30)
A_5	(40,50,60)	(60,70,80)	(30,40,50)	(60,70,80)	(40,50,60)	(60,70,80)

In this problem, the experts assume the following weighting vector: W = (0.2, 0.2, 0.2, 0.2, 0.1, 0.1). They assume that the WA that each state of nature will have is: V = (0.1, 0.2, 0.3, 0.2, 0.1, 0.1). Note that the FOWA operator has an importance of 40% and the FWA an importance of 60%. For doing so, we will use Eq. (3) to calculate the FOWAWA aggregation. The results are shown in Table 2.

Table 2: FOWAWA weights

	$\hat{v}_1$	$\hat{v}_2$	$\hat{v}_3$	$\hat{v}_4$	$\hat{v}_5$	$\hat{v}_6$
V*	0.14	0.2	0.26	0.2	0.1	0.1

With this information, we can aggregate the expected results for each state of nature in order to make a decision. In Table 3, we present different results obtained by using different types of FOWAWA operators.

Table 3: Aggregated results

	FA	FWA	FOWA	FWAM	FOWAWA
A_1	(48.3,58.3,68.3)	(46,56,66)	(53,63,73)	(50.18,60.18,70.18)	(48.8,58.8,68.8)
A_2	(53.3,63.3,73.3)	(53,63,73)	(55,65,75)	(53.98,63.98,73.98)	(53.8,63.8,73.8)
A_3	(50,60,70)	(43,53,63)	(54,64,74)	(51.6,61.6,71.6)	(47.4,57.4,67.4)
A_4	(46.6,56.6,66.6)	(44,54,64)	(54,64,74)	(49.56,59.56,69.56)	(48,58,68)
A_5	(48.3,58.3,68.3)	(47,57,67)	(51,61,71)	(49.38,59.38,69.38)	(48.6,58.6,68.6)

Note that we can also obtain these results by using Eq. (4). Then, we will calculate separately the FOWA and the FWA as shown in Table 4.

Table 4: First aggregation process

	FWA	FA	FOWA
A_1	(46,56,66)	(48.3,58.3,68.3)	(53,63,73)
A_2	(53,63,73)	(53.3,63.3,73.3)	(55,65,75)
A_3	(43,53,63)	(50,60,70)	(54,64,74)
A_4	(44,54,64)	(46.6,56.6,66.6)	(54,64,74)
A_5	(47,57,67)	(48.3,58.3,68.3)	(51,61,71)

After that, we will aggregate both models in the same process considering that the FOWA model has a degree of importance of 40% and the FWA 60% as shown in Table 5.

Table 5: Final aggregated results

	FA	FWA	FOWA	FWAM	FOWAWA
A_1	(48.3,58.3,68.3)	(46,56,66)	(53,63,73)	(50.18,60.18,70.18)	(48.8,58.8,68.8)
A_2	(53.3,63.3,73.3)	(53,63,73)	(55,65,75)	(53.98,63.98,73.98)	(53.8,63.8,73.8)
A_3	(50,60,70)	(43,53,63)	(54,64,74)	(51.6,61.6,71.6)	(47.4,57.4,67.4)
A_4	(46.6,56.6,66.6)	(44,54,64)	(54,64,74)	(49.56,59.56,69.56)	(48,58,68)
A_5	(48.3,58.3,68.3)	(47,57,67)	(51,61,71)	(49.38,59.38,69.38)	(48.6,58.6,68.6)

Obviously, we get the same results with both methods. If we establish an ordering of the alternatives, a typical situation if we want to consider more than one alternative, then, we get the results shown in Table 6. Note that the first alternative in each ordering is the optimal choice.

Table 6: Ordering of the policies

	Ordering
FA	$A_2 \succ A_3 \succ A_1 = A_5 \succ A_4$
FWA	$A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$
FOWA	$A_2 \succ A_3 = A_4 \succ A_1 \succ A_5$
FWAM	$A_2 \succ A_3 \succ A_1 \succ A_4 \succ A_5$
FOWAWA	$A_2 \succ A_1 \succ A_5 \succ A_4 \succ A_3$

As we can see, depending on the aggregation operator used, the ordering of the strategies may be different. Therefore, the decision about which policy select may be also different.

6. CONCLUSIONS

We have presented the FOWAWA operator. It is a new aggregation operator that unifies the FOWA operator with the FWA when the available information is uncertain and can be assessed with FNs. The main advantage of this operator is that it provides more complete information because it represents the information in a more complete way considering the maximum and the minimum and the possibility that the internal values will occur. We have studied some basic properties of this operator and different particular cases such as the OWAWA itself, the maximum, the minimum, the fuzzy average, the FOWA, etc.

We have analysed the applicability of the new approach and we have seen that it is very broad because it can be applied in a lot of problems where previously were studied with the WA or the OWA. In this paper, we have focussed on an application in decision making about selection of fiscal policies. We have seen that depending on the aggregation operator used the results may lead to different decisions.

