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Continuity of the asymptotic spectra for Toeplitz matrices

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I thank my family, teachers, and friends who took part in my vocational training and personal growth. With them I learned new ways to appreciate mathematics and life. Likewise, I want to show my esteem for the sense of belonging that my adviser had to contribute, correct, and improve the content of this document.

Introduction

On operator theory, one of the main subjects is the description and study of functions on spaces of infinite dimensions. According to our interests and academic foundation, it is possible to regard matrices as particular symbols that we manipulate to solve linear systems or as linear transformations relating to finite dimensional vector spaces, and conversely. Additionally, we can analyze how these objects behave when the range of a matrix tends to infinity.

In this branch of analysis, many concepts, theories, and problems have been introduced by numerous mathematicians in different epochs of time. Alongside this growth, mathematicians have been able to collaborate in other sciences. Under this atmosphere of scientific developments, Otto Toeplitz came into view as a professor in Germany and a disciple of David Hilbert.

His mathematical interests were wide and covered all branches of research, but mainly algebra. Most of his papers deal with problems of infinite matrices and the corresponding bilinear and quadratic forms.

Toeplitz matrices can be easily defined as matrices with constant diagonals parallel to the main diagonal. With this theory, it has been possible to illustrate abstract results and methods of linear algebra and functional analysis, such as C^*-algebras and index theory of Fredholm operators.

The continuous and increasing interest in analysis of Toeplitz operators can be explained by two reasons. On the one hand, these operators have an important connection with variety of problems in physics, probability theory, and several other fields. On the other hand, Toeplitz operators are one of the most important classes of non-selfadjoint operators and they are an interesting example in topics such as Banach algebras.

Our emphasis is on Toeplitz operators over the complex unitary circle ($\mathbb{T}$) and our aim is to establish a relation between the functional analytic properties of Toeplitz operators and their geometric behavior. That is how we divided our work in five chapters.

In Chapter 1, we will introduce the necessary concepts and theorems to understand Hilbert and Banach spaces. Many properties of the inner product and the norm appear, in detail, in [13]. Later, we will see Lebesgue spaces and Fourier transform. We will integrate this theory to incorporate the space $L^2(\mathbb{T})$, which will be important for us.

In Chapter 2, our purpose is to associate an infinite matrix with a linear bounded operator. Moreover, we will define the Banach algebra $\mathcal{B}(X)$, the collection of all bounded linear operators on X, and their norm. It is fundamental to discuss boundedness and invertibility inside this algebra and the meaning of the spectrum of a given matrix.

In Chapter 3, we are interested in Laurent matrices and their matrix representation. Again, the question when an operator on $\ell^2(\mathbb{Z})$ generates a Laurent matrix and viceversa, will be answered with the most important theorem of the chapter. We will characterize the multiplication operator and the range of a symbol on $L^\infty(\mathbb{T})$. Finally, we illustrate some examples of symbols.

Chapter 4 is the last one before we present our main result. We will define Toeplitz matrices and bring together all the study to explain under which conditions a Toeplitz matrix generates a bounded operator on $\ell^2(\mathbb{N})$. The Banach and Steinhaus theorem is necessary to introduce the norm of a Toeplitz operator. Besides, we will learn the connection between Fredholmness and invertibility, and we will use Hardy spaces to prove Coburn's Lemma.

In Chapter 5, we will give a review of the paper *"Asymptotic spectra of dense Toeplitz matrices are unstable"* [2] written by Albrecht Böttcher and Sergei M. Grudsky. In addition, we will use a specific symbol. Handling this symbol, we will calculate some truncated Toeplitz matrices and deal with the limiting set of the eigenvalues. We will announce two lemmas that help us to prove our main theorem which discusses the convergence of the limiting set of family of functions in the Hausdorff metric.

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Chapter 1

Preliminaries

In this chapter we deal with some concepts taken from the first chapter of [4]. These will help us to understand the theory of infinite matrices and bounded linear operators. The purpose is to set up the fundamentals and show basic results. We will study the Toeplitz matrices and finally we present various abstract results and methods of linear algebra and functional analysis.

1.1 Elementary concepts

Definition 1.1.1 (Metric). *A metric on a set X is a function*

$$d\colon X \times X \longrightarrow \mathbb{R}$$

having the following properties:

(i) $d(x, y) \geq 0$ *for all* $x, y \in X$*; equality holds if and only if* $x = y$.

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(iii) (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$, *for all* $x, y, z \in X$.

Definition 1.1.2 (Hausdorff metric). *Let X and Y be two nonempty subsets of a metric space. The Hausdorff metric is defined by*

$$d_H(X, Y) \equiv \max\left\{\sup_{x\in X}\inf_{y\in Y} d(x, y), \sup_{y\in Y}\inf_{x\in X} d(x, y)\right\}.$$

Definition 1.1.3 (Normed space). *A normed space is a vector space V together with a function $\|\cdot\|$ defined on it, called a norm, satisfying:*

(i) $0 \leq \|f\| < \infty$.

(ii) $\|f\| = 0$ *if and only if* $f = 0$.

(iii) (Homogeneity) $\|cf\| = |c|\|f\|$ *for any scalar* c.

(iv) (Triangle inequality) $\|f+g\| \leq \|f\| + \|g\|$.

For all $f, g \in V$.

We want to know the relation between a norm and a metric. A norm and a metric are two different notions, a norm is measuring the size of an object and a metric is measuring the distance between two elements. That means, a norm is a property of an element while a metric assigns a distance. If we have a vector space with a norm, it is always possible to define a metric in terms of that norm by putting $d(f,g) \equiv \|f-g\|$. In other words, a normed space is automatically a metric space, by defining the metric in terms of the norm in the natural way. But a metric space may not be a vector space. Thus, the concept of metric space is a generalization of the concept of a normed vector space.

Example 1.1.4. Let $d\colon X \times X \longrightarrow \mathbb{R}$ be the function,

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

This function satisfies the three conditions mentioned in Definition 1.1.1, so d is a metric. But, there is no a norm satisfying $d(x,y) = \|x-y\|$. To check this, suppose there is such a norm and let $\alpha \neq 0$ with $|\alpha| \neq 1$. Choose any $x, y \in X$ such that $x \neq y$. Then, $\alpha x \neq \alpha y$ and we have

$$1 = d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = \|\alpha(x-y)\| = |\alpha|\|x-y\| = |\alpha| d(x,y) = |\alpha|,$$

which is a contradiction.

Definition 1.1.5 (Power set). *Let X be a set. The power set of X is the collection of all subsets of X (including $\emptyset$ and X). A standard notation for the power set of X is 2^X.*

Definition 1.1.6 (Algebra). *An algebra of subsets of X is a collection of sets $\mathcal{M} \subset 2^X$ which satisfies*

(i) $\emptyset \in X$.

(ii) *If $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$.*

(iii) *If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$.*

By using de Morgan's laws we have immediately that an algebra satisfies other properties: $X \in \mathcal{M}$; if $A, B \in \mathcal{M}$, then $A \cap B \in \mathcal{M}$, and $A \setminus B \in \mathcal{M}$. Note that $\mathcal{M}$ is called an algebra because it accomplishes the set operations and if they are performed finitely many times on sets in $\mathcal{M}$, this always produces sets which are in it.

We include the trivial examples of an algebra: $\{\emptyset, X\}$ (the smallest one) and 2^X (the biggest one).

Example 1.1.7. Let $X = \{a, b, c\}$. For this set the power set is

$$2^X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}.$$

Now, consider $\mathcal{M} = \{\emptyset, \{a,b\}, \{c,d\}, X\}$. It is easy to verify that $\mathcal{M}$ is an algebra.

Definition 1.1.8 (σ-algebra)**.** *Let $\mathcal{M} \subset 2^X$ be an algebra. Then, $\mathcal{M}$ is a σ-algebra if additionally*

$$A_1, A_2, A_3, \ldots \in \mathcal{M} \Longrightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}.$$

It follows from de Morgan's laws that $\bigcap_{k=1}^{\infty} A_k \in \mathcal{M}$ as well. Besides, in a σ-algebra all the set operations performed countably many times on sets in $\mathcal{M}$ result in sets which belong to it.

Definition 1.1.9 (Extended real number line)**.** *The extended real line $\dot{\mathbb{R}}$ is defined as*

$$\dot{\mathbb{R}} \equiv \mathbb{R} \cup \{+\infty, -\infty\}.$$

Definition 1.1.10 (Measure)**.** *Let X be a set and $\mathcal{M}$ a σ-algebra over X. A function μ from $\mathcal{M}$ to $\dot{\mathbb{R}}$ is called a measure if it satisfies the following properties:*

(i) $0 \leq \mu(A) \leq \infty$.

(ii) $\mu(\emptyset) = 0$.

(iii) *If $A_1, A_2, \ldots$ are disjoint sets in $\mathcal{M}$, then*

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Definition 1.1.11 (Measure space)**.** *A measure space consists of the following three things:*

(i) *A nonempty set X.*

(ii) *A σ-algebra $\mathcal{M} \subset 2^X$.*

(iii) *A measure μ defined on $\mathcal{M}$.*

We call the triple $(X, \mathcal{M}, \mu)$ a measure space. When this is clear, we will simply write X.

Definition 1.1.12 (Almost everywhere)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space and P a property, when it can be applied to X we say that P is true almost everywhere in X if*

$$\mu(\{x \in X \colon P(x) \text{ is false}\}) = 0.$$

Definition 1.1.13 (σ-finite measure)**.** *Let μ be a positive measure of X. We say that μ is σ-finite if X is the countable union of numerable sets with finite measure.*

The term "almost everywhere" (a.e.) has a notion of a negligible set. For our context, if we are working with a measure space $(X, \mathcal{M}, \mu)$, we do not pay attention to sets of measure zero. In the context of measure theory we use the term "μ-almost everywhere" (μ-a.e.).

Definition 1.1.14 (Null set)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space and $N \subset X$ be a measurable set. N is a null set if $\mu(N) = 0$ and we say that N has measure zero.*

Definition 1.1.15 (Field)**.** *A field $\mathbb{F}$ is any set of elements that satisfies associativity, commutativity, distributivity, identity, and invertibility properties for both addition and multiplication operations. It is an algebraic structure: a nonzero commutative ring.*

From now on $\mathbb{F}$ is a field.

Definition 1.1.16 (Supremum norm)**.** *Let S be a set. Let $(X, \|\cdot\|)$ be a normed space over a field $\mathbb{F}$. Let $f\colon S \longrightarrow X$, then the supremum norm of f is*

$$\|f\|_\infty \equiv \inf\{M\colon \|f(x)\| \leq M \text{ for } \mu\text{-a.e. } x \in S\}.$$

Thus, there exists a null set N_f such that $\|f\|_\infty = \sup\{\|f(x)\|\colon x \in N_f^c\}$.

For example, consider the function $\chi_{\mathbb{Q}}\colon [0,1] \longrightarrow \{0,1\}$ defined by

$$\chi_{\mathbb{Q}}(x) \equiv \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then, $\|\chi_{\mathbb{Q}}\|_\infty = 0$.

For Proposition 1.1.17 and Theorem 1.1.18 we introduce $\mathcal{A}$ as the set of all the mappings $f\colon S \longrightarrow X$ such that $\|f\|_\infty < \infty$. Note that $\mathcal{A}$ is a vector space because addition of two bounded functions gives us a new bounded function, which belongs to $\mathcal{A}$, and the pointwise scalar multiplication gives us again a bounded function on $\mathcal{A}$; it is similar for their multiplication. It is important to mention that $\mathcal{A}$ is a function space with a geometric property which allows us to talk about topological and analytical concepts.

