Quantitative Fattorini-Hautus test and minimal null control time for parabolic problems

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Abstract

This paper investigates the link between the null controllability property for some abstract parabolic problems and an inequality that can be seen as a quantified Fattorini-Hautus test. Depending on the hypotheses made on the abstract setting considered we prove that this inequality either gives the exact minimal null control time or at least gives the qualitative property of existence of such a minimal time. We also prove that for many known examples of minimal time in the parabolic setting, this inequality recovers the value of this minimal time.

1 Introduction

1.1 Presentation of the minimal null control time problem

We consider an abstract control problem written in the form

$$\begin{cases} y'(t) = Ay(t) + Bu(t), & t \geq 0, \\ y(0) = y_0. \end{cases} \tag{1.1}$$

In this equation, u is a control that modifies the evolution of the state y. In all what follows we consider that the operator A is the infinitesimal generator of a C^0-semigroup on the Hilbert space $(H, \|\cdot\|)$ denoted by $t \mapsto e^{tA}$.

The space of controls will be denoted by $(U, \|\cdot\|_U)$. It is assumed to be a separable Hilbert space and is identified with its dual. The inner products on H and U are respectively denoted by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_U$. To take into account as many situations as possible we consider admissible control operators (see [29, Sec. 4.2] for instance) i.e. $B \in \mathcal{L}(U, D(A^*)')$ and its adjoint operator satisfies

$$\exists K_T > 0, \forall y \in D(A^*), \quad \int_0^T \|B^* e^{tA^*} y\|_U^2 \mathrm{d}t \leq K_T \|y\|^2. \tag{1.2}$$

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This setting allows to consider, for any $y_0 \in H$ and any $u \in L^2(0, T; U)$, the unique solution $y \in C^0([0, T], H)$ of system (1.1) defined by the following identity,

$$\langle y(t), z^t \rangle - \langle y_0, e^{tA^*} z^t \rangle = \int_0^t \langle u(\tau), B^* e^{(t-\tau)A^*} z^t \rangle_U \mathrm{d}\tau, \quad \forall t \in [0, T], \quad (1.3)$$

for any $z^t \in H$ (see for instance [12, Sec 2.3.1.] or [29, Sec 4.2]).

The property under investigation in this article is null controllability. Recall that system (1.1) is null controllable in time $T > 0$ if for any $y_0 \in H$ there exists $u \in L^2(0, T; U)$ such that the associated solution of (1.1) satisfies $y(T) = 0$. We will also mention another notion of controllability: system (1.1) is approximately controllable in time $T > 0$ if for any $y_0, y_1 \in H$, for any $\varepsilon > 0$, there exists $u \in L^2(0, T; U)$ such that the associated solution of (1.1) satifies $\|y(T) - y_1\| \leq \varepsilon$.

The following necessary condition for null controllability has been proved in [15, Prop 2.3].

Theorem 1.1. *Assume that problem* (1.1) *is null controllable in time* T. *Then, there exists* $C_T > 0$ *such that for any* $y \in D(A^*)$, *for any* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) > 0$, *one has,*

$$\|y\|^2 \leq C_T e^{2T\mathrm{Re}(\lambda)} \left(\frac{\|(A^* + \lambda)y\|^2}{\mathrm{Re}(\lambda)^2} + \frac{\|B^* y\|_U^2}{\mathrm{Re}(\lambda)} \right). \quad (1.4)$$

The aim of this article is to prove that this necessary condition is also sufficient for certain classes of parabolic control problems.

As detailed in Sec. 1.3, let us mention that this inequality can be seen as a quantified version of the Fattorini-Hautus test. In all this article we define

$$T_0 = \inf \left\{ T > 0 \; ; \; \exists C_T > 0; \forall y \in D(A^*), \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}(\lambda) > 0, \right.$$
$$\left. \|y\|^2 \leq C_T e^{2T\mathrm{Re}(\lambda)} \left(\frac{\|(A^* + \lambda)y\|^2}{\mathrm{Re}(\lambda)^2} + \frac{\|B^* y\|_U^2}{\mathrm{Re}(\lambda)} \right) \right\} \quad (1.5)$$

and $T_0 = +\infty$ when the previous set is empty.

The null controllability of parabolic partial differential equations has been widely studied since the pioneering work of FATTORINI and RUSSEL [17]. From the works of FURSIKOV and IMANUVILOV [19] and LEBEAU and ROBBIANO [23], it was commonly admitted that, in the context of parabolic partial differential equations, there is no restriction on the final time T and no geometric restriction on the control domain (in case of internal or boundary control).

But recently the study of particular examples highlighted the existence of a positive minimal time for null controllability or a geometric condition on the control domain. Actually, such an example was already provided in the 70s in [13] but the full scope of this result was not understood at that time due to the particular pointwise control exerted. The more recent results concerning such a minimal time have been proved in contexts that have also been considered as specific, namely control of coupled parabolic equations [2, 4, 5, 14] or control of degenerate parabolic equations [7, 8, 9, 6]. Though these three settings exhibit the same qualitative behavior, no precise link between them have been established so far.

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Our aim in this article is to give an abstract framework encompassing those different frameworks to study the minimal null control time property. More precisely, we will relate this minimal time to the time T_0 defined in (1.5). We will emphasize that this minimal time can have different origins. It can be created by some localization of (generalised) eigenfunctions with respect to the observation operator B^* (as in [13, 5, 14, 7, 8, 9, 6]). This aspect is dealt with in Theorem 1.2. But the minimal time can also be created by a condensation of the eigenvalues of the underlying operator as in [2, 4]. This aspect is dealt with in Theorem 1.3. In those two abstract settings the minimal null control time will exactly be given by T_0.

We will also propose a more general setting (encompassing the two previous ones) to deal with the situation where the minimal time comes from both the localization of the eigenfunctions and the condensation of the spectrum. In this case (see Theorem 1.4), we will prove that the existence of such a minimal time is related to T_0 but the exact value of this minimal time will remain an open problem.

Finally there are still examples that will not fit into the different settings we study. For some of these examples (see Sec. 4) we will still be able to prove that the minimal null control time is given by T_0. This analysis of particular examples will require to know a priori the value of the minimal time and thus, for the moment, the degenerate parabolic equations studied in [7, 8, 6] will remain out of the scope of this article.

1.2 Main results

Let us now define the setting that we will deal with in the next sections.

Hypothesis 1. *Assume that the operator* $-A^*$ *admits a sequence of eigenvalues* $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ *such that*

$$\exists \delta > 0, \ \text{Re}(\lambda_k) \geq \delta |\lambda_k|, \ \forall k \in \mathbb{N}^* \quad and \quad \sum_{k=1}^{+\infty} \frac{1}{|\lambda_k|} < +\infty. \qquad (1.6)$$

For any $k \in \mathbb{N}^*$*, we denote by* $r_k = \dim(\text{Ker}(A^* + \lambda_k))$ *the geometric multiplicity of the eigenvalue* λ_k *and assume that* $\sup_{k \in \mathbb{N}^*} r_k < +\infty$*. We denote by* $(\varphi_{k,j})_{k \in \mathbb{N}^*, 1 \leq j \leq r_k}$ *the associated sequence of normalised eigenfunctions and we assume that it forms a complete sequence in* H *i.e.*

$$\left(\langle \Phi, \varphi_{k,j} \rangle = 0, \quad \forall k \in \mathbb{N}^*, \forall j \in \{1, \ldots, r_k\} \right) \implies \Phi = 0.$$

Remark 1.1. As we will study parabolic problems, the hypothesis (1.6) mainly restricts our study to the one dimensional case. However, as presented in Sec. 4, inequality (1.4) is also a sufficient condition for certain examples in higher dimensions such as the 2D Grushin equation or a generalisation in any dimension of the academic example studied in [2].

Moreover, as underlined in Sec. 4.3, hypothesis (1.6) is not a technical assumption due to the technics at stake in this article (namely the moment method) but is deeply related to the characterization we are studying.

Remark 1.2. As it will be precised in Remark 2.2, one can weaken the assumption $\sup_{k \in \mathbb{N}^*} r_k < +\infty$ by authorizing a suitable growth.

Remark 1.3. We will always assume that the sequence of eigenvalues is normally ordered that is

$$\begin{cases} |\lambda_k| \leq |\lambda_{k+1}|, & \forall k \geq 1, \\ -\dfrac{\pi}{2} < \arg(\lambda_k) < \arg(\lambda_{k+1}) < \dfrac{\pi}{2} \text{ when } |\lambda_k| = |\lambda_{k+1}|. \end{cases} \tag{1.7}$$

Remark 1.4. Without loss of generality (through an orthonormalization process) we can always assume in Hypothesis 1 that for any $k \in \mathbb{N}^*$,

$$\langle \varphi_{k,i}, \varphi_{k,j} \rangle = \delta_{i,j}, \quad \forall i,j \in \{1,\ldots,r_k\}.$$

Here and in all this article $\delta_{i,j}$ denotes the Kronecker symbol i.e. $\delta_{i,j}$ is equal to 1 if $i = j$ and is equal to 0 otherwise.

Before stating our results, let us recall the following definition of the condensation index of a complex sequence in this setting.

Definition 1.1. Let $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ be a complex normally ordered sequence satisfying (1.6). Then the condensation index of this sequence is defined as

$$c(\Lambda) = \limsup_{k \to +\infty} \frac{-\ln|E'(\lambda_k)|}{\mathrm{Re}(\lambda_k)}$$

where the function E is defined by

$$E : z \in \mathbb{C} \mapsto \prod_{k=1}^{+\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right). \tag{1.8}$$

This definition can be found in [27]. The original (and of course equivalent) definition can be found in [10]. The first result of this article is the following theorem.

Theorem 1.2. *Assume that A^* satisfies Hypothesis 1 and that the control operator B is admissible. We also assume that the condensation index of the sequence $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ satisfies $c(\Lambda) = 0$. Let T_0 be defined by (1.5). Then:*

- *If $T_0 > 0$ and $T < T_0$, system (1.1) is not null controllable in time T;*

- *If $T_0 < +\infty$ and $T > T_0$, system (1.1) is null controllable in time T.*

Remark 1.5. Notice that the negative result in time $T < T_0$ directly comes from Theorem 1.1.

Remark 1.6. It has to be highlighted that the condition $c(\Lambda) = 0$ is not a strong condition. It is more general than many situations previously considered when applying the moment method in the parabolic setting where a gap-like condition was assumed (see [3, Proposition 3.11]). For example, the sequence $\Lambda = \{k^2, k^2 + \frac{1}{k}; k \in \mathbb{N}\}$ does not satisfy a gap-like condition whereas $c(\Lambda) = 0$.

Remark 1.7. As it will be discussed in Sec. 4.1.1 (on the particular case of system (2.19)), the minimal time T_0 given by inequality (1.4) can also turn out to be the minimal time for null controllability even if generalised eigenfunctions need to be taken into account to obtain a complete family. In the abstract setting we are considering, we were not able to prove (neither to disprove) this result. Instead, we prove in Proposition 2.1 a qualitative property: if a minimal time is needed for inequaltiy (1.4) to hold, then a minimal time is also needed for null controllability.

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Using inequality (1.4), we may also characterize null controllability in certain settings where we allow the spectrum to condensate. First, we define the Bohr index (adapted from [10] for a real-valued sequence) of a normally ordered sequence $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ by

$$\mathrm{Bohr}(\Lambda) = \limsup_{k \to +\infty} \frac{-\ln \inf_{j \neq k} |\lambda_k - \lambda_j|}{\mathrm{Re}(\lambda_k)}.$$

The Bohr index measures how two eigenvalues can get exponentially closer. The condensation index is in general a more subtle notion as it measures how packets of eigenvalues (with possibly increasing cardinality) get closer. Thus for any sequence Λ, it comes that $c(\Lambda) \geq \mathrm{Bohr}(\Lambda)$. Under some additional assumptions, we prove that the equivalent of Theorem 1.2 still holds when $c(\Lambda) = \mathrm{Bohr}(\Lambda)$, that is, that we allow eigenvalues to get closer two by two.

Theorem 1.3. *Assume that A^* satisfies Hypothesis 1 and that the eigenfunctions form a Riesz basis of H. Assume that the control operator B is admissible and that for any $k \neq j \in \mathbb{N}^*$,*

$$\mathrm{Ker}(B^*) \cap \mathrm{Span}(\varphi_k, \varphi_j) \neq \{0\} \tag{1.9}$$

and for any $\varepsilon > 0$

$$\|B^* \varphi_k\|_U e^{\varepsilon \mathrm{Re}(\lambda_k)} \xrightarrow[k \to +\infty]{} +\infty. \tag{1.10}$$

Finally, assume that the condensation index of the sequence $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ satisfies $c(\Lambda) = \mathrm{Bohr}(\Lambda)$.

Let T_0 be defined by (1.5). Then:

- *If $T_0 > 0$ and $T < T_0$, system (1.1) is not null controllable in time T;*

- *If $T_0 < +\infty$ and $T > T_0$, system (1.1) is null controllable in time T.*

In this result we obviously only consider the case $c(\Lambda) > 0$. Otherwise if $c(\Lambda) = 0$ the result stated in Theorem 1.2 is the same with less hypotheses.

Remark 1.8. In the case $T_0 < +\infty$, assumption (1.9) implies that the eigenvalues are geometrically simple. In fact, assume that for some $k \in \mathbb{N}^*$ we have $r_k \geq 2$. Then, it comes that there exist $\alpha_{k,1}, \alpha_{k,2} \in \mathbb{C}$ such that

$$v_k := \alpha_{k,1} \varphi_{k,1} + \alpha_{k,2} \varphi_{k,2} \in \mathrm{Ker}(B^*) \backslash \{0\}.$$

But applying inequality (1.4) with $v = v_k$ and $\lambda = \lambda_k$ implies $v_k = 0$. This is why we denoted simply the eigenfunctions by φ_k instead of $\varphi_{k,1}$.

Remark 1.9. Notice that assumption (1.9) is automatically satisfied when we consider a scalar time-dependent control (for example a boundary control in dimension one) i.e. $U = \mathbb{C}$. However this assumption can also be satisfied by systems of coupled parabolic equations with a particular structure (see for instance (3.7)). Actually, assumption (1.9) does not need to be satisfied for any k and any j. This is detailed in Remark 3.3.