In future research, we expect to develop further developments by using other types of information such as interval numbers, linguistic variables, expertons, etc. We will also add other characteristics in order to obtain a more complete formulation such as inducing variables, generalized and quasi-arithmetic means, distance measures, t-norms and t-conorms, etc. Finally, we will also develop different types of applications especially in decision theory but also in other fields such as statistics, business and economics, etc.

REFERENCES

Beliakov, G., A. Pradera, T. Calvo, 2007. *Aggregation Functions: A guide for practicioners*, Springer-Verlag, Berlin.

Calvo, T., G. Mayor, R. Mesiar, 2002. *Aggregation Operators: New Trends and Applications*, Physica-Verlag, New York.

Casanovas, M., J.M. Merigó, 2007. Using fuzzy OWA operators in decision making with Dempster-Shafer belief structure, in: *Proceedings of the AEDEM International Conference*, Krakow, Poland, pp. 475-486.

Chen, S.J., S.M. Chen, 2003. Fuzzy Risk Analysis Based on Similarity Measures of Generalized Fuzzy Numbers. *IEEE Transactions on Fuzzy Systems* 11: 45-56.

Chen, S.J., S.M. Chen, 2008. Fuzzy risk analysis based on measures of similarity between interval-valued fuzzy numbers. *Computers & Mathematics with Applications* 55: 1670-1685.

Gil-Aluja, J., 1998. *The interactive management of human resources in uncertainty*. Dordrecht: Kluwer Academic Publishers.

Kaufmann, A., J. Gil-Aluja, 1987. *Técnicas operativas de gestión para el tratamiento de la incertidumbre* (In Spanish), Ed. Hispano-europea, Barcelona.

Kaufmann, A., M.M. Gupta, 1985. *Introduction to fuzzy arithmetic*, Publications Van Nostrand, Rheinhold.

Merigó, J.M., 2008. *Nuevas extensiones a los operadores OWA y su aplicación en los métodos de decisión*, PhD Thesis (In Spanish), Department of Business Administration, University of Barcelona.

Merigó, J.M., M. Casanovas, 2007. The fuzzy generalized ordered weighted averaging operator, in: *Proceedings of the 14th SIGEF Congress*, Poiana-Brasov, Romania, pp. 504-517.

Merigó, J.M., M. Casanovas, 2008a. Using fuzzy numbers in heavy aggregation operators. *International Journal of Information Technology* 4: 177-182.

Merigó, J.M., M. Casanovas, 2008b. Uncertain decision making with Dempster-Shafer theory. In: *12th IPMU International Conference*, pp. 425-432, Torremolinos, Spain.

Merigó, J.M., A.M. Gil-Lafuente, 2009. The induced generalized OWA operator. *Information Sciences* 179: 729-741.

Moore, R.E., 1966. *Interval Analysis*, Prentice Hall, Englewood Cliffs, NJ.

Sadiq, R., S. Tesfamariam, 2008. Developing environmental indices using fuzzy numbers ordered weighted averaging (FN-OWA) operators. *Stochastic Environmental Research and Risk Assessment* 22: 495-505.

Torra, V., 1997. The weighted OWA operator. *International Journal of Intelligent Systems*, 12:153-166.

Torra, V., Y. Narukawa, 2007. *Modeling Decisions: Information Fusion and Aggregation Operators*. Berlin: Springer-Verlag.

Xu, Z.S., 2006. Induced uncertain linguistic OWA operators applied to group decision making, *Information Fusion* 7: 231-238.

Xu, Z.S., 2007. Multi-person multi-attribute decision making models under intuitionistic fuzzy environment. *Fuzzy Optimization and Decision Making* 6: 221-236.

Xu, Z.S., Q.L. Da, 2003. An Overview of Operators for Aggregating the Information, *International Journal of Intelligent Systems* 18: 953-969.

Xu, Z.S., R.R. Yager, 2008. Dynamic intuitionistic fuzzy multi-attribute decision making. *International Journal of Approximate Reasoning* 48: 246-262.

Yager, R.R., 1988. On Ordered Weighted Averaging Aggregation Operators in Multi-Criteria Decision Making, *IEEE Transactions on Systems, Man and Cybernetics*, B 18: 183-190.

Yager, R.R., 1993. Families of OWA operators, *Fuzzy Sets and Systems* 59: 125-148.

Yager, R.R., 1996. Quantifier guided aggregation using OWA operators, *International Journal of Intelligent Systems* 11: 49-73.

Yager, R.R., 2007. Centered OWA operators, *Soft Computing* 11: 631-639.

Yager, R.R., D.P. Filev, 1994. Parameterized "andlike" and "orlike" OWA Operators, *International Journal of General Systems* 22: 297-316.

Yager, R.R., J. Kacprzyk, 1997. *The Ordered Weighted Averaging Operators: Theory and Applications*, Kluwer Academic Publishers, Norwell, MA.