Proposition 1.1.17. *Let $f \in \mathcal{A}$, then $|f(x)| \leq \|f\|_\infty$ for μ-a.e. $x \in S$.*

Proof. By definition of infimum, there exists a sequence $(M_j)_{j\in\mathbb{N}}$ such that $|f(x)| \leq M_j$ for μ-a.e. $x \in S$ and $M_j \longrightarrow \|f\|_\infty$ whenever j goes to infinity. This means that, there exist null sets $N_j \subsetneq S$ satisfying $|f(x)| \leq M_j$ for $x \in N_j^c$. Now, we define $N \equiv \bigcup_{j=1}^\infty N_j$. Thus, N is a null set and if $x \in N^c$, then $|f(x)| \leq M_j$ for all j. Therefore, $x \in N^c$ implies that $|f(x)| \leq \|f\|_\infty$. ■

Theorem 1.1.18. *$\|\cdot\|_\infty$ is a norm on $\mathcal{A}$.*

Proof. Let $f, g \in \mathcal{A}$.

(i) Suppose that $\|f\|_\infty = 0$. Then, $\|f(x)\| = 0$ μ-a.e. for $x \in S$. Since $\|x\| = 0$ if and only if $x = 0$, we have that $f(x) = 0$ μ-a.e. for $x \in S$.

(ii) Now, let $\alpha \in \mathbb{F}$. We have,

$$\begin{aligned}\|\alpha f\|_\infty &= \inf\{M\colon \|\alpha f(x)\| \leq M \text{ for } \mu\text{-a.e. } x \in S\} \\ &= \inf\{M\colon \|f(x)\| \leq M/\alpha \text{ for } \mu\text{-a.e. } x \in S\}.\end{aligned}$$

Taking $M' = \frac{M}{|\alpha|}$, we obtain

$$\|\alpha f\|_\infty = \inf\{|\alpha| M'\colon \|f(x)\| \leq M' \text{ for } \mu\text{-a.e. } x \in S\} = |\alpha|\|f\|_\infty.$$

(iii) At last, by Proposition 1.1.17 we know that $f, g \in \mathcal{A}$ satisfy that $|f(x)| \leq \|f\|_\infty$ μ-a.e. and $|g(x)| \leq \|g\|_\infty$ μ-a.e., then

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty \ \mu\text{-a.e.}$$

Therefore, $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

We claim that $\|\cdot\|_\infty$ is a norm. ■

Definition 1.1.19 (Inner product). *Let V be a vector space. An inner product is a function from $V \times V$ to $\mathbb{F}$, denoted by $\langle \cdot, \cdot \rangle$, satisfying:*

(i) $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$ *for all* $f, g, h \in V$.

(ii) $\langle cf, g\rangle = c\langle f, g\rangle$ *for all* $f, g \in V$ *and any scalar* c.

(iii) $\overline{\langle f, g\rangle} = \langle g, f\rangle$ *for all* $f, g \in V$.

(iv) $0 \leq \langle f, f\rangle < \infty$ *for all* $f \in V$.

(v) $\langle f, f\rangle = 0$ *if and only if* $f = 0$.

Note that if $\mathbb{F} = \mathbb{C}$, we call $\langle \cdot, \cdot \rangle$ a complex inner product and if $\mathbb{F} = \mathbb{R}$, we call $\langle \cdot, \cdot \rangle$ a real inner product.

Definition 1.1.20 (Inner product space). *An inner product space is a vector space V together with an inner product. Any inner product space gives rise to a normed space by defining*

$$\|f\|^2 \equiv \langle f, f\rangle. \tag{1.1.1}$$

Theorem 1.1.24 will show that $\|\cdot\|$ is a norm.

Theorem 1.1.21 (Pythagorean theorem). *If u, v are orthogonal vectors in V, then*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof.

$$\|u + v\|^2 = \langle u + v, u + v\rangle = \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \langle v, u\rangle = \|u\|^2 + \|v\|^2.$$ ■

The inner product is the generalization of the dot product. In a vector space V, it is a way to multiply vectors and it associates to each pair of vectors in the space a scalar quantity. Geometrically, inner products measure the length of a vector or the angle between two vectors. Then, we can define the orthogonality between vectors $u, v \in V$ requiring $\langle u, v\rangle = 0$ and we denote it by $u \perp v$. On the other hand, if two vectors $u, v \in V$ are parallel we have the following situation: $u = c \cdot v$ for some scalar c, and we write $u \parallel v$.

Definition 1.1.22 (Orthogonal decomposition). *Let V be a vector space. Let $u, v \in V$ and $a \in \mathbb{F}$. The orthogonal decomposition of u is a pair u_1, u_2 satisfying:*

(i) u_1 *and* v *are parallel, and we have* $u_1 = \frac{\langle u, v\rangle}{\|v\|^2} v$.

(ii) u_2 *and* v *are orthogonal, and we have* $u_2 = u - \frac{\langle u,v\rangle}{\|v\|^2}v$.

(iii) $u = u_1 + u_2$.

It is easy to see that this decomposition is unique.

Theorem 1.1.23 (Cauchy-Schwarz inequality)**.** *If* $u, v \in V$*, then*

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Proof. Let $u, v \in V$. If $v = 0$, then the desired inequality holds. Thus, we can assume that $v \neq 0$. Consider the orthogonal decomposition

$$u = \frac{\langle u, v\rangle}{\|v\|^2}v + w,$$

where $w \perp v$. By the Pythagorean Theorem 1.1.21,

$$\|u\|^2 = \left\|\frac{\langle u, v\rangle}{\|v\|^2}v\right\|^2 + \|w\|^2 = \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \|w\|^2 \geq \frac{|\langle u, v\rangle|^2}{\|v\|^2}.$$

Multiplying both sides of this inequality by $\|v\|^2$ and taking square roots, we obtain Cauchy-Schwarz inequality. ■

Theorem 1.1.24. *Let* V *be a vector space with an inner product. Then,* $\|\cdot\|$ *given by Equation (1.1.1) is a norm.*

Proof. Let $f, g \in V$. We know that $\|f\|^2 = \langle f, f\rangle$, by Definition 1.1.19 we immediately obtain that $0 \leq \|f\| < \infty$, and $\|f\| = 0$ if and only if $f = 0$. Now,

(i) Let c be any scalar. $\|cf\|^2 = \langle cf, cf\rangle = c\bar{c}\langle f, f\rangle = |c|^2\|f\|^2$. Then, $\|cf\| = |c|\|f\|$.

(ii) Triangle inequality:

$$\begin{aligned}
\|f+g\|^2 = \langle f+g, f+g\rangle &= \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle \\
&= \langle f, f\rangle + \langle f, g\rangle + \overline{\langle f, g\rangle} + \langle g, g\rangle \\
&= \|f\|^2 + 2\mathrm{Re}(\langle f, g\rangle) + \|g\|^2 \\
&\leq \|f\|^2 + 2|\langle f, g\rangle| + \|g\|^2 \\
&\leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2, \text{ due to Theorem 1.1.23.} \\
&= (\|f\| + \|g\|)^2.
\end{aligned}$$

Therefore, $\|f\|^2 = \langle f, f\rangle$ is a norm. ■

Before introducing new concepts, we will discuss the completeness idea. It is a property of a space. In general, it can be interpreted as the idea that there are no gaps or missing points. One approach of completeness is known as the Least Upper Bound property (LUB), it states that any nonempty set of $\mathbb{R}$ which has an upper bound necessarily has a supremum in $\mathbb{R}$. Another approach to completeness is through the Bolzano–Weierstrass theorem which states that every bounded sequence has a convergent subsequence, and as a third approach we have the monotone convergence theorem, saying that a monotone sequence of real numbers is convergent if and only if it is bounded. The last notion is the Nested Intervals Property that states that if there is a nested sequence of closed bounded intervals, then their intersection is not empty.

We mentioned all the properties that describe completeness of the real numbers and permit us to take limits, which is central in everything done in real analysis.

In the idea of limit and its calculation, it is relevant to know the value to which a function or sequence is approaching, this is the purpose of the (ε, δ)-definition of limit. But, Cauchy's definition of a limit says that it is not necessary to know its value. Because, at the moment of doing calculations, the values that are being found are approaching a fixed value; they differ eventually of the limit by as little as one could wish. That fixed value is called the limit.

Definition 1.1.25 (Cauchy sequence)**.** *A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that if $m, n \geq H(\varepsilon)$, then $|x_n - x_m| < \varepsilon$.*

Definition 1.1.26 (Complete metric space)**.** *A complete metric space is a metric space in which every Cauchy sequence is convergent.*

Definition 1.1.27 (Uniform convergence)**.** *Suppose $\mathcal{S}$ is a set and $f_n \colon \mathcal{S} \longrightarrow \mathbb{F}$ is a function for every natural number n. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to a limit $f \colon \mathcal{S} \longrightarrow \mathbb{F}$ if for every $\varepsilon > 0$, there exists a natural number $N(\varepsilon)$ such that for all $x \in \mathcal{S}$ and all $n \geq N(\varepsilon)$ we have $|f_n(x) - f(x)| < \varepsilon$, for all $x \in \mathcal{S}$.*

Example 1.1.28.

(i) Let $x_n = \frac{1}{n}\sqrt{2}$ for each $n \in \mathbb{N}$. Note that, each x_n is an irrational number, i.e., $x_n \notin \mathbb{Q}$ and notice that $(x_n)_{n\in\mathbb{N}}$ converges to $0 \in \mathbb{Q}$. Therefore, $(x_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, so although the sequence is entirely in $\mathbb{I}$, it does not converge in $\mathbb{I}$. Moreover, we know that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Finally, $\mathbb{I}$ is not a complete metric space.

(ii) Let $\hat{x} \in \mathbb{I}$. For each $n \in \mathbb{N}$, let x_n be a rational number in the interval $(\hat{x} - \frac{1}{n}, \hat{x} + \frac{1}{n})$. Then, $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Q}$ that converges to $\hat{x} \notin \mathbb{Q}$. Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$. It means that $\mathbb{Q}$, is not a complete metric space.

Definition 1.1.29 (Hilbert space)**.** *A complete space with an inner product on which the norm is induced by its inner product is called a Hilbert space.*

Theorem 1.1.30 (Parallelogram law)**.** *Let $\mathcal{H}$ be a Hilbert space with the associated norm $\|\cdot\|$. Let $f, g \in \mathcal{H}$ be arbitrary. Then,*

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

Proof.

$$\begin{aligned}
\|f+g\|^2 + \|f-g\|^2 &= \langle f+g, f+g\rangle + \langle f-g, f-g\rangle \\
&= \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle + \langle f,f\rangle - \langle f,g\rangle - \langle g,f\rangle + \langle g,g\rangle \\
&= 2\langle f,f\rangle + 2\langle g,g\rangle \\
&= 2(\|f\|^2 + \|g\|^2).
\end{aligned}$$

∎

Definition 1.1.31 (Banach space)**.** *Let V be a normed space. If V is complete, then it is a Banach space.*

Now we distinguish between Hilbert and Banach spaces, we will mentioned how they are associated. Firstly, the essential difference between these two spaces is how the norm is defined. In Hilbert spaces the norm is defined via the inner product, $\|f\|^2 = \langle f, f\rangle$ while in Banach spaces the norm is defined directly by Definition 1.1.3. Thus, each Hilbert space is a Banach space. Something else to notice is that Hilbert spaces have a geometrical structure which allows us to talk about orthonormal bases, unitary operator, and so on (these terms will be defined later). However, many spaces of interest that are Banach spaces are not Hilbert spaces.

Theorem 1.1.34 presents a property that a Banach space needs to be a Hilbert space.