Remark 1.10. Some examples of problems satisfying these assumptions (as well as verifications of the hypothesis $\mathrm{Bohr}(\Lambda) = c(\Lambda)$) are discussed in Sec. 3.3.

Remark 1.11. It seems that hypothesis (1.10) is only technical; it allows us to study the effect of condensation of the spectrum when there is no localization of the eigenfunctions (with respect to B^*). However, if we do not assume this behavior of the observation operator B^* on the eigenfunctions, we would obtain only a qualitative result of existence of a minimal null controllability time belonging to $[T_0, 2T_0]$ (see Remark 3.2). Such a qualitative result is stated below in Theorem 1.4 in a more general setting.

Finally, in the most general setting we study, we higlight a striking phenomenon: in the case where a positive minimal time is needed for inequality (1.4) to hold (i.e. $T_0 \in (0, +\infty)$) then a positive minimal time is also needed for null controllability of system (1.1). More precisely, we prove the following result.

Theorem 1.4. *Assume that A^* satisfies Hypothesis 1. Assume that the control operator B is admissible. Let T_0 be defined by (1.5). Then, there exists $\widetilde{T} \in [T_0, T_0 + \mathrm{c}(\Lambda)]$ such that*

- *if $\widetilde{T} > 0$ and $T < \widetilde{T}$, system (1.1) is not null controllable in time T;*

- *if $\widetilde{T} < +\infty$ and $\mathrm{c}(\Lambda) < +\infty$, for $T > \widetilde{T}$, system (1.1) is null controllable in time T.*

Remark 1.12. Notice that this setting is rather general: no restriction is imposed on the condensation of eigenvalues, we allow geometric multiplicity for the eigenvalues and the control operator is only assumed to be admissible.

1.3 A brief review of previous results

As mentioned, inequality (1.4) can be seen as a quantified Hautus test. The Hautus test refers to a well known controllability necessary and sufficient condition in finite dimension [20]. In the case where H and U have finite dimensions and the operators A and B are matrices then system (1.1) is controllable in time $T > 0$ if there are no eigenfunction of A^* in the kernel of B^*. This can be rewritten as

$$\mathrm{Ker}(A^* + \lambda) \cap \mathrm{Ker}(B^*) = \{0\}, \qquad \forall \lambda \in \mathbb{C}. \tag{1.11}$$

Actually, a more general result was proved by FATTORINI in [16] before the work of Hautus. Consider again that H and U are Hilbert spaces (actually the result can be extended to Banach spaces) and that A generates a C^0-semigroup, has a compact resolvent and that the generalised eigenfunctions of A^* form a complete sequence. Consider that the control operator B is bounded. Then, the criterion (1.11) is proved to be equivalent to some unique continuation property (in infinite time) that turns out to be equivalent to approximate controllability when the semigroup generated by A is analytic. We also mention the work of OLIVE [24] where the hypothesis on B is weakened.

Of course, inequality (1.4) implies (1.11). This is why we considered it as quantified Fattorini-Hautus test.

The origin of this work is a conjecture of RUSSELL and WEISS [25]. To describe it briefly, let H and U be two Hilbert spaces and $A : D(A) \subset H \to H$ the generator of a C^0- semigroup (denoted by e^{tA}) on H. Consider the observability operator $C : D(A^*) \to U$ which is assumed A^*-bounded and admissible

in infinite time: there are two positive constants σ_∞ and γ_∞ such that for all $y \in D(A^*)$,

$$\|Cy\|_U \leq \sigma_\infty \|A^*y\|$$

and

$$\int_0^\infty \left\| Ce^{tA^*}y \right\|_U \, dt \leq \gamma_\infty \|y\|.$$

The cited authors proved that, with the previous assumptions, if the pair (A^*, C) is exactly observable on $(0, \infty)$, namely

$$\exists m_\infty > 0, \ \int_0^\infty \left\| Ce^{tA^*}y \right\|_U \, dt \geq m_\infty \|y\|, \quad \forall y \in D(A^*),$$

then there exists $m > 0$ such that for any complex number λ with $\mathrm{Re}(\lambda) > 0$ and any $y \in D(A^*)$,

$$\frac{\|(A^* + \lambda)y\|^2}{\mathrm{Re}(\lambda)^2} + \frac{\|Cy\|_U^2}{\mathrm{Re}(\lambda)} \geq m \|y\|^2. \tag{1.12}$$

The conjecture was then (see [25, p. 2]): if (1.12) holds then the pair (A^*, C) is exactly observable on $(0, \infty)$.

In the same paper, they showed that if A^* has a Riesz basis of eigenfunctions and a strong extra condition on the eigenvalues is satisfied, then the conjecture is true. But, some years later, JACOB and ZWART [21] proved by mean of a counterexample that the condition (1.12) is not sufficient in general for exact observability on $(0, \infty)$.

Exact observability on $(0, \infty)$ implies exact observability in some time $T_0 > 0$ (see [29, Proposition 6.5.2. p. 194]). For skew-adjoint operators A, a lot of works with various assumptions show that the condition (1.12) is sufficient for exact observability on $(0, \infty)$. They also give an estimate of this finite time T_0 of observability (for more details, see [29, Chapter 6 and the comments of this chapter]).

To our knowledge, only few papers deal with observability in finite time. The corresponding necessary condition for observability in time $T > 0$ (see Theorem 1.1, relation (1.4) in this paper), due to DUYCKAERTS and MILLER [15], asks again about the sufficiency of this condition for observability in time $T > 0$. In [15], the authors give, in a parabolic setting, an example showing that (1.4) can be true for any $T > 0$ without the pair (A^*, B^*) being observable at *any time* $T > 0$. This example together with its implications in our context are given in Sec. 4.3. Let us mention that these authors provide in the cited paper some positive results (sufficiency of condition (1.4) for any time $T > 0$) for abstract parabolic equations. The remark that this inequality can also identify a positive minimal time of observability in a parabolic setting seems to be new.

Structure of the article

We end this introduction by giving the strategy of proof in the simpler case where the eigenvalues of A^* are simple. Then, in Sec. 2 we prove the general case stated in Theorem 1.2. In this case, the minimal time will be inherited from a localization of the eigfunctions with respect to the control operator. Section 3 is dedicated to the proof of Theorem 1.3 where we allow some condensation of the spectrum of the underlying operator. We also prove in this section the more

general result of this article stated in Theorem 1.4. Finally, in Sec. 4 we prove that the minimal null control time is given by T_0 in some examples that do not fit into the different abstract results and discuss the assumption (1.6).

1.4 Strategy of proof

To detail the general idea used in this article, let us start by proving Theorem 1.2 in the simpler case of simple eigenvalues ($r_k = 1$ for all $k \in \mathbb{N}^*$). For the sake of simplicity the associated eigenfunction $\varphi_{k,1}$ will simply be denoted by φ_k. The negative result follows directly from Theorem 1.1. Thus, assume that $T_0 < +\infty$ and set $T > T_0$.

The proof of null controllability in time $T > T_0$ will rely on the moment method (initially developed in [17, 18]) although the considered space of controls may not be of finite dimension. From (1.3), it comes that for any $k \in \mathbb{N}^*$,

$$\langle y(T), \varphi_k \rangle - \langle y_0, e^{-\lambda_k T} \varphi_k \rangle = \int_0^T e^{-\lambda_k(T-t)} \langle u(t), B^* \varphi_k \rangle_U \mathrm{d}t. \qquad (1.13)$$

Thus, as the family $(\varphi_k)_{k \in \mathbb{N}^*}$ is assumed to be complete, it comes that $y(T) = 0$ if and only if for every $k \in \mathbb{N}^*$, the control u satisfies

$$\int_0^T e^{-\lambda_k(T-t)} \langle u(t), B^* \varphi_k \rangle_U \mathrm{d}t = -e^{-\lambda_k T} \langle y_0, \varphi_k \rangle. \qquad (1.14)$$

To apply the moment method let us recall the concept of biorthogonal family.

Definition 1.2. Let $\sigma = (\sigma_k)_{k \in \mathbb{N}}$ be a complex sequence and $T > 0$. We say that the family of functions $(q_k)_{k \in \mathbb{N}} \subset L^2(0, T; \mathbb{C})$ is a biorthogonal family to the exponentials associated with σ if for any $k, j \in \mathbb{N}$

$$\int_0^T e^{-\sigma_j t} q_k(t) \mathrm{d}t = \delta_{k,j}.$$

The following result is proved in [4, Theorem 4.1].

Proposition 1.1. *Let $T > 0$ and let $\sigma = (\sigma_k)_{k \in \mathbb{N}}$ be a normally ordered complex sequence satisfying (1.6). Then, there exists a biorthogonal family $(q_k)_{k \in \mathbb{N}}$ to the exponentials associated with σ such that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\|q_k\|_{L^2(0,T;\mathbb{C})} \leq C_\varepsilon e^{\mathrm{Re}(\sigma_k)(\mathrm{c}(\sigma)+\varepsilon)}, \quad \text{for } k \text{ sufficiently large,}$$

where $\mathrm{c}(\sigma)$ is the condensation index of the sequence σ.

Remark 1.13. In all what follows, C_ε will always denote a constant that may vary from one line to another which only depends on the parameter ε.

We now generalise the strategy previously used for instance in [22, 11, 1], that is, we seek for a control u solving (1.14) in the following form

$$u(t) = -\sum_{k \in \mathbb{N}^*} \alpha_k q_k(T-t) \frac{B^* \varphi_k}{\|B^* \varphi_k\|_U^2}, \qquad (1.15)$$

where the complex coefficients $(\alpha_k)_{k \in \mathbb{N}^*}$ have to be determined.

Remark 1.14. Notice that when u is a scalar control only depending on time (i.e. $B^* : H \to \mathbb{C}$) then this is the classical form of controls used in the moment method. When B is a localization operator and u depends both on time and space, this is different from the usual strategy where one looks for a fixed space profile multiplied by a control depending only on time. In any other situation, this is a new way of applying the moment method.

First of all notice that in the previous form we divided by $\|B^*\varphi_k\|_U$. This is done only by convenience, as it could be incorporated in the constant α_k, but we should justify that $\|B^*\varphi_k\|_U \neq 0$ for every $k \in \mathbb{N}^*$.

Since we assumed $T_0 < +\infty$ it comes that for any $\varepsilon > 0$, for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ and any $v \in D(A^*)$,

$$\|v\|^2 \leq C_\varepsilon e^{2\mathrm{Re}(\lambda)(T_0+\varepsilon)} \left(\frac{\|(A^*+\lambda)v\|^2}{\mathrm{Re}(\lambda)^2} + \frac{\|B^*v\|_U^2}{\mathrm{Re}(\lambda)} \right).$$

The choice $\lambda = \lambda_k$ and $v = \varphi_k$ in the previous inequality implies that for all $k \in \mathbb{N}^*$,

$$\|B^*\varphi_k\|_U \geq C_\varepsilon \sqrt{\lambda_k} e^{-\mathrm{Re}(\lambda_k)(T_0+\varepsilon)} > 0. \tag{1.16}$$

Now, using the biorthogonality in the time variable, plugging u given by (1.15) in the moment problem (1.14) leads to

$$\alpha_k = -e^{-\lambda_k T}\langle y_0, \varphi_k \rangle,$$

and thus a formal solution of (1.13) is given by

$$u(t) = -\sum_{k \in \mathbb{N}^*} e^{-\lambda_k T} \frac{\langle y_0, \varphi_k \rangle}{\|B^*\varphi_k\|_U^2} q_k(T-t)B^*\varphi_k. \tag{1.17}$$

The only remaining point is to prove that this series is indeed convergent in $L^2(0,T;U)$. This comes directly from the estimate (1.16) and Proposition 1.1. Indeed, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ independent of k such that

$$\left\| \sum_{k \in \mathbb{N}^*} e^{-\lambda_k T} \frac{\langle y_0, \varphi_k \rangle}{\|B^*\varphi_k\|_U^2} q_k(T - \cdot)B^*\varphi_k \right\|_{L^2(0,T;U)}$$

$$\leq \sum_{k \in \mathbb{N}^*} |\langle y_0, \varphi_k \rangle| e^{-\mathrm{Re}(\lambda_k)T} \frac{\|q_k\|_{L^2(0,T)}}{\|B^*\varphi_k\|_U}$$

$$\leq C_\varepsilon \sum_{k \in \mathbb{N}^*} |\langle y_0, \varphi_k \rangle| e^{-\mathrm{Re}(\lambda_k)T} \frac{e^{\varepsilon \mathrm{Re}(\lambda_k)}}{\sqrt{\mathrm{Re}(\lambda_k)}e^{-\mathrm{Re}(\lambda_k)(T_0+\varepsilon)}}$$

which converges for ε sufficiently small, since $T > T_0$.

The strategy we develop in this article will be the same to prove Theorems 1.2 and 1.3. Namely we seek for a control in a suitable form and use inequality (1.4) to obtain sufficient estimates to prove that this control is well defined.

2 Localization of eigenfunctions

In the first subsection we prove Theorem 1.2. We then end this section by giving some examples in the literature of parabolic partial differential equations (with a zero condensation index for the underlying operator) exhibiting a minimal null control time given by (1.5).

2.1 Proof of Theorem 1.2

The case $T < T_0$ follows directly from Theorem 1.1. Thus we focus on the case $T_0 < +\infty$ and we fix $T > T_0$.

As the moment problem is solved component by component, we distinguish two sets

$$\Sigma_S = \{k \in \mathbb{N}^*; r_k = 1\}, \qquad \Sigma_M = \{k \in \mathbb{N}^*; r_k \geq 2\} = \mathbb{N}^* \backslash \Sigma_S,$$

where r_k is the geometric multiplicity of the eigenvalue λ_k defined in Hypothesis 1. For the sake of simplicity, when $r_k = 1$, the associated eigenfunction $\varphi_{k,1}$ will simply be denoted by φ_k.

In Sec. 1.4, we already proved Theorem 1.2 in the case where $\Sigma_S = \mathbb{N}^*$ thus we will assume in the following that $\Sigma_M \neq \varnothing$.

