# Data-Driven Reachability Analysis for the Reconfiguration of Vehicle Control Systems

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**Abstract:** The paper presents a reconfigurable control strategy for the lateral stability of autonomous vehicles. The control strategy is based on the analysis of big data, which are provided by the sensor networks of autonomous vehicles. The core of the analysis method is a machine learning algorithm, with which the impacts of various vehicle signals on the lateral dynamics have been examined. In the analysis several scenarios with faults in the steering and in-wheel systems are considered using a high-fidelity simulation software. The results of the examination are built into the fault-tolerant reconfiguration strategy.

*Keywords:* reconfiguration strategy, big data analysis, autonomous vehicle systems

## 1. INTRODUCTION AND MOTIVATION

In the last years the autonomous vehicle functionalities have high impact on the trends of the vehicle control research area. Novel perspectives and technological approaches have been appeared in this field (e.g. artificial intelligence, deep learning and big data), while some conventional vehicle dynamic problems have been transformed. For example, in most advanced safety systems the purposes of the lateral stability analysis are to detect the critical interventions of drivers, and perform control actions by using control systems and actuators to avoid the loss of stability, see Gáspár et al. (2017); Palmieri et al. (2011). However, the purpose of the stability analysis in the field of autonomous vehicles is to avoid all of the critical situations by monitoring the environment and the traffic and apply the appropriate coordinated control systems of the vehicle, see Carvalho et al. (2013); Funke et al. (2017).

Ensuring stability is a crucial task in the reconfigurable and fault-tolerant control of autonomous vehicles. A reconfigurable control methodology has been developed to guarantee the stability and the performances of the system against various fault scenarious Gáspár et al. (2017). In the reconfigurable control the control systems and the actuators have similar impact on the dynamics of the vehicle. In the case of automated lateral control systems the most common faults might be the degradation of electric power steering mechanisms or wiring failures. Moreover, when torque-vectoring control is used, the fault or performance degradation of in-wheel hub motors due to overheating, mechanical failures, or motor control faults may result in hazardous vehicle instability Ifedi et al. (2013). In the reconfigurable control the coordination of actuators is modified by a so-called selection strategy. The appropriate actuator selection requires information about the impact of the interventions on vehicle dynamics. A theoretical background of the actuator selection in the reconfiguration strategy is proposed in Gáspár and Németh (2015). In this paper the reconfiguration strategy is based on the controllability analysis of the control interventions using the Sum-of-Squares method. Although the methods yield promising results, due to the numerical complexity of the controllability problem the application often leads to difficult solutions.

Reachability has an important role in the derivation of the reconfiguration strategy Gáspár et al. (2017), which is defined as follows. It given is a continuous-time linear time invariant system $\dot{x} = A(\rho)x + B(\rho)u$ with initial condition $x(0) = 0$. It is considered the set of reachable states with inputs $u$ whose components have unit-energy $u^T u \leq 1$ as Boyd et al. (1997):

$$\mathcal{R} \triangleq \left\{ x(T) \left| \begin{array}{ll} (x(t), u(t)) \,|\, \dot{x}(t) = Ax(t) + Bu(t), & x(0) = 0, \\ u^T(t)u(t) \leq 1, & T \geq 0 \end{array} \right. \right\} \quad (1)$$

In this paper a data-driven reachability analysis is applied to the reconfiguration strategy. The numerous measured data concerning autonomous vehicles are used for different purposes in driverless vehicles and intelligent traffic control. The reconfiguration strategy is developed by using data-mining and machine learning algorithms. Big data have been used in the prediction of vehicle slip through the combination of individual measurements of the vehicle and database information, see Jeon et al. (2015). Deep learning methods using the adaptive neuro-fuzzy modeling framework together with big data analysis have been applied to vehicle velocity prediction in Cheng et al. (2017). Data-mining algorithms to process electric vehicle battery data for energy-consumption and driving range

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purposes have been utilized by Lee and Wu (2015). An optimal trajectory selection strategy focusing on the safety of the autonomous vehicles using cloud database is found in Najada and Mahgoub (2016). Zhu et al. (2016) has presented the idea of the path planning strategy of public vehicle systems, which uses the traffic data. Big data can also be used for safety purposes, as e.g. Ghomi et al. (2017) has presented an application example to identify vehicle passenger injury factors. A reachability analysis using machine learning approach for space aircraft and robotic vehicles is performed in Allen et al. (2014).