To continue with our analysis of these two spaces, we need to introduce the concept of smoothness that is related with how many derivatives of a function exist and are continuous. It is possible to classify functions according to the properties of their derivatives. Let k be a non-negative integer, we say that the function f belongs to the class $\mathcal{C}^k(X)$ if its derivatives $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ exist and are continuous. On the other hand, when $k = \infty$ the collection $\mathcal{C}^\infty(X)$ consist of all the functions that have continuous derivatives of all orders. The other case is $k = 0$, we denote $\mathcal{C}(X)$ instead of $\mathcal{C}^0(X)$ and it consists of all the continuous functions.

Note that, $\mathcal{C}^\infty(X) = \bigcap_{k=0}^\infty \mathcal{C}^k(X)$ and we have the inclusions:

$$\mathcal{C}^\infty(X) \subsetneq \cdots \subsetneq \mathcal{C}^3(X) \subsetneq \mathcal{C}^2(X) \subsetneq \mathcal{C}^1(X) \subsetneq \mathcal{C}(X).$$

Lemma 1.1.32. *Let V be a real inner product space, then*

$$\langle u, v\rangle = \frac{\|u+v\|^2 + \|u-v\|^2}{4},$$

for all $u, v \in V$.

Proof. Let $u, v \in V$,

$$\begin{aligned}\frac{\|u+v\|^2 + \|u-v\|^2}{4} &= \frac{\langle u+v, u+v\rangle + \langle u-v, u-v\rangle}{4}\\ &= \frac{\|u\|^2 + 2\langle u, v\rangle + \|v\|^2 - (\|u\|^2 - 2\langle u, v\rangle + \|v\|^2)}{4}\\ &= \langle u, v\rangle.\end{aligned}$$

■

Lemma 1.1.33 (Polarization identity)**.** *Let V be a complex inner product space, then*

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2}{4}, \tag{1.1.2}$$

for all $u, v \in V$.

Proof. Let $u, v \in V$,

$$\|u+v\|^2 = \|u\|^2 + 2\mathrm{Re}(\langle u, v\rangle) + \|v\|^2. \tag{1.1.3}$$

$$\|u-v\|^2 = \|u\|^2 - 2\mathrm{Re}(\langle u, v\rangle) + \|v\|^2. \tag{1.1.4}$$

Now subtracting Equations (1.1.3) and (1.1.4) we get,

$$\|u+v\|^2 - \|u-v\|^2 = 4\mathrm{Re}(\langle u, v\rangle). \tag{1.1.5}$$

Further,

$$\begin{aligned} i\|u+iv\|^2 = i\langle u+iv, u+iv\rangle &= i\langle u,u\rangle + i\langle u, iv\rangle + i\langle iv, u\rangle + i\langle iv, iv\rangle \\ &= i\|u\|^2 + |i|^2\langle u,v\rangle + i^2\langle v,u\rangle + i\|v\|^2 \\ &= i\|u\|^2 + \langle u,v\rangle - \overline{\langle u,v\rangle} + i\|v\|^2. \end{aligned} \tag{1.1.6}$$

$$\begin{aligned} i\|u-iv\|^2 = i\langle u-iv, u-iv\rangle &= i\langle u,u\rangle + i\langle u, -iv\rangle + i\langle -iv, u\rangle + i\langle -iv, -iv\rangle \\ &= i\|u\|^2 - |i|^2\langle u,v\rangle - i^2\langle v,u\rangle + i|-i|^2\|v\|^2 \\ &= i\|u\|^2 - \langle u,v\rangle + \overline{\langle u,v\rangle} + i\|v\|^2. \end{aligned} \tag{1.1.7}$$

If we subtract Equations (1.1.6) and (1.1.7), we obtain

$$2\langle u,v\rangle - 2\overline{\langle u,v\rangle} = 4i\,\mathrm{Im}(\langle u,v\rangle). \tag{1.1.8}$$

Finally, adding Equations (1.1.5) and (1.1.8) we have

$$4(\mathrm{Re}(\langle u,v\rangle) + 4i\,\mathrm{Im}(\langle u,v\rangle)) = 4\langle u,v\rangle. \qquad \blacksquare$$

After these lemmas we are ready to present Theorem 1.1.34 which states the relation between Banach and Hilbert spaces.

Theorem 1.1.34. *Suppose X is a Banach space. The norm $\|\cdot\|$ is induced by an inner product if and only if Parallelogram law 1.1.30 holds on X.*

Proof. Let X be a Banach space.
($\Longrightarrow$) Suppose that the norm on X is induced by an inner product, this means $\|x\|^2 = \langle x, x\rangle$ for $x \in X$. Then, the parallelogram law holds and we already checked it.
($\Longleftarrow$) Suppose that the parallelogram law holds on X. We need to introduce an inner product which will induce the norm in this space and we use Polarization identity 1.1.33. We define $\langle\cdot,\cdot\rangle$ by Equation (1.1.2),

$$\langle u,u\rangle \equiv \frac{4\|u\|^2}{4} + \frac{i|1+i|^2\|u\|^2}{4} - \frac{i|1-i|^2\|u\|^2}{4} = \left(1 + \frac{i}{2} - \frac{i}{2}\right)\|u\|^2 = \|u\|^2.$$

Immediately, we obtain $\|u\|^2 = \langle u,u\rangle$ for $u \in X$. Besides, by Definition 1.1.19 we find that this inner product will satisfy conjugate symmetry, linearity in the first argument, and the remaining properties. $\blacksquare$

Theorem 1.1.35. *Let $K \subsetneq \mathbb{R}$ be a compact set. $(\mathcal{C}(K), \|\cdot\|_\infty)$ is a Banach space.*

Proof. $\mathcal{C}(K)$ is the space of all continuous functions on K. Here, we have

$$\|f\|_\infty = \sup\{|f(x)| \colon x \in K\}.$$

By Theorem 1.1.18, we easily obtain that $\mathcal{C}(K)$ is a normed space.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}(K)$, we need to show that f_n is convergent.

Let $x \in K$ fixed. Now, consider $(f_n(x))_{n\in\mathbb{N}}$ which is a real number sequence. Let $\varepsilon > 0$, there exists $N > 0$ such that

$$n, m \geq N \Longrightarrow |f_m(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \frac{\varepsilon}{2}.$$

What we did shows that for every $x \in K$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. But, $\mathbb{R}$ equipped with the euclidean metric is a complete metric space. Then, the limit of $f_n(x)$ exists in $\mathbb{R}$, we denote this limit by $f(x)$.

Now, we want to show that f_n converges to f in the supremum norm. By assumption f_n is Cauchy, from the triangle inequality we get

$$\|f_n - f\|_\infty \leq \|f_n - f_N\|_\infty + \|f - f_N\|_\infty,$$

by how we picked N, we have that $\|f_n - f_N\|_\infty < \frac{\varepsilon}{2}$. Taking the limit when n goes to infinity we obtain,

$$\lim_{n\to\infty} \|f_n - f_N\|_\infty = \|f - f_N\|_\infty < \frac{\varepsilon}{2}.$$

Thus, $\|f_n - f\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \overset{\star}{=} \varepsilon$ for $n \geq N$, accomplishing the aim.

To conclude, we will see that f belongs to $\mathcal{C}(K)$. Let $\varepsilon > 0$ and $x \in K$ fixed. Further, by $\star$ we get,

$$|f_n(x) - f(x)| \leq \|f_n - f\| < \frac{\varepsilon}{3} \quad \text{and} \quad |f_n(y) - f(y)| \leq \|f_n - f\| < \frac{\varepsilon}{3}, \text{ for all } y \in K. \tag{1.1.9}$$

In addition, f_n is continuous, so there exists $\delta(x, n) > 0$ such that

$$|x - y| < \delta(x, n) \Longrightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}, \text{ for every } y \in K.$$

Finally, if $n > N$ and $|x - y| < \delta(x, n)$ then,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

We have used the inequalities of (1.1.9). Therefore, f is continuous on K and it implies that f is uniformly continuous on K. ■

Now, we exhibit an example that reveals a Banach space which is not a Hilbert space. Consider $(\mathcal{C}([0, 1]), \|\cdot\|_\infty)$, by Theorem 1.1.35 this is a Banach space. Consider the functions $f(x) = 1 - x$ and $g(x) = x$, they belong to $\mathcal{C}([0, 1])$. However,

$$\|f - g\|_\infty^2 + \|f + g\|_\infty^2 = \|1 - 2x\|_\infty^2 + \|1\|_\infty^2 = 2$$

and

$$2(\|f\|_\infty^2 + \|g\|_\infty^2) = 2\|1 - x\|_\infty^2 + 2\|x\|_\infty^2 = 2 + 2 = 4.$$

Whence, these functions do not accomplish Parallelogram law 1.1.30.

Definition 1.1.36 (Borel sets)**.** *The class of Borel sets in $\mathbb{R}^n$ is the σ-algebra generated by the collection of open sets.*

Definition 1.1.37 (Measurable functions)**.** *Let $\mathcal{M}$ be a σ-algebra of subsets of X. Suppose that $f : X \to \dot{\mathbb{R}}$. Then, f is $\mathcal{M}$-measurable if for all $t \in \dot{\mathbb{R}}$ the set $f^{-1}([-\infty, t])$ belongs to $\mathcal{M}$. In other words,*

$$\{x \in X : f(x) \leq t\} \in \mathcal{M}, \quad \text{for every } t \in \dot{\mathbb{R}}.$$

Some very well known equivalent definitions are:

(i) $f^{-1}([-\infty, t]) \in \mathcal{M}$ for every $t \in \dot{\mathbb{R}}$.

(ii) $f^{-1}([-\infty, t)) \in \mathcal{M}$ for every $t \in (-\infty, \infty]$.

(iii) $f^{-1}([t, \infty]) \in \mathcal{M}$ for every $t \in \dot{\mathbb{R}}$.

(iv) $f^{-1}((t, \infty]) \in \mathcal{M}$ for every $t \in [-\infty, \infty)$.

(v) $f^{-1}(\{-\infty\})$, $f^{-1}(\{\infty\})$ belong to $\mathcal{M}$, and $f^{-1}(E) \in \mathcal{M}$ for every Borel set $E \subset \mathbb{R}$.

At this point, we would like to share some Lebesgue contributions. Lebesgue may be said to have created the first genuine theory of integration. His important contributions are (1) a fully developed measure-theoretic point of view that provided new ways of looking at the Cauchy-Riemann definition of the definite integral and (2) theoretical problems that had been found out within the context of Riemann's definition of the integral.

Building on the work of others, including those of Émile Borel and Camille Jordan, Lebesgue formulated the theory of measure in the early twentieth century. Also, he generalized the notion of the Riemann integral and revolutionized the integral calculus. Finally, he extended the concept of the area below a curve including many discontinuous functions. It was one of his main achievements in modern analysis.

Definition 1.1.38 (Lebesgue spaces L^p)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < \infty$. The space $L^p(X)$ consists of the equivalence classes of measurable functions $f : X \to \mathbb{C}$ such that*

$$\int_X |f|^p \, d\mu < \infty,$$

where two measurable functions are equivalent if they are equal μ-a.e. The L^p-norm of f is defined by

$$\|f\|_p^p \equiv \int_X |f|^p \, d\mu.$$

Theorem 1.1.39. *$(L^p(X), \|\cdot\|_p)$ is a normed space. It satisfies:*

(i) $0 \leq \|f\|_p < \infty$.

(ii) $\|f\|_p = 0$ *if and only if* $f = 0$ (μ*-a.e.*).

(iii) $\|cf\|_p = |c|\|f\|_p$ *for c any scalar.*

(iv) $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

The last property is known as Minkowski's inequality 1.1.43. We will prove it later.

Proof. Properties (i) and (iii) are easy to check.

(ii) ($\Longleftarrow$) Let $C = \{x \in X : f(x) \neq 0\}$. By assumption $\mu(C) = 0$. Now, it is possible to write

$$\int_X |f|^p \, d\mu = A + B,$$

where $A = \int_C |f|^p \, d\mu$ and $B = \int_{C'} |f|^p \, d\mu$, where $C' = X \setminus C$. Since C has measure zero, $A = 0$. Additionally, $f \restriction_{C'} \equiv 0$ so we get that $B = 0$. Therefore, $\|f\|_p = 0$.