From a simple computation it comes that for any $k \in \mathbb{N}^*$, for any $i \in \{1, \ldots, r_k\}$,

$$\langle y(T), \varphi_{k,i} \rangle - \langle y_0, e^{-\lambda_k T} \varphi_{k,i} \rangle = \int_0^T e^{-\lambda_k (T-t)} \langle u(t), B^* \varphi_{k,i} \rangle_U \mathrm{d}t. \qquad (2.1)$$

Thus as the family $(\varphi_{k,i})_{k \in \mathbb{N}^*, 1 \leq i \leq r_k}$ is assumed to be complete, it comes that $y(T) = 0$ if and only if for every $k \in \mathbb{N}^*$ and every $i \in \{1, \ldots, r_k\}$, the control u satisfies

$$\int_0^T e^{-\lambda_k (T-t)} \langle u(t), B^* \varphi_{k,i} \rangle_U \mathrm{d}t = -e^{-\lambda_k T} \langle y_0, \varphi_{k,i} \rangle. \qquad (2.2)$$

Using Proposition 1.1, let $(q_k)_{k \in \mathbb{N}}$ be the biorthogonal family to the exponentials associated with $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$. Following Sec. 1.4, we define

$$u^1(t) = -\sum_{k \in \Sigma_S} e^{-\lambda_k T} \frac{\langle y_0, \varphi_k \rangle}{\|B^* \varphi_k\|_U^2} q_k(T - t) B^* \varphi_k. \qquad (2.3)$$

We already proved that u^1 is well defined in $L^2(0, T; U)$ and with this choice of control, the moment problem (2.2) is satisfied for every $k \in \Sigma_S$.

To deal with multiple eigenvalues we will need the following lemma whose proof is postponed at the end of this subsection.

Lemma 2.1. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $r \in \mathbb{N}^*$. Let us consider $v = (v_1, \ldots, v_r) \in \mathcal{H}^r$ a linearly independent family. We denote by*

$$G = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq r}$$

the associated Gram matrix. Then the matrix G is invertible and the family $w = (w_1, \ldots, w_r)$ defined by $w^t = G^{-1} v^t$ is biorthogonal to v i.e.

$$\langle v_i, w_j \rangle = \delta_{i,j}, \quad \forall i, j \in \{1, \ldots, r\}.$$

Moreover, this biorthogonal family satisfies

$$\|w\| \leq \frac{\sqrt{r}}{\sigma}, \quad \text{where } \sigma^2 \text{ is the smallest eigenvalue of } G.$$

Let $k \in \Sigma_M$. To design our control we will apply this lemma to the family $(B^* \varphi_{k,1}, \ldots, B^* \varphi_{k,r_k})$.

To this end we prove that it is linearly independent. Assume that there exist $c_1, \ldots, c_{r_k} \in \mathbb{C}$ such that

$$0 = c_1 B^* \varphi_{k,1} + \cdots + c_{r_k} B^* \varphi_{k,r_k} = B^* \varphi,$$

where $\varphi = c_1 \varphi_{k,1} + \cdots + c_{r_k} \varphi_{k,r_k}$. Since $\varphi \in \mathrm{Ker}(A^* + \lambda_k)$, from (1.5) and the assumption $T_0 < +\infty$ it comes that $\varphi = 0$ which proves the linear independence.

Remark 2.1. Notice that this construction would not be possible for a scalar control depending only on the time variable (i.e. $B^* : H \to \mathbb{C}$). This case is excluded in the presence of multiple eigenvalues thanks to the assumption $T_0 < +\infty$. In this situation, system (1.1) is not approximately controllable since one could combine two eigenfunctions (associated with the same eigenvalue) to design a solution which would not be observable.

Using Lemma 2.1, we denote by $\Psi_k = (\Psi_{k,1}, \ldots, \Psi_{k,r_k})$ such a biorthogonal family to $(B^* \varphi_{k,1}, \ldots, B^* \varphi_{k,r_k})$. We seek for a control u^2 in the form

$$u^2(t) = \sum_{k \in \Sigma_M} q_k(T - t) \left(\sum_{1 \le i \le r_k} \alpha_{k,i} \Psi_{k,i} \right)$$

such that with this choice of control, the moment problem (2.2) is satisfied for every $k \in \Sigma_M$. Plugging u^2 in (2.2) and using biorthogonality both in time and space variables leads to

$$\alpha_{k,i} = -\langle y_0, \varphi_{k,i} \rangle e^{-\lambda_k T}.$$

The only remaining point to prove is the convergence of the series defining u^2 in $L^2(0, T; U)$. The estimate on q_k comes directly from Proposition 1.1.

From Lemma 2.1, to estimate $\|\Psi_k\|_{U^{r_k}}$ one can simply give a lower bound on the smallest eigenvalue of the matrix

$$G_k = (\langle B^* \varphi_{k,i}, B^* \varphi_{k,j} \rangle_U)_{1 \le i,j \le r_k}.$$

For any $c = (c_1, \ldots, c_{r_k})^t \in \mathbb{C}^{r_k}$ it comes that the vector defined by

$$\varphi = c_1 \varphi_{k,1} + \cdots + c_{r_k} \varphi_{k,r_k}$$

belongs to $\mathrm{Ker}(A^* + \lambda_k)$. From (1.5) it comes that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any $v \in \mathrm{Ker}(A^* + \lambda_k)$,

$$\|B^* v\|_U^2 \ge C_\varepsilon \mathrm{Re}(\lambda_k) e^{-2\mathrm{Re}(\lambda_k)(T_0 + \varepsilon)} \|v\|^2.$$

On the other hand, a direct computation gives that $\langle G_k c, c \rangle_{\mathbb{C}^{r_k}} = \|B^* \varphi\|_U^2$. Gathering these two estimates it comes that for any $c \in \mathbb{C}^{r_k}$,

$$\langle G_k c, c \rangle_{\mathbb{C}^{r_k}} \ge C_\varepsilon \mathrm{Re}(\lambda_k) e^{-2\mathrm{Re}(\lambda_k)(T_0 + \varepsilon)} \|c_1 \varphi_{k,1} + \cdots + c_{r_k} \varphi_{k,r_k}\|^2$$
$$\ge C_\varepsilon \mathrm{Re}(\lambda_k) e^{-2\mathrm{Re}(\lambda_k)(T_0 + \varepsilon)} \left(c_1^2 + \cdots + c_{r_k}^2 \right).$$

Finally we obtain that for any $i \in \{1, \ldots, r_k\}$,

$$\|\Psi_{k,i}\|_U \le C_\varepsilon \frac{\sqrt{r_k}}{\sqrt{\mathrm{Re}(\lambda_k)}} e^{\mathrm{Re}(\lambda_k)(T_0 + \varepsilon)} \tag{2.4}$$

Indeed, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left\| \sum_{k \in \mathbb{N}^*} q_k(T - \cdot) \left(\sum_{1 \le i \le r_k} e^{-\lambda_k T} \langle y_0, \varphi_{k,i} \rangle \Psi_{k,i} \right) \right\|_{L^2(0,T;U)}$$

$$\le \|y_0\| \sum_{k \in \mathbb{N}^*} e^{-\mathrm{Re}(\lambda_k)T} \|q_k\|_{L^2(0,T)} \sum_{1 \le i \le r_k} \|\Psi_{k,i}\|_U$$

$$\le C_\varepsilon \|y_0\| \sup_{k \in \mathbb{N}^*} r_k^{\frac{3}{2}} \sum_{k \in \mathbb{N}^*} \frac{1}{\sqrt{\mathrm{Re}(\lambda_k)}} e^{-\mathrm{Re}(\lambda_k)T} e^{\varepsilon \mathrm{Re}(\lambda_k)} e^{\mathrm{Re}(\lambda_k)(T_0 + \varepsilon)}$$

which converges since $T > T_0$ and $\sup\limits_{k \in \mathbb{N}^*} r_k < \infty$.

Finally, notice that using biorthogonality with respect to the time variable u^2 does not affect the moment problem (2.2) for $k \in \Sigma_S$ and vice-versa. Thus, setting $u = u^1 + u^2$ ends the proof of Theorem 1.2.

$\square$

Remark 2.2. Following closely the lines of the proof of Theorem 1.2 it is direct to remark that the hypothesis $\sup_{k \in \mathbb{N}^*} r_k < +\infty$ can be replaced by the following: for any $\varepsilon > 0$,

$$r_k e^{-\varepsilon \mathrm{Re}(\lambda_k)} \xrightarrow[k \to +\infty]{} 0.$$

Nevertheless, we stick with $\sup_{k \in \mathbb{N}^*} r_k < +\infty$ for the sake of simplicity.

Proof of Lemma 2.1. The matrix G is hermitian. Since we assumed that the family $(v_1, \ldots, v_r)$ is linearly independent, the Gram matrix G is also positive definite. We define $w^t = G^{-1} v^t$. Let us notice that w is biorthogonal to v. Indeed, for any $i, j \in \{1, \ldots, r\}$

$$\langle w_i, v_j \rangle = \langle \sum_{k=1}^{r} (G^{-1})_{i,k} v_k, v_j \rangle$$

$$= \sum_{k=1}^{r} (G^{-1})_{i,k} \langle v_k, v_j \rangle$$

$$= (G^{-1} G)_{i,j} = \delta_{i,j}.$$

Finally the estimate on this biorthogonal family comes from the following identity

$$r = \langle v, w \rangle = \langle G w, w \rangle \ge \sigma^2 \|w\|^2.$$

$\square$

2.2 Dealing with algebraically double eigenvalues

In this subsection we state and prove the qualitative result mentioned in Remark 1.7. We will use the following hypotheses.

Hypothesis 2. *Assume that the operator $-A^*$ admits a sequence of eigenvalues $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ satisfying (1.6).*

For any $k \in \mathbb{N}^*$ we assume that the algebraic multiplicity of the eigenvalue λ_k is equal to 2. We denote by $\varphi_{k,1}$ and $\varphi_{k,2}$, respectively, the associated normalised eigenfunction and generalised eigenfunction i.e.

$$\begin{cases} -A^*\varphi_{k,1} = \lambda_k \varphi_{k,1}, \\ -A^*\varphi_{k,2} = \lambda_k \varphi_{k,2} + \mu_k \varphi_{k,1}, \\ \|\varphi_{k,1}\| = \|\varphi_{k,2}\| = 1. \end{cases} \quad (2.5)$$

We assume that these eigenfunctions and generalised eigenfunctions form a complete sequence in H i.e.

$$\left(\langle \Phi, \varphi_{k,j} \rangle = 0, \quad \forall k \in \mathbb{N}^*, \forall j \in \{1,2\} \right) \implies \Phi = 0.$$

Remark 2.3. Without loss of generality we will always assume that $\langle \varphi_{k,1}, \varphi_{k,2} \rangle = 0$. The orthonormalization process only affects the value of μ_k.

Proposition 2.1. *Assume that A^* satisfies Hypothesis 2 and that $c(\Lambda) = 0$. Assume that B is an admissible control operator and for any $k \in \mathbb{N}^*$,*

$$\mathrm{Ker}(B^*) \cap \mathrm{Span}(\varphi_{k,1}, \varphi_{k,2}) \neq \{0\}. \quad (2.6)$$

Let T_0 be defined by (1.5). Then, there exists $\widetilde{T} \in [T_0, 2T_0]$ such that

- *If $T_0 > 0$ and $T < \widetilde{T}$, system (1.1) is not null controllable in time T;*

- *If $T_0 < +\infty$ and $T > \widetilde{T}$, system (1.1) is null controllable in time T.*

Remark 2.4. As for Theorem 1.2, the proof relies on the moment method and the control is built separately for each eigenvalue. Thus, the same result also holds if we have a complete sequence which is made with

- either eigenfunctions satisfying Hypothesis 1;

- or eigenfunctions and generalised eigenfunctions satisfying Hypothesis 2.

Moreover as the main concern in the proof is the convergence of the series defining the control, if the latter has a finite occurrence, one recovers that T_0 is the minimal time for null controllability.

Remark 2.5. The structural hypothesis made on B^* is similar to (1.9) thus the comments of Remark 1.9 still hold in this setting

Proof. As the construction of the control will be done separately for each eigenvalue, we can assume that $\mu_k \neq 0$ for any $k \in \mathbb{N}^*$: otherwise, for these frequencies, one simply follows the lines of Theorem 1.2.

Now, as we are dealing with generalised eigenfunctions, the moment problem is different from (1.14) or (2.2). Let y be the solution of (1.1). From (1.3), it comes that for any $k \in \mathbb{N}^*$ and any $j \in \{1,2\}$

$$\langle y(T), \varphi_{k,j} \rangle - \langle y_0, e^{TA^*}\varphi_{k,j} \rangle = \int_0^T \langle Bu(t), e^{(T-t)A^*}\varphi_{k,j} \rangle_U. \quad (2.7)$$

Using (2.5), we obtain

$$
\begin{aligned}
e^{(T-t)A^*}\varphi_{k,1} &= e^{-\lambda_k(T-t)}\varphi_{k,1}, \\
e^{(T-t)A^*}\varphi_{k,2} &= e^{-\lambda_k(T-t)}\varphi_{k,2} - (T-t)\mu_k e^{-\lambda_k(T-t)}\varphi_{k,1}.
\end{aligned}
\tag{2.8}
$$

Using the assumption that the generalised eigenfunctions form a complete family, it comes that $y(T) = 0$ if and only if, for any $k \in \mathbb{N}^*$

$$
\begin{cases}
\displaystyle\int_0^T e^{-\lambda_k(T-t)}\langle u(t), B^*\varphi_{k,1}\rangle_U \mathrm{d}t = -e^{-\lambda_k T}\langle y_0, \varphi_{k,1}\rangle, \\[2mm]
\displaystyle\int_0^T e^{-\lambda_k(T-t)}\langle u(t), B^*\varphi_{k,2}\rangle_U \mathrm{d}t - \mu_k \int_0^T (T-t)e^{-\lambda_k(T-t)}\langle u(t), B^*\varphi_{k,1}\rangle_U \mathrm{d}t \\[2mm]
\quad = -e^{-\lambda_k T}\langle y_0, \varphi_{k,2} - \mu_k T\varphi_{k,1}\rangle.
\end{cases}
\tag{2.9}
$$

The main difference with respect to the previous moment problem (1.14) is the presence of the function $s \mapsto se^{-\lambda_k s}$ in the second integral of the second equation. To deal with this, we will use an adapted biorthogonal family.