This paper proposes the analysis of autonomous lateral vehicle dynamics using machine learning and big data mining methods. The results of the analysis are used to derive a reconfiguration strategy which handles the fault scenarios of different actuators, such as steering and in-wheel electric motors. The reconfigurable control improves the lateral stability of the vehicle and guarantees various performances to be specified. The application of the big data mining methodology provides new possibilities and challenges in the field of autonomous vehicle control systems, especially on the reconfigurable control strategy to handle fault scenarios.

The structure of the paper is the following. Section 2 proposes the fundamentals of the applied machine learning algorithm. It also presents the core of the applied machine learning algorithm with which the relationships are explored. The results are demonstrated in Section 3 through various figures. Section 4 shows the reachability analysis in the reconfiguration strategy. Finally, the contributions of the paper and the further challenges are summarized in Section 5. Furthermore, Appendix A provides a brief summary of the background of the reachability set computation.

## 2. FUNDAMENTALS OF THE APPLIED MACHINE LEARNING ALGORITHM

The following section presents the fundamentals of the applied machine learning algorithm, which is used for the reachability analysis. In this paper C4.5 machine learning algorithm is used to perform the analysis. This widely used machine learning algorithm is able to generate decision trees for the classification of large amounts of data. The original algorithm was developed in 1960 by Hunt (1962). Over the past decades, the original method has been significantly improved, see e.g. Mitchell (1997); Quinlan (1993). In the following the basic concept of C4.5 method is presented.

The initial step of the algorithm is the collection of data from varying instances. In general, an instance has several types of values called attributes $A = \{A_1, A_2, ..., A_k\}$. An attribute can be an independent variable or a dependent variable called class. A dependent, class variable $C$ is always discrete with a predefined set of values $C = C_1, C_2...C_m$ with $m$ members. The collected data are divided into two parts:

(1) a training set, which is used for teaching the algorithm,
(2) a test set, which is used for evaluating the results.

The aim of the algorithm is to create a function based on the training set which is able to classify the instances by the selected class

$$DOM(A_1) \times DOM(A_2) \times ... \times DOM(A_k) \to DOM(C), \quad (2)$$

where $DOM(*)$ denotes the selected domain of function. The created function is ordered into a tree structure, as illustrated in an example, see Figure **??**. A tree consists of nodes and leaves. A node is associated with an attribute and a condition, and has at least two outcomes, which depend on the current value of the attribute. A leaf determines the value of the class for the current instance. The size of the resulting tree is a crucial part of the algorithm, since a large and complex tree makes it difficult to understand and use the results. Thus, C4.5 algorithm uses the greedy search method to produce the decision tree. Moreover, C4.5 algorithm considers the information gain and gain ratio criteria in the generation of the decision tree.

In the method, the information content $I(S)$ of a training set is determined as

$$I(S) = -\sum_{j=1}^{m} RF = (C_j, S) log(RF((C_j, S)), \quad (3)$$

where $S$ is a training set that belongs to $C_j$ and $RF(C_j, S)$ denotes the relative frequency of the instances. Let $B$ be a test that divides $S$ into subsets $S_1, S_2..S_t$. Then the information gain $G(S, B)$ can be calculated in the following form Quinlan (1993):

$$G(S, B) = I(S) - \sum_{i=1}^{t} \frac{|S_i|}{|S|} I(S_i). \quad (4)$$

The purpose of the gain criterion is to select the best test $B$ that maximizes $G(S, B)$. However, the maximization of $G(S, B)$ leads to a large number of outcomes in test $B$, which can lead to numerical difficulties. It can be avoided through the consideration of the potential information $P(S, B)$, such as

$$P(S, B) = -\sum_{i=1}^{t} \frac{|S_i|}{|S|} log \frac{|S_i|}{|S|}. \quad (5)$$

The ratio of $G(S, B)$ and $P(S, B)$ must be maximized by a test $B$:

$$max\left(\frac{G(S, B)}{P(S, B)}\right). \quad (6)$$

Finally, C4.5 algorithm builds up the appropriate decision tree using the optimized test $B$.

## 3. DATA COLLECTION AND THE DEMONSTRATION OF THE ANALYSIS

Since machine learning methods request large number of data to train their algorithms, several vehicle dynamic signals must be provided for the reachability analysis. Generally, these data are reached through measurements, but in the design and analysis of the vehicle control systems the simulations also have important role. Therefore, in the paper, the training data have been generated using CarSim, which is a high-fidelity vehicle dynamics simulation software.