($\Longrightarrow$) Suppose that $f \neq 0$ (μ-a.e.). Then, there exists $x_\circ \in X$ and $\varepsilon > 0$ such that $|f(x)| \neq 0$ for every $x \in B_\varepsilon(x_\circ)$. Without loss of generality (w.l.o.g.), we take $\varepsilon > 0$ small enough such that

$$\inf\{|f(x)| : x \in B_\varepsilon(x_\circ)\} \equiv \alpha \gneq 0 \Longrightarrow \int_X |f|^p \, d\mu \geq \alpha \cdot \mu(B_\varepsilon(x_\circ)) \gneq 0.$$

Finally, we conclude that $\|f\|_p \neq 0$. ■

Straightaway, we will introduce a new concept and two relevant results which were given by the German mathematician Otto Hölder. These are necessary to prove Minkowski's inequality 1.1.43.

Definition 1.1.40 (Hölder conjugate)**.** *Let $1 < p < \infty$. Then, $p' \equiv \frac{p}{p-1}$ and it satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. p' is known as the Hölder conjugate of p.*

Lemma 1.1.41. *Let $a \geq 0$, $b \geq 0$ and $p \in (1, \infty)$. Then,*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Proof. Let a, b be fixed. We define $f(a) \equiv ab - \frac{a^p}{p}$. We will find its minimum value, we get

$$f'(a) = b - a^{p-1} = 0.$$

This means that, at $a = b^{\frac{1}{p-1}} = b^{\frac{p'}{p}}$ f gets its minimum. Thus, using Definition 1.1.40 we obtain,

$$f(a) = ab - \frac{a^p}{p} \leq b^{\frac{1}{p-1}} b - \frac{b^{\frac{p}{p-1}}}{p} = b^{\frac{p}{p-1}} - \frac{b^{\frac{p}{p-1}}}{p} = b^{p'} \left(1 - \frac{1}{p}\right) = \frac{b^{p'}}{p'}.$$

Arriving at, $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$. ■

Theorem 1.1.42 (Hölder's inequality)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that $1 < p < \infty$ and $1 < q < \infty$ are Hölder conjugates. In addition, $u \in L^p(X)$ and $v \in L^q(X)$. Then, $uv \in L^1(X)$ and $\|uv\|_1 \leq \|u\|_p \|v\|_q$.*

Proof. If $u \equiv 0$ or $v \equiv 0$ (μ-a.e.) then, the result is trivial. Consequently, we can assume that $\|u\|_p > 0$ and $\|v\|_q > 0$. Now, note that by Lemma 1.1.41,

$$\frac{|u(x)v(x)|}{\|u\|_p\|v\|_q} \leq \frac{|u(x)|^p}{p\|u\|_p^p} + \frac{|v(x)|^q}{q\|v\|_q^q}.$$

Taking integrals, we get

$$\begin{aligned}
\int_X \frac{|u(x)v(x)|}{\|u\|_p\|v\|_q} d\mu &\leq \int_X \frac{|u(x)|^p}{p\|u\|_p^p} d\mu + \int_X \frac{|v(x)|^q}{q\|v\|_q^q} d\mu, \\
\frac{1}{\|u\|_p\|v\|_q}\|uv\|_1 &\leq \frac{1}{p\|u\|_p^p}\|u\|_p^p + \frac{1}{q\|v\|_q^q}\|v\|_q^q, \\
\frac{1}{\|u\|_p\|v\|_q}\|uv\|_1 &\leq \frac{1}{p} + \frac{1}{q} = 1.
\end{aligned}$$

Therefore, $\|uv\|_1 \leq \|u\|_p\|v\|_q$. ∎

Theorem 1.1.43 (Minkowski's inequality)**.** *Let $f, g \in L^p(X)$ and $p \geq 1$. Then,*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. If $f + g \equiv 0$ (μ-a.e.), we immediately get the result. Suppose that $f + g \not\equiv 0$ (μ-a.e.). Using the triangle inequality, we obtain

$$\begin{aligned}
\|f+g\|_p^p &= \int_X |f+g|^p \, d\mu = \int_X |f+g| \cdot |f+g|^{p-1} \, d\mu \\
&\leq \int_X (|f| + |g|) \cdot |f+g|^{p-1} \, d\mu = \int_X |f| \cdot |f+g|^{p-1} \, d\mu + \int_X |g| \cdot |f+g|^{p-1} \, d\mu,
\end{aligned}$$

taking $u = f$ or $u = g$, and $v = (f+g)^{p-1}$, Hölder's inequality 1.1.42 gives us,

$$\begin{aligned}
\|f+g\|_p^p &\leq \left(\left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} + \left(\int_X |g|^p \, d\mu \right)^{\frac{1}{p}} \right) \cdot \left(\int_X |f+g|^{(p-1)\left(\frac{p}{p-1}\right)} \, d\mu \right)^{1-\frac{1}{p}} \\
&= (\|f\|_p + \|g\|_p) \cdot \frac{\|f+g\|_p^p}{\|f+g\|_p}.
\end{aligned}$$

Thus, multiplying both sides by $\frac{\|f+g\|_p}{\|f+g\|_p^p}$ we get Minkowski's inequality.

To prove that $f + g \in L^p(X)$ we need this observation:

$$|f+g|^p \leq (|f| + |g|)^p \leq (2\max\{|f|, |g|\})^p = 2^p \max\{|f|^p, |g|^p\} \leq 2^p(|f|^p + |g|^p).$$

The functions $|f|^p$ and $|g|^p$ are integrable. Therefore, $|f+g|^p$ is integrable and $f+g \in L^p(X)$ as we desire. ∎

To talk about completeness of $L^p(X)$ we will establish a few lemmas which we found in [8].

Lemma 1.1.44. *If $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence on $\mathcal{M}$, a metric space with metric d, and there exists a subsequence $(f_{k_j})_{j\in\mathbb{N}}$ convergent, then f_k converges.*

Proof. Let $\varepsilon > 0$. There exist N_1, N_2, and $f \in \mathcal{M}$ such that

$$m, n > N_1 \Longrightarrow d(f_n, f_m) < \frac{\varepsilon}{2} \quad \text{and} \quad j > N_2 \Longrightarrow d(f_{k_j}, f) < \frac{\varepsilon}{2}.$$

W.l.o.g., we can assume that $k_j > N_1$ for $j > N_2$ and with this assumption we have:

$$d(f_n, f) \leq d(f_n, f_{k_j}) + d(f_{k_j}, f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, f_k converges to f. ■

Lemma 1.1.45. *Suppose that $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence on $\mathcal{M}$, a metric space with a metric d. Then, there exists a subsequence $(f_{k_j})_{j\in\mathbb{N}}$ such that $d(f_{k_{j+1}}, f_{k_j}) < \frac{1}{2^j}$.*

Proof. Let $\varepsilon > 0$. By assumption, there exists $N(\varepsilon)$ such that if $n, k > N$, then $|f_n - f_k| < \varepsilon$. We attempt to construct a sequence $(n_j)_{j\in\mathbb{N}}$ as follows:

$$n_1, k > N(2^{-1}) \Longrightarrow d(f_{n_1}, f_k) < \frac{1}{2},$$

$$n_2, k > N(2^{-2}) \Longrightarrow d(f_{n_2}, f_k) < \frac{1}{2^2},$$

$$\vdots$$

In general, we require:

$$n_j \geq N(2^{-j}) \text{ and } n_{j+1} > n_j \text{ for } j \in \mathbb{N}.$$

Since $n_{j+1} > n_j \geq N(2^{-j})$, we deduce that $d(f_{k_{j+1}}, f_{k_j}) \leq \frac{1}{2^j}$. ■

Lemma 1.1.46 (Triangle inequality for infinite factors)**.** *Assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence of functions in $L^p(X)$ and each $f_k \geq 0$. Then,*

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_p \leq \sum_{k=1}^{\infty} \|f_k\|_p.$$

Proof. First of all, we have to mention that the statement is similar to Minkowski's inequality for a countable sum. Define $F_N \equiv \sum_{k=1}^{N} f_k$. Besides, we know that

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_p = \| \lim_{N\to\infty} F_N \|_p = \Big(\int_X | \lim_{N\to\infty} F_N |^p \, d\mu \Big)^{\frac{1}{p}} = \Big(\int_X \lim_{N\to\infty} |F_N|^p \, d\mu \Big)^{\frac{1}{p}}.$$

By Lebesgue's dominated convergence theorem (LDCT),

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_p = \lim_{N\to\infty} \Big(\int_X |F_N|^p \, d\mu \Big)^{\frac{1}{p}} = \lim_{N\to\infty} \|F_N\|_p = \lim_{N\to\infty} \Big\| \sum_{k=1}^{N} f_k \Big\|_p.$$

By Minkowski's inequality 1.1.43 we get,

$$\Big\| \sum_{k=1}^{\infty} f_k \Big\|_p \leq \lim_{N\to\infty} \sum_{k=1}^{N} \|f_k\|_p = \sum_{k=1}^{\infty} \|f_k\|_p.$$ ■

The next theorem was found out and proven independently by Frigyes Riesz and Ernst Fischer.

Theorem 1.1.47 (Riesz-Fischer). *$(L^p(X), \|\cdot\|_p)$ is a Banach space $(p \geq 1)$.*

By Theorem 1.1.39 we know that $L^p(X)$ is a normed space. What remains is to prove the completeness of $L^p(X)$.

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $L^p(X)$. By Lemmas 1.1.44 and 1.1.45, we can assume that $\|f_{k+1} - f_k\|_p < \frac{1}{2^k}$ for $k \geq 1$.

We want to see that f_k converges. So, we define the function F as:

$$F(x) \equiv |f_1(x)| + \sum_{j=1}^{\infty} |f_{j+1}(x) - f_j(x)|.$$

Using Lemma 1.1.46 we shall prove that F is $\mathcal{M}$-measurable, where $\mathcal{M}$ is a σ-algebra. By Lemma 1.1.45, we could take a subsequence if it were necessary, we get

$$\|F\|_p \leq \|f_1\|_p + \sum_{j=1}^{\infty} \|f_{j+1} - f_j\|_p \leq \|f_1\|_p + \sum_{j=1}^{\infty} \frac{1}{2^j} = \|f_1\|_p + 1 < \infty.$$

This provides that $F \in L^p(X)$. Since $F < \infty$ (μ-a.e.), there exists a null set $N \subset X$ such that $|F(x)| \neq \infty$ for $x \in N^c$.

Considering such a set, if $x \in N^c$ we can establish that

$$F(x) = |f_1(x)| + \sum_{j=1}^{\infty} |f_{j+1}(x) - f_j(x)| \quad \text{converges pointwise.} \tag{1.1.10}$$

We know that a series converges if it is absolutely convergent. Then, by Equation (1.1.10)

- ◇ $f_1(x) + \sum_{j=1}^{\infty}(f_{j+1}(x) - f_j(x))$ converges for every $x \in N^c$,
- ◇ $f_1(x) + \lim_{k\to\infty} \sum_{j=1}^{k}(f_{j+1}(x) - f_j(x))$ exists for every $x \in N^c$, and finally
- ◇ $\lim_{k\to\infty} f_k(x)$ also exists for every $x \in N^c$.

Therefore, it is possible to define

$$f(x) \equiv \begin{cases} \lim_{k\to\infty} f_k(x) & \text{if } x \notin N; \\ 0 & \text{if } x \in N. \end{cases}$$

Note that, $f(x) = f_1(x) + \sum_{j=1}^{\infty}(f_{j+1}(x) - f_j(x))$ and $|f(x)| \leq F(x)$, for $x \in N^c$.