We denote by $e_{k,1}$ and $e_{k,2}$ the $L^2(0,T;\mathbb{C})$ functions defined by

$$
e_{k,1}(t) = e^{-\lambda_k t}, \qquad e_{k,2}(t) = te^{-\lambda_k t}.
$$

To define and estimate a suitable biorthogonal family, we will use the following general proposition. Its proof is postponed to Appendix A and follows [3, 4].

Proposition 2.2. *Assume that $\Lambda = (\lambda_k)_{k\in\mathbb{N}^*}$ is a normally ordered sequence satisfying (1.6). Then, there exists a biorthogonal family $\{q_{k,1}, q_{k,2}\}_{k\in\mathbb{N}^*}$ to $\{e_{k,1}, e_{k,2}\}_{k\in\mathbb{N}^*}$ in $L^2(0,T;\mathbb{C})$, i.e.,*

$$
\int_0^T e_{k,j}(t)q_{l,i}(t)\mathrm{d}t = \delta_{k,l}\delta_{j,i}, \quad \forall k,l \in \mathbb{N}^*, \forall j,i \in \{1,2\}.
$$

Moreover, we have the following estimate: for any $\varepsilon > 0$,

$$
\|q_{k,j}\|_{L^2(0,T)} \leq C_\varepsilon e^{(4c(\Lambda)+\varepsilon)\mathrm{Re}(\lambda_k)}, \quad \forall k \in \mathbb{N}^*, \forall j \in \{1,2\}.
\tag{2.10}
$$

Using this biorthogonal family we look for a control u satisfying (2.9) in the following form

$$
u(t) = \sum_{k\in\mathbb{N}^*} \alpha_k q_{k,1}(T-t)\frac{B^*\varphi_{k,1}}{\|B^*\varphi_{k,1}\|_U^2} + \beta_k q_{k,2}(T-t)\frac{B^*\varphi_{k,1}}{\|B^*\varphi_{k,1}\|_U^2}.
\tag{2.11}
$$

Thus, for u to solve (2.9), the coefficients α_k and β_k have to satisfy

$$
\begin{cases}
\alpha_k = -e^{-\lambda_k T}\langle y_0, \varphi_{k,1}\rangle, \\[2mm]
\alpha_k \dfrac{\langle B^*\varphi_{k,2}, B^*\varphi_{k,1}\rangle_U}{\|B^*\varphi_{k,1}\|_U^2} - \mu_k\beta_k = -e^{-\lambda_k T}\langle y_0, \varphi_{k,2} - T\mu_k\varphi_{k,1}\rangle.
\end{cases}
\tag{2.12}
$$

From assumption (2.6) it comes that there exist $\alpha_{k,1}, \alpha_{k,2} \in \mathbb{C}$ such that

$$
\alpha_{k,1}\varphi_{k,1} + \alpha_{k,2}\varphi_{k,2} \in \mathrm{Ker}(B^*).
$$

As $T_0 < +\infty$, inequality (1.4) implies that $B^*\varphi_{k,1} \neq 0$. Thus, there exists $\gamma_k \in \mathbb{C}$ such that $\gamma_k\varphi_{k,1} - \varphi_{k,2} \in \mathrm{Ker}(B^*)$ leading to

$$B^*\varphi_{k,2} = \gamma_k B^*\varphi_{k,1}.$$

Thus we obtain the following choice

$$\begin{cases} \alpha_k = -e^{-\lambda_k T}\langle y_0, \varphi_{k,1}\rangle, \\ \beta_k = -e^{-\lambda_k T}\dfrac{\gamma_k}{\mu_k}\langle y_0, \varphi_{k,1}\rangle - e^{-\lambda_k T}\dfrac{1}{\mu_k}\langle y_0, \varphi_{k,2} - T\mu_k\varphi_{k,1}\rangle. \end{cases} \tag{2.13}$$

To conclude we need to prove that this defines a control $u \in L^2(0,T;U)$ when $T > 2T_0$.

Using inequality (1.4) it comes that for any $\varepsilon > 0$,

$$\|B^*\varphi_{k,1}\|_U^2 \geq C_\varepsilon \mathrm{Re}(\lambda_k)e^{-2\mathrm{Re}(\lambda_k)(T_0+\varepsilon)}. \tag{2.14}$$

From the definition of γ_k, inequality (1.4) with $v = \gamma_k\varphi_{k,1} - \varphi_{k,2}$ implies for any $\varepsilon > 0$

$$\begin{aligned} \|\mu_k\varphi_{k,1}\|^2 &\geq C_\varepsilon \mathrm{Re}(\lambda_k)^2 e^{-2\mathrm{Re}(\lambda_k)(T_0+\varepsilon)}\|\gamma_k\varphi_{k,1} - \varphi_{k,2}\|^2 \\ &= C_\varepsilon \mathrm{Re}(\lambda_k)^2 e^{-2\mathrm{Re}(\lambda_k)(T_0+\varepsilon)}(1 + \gamma_k^2). \end{aligned}$$

Thus we deduce that

$$|\mu_k| \geq C_\varepsilon \mathrm{Re}(\lambda_k)e^{-\mathrm{Re}(\lambda_k)(T_0+\varepsilon)} \tag{2.15}$$

and

$$\frac{|\gamma_k|}{|\mu_k|} \leq C_\varepsilon \frac{1}{\mathrm{Re}(\lambda_k)}e^{\mathrm{Re}(\lambda_k)(T_0+\varepsilon)} \tag{2.16}$$

Using (2.10) with the assumption $c(\Lambda) = 0$ and (2.14) it comes that for any $\varepsilon > 0$,

$$\left\|\alpha_k q_{k,1}(T - \cdot)\frac{B^*\varphi_{k,1}}{\|B^*\varphi_{k,1}\|_U^2}\right\|_{L^2(0,T;U)} \leq C_\varepsilon \frac{|\langle y_0, \varphi_{k,1}\rangle|}{\sqrt{\mathrm{Re}(\lambda_k)}}e^{-\mathrm{Re}(\lambda_k)T}e^{\mathrm{Re}(\lambda_k)(T_0+2\varepsilon)}.$$

Similarly,

$$\begin{aligned} &\left\|\beta_k q_{k,2}(T - \cdot)\frac{B^*\varphi_{k,1}}{\|B^*\varphi_{k,1}\|_U^2}\right\|_{L^2(0,T;U)} \\ &\leq C_\varepsilon e^{-\mathrm{Re}(\lambda_k)T}\frac{e^{\mathrm{Re}(\lambda_k)(T_0+2\varepsilon)}}{\sqrt{\mathrm{Re}(\lambda_k)}}\left(\left(\frac{|\gamma_k|}{|\mu_k|} + T\right)|\langle y_0, \varphi_{k,1}\rangle| + \frac{1}{|\mu_k|}|\langle y_0, \varphi_{k,2}\rangle|\right) \\ &\leq C_\varepsilon \frac{1}{\mathrm{Re}(\lambda_k)^{3/2}}e^{-\mathrm{Re}(\lambda_k)(T-2T_0-3\varepsilon)}\left(|\langle y_0, \varphi_{k,1}\rangle| + |\langle y_0, \varphi_{k,2}\rangle|\right). \end{aligned}$$

As $T > 2T_0$, the last two estimates end the proof of Proposition 2.1.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

If we add another hypothesis on the behaviour of the control operator on the eigenfunctions then, we recover the exact value of the minimal time. More precisely, we get the following corollary.

Corollary 2.1. *Assume that the hypotheses of Proposition 2.1 hold. Assume moreover that for any $\varepsilon > 0$,*

$$\|B^*\varphi_{k,1}\|_U e^{\varepsilon \mathrm{Re}(\lambda_k)} \underset{k \to +\infty}{\longrightarrow} +\infty.$$

Then, the conclusion of Proposition 2.1 hold with $\widetilde{T} = T_0$.

Proof. The proof is exactly the same except that (2.14) is now replaced by

$$\|B^*\varphi_{k,1}\|_U \geq C_\varepsilon e^{-\varepsilon \mathrm{Re}(\lambda_k)},$$

for k sufficiently large. Following the lines leads to null controllability in time $T > T_0$. $\qquad\square$

2.3 Examples of application

2.3.1 Pointwise controllability of a one dimensional heat equation

Let us consider the system

$$\begin{cases} \partial_t y = \partial_{xx} y + \delta_{x_0} u(t), & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1). \end{cases} \tag{2.17}$$

The eigenfunctions and eigenvalues of the underlying operator are denoted by

$$\varphi_k(x) = \sqrt{2}\sin(k\pi x), \qquad \lambda_k = k^2\pi^2. \tag{2.18}$$

In this case, the condensation index of the sequence $\Lambda = \{\lambda_k; k \in \mathbb{N}\}$ is equal to 0 (see for instance Remark 1.6). It is proved in [13] that there is a minimal time for null controllability of (2.17) in $H = L^2(0,1)$ with controls in $L^2(0,T;\mathbb{R})$ given by

$$T_{min} = \limsup_{k \to +\infty} \frac{-\ln|\varphi_k(x_0)|}{\lambda_k}.$$

This is historically the first example of a minimal null control time in the parabolic framework.

The abstract setting to fit into the formalism of system (1.1) is given for example in [4, Sec. 6.1]. With this abstract setting it is straightforward to prove that this example satisfies the assumption of Theorem 1.2. Thus we deduce that $T_0 = T_{min}$.

2.3.2 A parabolic cascade system with a variable coupling coefficient: a particular case of internal control

Let $q \in L^\infty(0,1;\mathbb{R})$. Let us consider the following system studied in [5]

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & q(x) \\ 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 \\ u\mathbf{1}_\omega \end{pmatrix}, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases}$$
$$\tag{2.19}$$

with $\omega = (a, b) \subset (0, 1)$ and $\text{Supp}(q) \cap \omega = \emptyset$. For any $k \in \mathbb{N}^*$ let

$$I_k(q) = \int_0^1 q(x)\varphi_k(x)^2 \mathrm{d}x, \qquad I_{1,k}(q) = \int_0^a q(x)\varphi_k(x)^2 \mathrm{d}x. \qquad (2.20)$$

To fit into our abstract setting, we set $H = L^2(0, 1; \mathbb{R}^2)$, $U = L^2(\omega)$ and

$$A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{Id} - \begin{pmatrix} 0 & q(x) \\ 0 & 0 \end{pmatrix},$$
$$D(A) = H^2(0, 1; \mathbb{R}^2) \cap H_0^1(0, 1; \mathbb{R}^2).$$

The following proposition is proved in [5, Proposition 2.1, Lemma 2.3].

Proposition 2.3. *The spectrum of $-A^*$ is given by the sequence of eigenvalues $\lambda_k = k^2\pi^2$. Let*

$$\varphi_{k,1} = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \qquad \varphi_{k,2} = \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix},$$

with (λ_k, φ_k) is given by (2.18) and ψ_k is the unique solution of

$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) - \lambda_k\psi(x) = (I_k(q) - q(x))\,\varphi_k(x), & x \in (0, 1), \\ \psi(0) = \psi(1) = 0, \\ \displaystyle\int_0^1 \psi(x)\varphi_k(x)\mathrm{d}x = 0. \end{cases} \qquad (2.21)$$

Then,
$$(A^* + \lambda_k)\varphi_{k,1} = 0, \qquad (A^* + \lambda_k)\varphi_{k,2} = -I_k(q)\varphi_{k,1},$$

and the family $\{\varphi_{k,1}, \varphi_{k,2} : k \in \mathbb{N}^*\}$ forms a Riesz basis of H.

Moreover it is proved in [5, Theorem 1.3] that if

$$|I_k(q)| + |I_{1,k}(q)| \neq 0, \quad \forall k \in \mathbb{N}^*, \qquad (2.22)$$

which is a necessary and sufficient condition for approximate controllability, then there is a minimal time for null controllability of system (2.19) in H given by

$$T_{min} = \limsup_{k \to +\infty} \frac{\min\{-\ln|I_k(q)|, -\ln|I_{1,k}(q)|\}}{\lambda_k}.$$

Here, it is implicitly assumed that $\ln(0) = -\infty$.

From Proposition 2.3 it comes that if we assume that $I_k(q) = 0$ for any $k \in \mathbb{N}^*$ then the eigenvalue λ_k has geometric multiplicity equals to 2. Thus, in this case, system (2.19) fits into the setting of Theorem 1.2 (recall that $c(\Lambda) = 0$) and we deduce that

$$T_0 = T_{min} = \limsup_{k \to +\infty} \frac{-\ln|I_{1,k}(q)|}{\lambda_k}.$$

Remark 2.6. In this example, the general case where $I_k(q) \neq 0$ for some $k \in \mathbb{N}^*$, does not fit into our abstract setting. One needs to take into account generalised eigenfunctions to obtain a complete sequence but assumption (2.6) is not satisfied by the internal control operator. Nevertheless, this general case is discussed in Sec. 4.1.1 where we still prove that $T_0 = T_{min}$ (see Proposition 4.1).

2.3.3 A parabolic cascade system with a variable coupling coefficient: boundary control

Let $q \in L^{\infty}(0,1;\mathbb{R})$. Let us consider the following system studied in [5]

$$
\begin{cases}
\partial_t y + \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & q(x) \\ 0 & 0 \end{pmatrix} y = 0, & (t,x) \in (0,T) \times (0,1), \\
y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = 0, & t \in (0,T), \\
y(0,x) = y_0(x), & x \in (0,1),
\end{cases}
\tag{2.23}
$$

We keep the notations introduced in the previous example. It is proved in [5, Theorem 1.1] that if

$$
|I_k(q)| \neq 0, \quad \forall k \in \mathbb{N}^*,
\tag{2.24}
$$

which is a necessary and sufficient condition for approximate controllability, then there is a minimal time for null controllability of system (2.23) in $H^{-1}(0,1;\mathbb{R}^2)$ with controls in $L^2(0,T;\mathbb{R})$ given by

$$
T_{min} = \limsup_{k \to +\infty} \frac{-\ln|I_k(q)|}{\lambda_k}.
$$

From Proposition 2.3 it comes that system (2.23) fits into the setting of Proposition 2.1 (the assumption (2.6) being automatically satisfied as we are dealing with a scalar control depending only on the time variable). For a precise definition of the operator B as well as its admissibility we refer for example to [4, Sec. 6.2]. Thus $T_{min} \in [T_0, 2T_0]$.