During the simulation scenarios the autonomous vehicle in CarSim has been controlled by a lateral tracking controller, which is able to modify the lateral position and the yaw motion of the vehicle. It can result in huge number of simulations, which is the fundamental of the big data analysis. The results of this paper have been built on ten million simulations. In the simulations the amplitude and the frequency of the vehicle signals are sufficiently large to reach wide ranges of states of the vehicle, e.g. yaw-rate, vehicle and tyre side-slips. Thus, stable and unstable, controllable and uncontrollable regions of the vehicle are reached. The classification of the instances is performed in the following way.

There is a well-known kinematic relationship between the slip angle of the front wheel $\alpha_1$, the yaw-rate $\dot{\psi}$, the steering angle $\delta$ and the vehicle side slip $\beta$ Rajamani (2005):

$$\alpha_1 = \delta - \beta - \frac{l_1 \dot{\psi}}{v_x} \tag{7}$$

where $l_1$ parameter is the distance between the front wheels and the vehicle center of gravity and $v_x$ is the actual longitudinal velocity. During the CarSim simulations all of these signals are measured. The instances can be classified by the percentage of the deviation which is derived from (7), such as

$$\text{if } \frac{|1+\alpha_1|}{|1+\delta-\beta-\frac{l_1\dot{\psi}}{v_x}|} \leq \varepsilon, \text{ then } i^{th} \text{ instance is controllable,} \tag{8a}$$

$$\text{if } \frac{|1+\alpha_1|}{|1+\delta-\beta-\frac{l_1\dot{\psi}}{v_x}|} > \varepsilon, \text{ then } i^{th} \text{ instance is uncontrollable.} \tag{8b}$$

where $\varepsilon$ is an experimentally defined parameter.

The reachability is examined through the data-mining WEKA software, in which the C4.5 algorithm has been implemented by Witten and Frank (2005). The attributes of the instances are $\alpha_1$, $\alpha_2$, $\beta$ side-slip angles of the vehicle, $\dot{\psi}$ yaw rate, $v_x$ longitudinal velocity and $\mu$ adhesion coefficient. The $C$ class has two values, i.e., *good* and *bad*, and the instances are classified by the algorithm (8). During the analysis the training set contains approximately 1.2 million instances, while the test set for the validation has 2 million members.

The generated trees are evaluated by the cross-validation technique, the results can be found in Table 1. The first column in Table 1 shows the minimum number of instances which are contained in a leaf. The second column illustrates the percentage of the correctly classified instances. The sizes of the produced trees are in the last column. Note that the increasing number of the minimum objects decreases both the percentage of the correctly classified instances and the sizes of the trees.

Figure 1 shows the results of the decision tree and the classified test sets in the plane of $\dot{\psi}$ and $\beta$ at different velocities. These two attributes have high impacts on the resulting decision tree, which shows that the calculated sets fit well. Thus, the resulting sets from the machine learning algorithm approximate the experimental results appropriately. The sizes of the polytopic sets become larger with increasing velocity, which means that the vehicle can reach larger $\beta - \dot{\psi}$ regions at high velocities. This tendency is confirmed by the experience in vehicle dynamics.

Table 1. Relationship between the tree size and the object number

| Min. Objects | Correctly Classified Inst. | Size of Tree |
|---|---|---|
| 2 | 99.6797% | 2344 |
| 10 | 99.5885% | 1174 |
| 100 | 99.3214% | 255 |
| 500 | 99.1301% | 78 |
| 1000 | 98.9508% | 39 |
| 5000 | 98.8% | 11 |

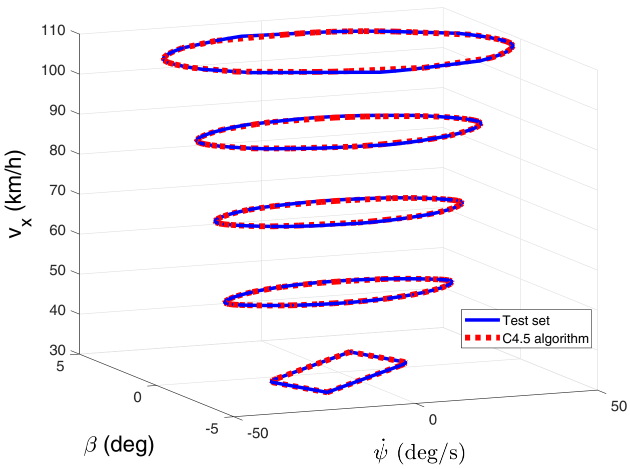



Fig. 1. $\dot{\psi}$ and $\beta$ sets depending on velocity $v_x$

In the following, the reachability sets of the vehicle are investigated in three different cases:

(1) Both front wheels and torque vectoring are used for steering (Torq. Vect. + 2WS), which means a fault-free scenario.
(2) The vehicle uses only its front wheels for steering (2WS), which means that torque vectoring functionality cannot be guaranteed due to the fault of an in-wheel motor.
(3) The vehicle uses only one wheel and torque vectoring for steering (Torq. Vect. + 1WS). This scenario is related to the fault of one wheel in the independent steering.

Figure 2 illustrates the reachability sets in the plane of the yaw rate $\dot{\psi}$ and side-slip angle $\beta$ at fixed adhesion coefficient $\mu = 0.5$ and at different velocities. It is concerned with a wet road, which may cause dangerous situations at high velocity. The figure shows that the largest sets belong to the combined steering case, in which both front wheels are used. The sets of the case 'Torq. Vect. + 1WS1' are close to the previous case, while the sets of the case '2WS' are significantly smaller. It means that the fault of one steered wheel has a lower impact on vehicle dynamics than the fault of the entire torque vectoring functionality.

In Figure 3 the reachability sets of $\dot{\psi}$ and $\beta$ are shown at a higher adhesion coefficient $\mu = 0.9$. Compared to the previous results, the sets of the case 'Torq. Vect +

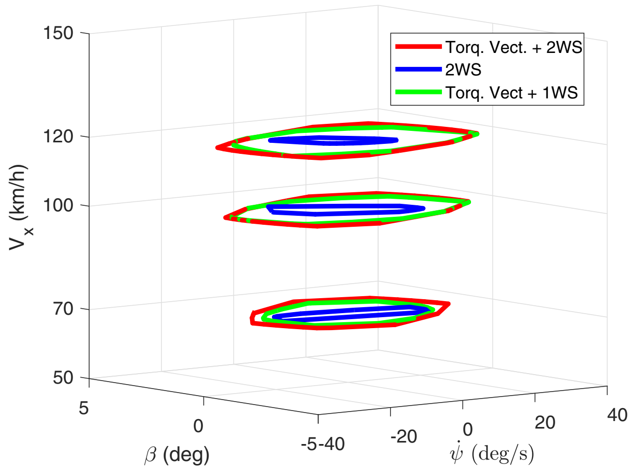



Fig. 2. $\dot{\psi}$ and $\beta$ sets at $\mu = 0.5$

2WS' vary at all velocities. The regions of the case '2WS' and 'Torq. Vect + 1WS' become smaller at all velocities, which means that the impact of the fault on the actuators of vehicle dynamics varies, e.g. the sets of the case 'Torq. Vect. + 1WS' are approximately half the size of the regions of the case 'Torq. Vect + 2WS'. However, at $\mu = 0.5$ these scenarios are almost the same.