Thus,

$$f(x) - f_k(x) = \sum_{j=k}^{\infty}(f_{j+1}(x) - f_j(x)), \text{ for } x \in N^c.$$

The last equality is true because $\sum_{j=k}^{\infty}(f_{j+1}(x)-f_j(x))$ is a telescoping series. This implies that,

$$|f(x)-f_k(x)| \leq \sum_{j=k}^{\infty} |f_{j+1}(x)-f_j(x)|.$$

By Lemma 1.1.46, we can assume that $\|f-f_k\|_p \leq \sum_{j=k}^{\infty}\|f_{j+1}-f_j\|_p \leq \sum_{j=k}^{\infty}\frac{1}{2^j} = 2^{1-k}$, this means that $\lim_{k\to\infty}\|f-f_k\|_p = 0$. Then, f_k converges to f.

Recall that $|f(x)| \leq F(x)$ for $x \in N^c$ and since $F \in L^p(X)$, we deduce $f \in L^p(X)$. ■

We want to know a little bit more about the space $L^\infty(X)$ and we will state two important definitions. First, we say that the class $L^\infty(X)$ consists of the bounded measurable functions.

Definition 1.1.48 (Essentially bounded). *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{F}$. Let f be a function on X. We say that f is essentially bounded on X if there exists $M > 0$ such that $\|f\| \leq M$ (μ-a.e.) on X.*

Definition 1.1.49 (Essential supremum). *Let $L^\infty(X)$ be the collection of all essentially bounded functions on X. Moreover, let $f \in L^\infty(X)$ then, we state that the essential supremum of f is*

$$\operatorname{ess\,sup} f \equiv \inf\{M\colon \|f\| \leq M \text{ for } \mu\text{-a.e. } x \in X\} = \|f\|_\infty.$$

By Theorem 1.1.18 we obtain that $\operatorname{ess\,sup} f$ is the L^∞-norm.

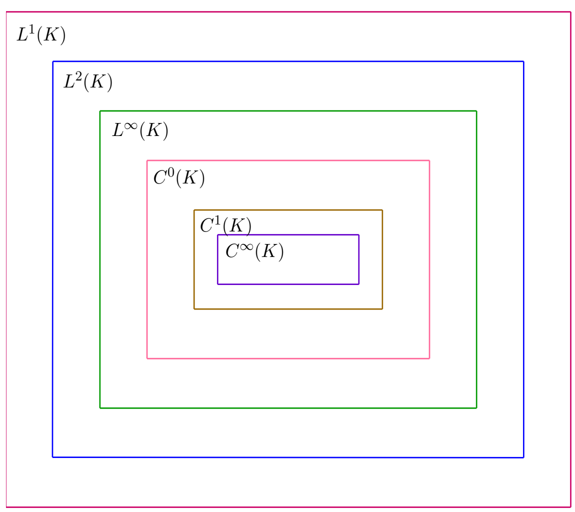


Figure 1.1: Nested spaces with K a compact set.

Now, we know that $L^p(X)$ is a Banach space. It is time to give some examples:

1. $\mathbb{R}^n$ with the norm defined thereby $\|x\|^2 \equiv \sum_{i=1}^{n}|x_i|^2$ is a Banach space.
2. $\mathbb{R}^n$ associated with the norm $\|x\| \equiv \max\{|x_1|, |x_2|, \ldots, |x_n|\}$ is a Banach space.

3. Let $C_b(X)$ be the space of bounded continuous functions ($f\colon X \longrightarrow \mathbb{C}$) with the supremum norm $\|f\|_\infty \equiv \sup\{|f(x)|\colon x \in X\}$. Then, $(C_b(X), \|\cdot\|_\infty)$ is a Banach space.

   By Theorem 1.1.18, $\|\cdot\|_\infty$ is a norm and mimicking the proof of Theorem 1.1.35 we can deduce the completeness.

4. Let $C_c(X)$ be the space of continuous functions ($f\colon X \longrightarrow \mathbb{C}$) with compact support. We define the support of f as $\operatorname{supp}(f) \equiv \overline{\{x \in X\colon f(x) \neq 0\}}$. With the supremum norm the space $(C_c(X), \|\cdot\|_\infty)$ is not a Banach space because it is not complete. Let ϕ be a nonzero continuous function with support inside $[0,1]$. We define the functions f_n as follows,

$$f_1(x) \equiv \phi(x),$$

$$f_2(x) \equiv \phi(x) + \frac{1}{2}\phi(x-1),$$

$$f_3(x) \equiv \phi(x) + \frac{1}{2}\phi(x-1) + \frac{1}{3}\phi(x-2),$$

$$\vdots$$

   In general, we require:

$$f_n(x) \equiv \sum_{i=0}^{n-1} \frac{1}{i+1}\phi(x-i).$$

   In particular, if we take $\phi(x) = \sin(\pi x)$ the graphics of the f_n's are:

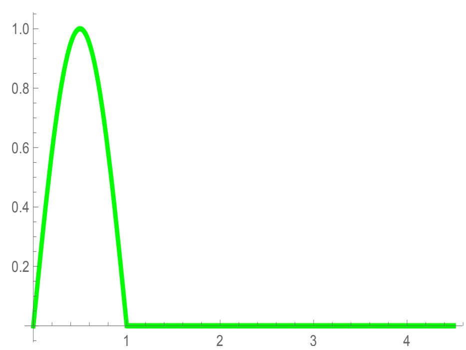


Figure 1.2: Graphic of f_1.

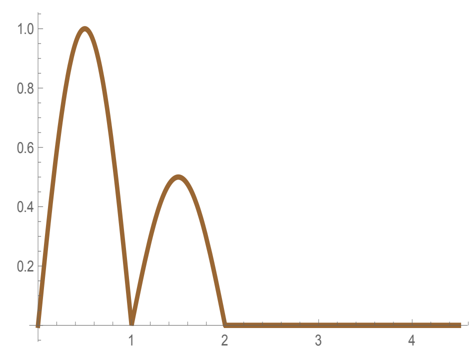


Figure 1.3: Graphic of f_2.

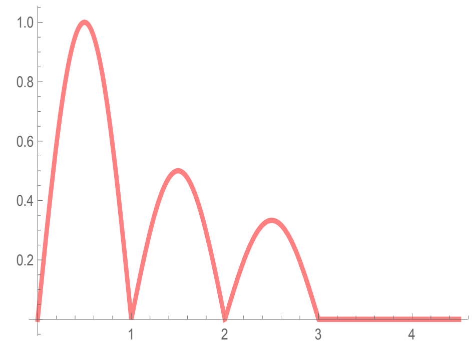


Figure 1.4: Graphic of f_3.

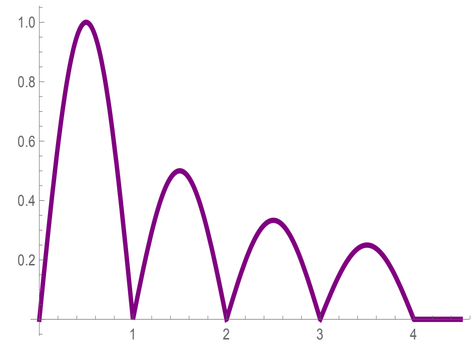


Figure 1.5: Graphic of f_4.

Then, for every n we get that $f_n \in C_c(\mathbb{R})$. We need prove that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

W.l.o.g., suppose that $n < m$. Let $\mathcal{J} \equiv |f_m(x) - f_n(x)|$.

$$\mathcal{J} = \Big| \sum_{i=n}^{m-1} \frac{1}{i+1}\phi(x-i) \Big| \leq \sum_{i=n}^{m-1} \frac{1}{i+1}|\phi(x-i)|.$$

Note that, $x - i \in [0,1]$ if and only if $i = \lfloor x \rfloor$. Then, we get

$$\mathcal{J} = \begin{cases} \frac{1}{\lfloor x \rfloor + 1}|\phi(x - \lfloor x \rfloor)|, & \text{if } \lfloor x \rfloor \in [n, m-1], \\ 0, & \text{otherwise.} \end{cases}$$

In either case, $\mathcal{J} \leq \frac{1}{n+1}\|\phi\|_\infty$ tends to zero as n goes to ∞. Therefore, $(f_i)_{i\in\mathbb{N}}$ is a Cauchy sequence.

Additionally, f_n converges uniformly to the continuous function

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{i+1}\phi(x-i).$$

However, f does not have compact support. Therefore, $C_c(\mathbb{R})$ is not a complete space with the supremum norm.

5. (Continuous functions which vanish at infinity).

$$C_0(X) \equiv \Big\{ f\colon X \longrightarrow \mathbb{C}\colon \lim_{|x|\to\infty} f(x) = 0 \Big\}.$$

Vanishing at infinity means that for every $\varepsilon > 0$ there exists a compact subset $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \in K^c$. Moreover, notice that $C_0(X) \subseteq C_b(X)$. It is easy to prove that $C_0(X)$ is a subspace of $C_b(X)$.

For example, consider the function $f_n \colon \mathbb{R} \longrightarrow [0,1]$ defined by

$$f_n(x) = \begin{cases} \frac{1}{n}, & \text{if } |x| < n, \\ \frac{(n+1-|x|)}{n}, & \text{if } n \leq |x| \leq n+1, \\ 0, & \text{if } |x| > n+1. \end{cases}$$

To be clear, we illustrate some $f_n's$ as follows:

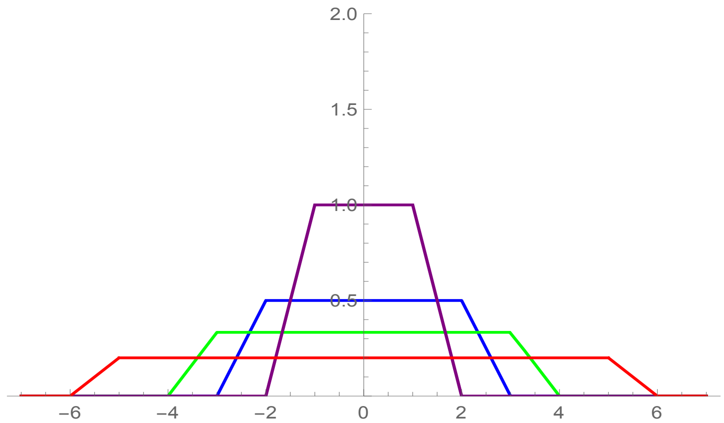


Figure 1.6: f_1 in purple, f_2 in blue, f_3 in green and f_5 in red.

For every $n \in \mathbb{N}$, $f_n \in C_0(\mathbb{R})$ but notice that $\lim_{n\to\infty} f_n \not\equiv 0$. We must be careful at $x = 0$ while we are finding out the limit of this sequence of functions, because $\lim_{n\to\infty} f_n(0) = \infty$. Hence, $\lim_{n\to\infty} f_n(x) \notin C_0(\mathbb{R})$ and $C_0(X)$ is not a complete subspace of $C_b(X)$.

Example 1.1.50. Consider real functions. The function $f(x) = e^{-x^2}$ belongs to $C_0(\mathbb{R})$ but not to $C_c(\mathbb{R})$. Moreover, the function

$$f(x) = \begin{cases} 1 - x^2, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

belongs to $C_c(\mathbb{R})$.

Finally, we obtain that $C_c(X) \subseteq C_0(X) \subseteq C_b(X)$. These are vector spaces but only $C_b(X)$ is a Banach space.

We will be particularly interested in the Lebesgue space $L^2(\mathbb{T})$ during the development of this document.

Now, we will introduce other vector spaces (linear spaces) that are important to us. These are the "sequence spaces" which are vector spaces whose elements are infinite sequences, they can be real or complex. Here, addition and scalar multiplication are defined componentwise. Sequence spaces have similar properties as those mentioned for Lebesgue spaces.