Moreover it comes that

$$
\|B^* \varphi_{k,1}\|_U = |\varphi_k'(0)| = \sqrt{2}k\pi.
$$

Thus, from Corollary 2.1, one has $T_0 = T_{min}$.

3 Condensation of the spectrum

In this section we prove Theorems 1.3 and 1.4. The sketch of proof is the same as Theorem 1.2 except that the condensation of eigenvalues has to be taken into account while estimating the biorthogonal family.

3.1 Condensation of eigenvalues given by the Bohr index

The case $T < T_0$ follows directly from Theorem 1.1 thus we focus on the case $T_0 < +\infty$ and we fix $T > T_0$.

We start by proving that in this setting the condensation index is necessarily finite.

Proposition 3.1. *Assume that the hypotheses of Theorem 1.3 hold. Then, if $T_0 < +\infty$ it comes that*

$$
c(\Lambda) \leq T_0.
$$

Proof. From Hypothesis (1.9) it comes that there exist $\alpha_k, \alpha_j \in \mathbb{C}$ such that

$$\alpha_k \varphi_k + \alpha_j \varphi_j \in \mathrm{Ker}(B^*).$$

Inequality (1.4) implies that $B^* \varphi_k \neq 0$. Thus, there exists $\alpha_{k,j} \in \mathbb{C}$ such that

$$v_{k,j} := \alpha_{k,j} \varphi_k + \varphi_j \in \mathrm{Ker}(B^*).$$

Applying (1.4) it comes that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any $k \in \mathbb{N}^*$ and any $j \in \mathbb{N}^*$,

$$\|v_{k,j}\|^2 \leq C_\varepsilon e^{2(T_0+\varepsilon)\mathrm{Re}(\lambda_k)} \frac{\|(A^* + \lambda_k)v_{k,j}\|^2}{\mathrm{Re}(\lambda_k)^2}. \tag{3.1}$$

A direct computation leads to $(A^* + \lambda_k)v_{k,j} = (\lambda_k - \lambda_j)\varphi_j$. Thus, as we assumed that the eigenfunctions form a Riesz basis, there exists $c > 0$ such that for any $k, j \in \mathbb{N}^*$

$$\|v_{k,j}\|^2 \geq c(1 + \alpha_{k,j}^2) \geq c. \tag{3.2}$$

This leads to

$$|\lambda_k - \lambda_j| \geq C_\varepsilon \mathrm{Re}(\lambda_k) e^{-\mathrm{Re}(\lambda_k)(T_0+\varepsilon)}. \tag{3.3}$$

Thus, estimates (3.1), (3.2) and (3.3) imply $\mathrm{Bohr}(\Lambda) \leq T_0$. The conclusion follows from the assumption $\mathrm{Bohr}(\Lambda) = \mathrm{c}(\Lambda)$. $\qquad \square$

Remark 3.1. As seen in the previous proof, inequality (1.4) allows to estimate, under the structural assumption (1.9), the Bohr index of the sequence of eigenvalues. However, it is not clear if inequality (1.4) leads to an estimate for the condensation index in the general case. This is the main reason why we imposed the extra condition $\mathrm{Bohr}(\Lambda) = \mathrm{c}(\Lambda)$.

One should notice that the structural assumption (1.9) is necessary for Proposition 3.1 to hold. Consider the following system

$$\begin{cases} \partial_t y = \begin{pmatrix} \partial_{xx} & 0 \\ 0 & d\partial_{xx} \end{pmatrix} y, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, & y(t,1) = 0, \\ y(0,x) = y_0(x), \end{cases}$$

with $d > 0$. It is direct to notice that controllability in $H^{-1}(0,1;\mathbb{R}^2)$ holds in any time $T > 0$ and thus $T_0 = 0$. However, from [4, Proposition 6.20], one can choose d to prescribe any value in $[0, +\infty]$ for the condensation index of the spectrum of the underlying operator.

Proof of Theorem 1.3. Let $T > T_0$. As we assumed that the eigenvalues are all simple the strategy is exactly the one detailed in Sec. 1.4. Indeed the control given by (1.17) is a formal solution of the moment problem (1.14). Thus, in view of (1.17), the only remaining point is to prove that the following series

$$\sum_{k \in \mathbb{N}^*} e^{-\lambda_k T} \frac{\langle y_0, \varphi_k \rangle}{\|B^* \varphi_k\|_U^2} q_k(T-t) B^* \varphi_k$$

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converges in $L^2(0, T; U)$.

Recall that, from Proposition 1.1, we deduce that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any $k \in \mathbb{N}^*$

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{\mathrm{Re}(\lambda_k)(\mathrm{c}(\Lambda)+\varepsilon)}. \tag{3.4}$$

Thus, using Proposition 3.1, it comes that

$$\|q_k\|_{L^2(0,T)} \leq C_\varepsilon e^{\mathrm{Re}(\lambda_k)(T_0+\varepsilon)}. \tag{3.5}$$

As a consequence, we obtain that

$$\left\| e^{-\lambda_k T} \frac{\langle y_0, \varphi_k \rangle}{\|B^*\varphi_k\|_U^2} q_k(T - \cdot) B^* \varphi_k \right\|_{L^2(0,T;U)}$$
$$\leq C_\varepsilon \frac{|\langle y_0, \varphi_k \rangle|}{e^{\varepsilon \mathrm{Re}(\lambda_k)} \|B^*\varphi_k\|_U} e^{-\mathrm{Re}(\lambda_k)(T-T_0-2\varepsilon)}.$$

As $T > T_0$ the choice of ε sufficiently small and assumption (1.10) end the proof. $\qquad \square$

Remark 3.2. Notice that if in the previous proof, instead of assuming (1.10), we had applied inequality (1.4) with $v = \varphi_k$ and $\lambda = \lambda_k$ we would have obtained the following estimate

$$\|B^*\varphi_k\|_U \geq C_\varepsilon \sqrt{\mathrm{Re}(\lambda_k)} e^{-\mathrm{Re}(\lambda_k)(T_0+\varepsilon)}.$$

This would have led to controllability in time $T > 2T_0$. Using Proposition 3.1, this result is contained in Theorem 1.4 proved in the next subsection.

Remark 3.3. A careful inspection of the proof of Theorem 1.3 shows that assumption (1.9) can be replaced by the less explicit assumption

$$\mathrm{Ker}(B^*) \cap \mathrm{Span}(\varphi_{k_n}, \varphi_{j_n}) \neq \{0\}, \quad \forall n \in \mathbb{N}^*, \tag{3.6}$$

where the sequence $(k_n)_{n \in \mathbb{N}^*}$ is such that

$$\lim_{n \to +\infty} \frac{-\ln \inf\limits_{j \neq k_n} |\lambda_{k_n} - \lambda_j|}{\mathrm{Re}(\lambda_{k_n})} = \limsup_{k \to +\infty} \frac{-\ln \inf\limits_{j \neq k} |\lambda_k - \lambda_j|}{\mathrm{Re}(\lambda_k)}$$

and the sequence $(j_n)_{n \in \mathbb{N}^*}$ is such that

$$|\lambda_{k_n} - \lambda_{j_n}| = \inf_{j \neq k_n} |\lambda_{k_n} - \lambda_j|.$$

3.2   A qualitative result in the general setting

We prove in this subsection the qualitative result in the most general setting considered, namely Theorem 1.4. The proof follows the lines of the proof of Theorem 1.2 except that we take into account the effect of the condensation index in the estimate of the biorthogonal family (in time).

Proof of Theorem 1.4. The lack of null controllability when $T < T_0$ follows directly from Theorem 1.1. We thus focus on the case $T_0 < +\infty$ and we set $T > T_0 + \mathrm{c}(\Lambda)$.

• To clarify the presentation let us start with the simplified case where the eigenvalues are geometrically simple i.e. $r_k = 1$.

As already used the control given by (1.17) is a formal solution of the moment problem (1.14). Thus, in view of (1.17), the only thing to prove is that the following series

$$\sum_{k \in \mathbb{N}^*} e^{-\lambda_k T} \frac{\langle y_0, \varphi_k \rangle}{\|B^* \varphi_k\|_U^2} q_k(T - t) B^* \varphi_k$$

converges in $L^2(0, T; U)$.

For any $\varepsilon > 0$, inequality (1.4), implies that

$$\|B^* \varphi_k\|_U \geq C_\varepsilon \sqrt{\mathrm{Re}(\lambda_k)} e^{-\mathrm{Re}(\lambda_k)(T_0 + \varepsilon)}$$

and Proposition 1.1 implies that

$$\|q_k(T - \cdot)\|_{L^2(0,T)} \leq C_\varepsilon e^{\mathrm{Re}(\lambda_k)(\mathrm{c}(\Lambda) + \varepsilon)}.$$

Thus, the formal solution is actually a solution of the moment problem in $L^2(0, T; U)$ as soon as $T > T_0 + \mathrm{c}(\Lambda)$.

• We now turn to the case of geometrically multiple eigenvalues. As in the proof of Theorem 1.2 we define the sets

$$\Sigma_S = \{k \in \mathbb{N}^*; r_k = 1\}, \qquad \Sigma_M = \{k \in \mathbb{N}^*; r_k \geq 2\}$$

and the control $u = u_1 + u_2$ where u_1 is given by (2.3) and

$$u_2(t) = -\sum_{k \in \Sigma_M} q_k(T - t) \left(\sum_{1 \leq i \leq r_k} \langle y_0, \varphi_{k,i} \rangle e^{-\lambda_k T} \Psi_{k,i} \right),$$

with $(\Psi_{k,1}, \ldots, \Psi_{k,r_k})$ the biorthogonal family to $(B^* \varphi_{k,1}, \ldots, B^* \varphi_{k,r_k})$ given by Lemma 2.1. The same computations as in the proof of Theorem 1.2 imply the same estimate (2.4) for this biorthogonal family. This, together with Proposition 1.1, directly gives that $u \in L^2(0, T; U)$ if $T > T_0 + \mathrm{c}(\Lambda)$ and u is a solution of the moment problem.

$\square$

3.3 Examples

3.3.1 An academic example

Let (λ_k, φ_k) be defined as in (2.18). Let $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$0 < |f(s)| < s, \ \forall s \quad \text{and} \quad f(s) = \underset{s \to +\infty}{o}(s).$$

Consider the operator L_f defined by

$$L_f = \sum_{k \in \mathbb{N}^*} f(\lambda_k) \langle \cdot, \varphi_k \rangle \varphi_k.$$

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Let us consider the following system studied in [2, Sec. 2.2]

$$\begin{cases} \partial_t y = \begin{pmatrix} \partial_{xx} & L_f \\ L_f & \partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 \\ u(t) \end{pmatrix} & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1), \end{cases} \quad (3.7)$$

Let $H = L^2(0,1;\mathbb{R}^2)$ and $U = \mathbb{R}$. Define the operator A by

$$Ay = \begin{pmatrix} \partial_{xx} & L_f \\ L_f & \partial_{xx} \end{pmatrix} y,$$

$$D(A) = H^2(0,1;\mathbb{R}^2) \cap H_0^1(0,1;\mathbb{R}^2),$$

and the bounded control operator $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The spectrum of the operator $-A^*$ is given by $\Lambda = \{\lambda_k \pm f(\lambda_k); k \in \mathbb{N}^*\}$ and the associated eigenfunctions are given by

$$\varphi_k^\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_k \\ \mp \varphi_k \end{pmatrix}.$$

Moreover from [2] it can be proved that there exists a minimal null control time for system (3.7) given by

$$T_{min} = \limsup_{k \to +\infty} \frac{-\ln|f(\lambda_k)|}{\lambda_k}.$$

For k sufficiently large the eigenvalues are ordered in the following order

$$\lambda_k - f(\lambda_k) < \lambda_k + f(\lambda_k) < \lambda_{k+1} - f(\lambda_{k+1}).$$

Thus the spectrum Λ can be split into the following condensation grouping $G_k = \{\lambda_k \pm f(\lambda_k)\}$. From [27] we deduce that

$$c(\Lambda) = \mathrm{Bohr}(\Lambda) = \limsup_{k \to +\infty} \frac{-\ln|f(\lambda_k)|}{\lambda_k}.$$

Given the particular structure of the eigenfunctions it comes that

$$\varphi_k^+ + \varphi_k^- = \begin{pmatrix} \sqrt{2}\varphi_k \\ 0 \end{pmatrix}$$

and thus $B^*(\varphi_k^+ + \varphi_k^-) = 0$. According to Remark 3.3 it comes that system (3.7) fits into the setting of Theorem 1.3. Thus, it comes that

$$T_0 = T_{min} = \limsup_{k \to +\infty} \frac{-\ln|f(\lambda_k)|}{\lambda_k}.$$

Remark 3.4. This example is generalised in Sec. 4.2 in any space dimension. Though this generalisation will not fit into our abstract settings it will exhibit a minimal null control time still given by T_0.

3.3.2 A parabolic cascade system with different diffusions: boundary control

Let us consider the following system studied in [4, Sec. 6.2]

$$\begin{cases} \partial_t y = \begin{pmatrix} \partial_{xx} & 0 \\ 0 & d\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y, & (t,x) \in (0,T) \times (0,1), \\[2mm] y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = 0, & t \in (0,T), \\[2mm] y(0,x) = y_0(x), & x \in (0,1), \end{cases} \tag{3.8}$$

with $d \in (0, +\infty)$, $d \neq 1$.