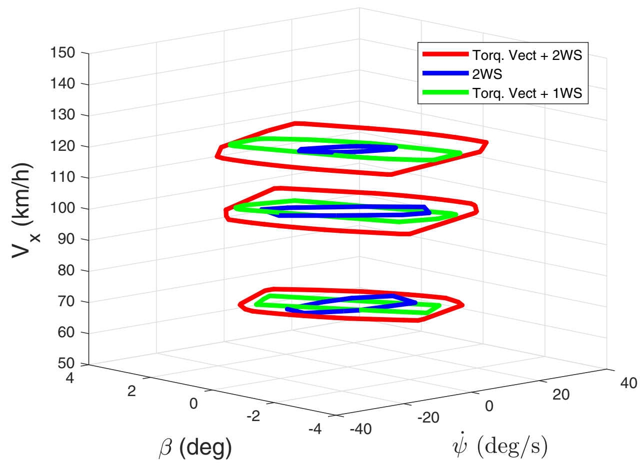
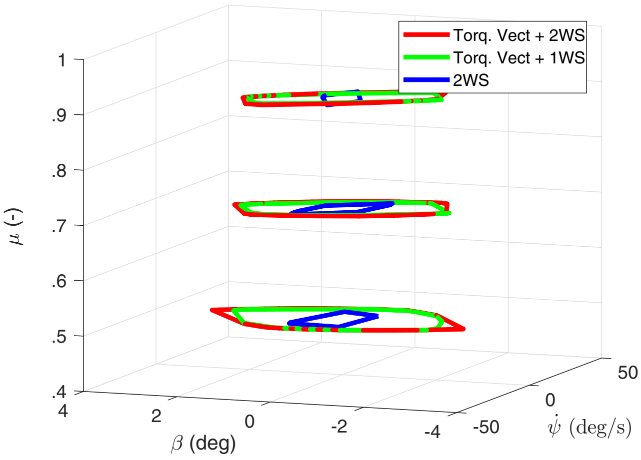
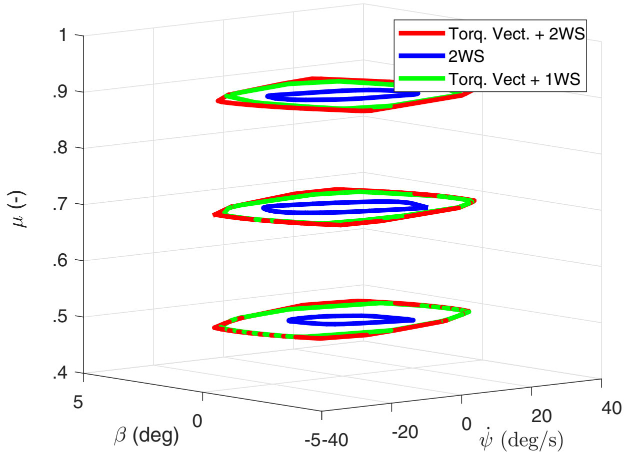



Fig. 3. $\dot{\psi}$ and $\beta$ sets at $\mu = 0.9$

In the following examinations the velocities are fixed, while the adhesion coefficients are modified, see Figure 4. Figure 4(a) shows the reachability sets of $\dot{\psi}$ and $\beta$ at different adhesion coefficients and at the fixed velocity of 70km/h. The sizes of the sets become smaller at high adhesion coefficients and larger at low adhesion coefficients. The same trend can be observed at velocity $v = 120km/h$, see Figure 4(b). In this case the sets of '2WS' become larger at high adhesion coefficients, while the the sets of the other two cases are the same at all adhesion coefficients. Thus, the effect of the faults must be considered dependent on $\mu$ and $v$ jointly.




(a) Reachable states at $v = 70km/h$




(b) Reachable states at $v = 120km/h$

Fig. 4. Reachability sets with varying $\mu$

## 4. APPLICATION POSSIBILITIES IN THE RECONFIGURATION STRATEGY

In this section the results of the reachability analysis using big data for the control interventions of the lateral dynamics will be exploited. The results will be built into the coordinated control of steering and torque vectoring. In the following the application possibilities of the reachability analysis in the design of the fault-tolerant control will be illustrated.

Autonomous vehicles must guarantee a large number of performances. Results obtained by the model-based reachability and controllability analysis have yielded exact relationships between the interventions and the vehicle signals, see e.g. Németh et al. (2017). However, due to the complexity of the model-based methods the formulation of the relationships may be difficult. Data analysis provides a possibility to extend the model-based results in the coordination and reconfiguration strategies of autonomous vehicle control systems. It means that the measured and estimated signals (velocity, adhesion coefficient, side-slip angle, etc.) provide inputs for the supervisory control, whose decision concerning the coordinated actuation can be made. Through this extension the reachability and the controllability of the system are improved.

The reachability sets provide information about the states, which are reached by a given control intervention. The size and shape of these sets can be different from each other, as it has been shown in Section 3. The sets vary depending on the current intervention, e.g. at fault-free or fault cases. The fault of an actuator modifies the reachable set, which means that reconfiguration must be performed to guarantee the stability and the performances of the system.