Definition 1.1.51 (Lebesgue spaces ℓ^p)**.** *Let $1 \leq p < \infty$. The space $\ell^p(\mathbb{I})$, with $\mathbb{I}$ a countable set of indices, consists of all sequences $\boldsymbol{x} : \mathbb{I} \to \mathbb{F}$ such that $\sum_{i\in\mathbb{I}} |x_i|^p < \infty$. The ℓ^p-norm of $\boldsymbol{x} \in \ell^p(\mathbb{I})$ is defined by*

$$\|\boldsymbol{x}\|_p^p \equiv \sum_{i\in\mathbb{I}} |x_i|^p.$$

The Lebesgue space $\ell^p(\mathbb{I})$ is a sequence space. The sequence $\boldsymbol{x} \in \ell^p(\mathbb{I})$ can be indexed by $\mathbb{N}$ or $\mathbb{Z}$ and these are the most common cases.

Proposition 1.1.52. *The ℓ^p-norm of $\boldsymbol{x} \in \ell^p(\mathbb{I})$ is indeed a norm. We have,*

(i) *For any* $\boldsymbol{x} \in \ell^p(\mathbb{I})$ *it holds that* $\|\boldsymbol{x}\|_p = 0$ *if and only if* $\boldsymbol{x} = 0$, *i.e., each* x_i *must be zero for* $i \in \mathbb{I}$.

(ii) *For any* $\alpha \in \mathbb{F}$ *and* $\boldsymbol{x} \in \ell^p(\mathbb{I})$ *we have that* $\|\alpha \cdot \boldsymbol{x}\|_p^p = |\alpha|^p \|\boldsymbol{x}\|_p^p$.

(iii) *For any* $\boldsymbol{x}, \boldsymbol{y} \in \ell^p(\mathbb{I})$, Theorem 1.1.54 *will show that* $\|\boldsymbol{x} + \boldsymbol{y}\|_p \leq \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$.

Then, we state that $\ell^p(\mathbb{I})$ *is a normed space.*

The next theorems allow us to talk about completeness in $\ell^p(\mathbb{I})$. Besides, they present properties which we should be familiar with.

The counting measure permits us to obtain the properties of $\ell^p(\mathbb{I})$ and their proofs.

Theorem 1.1.53 (Hölder's inequality for sequences)**.** *Assume that* $1 < p, q < \infty$ *are Hölder conjugates. In addition,* $\boldsymbol{u} \in \ell^p(\mathbb{I})$ *and* $\boldsymbol{v} \in \ell^q(\mathbb{I})$. *Then,* $\boldsymbol{uv} \in \ell^1(\mathbb{I})$ *and* $\|\boldsymbol{uv}\|_1 \leq \|\boldsymbol{u}\|_p \|\boldsymbol{v}\|_q$.

Proof. Let $\boldsymbol{u} \equiv (u_i)_{i\in\mathbb{I}}$ and $\boldsymbol{v} \equiv (v_j)_{j\in\mathbb{I}}$. If we have $\boldsymbol{u} \equiv 0$ or $\boldsymbol{v} \equiv 0$, then theorem holds trivially. Assume that $\boldsymbol{u} \not\equiv 0$ and $\boldsymbol{v} \not\equiv 0$. W.l.o.g., suppose that $\sum_{i\in\mathbb{I}} |u_i|^p = \sum_{j\in\mathbb{I}} |v_j|^q = 1$. Due to Lemma 1.1.41 we have,

$$|u_i v_j| \leq \frac{1}{p}|u_i|^p + \frac{1}{q}|v_j|^q. \tag{1.1.11}$$

Adding over $\mathbb{I}$, we get

$$\sum_{i,j\in\mathbb{I}} |u_i v_j| \leq \frac{1}{p}\sum_{i\in\mathbb{I}} |u_i|^p + \frac{1}{q}\sum_{j\in\mathbb{I}} |v_j|^q = \frac{1}{p} + \frac{1}{q} = 1.$$

When $\boldsymbol{u}$ and $\boldsymbol{v}$ converge to a number different of 1. We construct two new sequences as follows:

$$(a_i)_{i\in\mathbb{I}} \equiv \frac{(u_i)_{i\in\mathbb{I}}}{\|\boldsymbol{u}\|_p} \quad \text{and} \quad (b_j)_{j\in\mathbb{I}} \equiv \frac{(v_j)_{j\in\mathbb{I}}}{\|\boldsymbol{v}\|_q}.$$

Note that, $\sum_{i\in\mathbb{I}} |a_i|^p = 1$ and $\sum_{j\in\mathbb{I}} |b_j|^q = 1$. Then, we can apply Equation (1.1.11),

$$\sum_{i,j\in\mathbb{I}} |a_i b_j| = \sum_{i,j\in\mathbb{I}} \frac{|u_i v_j|}{\|\boldsymbol{u}\|_p \cdot \|\boldsymbol{v}\|_q} \leq 1.$$

Therefore, $\sum_{i,j\in\mathbb{I}} |a_i b_j| \leq (\sum_{i\in\mathbb{I}} |u_i|^p)^{\frac{1}{p}} (\sum_{j\in\mathbb{I}} |v_j|^q)^{\frac{1}{q}}$ as we desired. ■

Theorem 1.1.54 (Minkowski's inequality for series)**.** *Let* $\boldsymbol{x}, \boldsymbol{y} \in \ell^p(\mathbb{I})$ *and* $p \geq 1$. *Then,* $\boldsymbol{x} + \boldsymbol{y} \in \ell^p(\mathbb{I})$ *and*

$$\|\boldsymbol{x} + \boldsymbol{y}\|_p \leq \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p.$$

Proof. Firstly, we know that $\boldsymbol{x} + \boldsymbol{y} \equiv (x_i + y_i)_{i\in\mathbb{I}}$. If $\boldsymbol{x} + \boldsymbol{y} \equiv 0$, we get Minkowski's inequality. Assume that $\boldsymbol{x} + \boldsymbol{y} \not\equiv 0$. Using triangle inequality we obtain,

$$\begin{aligned}\|\boldsymbol{x} + \boldsymbol{y}\|_p^p &= \sum_{i\in\mathbb{I}} |x_i + y_i|^p = \sum_{i\in\mathbb{I}} |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &\leq \sum_{i\in\mathbb{I}} |x_i + y_i|^{p-1}|x_i| + \sum_{i\in\mathbb{I}} |x_i + y_i|^{p-1}|y_i|.\end{aligned}$$

Taking $\boldsymbol{u} = \boldsymbol{x}$ or $\boldsymbol{u} = \boldsymbol{y}$, and $\boldsymbol{v} = (\boldsymbol{x}+\boldsymbol{y})^{p-1}$ Hölder's inequality for sequences 1.1.53 give us,

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|_p^p &\leq \left(\Big(\sum_{i\in\mathbb{I}} |x_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i\in\mathbb{I}} |y_i|^p\Big)^{\frac{1}{p}}\right) \cdot \Big(\sum_{i\in\mathbb{I}} |x_i+y_i|^p\Big)^{\frac{p-1}{p}} \\
&= (\|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p) \cdot \Big(\sum_{i\in\mathbb{I}} |x_i+y_i|^{(p-1)\frac{p}{p-1}}\Big)^{1-\frac{1}{p}} \\
&= (\|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p) \cdot \frac{\|\boldsymbol{x}+\boldsymbol{y}\|_p^p}{\|\boldsymbol{x}+\boldsymbol{y}\|_p}.
\end{aligned}$$

Finally, multiplying both sides by $\frac{\|\boldsymbol{x}+\boldsymbol{y}\|_p}{\|\boldsymbol{x}+\boldsymbol{y}\|_p^p}$ we obtain Minkowski's inequality for series.

To prove that $\boldsymbol{x}+\boldsymbol{y} \in \ell^p(\mathbb{I})$, notice that for $i \in \mathbb{I}$:

$$|x_i+y_i|^p \leq (|x_i|+|y_i|)^p \leq (2\max\{|x_i|,|y_i|\})^p = 2^p \max\{|x_i|^p, |y_i|^p\} \leq 2^p(|x_i|^p + |y_i|^p).$$

Therefore, for each $i \in \mathbb{I}$ we have $x_i + y_i \in \mathbb{F}$ and additionally $\boldsymbol{x}+\boldsymbol{y} \colon \mathbb{I} \longrightarrow \mathbb{F}$ is a new sequence satisfying $\sum_{i\in\mathbb{I}} |x_i+y_i|^p < \infty$. ■

Theorem 1.1.55. *$(\ell^p(\mathbb{I}), \|\cdot\|_p)$ is a Banach space.*

Proof. By Proposition 1.1.52 we know that $\ell^p(\mathbb{I})$ is a normed space. So, we will prove the completeness of this space. Suppose that $\boldsymbol{x}^{(n)} \equiv ((x_i^{(n)})_{i\in\mathbb{I}})_{n\in\mathbb{N}} \in \ell^p(\mathbb{I})$ is a Cauchy sequence respect to ℓ^p-norm. Then, for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that if $n, m \geq N(\varepsilon)$ then,

$$\sum_{i\in\mathbb{I}} |x_i^{(n)} - x_i^{(m)}|^p = \|\boldsymbol{x}^{(n)} - \boldsymbol{x}^{(m)}\|_p^p < \varepsilon. \tag{1.1.12}$$

In particular, for any $i \in \mathbb{I}$ we have,

$$|x_i^{(n)} - x_i^{(m)}|^p < \varepsilon.$$

Thus, for each fixed i, $(x_i^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{F}$ and so has a limit $x_i \in \mathbb{F}$.

Let's see that $\boldsymbol{x_i}^{(n)}$ converges to $(x_i)_{i\in\mathbb{I}}$.

If we take m goes to infinity in Equation (1.1.12), it gives us

$$\sum_{i\in\mathbb{I}} |x_i^{(n)} - x_i|^p \leq \varepsilon. \tag{1.1.13}$$

For any $n \geq N(\varepsilon)$ we have,

$$\sum_{i\in\mathbb{I}} |x_i|^p \leq \sum_{i\in\mathbb{I}} |x_i^{N(\varepsilon)} - x_i|^p + \sum_{i\in\mathbb{I}} |x_i^{N(\varepsilon)}|^p \leq \varepsilon + \|\boldsymbol{x_i}^{N(\varepsilon)}\|_p^p.$$

We conclude $\sum_{i\in\mathbb{I}} |x_i|^p$ is finite, it implies that $(x_i)_{i\in\mathbb{I}} \in \ell^p(\mathbb{I})$.

Finally, Equation (1.1.13) shows that $\lim_{n\to\infty} \|x_i^{(n)} - x_i\|_p^p = 0$, as we required. ■

On $\ell^2(\mathbb{Z})$ we define an inner product as follows:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \equiv \sum_{i \in \mathbb{Z}} x_i \overline{y_i} \quad \text{for} \quad \boldsymbol{x}, \boldsymbol{y} \in \ell^2(\mathbb{Z}).$$

It is easy to see that $\langle \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{w} \rangle = \langle \boldsymbol{x}, \boldsymbol{w} \rangle + \langle \boldsymbol{y}, \boldsymbol{w} \rangle$, $\langle c\boldsymbol{x}, \boldsymbol{y} \rangle = c\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, and $\overline{\langle \boldsymbol{x}, \boldsymbol{y} \rangle} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$, for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w} \in \ell^2(\mathbb{Z})$ and c any scalar. Furthermore, it holds that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \|\boldsymbol{x}\|_2^2$ and using Definition 1.1.51 with $d = 2$ and $\mathbb{I} = \mathbb{Z}$ we have that $0 \leq \langle \boldsymbol{x}, \boldsymbol{x} \rangle < \infty$. Finally, Proposition 1.1.52 gives us that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if $x_i = 0$ for every $i \in \mathbb{Z}$.