To fit into our abstract setting, we set $H = H^{-1}(0,1;\mathbb{R}^2)$, $U = \mathbb{R}$ and

$$A = \begin{pmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x^2} & 0 \\ 0 & d\frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$D(A) = H^2(0,1;\mathbb{R}^2) \cap H_0^1(0,1;\mathbb{R}^2).$$

The precise definition of the operator B as well as its admissibility are given in [4, Sec. 6.2].

Let (λ_k, φ_k) be as in (2.18). The following proposition is proved in [4, Propositions 6.11 and 6.12].

Proposition 3.2. *The spectrum of $-A^*$ is given by the sequence of eigenvalues $\Lambda = \{\lambda_{k,1} = \lambda_k, \lambda_{k,2} = d\lambda_k; k \in \mathbb{N}^*\}$. These eigenvalues are simple if and only if $\sqrt{d} \notin \mathbb{Q}$.*

If $\sqrt{d} \notin \mathbb{Q}$, the associated eigenfunctions are given by

$$\varphi_{k,1} = \sqrt{\lambda_k} \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}, \qquad \varphi_{k,2} = \sqrt{\lambda_k} \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix},$$

with $\psi_k = \frac{1}{\lambda_k(d-1)} \varphi_k$.

The family $\{\varphi_{k,1}, \varphi_{k,2} : k \in \mathbb{N}^*\}$ forms a Riesz basis of H and

$$B^* \varphi_{k,1} = \frac{\sqrt{2}}{d-1}, \qquad B^* \varphi_{k,2} = \sqrt{2}\lambda_k.$$

Moreover it is proved in [4, Theorem 6.17] that if

$$\sqrt{d} \notin \mathbb{Q} \tag{3.9}$$

which is a necessary and sufficient condition for approximate controllability, then there is a minimal time for null controllability of system (3.8) in H given by

$$T_{min} = c(\Lambda).$$

From Proposition 3.2, it comes that the only remaining point to prove to fit system (3.8) into the setting of Theorem 1.3 is $c(\Lambda) = \text{Bohr}(\Lambda)$.

This property is proved in [26]. For the sake of completeness, we reproduce the proof in Appendix B.

Thus we deduce that

$$T_0 = T_{min} = c(\Lambda).$$

3.3.3 A parabolic cascade system with different diffusions: pointwise control

Let us consider the following system studied in [4, Sec. 6.3]

$$
\begin{cases}
\partial_t y = \begin{pmatrix} \partial_{xx} & 1 \\ 0 & d\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 \\ \delta_{x_0} u(t) \end{pmatrix} & (t,x) \in (0,T) \times (0,1), \\
y(t,0) = y(t,1) = 0, & t \in (0,T), \\
y(0,x) = y_0(x), & x \in (0,1),
\end{cases} \tag{3.10}
$$

with $d \in (0,+\infty)$, $d \neq 1$.

The abstract setting for the operator A is the same as the one introduced in Sec. 3.3.2. Thus the spectral analysis is the one given in Proposition 3.2. The abstract setting for the operator B is the one introduced in Sec. 2.3.1.

Moreover it is proved in [4, Theorem 6.31] that if

$$
\sqrt{d} \notin \mathbb{Q} \quad \text{and} \quad \varphi_k(x_0) \neq 0, \ \forall k \in \mathbb{N}^* \tag{3.11}
$$

which is a necessary and sufficient condition for approximate controllability, then there is a minimal time for null controllability of system (3.10) in H given by

$$
T_{min} = \max_{i=1,2} \limsup_{k \to +\infty} \frac{-\ln|\varphi_k(x_0)| - \ln|E'(\lambda_{k,i})|}{\lambda_{k,i}}.
$$

Thus, if d is such that $c(\Lambda) < +\infty$, system (3.10) directly fits into the setting of Theorem 1.4 and we deduce that

$$
T_0 \leq T_{min} \leq T_0 + c(\Lambda).
$$

4 Further results

This section is dedicated to further results. First we prove that for some examples of parabolic problems exhibiting a (positive finite) minimal null control time (that does not fit into the abstract setting developed in this article), this minimal time is still given by T_0. Then, we propose an academic example (inspired by (3.7)) to give a multi-dimensional setting where the minimal time is given by T_0. Contrarily to the rest of this article, the proof does not rely on the moment method. Finally we end this article with a (counter-) example showing that hypothesis (1.6) is not a technical assumption due to the use of moment method but is deeply related to the validity of the presented results.

4.1 Known examples of minimal time not covered by the abstract setting.

4.1.1 A parabolic cascade system with a variable coupling coefficient: the general case of internal control

Let us consider the setting and notations detailed in Sec. 2.3.2 for the study of system (2.19) but with the general hypothesis

$$
|I_k(q)| + |I_{1,k}(q)| \neq 0.
$$

Recall that, in this setting, the eigenvalue λ_k could have algebraic multiplicity equal to 2. One has,

Proposition 4.1. *Let T_0 be the minimal time defined by (1.5). Then*

$$T_0 = T_{min} = \limsup_{k \to +\infty} \frac{\min\{-\ln|I_k(q)|, -\ln|I_{1,k}(q)|\}}{\lambda_k}.$$

Proof. Assume that $T > T_{min}$. Then, from [5], system (2.19) is null controllable. Thus, Theorem 1.1 implies that inequality (1.4) holds and thus $T \geq T_0$ which gives $T_0 \leq T_{min}$.

Assume now that T is such that inequality (1.4) holds. From the definition of T_{min} we deduce that, for a subsequence still denoted k, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}^*$ such that

$$\max\{|I_k(q)|, |I_{1,k}(q)|\} \leq e^{-\lambda_k(T_{min}-\varepsilon)}, \quad \forall k \geq K. \tag{4.1}$$

We will use the following lemma which gives details on the structure of the generalised eigenfunctions. Its proof is given in [5, Proposition 2.6].

Lemma 4.1. *For any $k \in \mathbb{N}^*$, there exists $\tau_k \in \mathbb{R}$ such that*

$$\psi_k(x) = \tau_k \varphi_k(x) + \xi_k(x), \qquad \forall x \in \omega,$$

with ψ_k given in Proposition 2.3 and

$$\|\xi_k\|_{L^2(\omega)} \leq C\left(|I_k(q)| + |I_{1,k}(q)|\right).$$

Then, we apply inequality (1.4) with $\lambda = \lambda_k$ and $v = \varphi_{k,2} - \tau_k\varphi_{k,1} = \begin{pmatrix} \varphi_k \\ \psi_k - \tau_k\varphi_k \end{pmatrix}$. Using the previous lemma and Proposition 2.3, we deduce that,

$$\|v\|^2 = 1 + \|\psi_k - \tau_k\varphi_k\|^2 \geq 1,$$
$$\|(A^* + \lambda)v\|^2 = \|I_k(q)\varphi_{k,1}\|^2 = |I_k(q)|^2,$$
$$\|B^*v\|_U^2 = \|\psi_k - \tau_k\varphi_k\|_{L^2(\omega)}^2 = \|\xi_k\|_{L^2(\omega)}^2 \leq C\left(|I_k(q)|^2 + |I_{1,k}(q)|^2\right),$$

it comes that

$$1 \leq \frac{C}{\lambda_k}e^{2\lambda_k T}\left(|I_k(q)|^2 + |I_{1,k}(q)|^2\right).$$

Using the estimate (4.1) it comes that

$$1 \leq \frac{C}{\lambda_k}e^{2\lambda_k(T-(T_{min}-\varepsilon))}.$$

Letting k go to $+\infty$ implies $T \geq T_{min} - \varepsilon$ and thus $T_0 \geq T_{min}$ which ends the proof.

$\square$

4.1.2 Extension to cascade systems with a first order coupling term

In [14] the results of [5] concerning the controllability of system (2.19) and system (2.23) are extended (using the same technics) to the following systems

$$
\begin{cases}
\partial_t y + \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & p(x)\partial_x + q(x) \\ 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 \\ u\mathbf{1}_\omega \end{pmatrix}, \\
y(t,0) = y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
y(0,x) = y_0(x),
\end{cases}
$$

and

$$
\begin{cases}
\partial_t y + \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & p(x)\partial_x + q(x) \\ 0 & 0 \end{pmatrix} y = 0, \\
y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \\
y(t,1) = 0, \\
y(0,x) = y_0(x).
\end{cases}
$$

The results stated in Sec. 2.3.2, Sec. 2.3.3 and Proposition 4.1 directly extend to these systems.

4.1.3 Degenerate parabolic equation of Grushin-type

In [9] it is proved that the following system

$$
\begin{cases}
\partial_t y - \partial_{x_1 x_1} y - x_1^2 \partial_{x_2 x_2} y = \mathbf{1}_\omega u(t, x_1, x_1), & (t,(x_1,x_2)) \in (0,T) \times \Omega, \\
y(t, x_1, x_2) = 0, & (t,(x_1,x_2)) \in (0,T) \times \partial\Omega, \\
y(0, x_1, x_2) = y_0(x_1, x_2),
\end{cases}
$$

$$(4.2)$$

with $\Omega = (-1,1) \times (0,1)$ and $\omega = \left((-b,-a) \cup (a,b) \right) \times (0,1)$ is null controllable in $L^2(\Omega)$ if and only if $T > T_{min} = \frac{a^2}{2}$. Contrarily to previously mentioned one-dimensional examples, this is a two dimensional degenerate parabolic problem.

Let us specify the abstract setting to express system (4.2) as (1.1). We set $H = L^2(\Omega)$ and $U = L^2(\omega)$. For any $y, z \in C_0^\infty(\Omega)$, let

$$
(y, z) = \int_\Omega \left(\partial_{x_1} y \partial_{x_1} z + x_1^2 \partial_{x_2} y \partial_{x_2} \right) \mathrm{d}x_1 \mathrm{d}x_2
$$

$\| \cdot \|_V^2 = (\cdot, \cdot)$ and $V = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_V}$. The operator A is defined by

$$
D(A) = \left\{ y \in V ; \exists c > 0 \text{ such that } |(y, z)| \leq c\|z\|_{L^2(\Omega)}, \forall z \in V \right\},
$$
$$
\langle Ay, z \rangle = -(y, z), \quad \forall z \in V.
$$

The operator B is defined by $B = \mathbf{1}_\omega$.

Although this system does not fit into the considered settings (assumption (1.6) is not proved) the time T_0 is the minimal null control time.

Proposition 4.2. *Let T_0 be the minimal time defined by (1.5) with the operators A and B defined above. Then,*

$$
T_0 = T_{min} = \frac{a^2}{2}.
$$

Remark 4.1. The existence of a minimal null control time for this system was already proved in [7] in the case where $\omega = (a, b) \times (0, 1)$ but its value is not yet known. The following proof needs the value of the minimal time to be exactly $\frac{a^2}{2}$. This is why we considered a control domain made of two symmetric strips $\omega = \left((-b, -a) \cup (a, b)\right) \times (0, 1)$.

Proof. Assume that $T > \frac{a^2}{2}$. Then as problem (4.2) is null controllable, Theorem 1.1 implies that (1.4) holds and thus $T \geq T_0$ which gives $T_0 \leq \frac{a^2}{2}$.

Assume now that T is such that inequality (1.4) holds. The following properties are proved in [7, Lemma 2, Lemma 4].

Let λ_n be the smallest eigenvalue of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (n\pi)^2 x^2$ on the domain $H^2(-1, 1; \mathbb{R}) \cap H_0^1(-1, 1; \mathbb{R})$ and v_n the associated $L^2(-1, 1; \mathbb{R})$ normalised eigenfunction. Then, there exists $c > 0$ such that for any $n \in \mathbb{N}^*$,

$$n\pi \leq \lambda_n \leq n\pi + c.$$

Moreover, v_n is even and

$$\int_a^b v_n^2(x)\mathrm{d}x \underset{n \to +\infty}{\sim} \frac{e^{-a^2 n\pi}}{2a\pi\sqrt{n}}.$$

Thus applying inequality (1.4) with $\lambda = \lambda_n$ and $v(x_1, x_2) = \sqrt{2}v_n(x_1)\sin(n\pi x_2)$ it comes that

$$
\begin{aligned}
1 \leq C_T \frac{e^{2\lambda_n T}}{\lambda_n}\|B^*v\|_U^2 &= \frac{2C_T e^{2\lambda_n T}}{\lambda_n}\int_a^b v_n^2(x)\mathrm{d}x \\
&\leq \frac{C}{\lambda_n}e^{2n\pi T}\int_a^b v_n^2(x)\mathrm{d}x \\
&\underset{n \to +\infty}{\sim} \frac{C}{\sqrt{n}\lambda_n}e^{2n\pi\left(T - \frac{a^2}{2}\right)}.
\end{aligned}
\tag{4.3}
$$

Thus $T \geq \frac{a^2}{2}$. This leads to $T_0 \geq \frac{a^2}{2}$ and ends the proof. $\square$

Remark 4.2. As it may not appear in the previous proposition (due to the formulation using the infimum), in this example, inequality (1.4) even captures the controllability property when the time is equal to the minimal null control time. Indeed inequality (4.3) implies that (1.4) cannot hold in time $T = \frac{a^2}{2}$. From Theorem 1.1 it implies that system (4.2) cannot be controllable in time $T = \frac{a^2}{2}$.

4.2 An academic example in any dimension

In this section we give an abstract system (inspired by system (3.7)), in any space dimension, which exhibits a minimal null control time given by T_0 defined by (1.5).

Assume that $H = X \times X$ with X being a Hilbert space and that $U = X$. Assume that the operator $-A$ is positive self-adjoint on H and denote by $\{\lambda_{k,i}; i = 1, 2, k \in \mathbb{N}^*\}$ its sequence of eigenvalues sorted in the following order

$$\cdots < \lambda_{k,1} < \lambda_{k,2} < \lambda_{k+1,1} < \cdots$$

Assume moreover that the corresponding eigenfunctions have the following form

$$\varphi_{k,i} = \varphi_k e_i,$$

where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$ and $\{\varphi_k; k \in \mathbb{N}^*\}$ is a Hilbert basis of X.

We consider the bounded control operator $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$.