Figure 5 illustrates three reachability sets, which are related to two actuators $A_1$ and $A_2$. $R_1$ is the reachable set of the actuation of $A_1$, while $R_2$ is related to $A_2$. The goal of the control problem is to guarantee the following inequality: $\dot{\psi}_{min} \leq \dot{\psi} \leq \dot{\psi}_{max}$, where $\dot{\psi}_{min}, \dot{\psi}_{max}$ are determined by a high level autonomous vehicle control algorithm. The healthy actuator $A_1$ guarantees the yaw-rate criterion, see the state $x_1$. However, in case of vehicle dynamic actuator faults the reachability set is generally reduced. Thus, if a fault occurs in the operation $A_1$, the reachable set $R_{1,f}$ is smaller than $R_1$ and the yaw-rate criterion cannot be guaranteed. Figure 5 shows that the yaw-rate criterion is also guaranteed by $A_2$. Since the impact of $A_2$ on the reachable set is a priori known from the data analysis, the reconfiguration to $A_2$ from $A_1$ can be performed, which leads to the new state $x_2$. Through the reconfiguration of the actuators, the performance of the entire system can be guaranteed. Since the shapes and the sizes of the steering, torque-vectoring and their combined intervention are different, the proposed method can be efficiently used in the reconfiguration strategy of the lateral control systems.

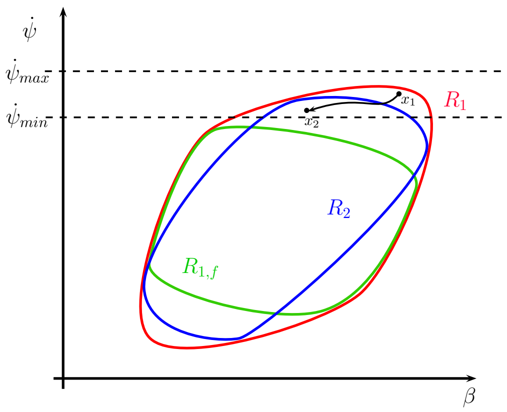



Fig. 5. Background of the reconfiguration strategy

## 5. CONCLUSIONS

The paper has proposed a data-based analysis on the lateral dynamics and the reconfiguration strategy for autonomous vehicles. In the analysis the big data from the signals of the vehicles have been processed through the machine learning algorithm C4.5. The paper has proposed the determination of the reachable regions of the vehicle, in which the impact of the actuator fault on the vehicle dynamics has been shown. The results of the analysis have been built into the reconfiguration strategy, which handles the fault scenarios of different actuators, such as steering and in-wheel electric motors. In the future the data-based method will be extended to analyse additional vehicle dynamics and actuators.

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## Appendix A. A BRIEF INTRODUCTION TO THE COMPUTATION OF REACHABLE SETS

The aim of the following description is to briefly summarize the mathematical background of some reachability set (1) computation problems. In the case of linear and LPV systems $\dot{x} = A(\rho)x + Bu$ the ellipsoidal approximation of the reachable sets can be applied $\varepsilon = \xi \left\{\xi^T X \xi \leq 1\right\}$, see Shin (2002), which leads to LMI conditions:

$$\begin{bmatrix} A^T(\rho)X(\rho) + X(\rho)A(\rho) + \alpha X(\rho) + \dot{X}(\rho) & X(\rho)B \\ B^T X(\rho) & -\alpha I \end{bmatrix} \leq 0 \tag{A.1a}$$

$$X(\rho) > 0 \tag{A.1b}$$

$$\alpha \geq 0 \tag{A.1c}$$

In the case of nonlinear systems an appropriate solution is the approximation in a polynomial form $\dot{x} = f(x) + g(x)u$, where $f, g$ are matrices with polynomial functions. Using the Sum-of-Squares (SOS) programming the reachable sets of the system can be inner approximation through higher-order ellipsoids, see Jarvis-Wloszek et al. (2005); Németh et al. (2017). The SOS conditions are formulated by Jarvis-Wloszek et al. (2005)

$$\min_V \beta \tag{A.2}$$

such that

$$V - l \in \Sigma_n \tag{A.3a}$$

$$-((1-V)s_4 + (p-\beta)s_5 + (1-V)(p-\beta)s_6 + (p-\beta)^2) \in \Sigma_n \tag{A.3b}$$

$$-((1-V)s_{10} + (\nabla^T(f(x) + g(x)u) - u^T u)) \in \Sigma_{n+n_i} \tag{A.3c}$$

where $s_i$ are polynomials in SOS form and $V$ is the polynomial Lyapunov function. The reachable set is yielded the level set $V = 1$. Moreover, $l$ and $p$ are fixed predefined polynomials.

Another efficient method for the computation of the reachable sets in the vehicle control is their approximation with convex polyhedral, see e.g. Althoff et al. (2007); Seron et al. (2008); Palmieri et al. (2012). In these reachability analyses the reachable regions are covered with polyhedral, which can guarantee a less conservative approximation compared to the ellipsoidal approach Stursberg and Krogh (2003); Girard (2005).