Notice that, ℓ^2-norm arises from this inner product. With the work we have already done about $\ell^p(\mathbb{I})$, we get that $(\ell^2(\mathbb{Z}), \|\cdot\|_2)$ is a Hilbert space.

With this result we have shown that Lebesgue spaces $\ell^p(\mathbb{I})$ are Banach spaces. The entire work we have done is intended to establish a relation between a function and its Fourier coefficients, and for that task we need to impose some requirements. Furthermore, we will be interested in $L^2(X)$ and $\ell^2(\mathbb{I})$ spaces. Our purpose is to understand the isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.

1.2 Fourier section

Initially, it is important to know that Fourier analysis will be done in Hilbert spaces. In this section we study how to construct a Fourier series and what it means.

Definition 1.2.1 (Orthonormal basis). *A subset $\{v_1, \dots, v_k\}$ of a vector space V with the inner product $\langle \cdot, \cdot \rangle$, is called orthonormal if $\langle v_i, v_j \rangle = 0$ when $i \neq j$. Moreover, they are all required to have length one: $\langle v_i, v_i \rangle = 1$. This subset must be linearly independent and must span V.*

Now, we can gather information about $L^2(X)$. First, recall that we are working with measurable functions $f \colon X \to \mathbb{C}$ that are square integrable on X. We can define an inner product on $(L^2(X), \|\cdot\|_2)$ via

$$\langle f, g \rangle \equiv \int_X f\bar{g}\, d\mu.$$

Proposition 1.2.2. *Let $(L^2(X), \|\cdot\|_2)$ and $f, g \in L^2(X)$. Then, $\langle f, g \rangle = \int_X f\bar{g}\, d\mu$ is an inner product.*

Proof. We will verify that it satisfies all the properties:

(i) Let $f, g, h \in L^2(X)$.

$$\langle f+g, h \rangle = \int_X (f+g)\bar{h}\, d\mu = \int_X (f\bar{h} + g\bar{h})\, d\mu = \int_X f\bar{h}\, d\mu + \int_X g\bar{h}\, d\mu = \langle f, h \rangle + \langle g, h \rangle.$$

(ii) Let c be any scalar.

$$\langle cf, g \rangle = \int_X (cf)\bar{g}\, d\mu = \int_X c(f\bar{g})\, d\mu = c\int_X f\bar{g}\, d\mu = c\langle f, g \rangle.$$

(iii) $\overline{\langle f, g\rangle} = \overline{\int_X f\bar{g}\,d\mu} = \int_X g\bar{f}\,d\mu = \langle g, f\rangle$.

(iv) $\langle f, f\rangle = \int_X f\bar{f}\,d\mu = \int_X |f|^2\,d\mu = \|f\|_2^2$. By how $L^2(X)$ is defined we obtain that $0 \leq \langle f, f\rangle < \infty$, as we desire.

(v) By the property (ii) of Theorem 1.1.39 we easily get that, $\langle f, f\rangle = 0$ if and only if $f = 0$ ($\mu-$a.e.).

Therefore, we conclude that $\langle\cdot,\cdot\rangle$ is an inner product on $L^2(X)$. ■

Note that, the norm does indeed come from this inner product. Furthermore, by Theorem 1.1.47 we know that $L^2(X)$ is complete with the norm $\|\cdot\|_2$. Thus, $(L^2(X), \|\cdot\|_2)$ is a Hilbert space.

Now, we deal with the space $L^2(\mathbb{T})$ which is the space of functions $f\colon \mathbb{T} \longrightarrow \mathbb{C}$ such that $\oint_{\mathbb{T}} |f(t)|^2\,dt < \infty$ and the norm for $f \in L^2(\mathbb{T})$ is defined as follows

$$\|f\|_2^2 \equiv \frac{1}{2\pi i}\oint_{\mathbb{T}} |f(t)|^2 \frac{dt}{t}.$$

In this case, for any $f, g \in L^2(\mathbb{T})$ we have

$$\langle f, g\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta = \frac{1}{2\pi i}\oint_{\mathbb{T}} f(t)\overline{g(t)}\frac{dt}{t}.$$

Definition 1.2.3 (Fourier coefficients)**.** *On $L^2(\mathbb{T})$ let $\mathfrak{B} = (t^k)_{k=-\infty}^{\infty}$. We define $f_k \equiv \langle f, t^k\rangle$ for all $f \in L^2(\mathbb{T})$ and $k \in \mathbb{Z}$, f_k is known as the k-th Fourier coefficient of f. Thus,*

$$f_k = \frac{1}{2\pi i}\oint_{\mathbb{T}} f(t)t^{-k-1}\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}d\theta.$$

Theorem 1.2.6 will show that $\mathfrak{B}$ is an orthonormal basis.

The mapping $\Phi\colon L^2(\mathbb{T}) \longrightarrow \ell^2(\mathbb{Z})$ given by $f \longmapsto (f_k)_{k\in\mathbb{Z}}$ is known as the Fourier transform for $\mathbb{T}$. It is the operator which sends a function to the sequence of its Fourier coefficients and defines an isomorphism between the spaces $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$. Besides, notice that if f belongs to $L^2(\mathbb{T})$ then, the Fourier transform is the function $f_k\colon \mathbb{Z} \longrightarrow \mathbb{C}$.

Theorem 1.2.4 (Parseval's identity)**.** *Let $f \in L^2(\mathbb{T})$ and f_k be the k-th Fourier coefficient of f. Then,*

$$\sum_{k\in\mathbb{Z}} |f_k|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\,d\theta.$$

Proof.

$$\|f\|_2^2 = \langle f, f\rangle = \left\langle \sum_{k\in\mathbb{Z}} f_k t^k, \sum_{j\in\mathbb{Z}} f_j t^j\right\rangle = \sum_{k,j\in\mathbb{Z}} f_k\overline{f_j}\langle t^k, t^j\rangle = \sum_{k\in\mathbb{Z}} |f_k|^2.$$ ■

Theorem 1.2.5. *The Fourier transform is an isometric isomorphism from $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$.*

Proof. Let $\Phi\colon L^2(\mathbb{T}) \longrightarrow \ell^2(\mathbb{Z})$ be the Fourier transform. By Parseval's identity we know that $\|\Phi f\|_2 = \|f\|_2$. Now, we want to show that Φ is linear.

Let $a, b \in \mathbb{C}$ and $f, g \in L^2(\mathbb{T})$.

$$\begin{aligned}
\Phi(af+bg)(t) =& \frac{1}{2\pi}\int_0^{2\pi} [af(e^{i\theta}) + bg(e^{i\theta})]e^{-ik\theta}\, d\theta \\
=& \frac{1}{2\pi}\int_0^{2\pi} [af(e^{i\theta})e^{-ik\theta} + bg(e^{i\theta})e^{-ik\theta}]\, d\theta \\
=& \frac{1}{2\pi}\int_0^{2\pi} af(e^{i\theta})e^{-ik\theta}\, d\theta + \frac{1}{2\pi}\int_0^{2\pi} bg(e^{i\theta})e^{-ik\theta}\, d\theta \\
=& a\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-i\theta}\, d\theta + b\frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})e^{-ik\theta}\, d\theta \\
=& a\,\Phi f(t) + b\,\Phi g(t).
\end{aligned}$$

What remains is to verify the isomorphism $L^2(\mathbb{T}) \simeq \ell^2(\mathbb{Z})$. First, Φ is bijective and admits an inverse by how we have defined it. We will see that these spaces are equivalent with respect to the inner product. For $f, g \in L^2(\mathbb{T})$ we assume that $f(t) = \sum_{k\in\mathbb{Z}} f_k t^k$ and $g(t) = \sum_{j\in\mathbb{Z}} g_j t^j$,

$$\begin{aligned}
\langle f, g\rangle =& \frac{1}{2\pi i}\oint_{\mathbb{T}} \Big(\sum_{k\in\mathbb{Z}} f_k t^k\Big)\Big(\sum_{j\in\mathbb{Z}} \overline{g_j} t^{-j}\Big)\frac{dt}{t} \\
=& \frac{1}{2\pi i}\oint_{\mathbb{T}} \sum_{s\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} f_k\overline{g_{k-s}}\Big)t^s\frac{dt}{t} \\
=& \sum_{s\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} f_k\overline{g_{k-s}}\Big)\frac{1}{2\pi i}\oint_{\mathbb{T}} t^s\frac{dt}{t}.
\end{aligned}$$

Calculating this last integral, we get that

$$\frac{1}{2\pi i}\oint_{\mathbb{T}} t^s\frac{dt}{t} = \begin{cases} 1, & \text{if } s = 0, \\ 0, & \text{if } s \neq 0. \end{cases}$$

Thus, $\langle f, g\rangle = \sum_{k\in\mathbb{Z}} f_k\overline{g_k} = \langle (f_k)_{k\in\mathbb{Z}}, (g_k)_{k\in\mathbb{Z}}\rangle = \langle \Phi f, \Phi g\rangle$. ■

Theorem 1.2.6. *$\mathfrak{B}$ is an orthonormal basis for $L^2(\mathbb{T})$.*

Proof. (i) $\langle t^k, t^k\rangle = \dfrac{1}{2\pi i}\oint_{\mathbb{T}} |t^k|^2\dfrac{dt}{t} = \dfrac{1}{2\pi}\int_0^{2\pi} |e^{ik\theta}|^2\, d\theta = 1$. Therefore, $\|t^k\|_2 = 1$ for every $k \in \mathbb{Z}$.

(ii) Let $m, n \in \mathbb{Z}$ with $m \neq n$.

$$\langle t^n, t^m\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\theta} d\theta = \frac{1}{2\pi i}\left[\frac{e^{i(n-m)\theta}}{n-m}\right]_0^{2\pi} = 0.$$

(iii) Let $f \in L^2(\mathbb{T})$. We require that $f(t) = \sum_{k\in\mathbb{Z}} f_k t^k$.

Now we denote $\mathcal{A} \equiv \left\| f - \sum_{k=-N}^{N} f_k t^k \right\|_2^2$ and a simple calculation reveals that:

$$\begin{aligned}\mathcal{A} &= \left\langle f - \sum_{k=-N}^{N} f_k t^k, f - \sum_{k=-N}^{N} f_k t^k \right\rangle \\ &= \langle f, f\rangle - \left\langle f, \sum_{k=-N}^{N} f_k t^k \right\rangle - \left\langle \sum_{k=-N}^{N} f_k t^k, f \right\rangle + \left\langle \sum_{k=-N}^{N} f_k t^k, \sum_{k=-N}^{N} f_k t^k \right\rangle \\ &= \|f\|_2^2 - \sum_{k=-N}^{N} \overline{f_k}\langle f, t^k\rangle - \sum_{k=-N}^{N} f_k \langle t^k, f\rangle + \sum_{j=-N}^{N}\sum_{l=-N}^{N} f_j \overline{f_l} \langle t^j, t^l\rangle.\end{aligned}$$

In the case that $j \neq l$ from the previous item we get $\langle t^j, t^l\rangle = 0$, then

$$\begin{aligned}\mathcal{A} &= \|f\|_2^2 - \sum_{k=-N}^{N} \overline{f_k}\langle f, t^k\rangle - \sum_{k=-N}^{N} f_k \overline{\langle f, t^k\rangle} + \sum_{k=-N}^{N} |f_k|^2 \\ &= \|f\|_2^2 - \sum_{k=-N}^{N} \overline{f_k} f_k - \sum_{k=-N}^{N} f_k \overline{f_k} + \sum_{k=-N}^{N} |f_k|^2 \\ &= \|f\|_2^2 - \sum_{k=-N}^{N} |f_k|^2.\end{aligned}$$