Proposition 4.3. *Assume that A and B are as defined above. Let T_0 be defined by (1.5).*

- *if $T_0 > 0$ and $T < T_0$ then system (1.1) is not null controllable in time T;*

- *if $T_0 < +\infty$ and $T > T_0$ then system (1.1) is null controllable in time T.*

Proof. Assume that $T_0 < T < +\infty$. Inequality (1.4) with $v = \varphi_{k,i}$ and $\lambda = \lambda_{k,i}$ implies $b_1 b_2 \neq 0$.

To prove null controllability of system (1.1) we prove that there is $C > 0$ such that for any $y_k^0 \in \mathbb{R}^2$ there exists $u_k \in L^2(0, T; \mathbb{R})$ with $\|u_k\|_{L^2} \leq C \|y_k^0\|$ such that the associated solution of

$$\begin{cases} y_k'(t) = A_k y_k(t) + B u_k(t), & \forall t > 0 \\ y_k(0) = y_k^0, \end{cases} \tag{4.4}$$

with $A_k := \begin{pmatrix} -\lambda_{k,1} & 0 \\ 0 & -\lambda_{k,2} \end{pmatrix}$ satisfies $y_k(T) = 0$. The Hilbert basis property of $\{\varphi_k; k \in \mathbb{N}^*\}$ will then end the proof.

As it is classical (see for instance [30]) this finite dimensional system is controllable and the minimal $L^2(0, T; \mathbb{R})$ norm control is given by

$$u_k(t) = -B^* e^{A_k(T-t)} Q_{k,T}^{-1} e^{A_k T} y_k^0. \tag{4.5}$$

where

$$Q_{k,T} = \int_0^T e^{A_k s} B B^* e^{A_k s} \mathrm{d}s.$$

We now need to estimate the norm of this control. As $BB^* = \begin{pmatrix} b_1^2 & b_1 b_2 \\ b_1 b_2 & b_2^2 \end{pmatrix}$, it comes that

$$Q_{k,T} = T \begin{pmatrix} b_1^2 \eta(-2T\lambda_{k,1}) & b_1 b_1 \eta(-T(\lambda_{k,1} + \lambda_{k,2})) \\ b_1 b_1 \eta(-T(\lambda_{k,1} + \lambda_{k,2})) & b_2^2 \eta(-2T\lambda_{k,2}) \end{pmatrix},$$

with $\eta : s \mapsto \frac{e^s - 1}{s}$. As,

$$\|u_k\|_{L^2(0,T)}^2 = \langle Q_{k,T}^{-1} e^{A_k T} y_k^0, e^{A_k T} y_k^0 \rangle \leq \max_{1 \leq i \leq 2} e^{-2\lambda_{i,k} T} \|Q_{k,T}^{-1}\| \|y_k^0\|^2, \tag{4.6}$$

we will estimate the norm of $Q_{k,T}^{-1}$. Let σ_k be its smallest eigenvalue. The following computations are closely related to those of [2, Sec. 2.2].

As $Q_{k,T} \in \mathcal{M}_2(\mathbb{R})$ we obtain

$$\frac{2 \det Q_{k,T}}{\operatorname{tr} Q_{k,T}} \geq \sigma_k \geq \frac{\det Q_{k,T}}{\operatorname{tr} Q_{k,T}}, \quad \forall k \geq 1, \tag{4.7}$$

where tr denotes the trace of a matrix.

For any $s_1, s_2 \in (-\infty, 0)$ there exists $\bar{s} \in (s_1, s_2)$ such that

$$\eta(2s_1)\eta(2s_2) = \eta(s_1 + s_2)^2 e^{(s_2 - s_1)^2 (\ln(\eta(\bar{s})))''}.$$

Explicit computations lead to

$$(\ln(\eta(s)))'' = \frac{1}{s^2} - \frac{1}{(e^{\frac{s}{2}} - e^{-\frac{s}{2}})^2}$$

and thus

$$\frac{1}{s^2 + 12} \leq (\ln(\eta(s)))'' \leq \frac{1}{s^2}.$$

Let $s_1 = -T\lambda_{k,1}$ and $s_2 = -T\lambda_{k,2}$. Thus,

$$\det Q_{k,T} = T^2 b_1^2 b_2^2 \eta(-2T\lambda_{k,1})\eta(-2T\lambda_{k,2})\left(1 - e^{-(\lambda_{k,1} - \lambda_{k,2})^2)\psi_k}\right)$$

with ψ_k satisfying

$$\frac{1}{T^2\lambda_{k,2}^2 + 12} \leq \psi_k \leq \frac{1}{4T^2\lambda_{k,1}^2}.$$

Now the trace of $Q_{k,T}$ is given by

$$\operatorname{tr}(Q_{k,T}) = T b_1^2 \eta(-2T\lambda_{k,1}) + T b_2^2 \eta(-2T\lambda_{k,2}).$$

Gathering these informations leads to

$$\sigma_k \geq \frac{T b_1^2 b_2^2 \eta(-2T\lambda_{k,1})\eta(-2T\lambda_{k,2})\left(1 - \exp\left(\frac{-(\lambda_{k,1} - \lambda_{k,2})^2)}{4\lambda_{k,1}^2}\right)\right)}{b_1^2 \eta(-2T\lambda_{k,1}) + b_2^2 \eta(-2T\lambda_{k,2})}$$

$$\geq C \frac{T b_1^2 b_2^2 \eta(-2T\lambda_{k,1})\eta(-2T\lambda_{k,2})(\lambda_{k,1} - \lambda_{k,2})^2}{\lambda_{k,1}^2 (b_1^2 \eta(-2T\lambda_{k,1}) + b_2^2 \eta(-2T\lambda_{k,2}))}$$

We now use inequality (1.4) to estimate $|\lambda_{k,2} - \lambda_{k,1}|$ in the same spirit as in the proof of Proposition 3.1. Applying inequality (1.4) with $v = \varphi_{k,2} - \frac{b_2}{b_1}\varphi_{k,1}$ and $\lambda = \lambda_{k,1}$ we obtain for any $\varepsilon > 0$,

$$|\lambda_{k,2} - \lambda_{k,1}|^2 \geq C_\varepsilon \lambda_{k,1}^2 e^{-2\lambda_{k,1}(T_0 + \varepsilon)}.$$

Finally using this inequality in the estimate of σ_k and using

$$\frac{1}{4T\lambda_{k,i}} \leq \eta(-2T\lambda_{k,i}) \leq \frac{1}{2T\lambda_{k,i}}$$

yield

$$\sigma_k \geq \frac{C_\varepsilon}{\lambda_{k,1} + \lambda_{k,2}} e^{-2\lambda_{k,1}(T_0 + \varepsilon)}.$$

From (4.6), we deduce

$$\|u_k\|_{L^2(0,T)}^2 \leq C_\varepsilon (\lambda_{k,1} + \lambda_{k,2}) e^{-2\lambda_{k,1}(T - (T_0 + \varepsilon))} \|y_k^0\|^2$$

which ends the proof.

$\square$

4.3 Comments on the behaviour of the spectrum

We recall some results on the harmonic oscillator proved in [15, Sec. 5.1].

Let $H = L^2(\mathbb{R})$ and $U = L^2(\mathbb{R})$. Define the operator

$$A = -\partial_{xx} + x^2, \qquad D(A) = \{y \in H^2(\mathbb{R}); x \mapsto x^2 y(x) \in L^2(\mathbb{R})\}.$$

The operator is self-adjoint and generates a C^0 semigroup. Its eigenvalue are $\{\lambda_k = 2k - 1; k \in \mathbb{N}^*\}$ and its eigenfunctions are the Hermite polynomials (see for instance [28]). Thus they form a Hilbert basis.

Consider the bounded operator B defined by $B = \mathbf{1}_{(-\infty, x_0)}$ with $x_0 \in \mathbb{R}$. In this setting it is proved in [15, Proposition 5.1] that system (1.1) is not null controllable for any time $T > 0$.

However, still in [15, Proposition 5.1] it is proved that the following inequality holds: there exists $M, m \in \mathbb{R}$ such that

$$\|v\|^2 \leq M\|(A^* + \lambda)v\|^2 + m\|B^*v\|_U^2, \qquad \forall v \in D(A), \ \forall \lambda \in \mathbb{R}.$$

As for any $T > 0$ the functions

$$\lambda \in (0, +\infty) \mapsto \frac{e^{2\lambda T}}{\lambda^2} \quad \text{and} \quad \lambda \in (0, +\infty) \mapsto \frac{e^{2\lambda T}}{\lambda}$$

are bounded from below, following definition (1.5), it comes that $T_0 = 0$. Thus inequality (1.4) is valid for any T whereas null controllability holds for no value of T.

It is fundamental to notice that this example satisfies every assumption of Theorem 1.4 except that in this situation

$$\sum_{k \in \mathbb{N}^*} \frac{1}{\lambda_k} = \sum_{k \in \mathbb{N}^*} \frac{1}{2k - 1} = +\infty.$$

Thus assumption (1.6) should be considered as a technical assumption due to the use of moment method but as a concrete limitation of the characterization of null controllability using inequality (1.4).

A General estimate of biorthogonal families

Let us start recalling that the Blaschke product associated with the sequence $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ is the function $W : \mathbb{C}_+ \to \mathbb{C}$ defined by:

$$\begin{cases} W(\lambda) = W(\lambda, \Lambda) = \displaystyle\prod_{k \geq 1} \delta_k \frac{1 - \lambda/\lambda_k}{1 + \lambda/\overline{\lambda_k}}, \quad \lambda \in \mathbb{C}_+, \\[2mm] \delta_k = \dfrac{\lambda_k}{\overline{\lambda_k}} \dfrac{|\lambda_k - 1|}{|\lambda_k + 1|} \dfrac{\overline{\lambda_k} + 1}{\overline{\lambda_k} - 1} \quad (\delta_k = 1 \text{ if } \lambda_k = 1). \end{cases}$$

Under assumption (1.6), the previous function W is well-defined, $W \in H^\infty(\mathbb{C}_+)$, the space of bounded and holomorphic functions defined on $\mathbb{C}_+$, is defined almost everywhere on $i\mathbb{R}$ and satisfies $|W(\lambda)| < 1$, for $\mathrm{Re}\,\lambda > 0$, and $|W(i\tau)| = 1$, for

almost every $\tau \in \mathbb{R}$ (see for instance [3, Lemma 4.2]). A direct computation also gives

$$|W'(\lambda_k)| = \frac{1}{2\mathrm{Re}(\lambda_k)}\mathcal{P}_k^{-1}, \quad \text{with } \mathcal{P}_k := \prod_{\substack{l \geq 1 \\ l \neq k}} \left| \frac{1 + \lambda_k / \overline{\lambda_l}}{1 - \lambda_k / \lambda_l} \right|.$$

On the other hand (see [4, Theorem 3.9]),

$$c(\Lambda) = \limsup_{k \to +\infty} \frac{-\ln|W'(\lambda_k)|}{\mathrm{Re}(\lambda_k)} = \limsup_{k \to +\infty} \frac{\ln \mathcal{P}_k}{\mathrm{Re}(\lambda_k)}. \qquad (\mathrm{A.1})$$

Let us now prove Proposition 2.2 when $T = \infty$. To this end, let us first consider functions $e_{k,1} = e^{-\lambda_k t}$ and $e_{k,2} = te^{-\lambda_k t}$ in $L^2(0, \infty; \mathbb{C})$. Assumption (1.6) allows us to apply [3, Proposition 4.1] (with $\eta = 2$), and deduce the existence of a biorthogonal family $\{\widetilde{q}_{k,1}, \widetilde{q}_{k,2}\}_{k \in \mathbb{N}^*} \subset A(\Lambda, \infty)$ to $\{e_{k,1}, e_{k,2}\}_{k \in \mathbb{N}^*}$ which satisfies for any $k \in \mathbb{N}^*$ and any $j \in \{1, 2\}$

$$\|\widetilde{q}_{k,j}\|_{L^2(0,\infty;\mathbb{C})} \leq C \left[1 + \frac{1}{\mathrm{Re}(\lambda_k)}\right] (\mathrm{Re}(\lambda_k))^4 |1 + \lambda_k|^8 \mathcal{P}_k^4, \qquad (\mathrm{A.2})$$

where $C > 0$ is a constant and, for $\tau \in (0, \infty]$, $A(\Lambda, \tau)$ is the closed subspace of $L^2(0, \tau; \mathbb{C})$ given by

$$A(\Lambda, \tau) = \overline{\mathrm{span}\ \{e_{k,1}, e_{k,2} : k \in \mathbb{N}^*\}}^{L^2(0,\tau;\mathbb{C})}.$$

From formulae (A.1) and (A.2), we deduce that, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|\widetilde{q}_{k,j}\|_{L^2(0,\infty;\mathbb{C})} \leq C_\varepsilon e^{(4c(\Lambda)+\varepsilon)\mathrm{Re}(\lambda_k)}, \quad \forall k \in \mathbb{N}^*, \quad \forall j \in \{1, 2\}. \qquad (\mathrm{A.3})$$

This proves Proposition 2.2 and (2.10) when $T = \infty$.

The general case $T \in (0, \infty)$ can be deduced from the previous result and

Lemma A.1. *Let $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ be a sequence satisfying (1.6). Then, for any $T \in (0, \infty)$, the restriction operator*

$$R_T : \varphi \in A(\Lambda, \infty) \longmapsto R_T \varphi = \varphi_{|(0,T)} \in A(\Lambda, T)$$

is a topological isomorphism. In particular, there exists a positive constant C_T, depending on the sequence Λ and T, such that

$$\|\varphi\|_{L^2(0,\infty;\mathbb{C})} \leq C_T \|\varphi\|_{L^2(0,T;\mathbb{C})}, \quad \forall \varphi \in A(\Lambda, \infty). \qquad \square$$

Before proving Lemma A.1, let us complete the proof of Proposition 2.2. Let us set

$$q_{k,j} = \left(R_T^{-1}\right)^* \widetilde{q}_{k,j} \in A(\Lambda, T), \quad \forall k \in \mathbb{N}^*, \forall j \in \{1, 2\}.$$

From Lemma A.1 and (A.3), it is clear that the function $q_{k,j}$ satisfies, for any $\varepsilon > 0$, inequality (2.10) for any $k \in \mathbb{N}^*$ and $j \in \{1, 2\}$.

On the other hand, we can write

$$\begin{aligned}
\delta_{k,l}\delta_{j,i} &= (e_{k,j}, \widetilde{q}_{l,i})_{L^2(0,\infty;\mathbb{C})} \\
&= (R_T^{-1} R_T e_{k,j}, \widetilde{q}_{l,i})_{L^2(0,\infty;\mathbb{C})} \\
&= (R_T e_{k,j}, \left(R_T^{-1}\right)^* \widetilde{q}_{l,i})_{L^2(0,T;\mathbb{C})} \\
&= (e_{k,j}, q_{l,i})_{L^2(0,T;\mathbb{C})}, \quad \forall k, l \in \mathbb{N}^*, \forall j, i \in \{1, 2\}
\end{aligned}$$

i.e., $\{q_{k,1}, q_{k,2}\}_{k \in \mathbb{N}^*} \subset A(\Lambda, T)$ is a biorthogonal family to $\{e_{k,1}, e_{k,2}\}_{k \in \mathbb{N}^*}$ in $L^2(0, T; \mathbb{C})$. This ends the proof of Proposition 2.2.

Proof of Lemma A.1. The proof follows the same arguments of the proof of [4, Lemma 4.2] and uses [4, Lemma 4.6].

From the definition of the space $A(\Lambda, T)$, it is clear that we will obtain the proof if we show that there exits a positive constant $C_T > 0$ such that

$$\|P\|_{L^2(0,\infty;\mathbb{C})} \leq C_T \|P\|_{L^2(0,T;\mathbb{C})}, \quad \forall P \in \mathfrak{P},$$

where

$$\mathfrak{P} := \left\{ P : P(z) = \sum_{j=1}^{N} \left(a_j e^{-\lambda_j z} + b_j z e^{-\lambda_j z} \right), \text{ with } N \geq 1, \ a_j, b_j \in \mathbb{C} \right\}.$$

We argue by contradiction. Therefore, assume that there exists a sequence $\{P_m\}_{m \geq 1} \subset \mathfrak{P}$ such that

$$\lim \|P_m\|_{L^2(0,T;\mathbb{C})} = 0 \quad \text{and} \quad \|P_m\|_{L^2(0,\infty;\mathbb{C})} = 1 \quad \forall m \geq 1. \tag{A.4}$$

Observe that condition (1.6) implies the existence of $\theta_\delta \in [0, \pi/2)$ such that

$$\Lambda = \{\lambda_k\}_{k \geq 1} \subset S_\delta := \{z = re^{i\theta} \in \mathbb{C} : r > 0, \ |\theta| \leq \theta_\delta\}. \tag{A.5}$$

Let $\theta_0 \in (\theta_\delta, \pi/2)$ where $\theta_\delta \in (0, \pi/2)$ is such that (A.5) holds. Let us also fix $\varepsilon > 0$, $\theta_0 \in (\theta_\delta, \pi/2)$ and $\tau \in (0, \cos\theta_0)$ and consider the set

$$S_{\varepsilon,\theta_0,\tau} = \left\{ z = x + iy : x \geq \varepsilon, \ \frac{|y|}{x} \leq \frac{\cos\theta_0 - \tau}{\sin\theta_0} \right\}.$$

Applying [4, Lemma 4.6] in the set $S_{\varepsilon,\theta_0,\tau}$ we deduce that there exists a constant $C_\varepsilon > 0$ such that, for any $P \in \mathfrak{P}$, one has

$$|P(z)| \leq C_\varepsilon \|P\|_{L^2(0,\infty;\mathbb{C})} \left(1 + |z|\right) e^{-\frac{1}{4}\alpha\tau \mathrm{Re}(z)}, \quad \forall z \in S_{\varepsilon,\theta_0,\tau},$$

where $\alpha = \min_{k \geq 1} |\lambda_k| > 0$. Using now (A.4), we can conclude that the sequence $\{P_m\}_{m \geq 1}$ is uniformly bounded on the domain $S_{\varepsilon,\theta_0,\tau}$. Therefore, it is a normal family of holomorphic functions on $S_{\varepsilon,\theta_0,\tau}$ and there exists a subsequence, still denoted by $\{P_m\}_{m \geq 1}$, and a holomorphic function P on $S_{\varepsilon,\theta_0,\tau}$ such that $P_m \to P$ uniformly on the compact sets of $S_{\varepsilon,\theta_0,\tau}$. Furthermore, from Lebesgue's theorem, $P_m \to P$ in $L^2(\eta, \infty; \mathbb{C})$ for any $\eta > \varepsilon$. Assumption (A.4) implies that $P \equiv 0$ on the interval (η, T) for any $\eta : 0 < \varepsilon < \eta < T$. Since P is holomorphic on $S_{\varepsilon,\theta_0,\tau}$, we get $P \equiv 0$ on (ε, ∞). Whence $\lim \|P_m\|_{L^2(T,\infty;\mathbb{C})} = 0$ and since, by our assumption, $\lim \|P_m\|_{L^2(0,T;\mathbb{C})} = 0$ it follows that

$$\lim \|P_m\|_{L^2(0,\infty;\mathbb{C})} = 0.$$

This contradicts (A.4) and provides the proof of Lemma A.1.

$\square$

B Condensation index of a particular sequence

Consider the sequence defined by $\Lambda = \{\lambda_k, d\lambda_k; k \in \mathbb{N}^*\}$ with $\lambda_k = k^2\pi^2$ and $\sqrt{d} \notin \mathbb{Q}$.

For the sake of completeness, we reproduce here the computations of [26] communicated by E.H. SAMB which proves that $c(\Lambda) = \mathrm{Bohr}(\Lambda)$.

From the computations in the proof of [4, Proposition 6.20], it comes that

$$c(\Lambda) = \max\{\ell_1, \ell_2\},$$

with

$$\ell_1 = \limsup_{k \to +\infty} \frac{-\ln|E'(\lambda_k)|}{\lambda_k}, \qquad \ell_2 = \limsup_{k \to +\infty} \frac{-\ln|E'(d\lambda_k)|}{d\lambda_k}.$$

It is also proved that

$$\ell_1 = \limsup_{k \to +\infty} \frac{-\ln\left|\sin\left(\frac{k\pi}{\sqrt{d}}\right)\right|}{\lambda_k}, \qquad \ell_2 = \limsup_{k \to +\infty} \frac{-\ln\left|\sin\left(k\pi\sqrt{d}\right)\right|}{d\lambda_k}.$$

The fact that the condensation index is indeed a Bohr index follows from the application of the next lemma to the particular sequence $(k_n)_{n \in \mathbb{N}^*}$ such that

$$\lim_{n \to +\infty} \frac{-\ln|E'(\lambda_{k_n})|}{\lambda_{k_n}} = \ell_1.$$

Lemma B.1 ([26]). *For any sequence of integers $(k_n)_{n \in \mathbb{N}^*}$ going to $+\infty$ as n goes to $+\infty$ there exists a sequence $(j_n)_{n \in \mathbb{N}^*}$ such that*

$$\lim_{n \to +\infty} \left| \frac{-\ln|\lambda_{k_n} - d\lambda_{j_n}|}{\lambda_{k_n}} - \frac{-\ln|E'(\lambda_{k_n})|}{\lambda_{k_n}} \right| = 0.$$

Proof. For any $n \in \mathbb{N}^*$, let j_n the nearest integer to $\frac{k_n}{\sqrt{d}}$. Since

$$\left| \frac{k_n\pi}{\sqrt{d}} - j_n\pi \right| \le \frac{\pi}{2},$$

we obtain

$$2\left| \frac{k_n}{\sqrt{d}} - j_n \right| \le \sin\left| \frac{k_n\pi}{\sqrt{d}} - j_n\pi \right| = \left|\sin\left(\frac{k_n\pi}{\sqrt{d}}\right)\right| \le \pi\left| \frac{k_n}{\sqrt{d}} - j_n \right|.$$

Using [4, Proposition 6.20] leads to

$$|E'(\lambda_k)| = \frac{d\sinh(k\pi)\sinh\left(\frac{k\pi}{\sqrt{d}}\right)}{2k^5\pi^3}\left|\sin\left(\frac{k\pi}{\sqrt{d}}\right)\right|, \quad \forall k \in \mathbb{N}^*.$$

Finally, this yields

$$
\left| \frac{-\ln|\lambda_{k_n} - d\lambda_{j_n}|}{\lambda_{k_n}} - \frac{-\ln|E'(\lambda_{k_n})|}{\lambda_{k_n}} \right| = \frac{1}{\lambda_{k_n}} \left| \ln \left| \frac{E'(\lambda_{k_n})}{\lambda_{k_n} - d\lambda_{j_n}} \right| \right|
$$

$$
= \frac{1}{\lambda_{k_n}} \left| \ln \left(\frac{d \sinh(k\pi) \sinh\left(\frac{k\pi}{\sqrt{d}} \right)}{2k^5 \pi^5 (k_n + \sqrt{d}j_n)} \left| \frac{\sin\left(\frac{k\pi}{\sqrt{d}} \right)}{k_n - \sqrt{d}j_n} \right| \right) \right|
$$

$$
\leq \frac{1}{\lambda_{k_n}} \left| \ln \left(\frac{d \sinh(k\pi) \sinh\left(\frac{k\pi}{\sqrt{d}} \right)}{2k^5 \pi^5 (k_n + \sqrt{d}j_n)} \right) \right| + \frac{1}{\lambda_{k_n}} \left| \ln \frac{\pi}{\sqrt{d}} \right|
$$

$$
\leq \frac{1}{\lambda_{k_n}} \left| \ln \frac{d e^{k_n \pi} e^{\frac{k_n \pi}{\sqrt{d}}}}{8 k_n^5 \pi^5 (k_n + \sqrt{d}j_n)} \right| + \frac{1}{\lambda_{k_n}} \left| \ln \frac{\pi}{\sqrt{d}} \right|
$$

The right-hand side goes to 0 as n goes to $+\infty$ which ends the proof.

$\square$

References

[1] D. Allonsius, F. Boyer, and M. Morancey. Spectral analysis of discrete elliptic operators and applications in control theory. Preprint, December 2016.

[2] F. Ammar Khodja, A. Benabdallah, C. Dupaix, and I. Kostin. Null-controllability of some systems of parabolic type by one control force. *ESAIM Control Optim. Calc. Var.*, 11(3):426–448, 2005.

[3] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials. *J. Math. Pures Appl. (9)*, 96(6):555–590, 2011.

[4] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences. *J. Funct. Anal.*, 267(7):2077–2151, 2014.

[5] F. Ammar Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. New phenomena for the null controllability of parabolic systems: minimal time and geometrical dependence. *J. Math. Anal. Appl.*, 444(2):1071–1113, 2016.

[6] K. Beauchard and P. Cannarsa. Heat equation on the Heisenberg group: observability and applications. *J. Differential Equations*, 262(8):4475–4521, 2017.

[7] K. Beauchard, P. Cannarsa, and R. Guglielmi. Null controllability of Grushin-type operators in dimension two. *J. Eur. Math. Soc. (JEMS)*, 16(1):67–101, 2014.

[8] K. Beauchard, B. Helffer, R. Henry, and L. Robbiano. Degenerate parabolic operators of Kolmogorov type with a geometric control condition. *ESAIM Control Optim. Calc. Var.*, 21(2):487–512, 2015.

[9] K. Beauchard, L. Miller, and M. Morancey. 2D Grushin-type equations: minimal time and null controllable data. *J. Differential Equations*, 259(11):5813–5845, 2015.

[10] V. Bernstein. *Leçons sur les Progrès Récents de la Théorie des Séries de Dirichlet.* Gauthier-Villars, 1933.

[11] P. Cannarsa, P. Martinez, and J. Vancostenoble. The cost of controlling weakly degenerate parabolic equations by boundary controls. *Math. Control Relat. Fields*, 7(2):171–211, 2017.

[12] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2007.

[13] S. Dolecki. Observability for the one-dimensional heat equation. *Studia Math.*, 48:291–305, 1973.

[14] M. Duprez. Controllability of a 2x2 parabolic system by one force with space-dependent coupling term of order one. *ESAIM: COCV*, 2017. Forthcoming paper.

[15] T. Duyckaerts and L. Miller. Resolvent conditions for the control of parabolic equations. *J. Funct. Anal.*, 263(11):3641–3673, 2012.

[16] H. O. Fattorini. Some remarks on complete controllability. *SIAM J. Control*, 4:686–694, 1966.

[17] H. O. Fattorini and D. L. Russell. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, 43:272–292, 1971.

[18] H. O. Fattorini and D. L. Russell. Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations. *Quart. Appl. Math.*, 32:45–69, 1974/75.

[19] A.V. Fursikov and O.Y. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series.* Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[20] M. L. J. Hautus. Controllability and observability conditions of linear autonomous systems. *Nederl. Akad. Wetensch. Proc. Ser. A 72 = Indag. Math.*, 31:443–448, 1969.

[21] B. Jacob and H. Zwart. Counterexamples concerning observation operators for C_0-semigroups. *SIAM J. Control Optim.*, 43(1):137–153, 2004.

[22] J. Lagnese. Control of wave processes with distributed controls supported on a subregion. *SIAM J. Control Optim.*, 21(1):68–85, 1983.

[23] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.

[24] G. Olive. Boundary approximate controllability of some linear parabolic systems. *Evol. Equ. Control Theory*, 3(1):167–189, 2014.

[25] D.L. Russell and G. Weiss. A general necessary condition for exact observability. *SIAM J. Control Optim.*, 32(1):1–23, 1994.

[26] E.H. Samb. Boundary null-controllability of a system of two parabolic equations with different diffusions and non constant coupling term. In Preparation, 2017.

[27] J.R. Shackell. Overconvergence of Dirichlet series with complex exponents. *J. Analyse Math.*, 22:135–170, 1969.

[28] G. Szego. *Orthogonal polynomials*. American Mathematical Society, Providence, R.I., third edition, 1967. American Mathematical Society Colloquium Publications, Vol. 23.

[29] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.

[30] J. Zabczyk. *Mathematical control theory*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. An introduction, Reprint of the 1995 edition.