Therefore, $\left\| f - \sum_{k=-N}^{N} f_k t^k \right\|_2^2 = \|f\|_2^2 - \sum_{k=-N}^{N} |f_k|^2$. At this point, we should analyze the limit $\lim_{N\to\infty} \sum_{k=-N}^{N} |f_k|^2$, but Parseval's identity gives us that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} |f_k|^2 = \|f\|_2^2.$$

We get that $\left\| f - \sum_{k=-N}^{N} f_k t^k \right\|_2^2 \xrightarrow[N\to\infty]{} 0$. Thus, the set of vectors $(t^k)_{k\in\mathbb{Z}}$ spans $L^2(\mathbb{T})$. Whence, $\mathfrak{B}$ is an orthonormal basis. ■

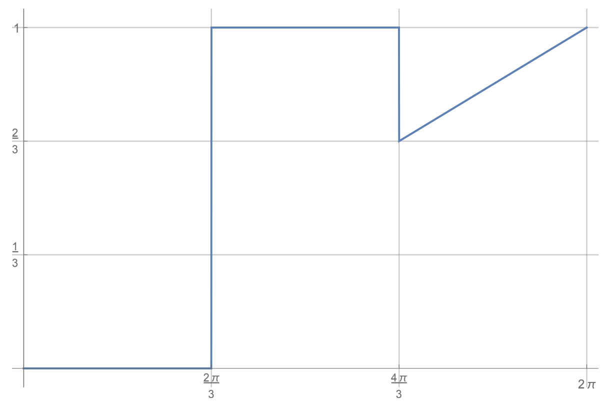
Remark 1.2.7. Note that $f(t) = \sum_{k\in\mathbb{Z}} f_k t^k$ means that $f - \sum_{k\in\mathbb{Z}} f_k t^k$ is a function with $\|\cdot\|_2 = 0$. Besides, the functions $f(t)$ and $\sum_{k\in\mathbb{Z}} f_k t^k$ are not necessarily equal for every value of t. That being so, we can interpret that given f it is possible to represent it as an infinite sum and this is done via Fourier transform which decomposes the function f.

Definition 1.2.8 (Fourier series)**.** *A Fourier series for a piecewise continuous function is an infinite series expansion of oscillating functions. We are referring to the study of wave motion, when a basic waveform repeats itself periodically. Then, for $f \in L^2(\mathbb{T})$ the Fourier series is*

$$\sum_{k\in\mathbb{Z}} f_k e^{ik\theta} = \sum_{k\in\mathbb{Z}} f_k t^k.$$

Example 1.2.9. Consider the function $f\colon \mathbb{T} \longrightarrow [0,1]$ given by

$$f(e^{i\theta}) = \begin{cases} 0, & \text{if } 0 \le \theta \le \frac{2\pi}{3}, \\ 1, & \text{if } \frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}, \\ \frac{1}{2\pi}\theta, & \text{if } \frac{4\pi}{3} \le \theta \le 2\pi. \end{cases}$$



Note that f is a piecewise continuous function. The k-th Fourier coefficient of f is

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta}\, d\theta.$$

We can write it as follows:

$$\begin{aligned} f_k &= \frac{1}{2\pi} \int_0^{\frac{2\pi}{3}} a(e^{i\theta}) e^{-ik\theta}\, d\theta + \frac{1}{2\pi} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} a(e^{i\theta}) e^{-ik\theta}\, d\theta + \frac{1}{2\pi} \int_{\frac{4\pi}{3}}^{2\pi} a(e^{i\theta}) e^{-ik\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} e^{-ik\theta}\, d\theta + \frac{1}{4\pi^2} \int_{\frac{4\pi}{3}}^{2\pi} \theta e^{-ik\theta}\, d\theta. \end{aligned}$$

After calculating these integrals we get,

$$f_k = \frac{e^{-2ik\pi}}{12k^2\pi^2}\left(3 + 6ik\pi - 6ik\pi e^{\frac{4ik\pi}{3}} + e^{\frac{2ik\pi}{3}}(-3 + 2ik\pi)\right), \quad k \neq 0 \quad \text{and} \quad f_0 = \frac{11}{18}.$$

Let $F_m(t) \equiv \sum_{k=-m}^{m} f_k t^k$. Clearly, $f = \lim_{m\to\infty} F_m$. The figures 1.7–1.9 exhibit the graphics of the functions F_5, F_{10}, and F_{40} for f as in Example 1.2.9.

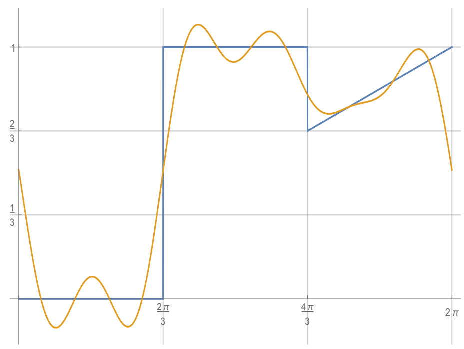


Figure 1.7: Graphic of F_5 (yellow) and f (blue) from Example 1.2.9.

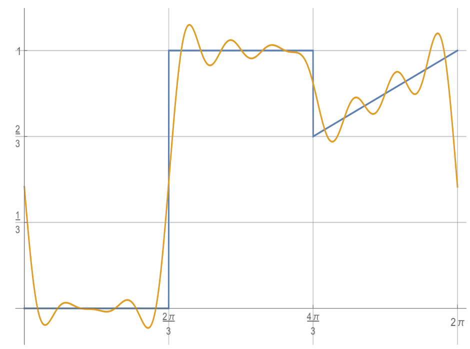


Figure 1.8: Graphic of F_{10} (yellow) and f (blue) from Example 1.2.9.

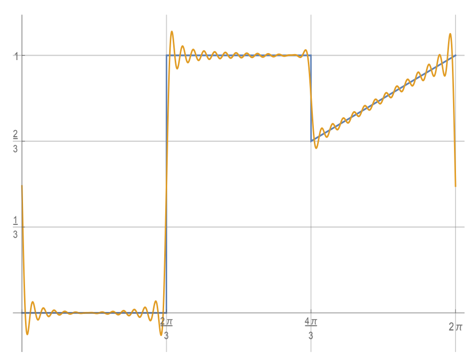


Figure 1.9: Graphic of F_{40} (yellow) and f (blue) from Example 1.2.9.

Chapter 2

Infinite matrices

2.1 Boundedness and invertibility

Boundedness of functions is a useful condition, but it is stronger on linear transformations. Besides, it has a profound effect all over operator theory. Regarding invertible transformations, we say that a bounded linear transformation A from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ is invertible if there exists a bounded linear transformation B (from $\mathcal{K}$ into $\mathcal{H}$) such that $AB \equiv \mathbb{I}_{\mathcal{K}}$ and $BA \equiv \mathbb{I}_{\mathcal{H}}$.

Definition 2.1.1 (Linear operator). *If E, F are vector spaces over a field $\mathbb{F}$, a linear operator from E to F is a mapping $T\colon E \longrightarrow F$ such that $T(\lambda x + \mu y) = \lambda Tx + \mu Ty$ for every $\lambda, \mu \in \mathbb{F}$ and every $x, y \in E$.*

Definition 2.1.2 (Bounded linear operator). *If E, F are normed spaces, then a linear operator $T\colon E \longrightarrow F$ is said to be bounded if there exists $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for every $x \in E$.*

Definition 2.1.3 (Topological vector space). *Let V be a vector space endowed with a topology τ. The pair $\langle V, \tau\rangle$ is called a topological vector space if the vector space operations are continuous with respect to τ.*

The condition means that the mappings:

$$V \times V \longrightarrow V \quad (x, y) \longmapsto x + y \quad \text{and} \quad \mathbb{F} \times V \longrightarrow V \quad (\alpha, x) \longmapsto \alpha x$$

are continuous. In addition, functions and linear operators belong to a topological vector space and the topology is defined to catch the notion of convergence of sequences of functions. Then, all Hilbert and Banach spaces are examples of topological vector spaces.

Given a Banach space X, we denote by $\mathcal{B}(X)$ the collection of all bounded linear operators on X. Besides, $\mathcal{B}(X)$ is equipped with $\|\cdot\|_{op}$, this is the norm of an operator $A \in \mathcal{B}(X)$ and it is defined as follows,

$$\|A\|_{op} \equiv \sup\left\{\frac{\|Ax\|_X}{\|x\|_X} : x \neq 0\right\}.$$

$\|A\|_{op}$ is the largest ratio of $\|A\|_X$ to $\|x\|_X$ by which A stretches an element of X. When the supremum exists it is the number by which this ratio is bounded.

Theorem 2.1.4. *Let* $D_1 = \sup\{\|Ax\|_X : \|x\|_X \leq 1\}$, $D_2 = \sup\{\|Ax\|_X : \|x\|_X = 1\}$, *and* $D_3 = \sup\{\|Ax\|_X / \|x\|_X : x \neq 0\}$. *Then,* $D_1 = D_2 = D_3$.

Proof. To begin with, notice that $D_2 \leq D_1$. Let $x \neq 0$, then $\left\| \frac{x}{\|x\|_X} \right\|_X = 1$. Moreover, $\frac{1}{\|x\|_X}(Ax) = A\left(\frac{x}{\|x\|_X}\right)$. Then, $\left\| A\left(\frac{x}{\|x\|_X}\right) \right\|_X = \frac{\|Ax\|_X}{\|x\|_X}$ and we get that $D_3 \leq D_2$.

Now, if we assume that $\|x\|_X \leq 1$, then $\|Ax\|_X \leq \frac{\|Ax\|_X}{\|x\|_X}$. Thus, $D_1 \leq D_3$.

Finally, we obtain that $D_1 = D_2 = D_3$. ■

Theorem 2.1.5. *Let* $A \in \mathcal{B}(X)$. *Then,* A *is a continuous linear operator.*

Proof. By assumption A is bounded, then there exists $M > 0$ such that $\|Ax\|_X \leq M\|x\|_X$ for all $x \in X$.

Let $\varepsilon > 0$. If $x, y \in X$ and $\|x - y\|_X < \frac{\varepsilon}{M}$, then

$$\|Ax - Ay\|_X = \|A(x - y)\|_X \leq M\|x - y\|_X < \varepsilon.$$

Therefore, A is not only continuous but uniformly continuous also. ■

Definition 2.1.6 (Algebra over a field)**.** *Let* $\mathbb{F}$ *be a field and* V *be a vector space over* $\mathbb{F}$ *equipped with an additional binary operation* $\cdot : V \times V \longrightarrow V$, *i.e., given two elements* $x, y \in V$, *then* $x \cdot y \in V$ *and it is the product of* x *and* y. *Thus,* V *is an algebra over* $\mathbb{F}$ *if the following hold for* $x, y, z \in V$ *and* $a \in \mathbb{F}$,

(i) $x(yz) = (xy)z$.

(ii) $x(y + z) = xy + xz$ *and* $(x + y)z = xz + yz$.

(iii) $a(xy) = (ax)y = x(ay)$.

Definition 2.1.7 (Normed algebra)**.** *Let* V *be an algebra over a field which has a submultiplicative norm, i.e., for all* $x, y \in V$ *it satisfies*

$$\|xy\|_V \leq \|x\|_V \|y\|_V.$$

Now, we consider the structure of $\mathcal{B}(X)$. In general terms, $\mathcal{B}(X)$ is a vector space with the usual operations: addition, scalar multiplication, and associativity for the composition. Additionally, there is an identity operator, denoted by $\mathbb{I}_X$. More precisely, $\mathcal{B}(X)$ is an algebra of continuous linear operators on X and a topological vector space.

Proposition 2.1.8. *Let* $\mathcal{B}(X)$ *be the space of bounded linear operators. Then,* $\|\cdot\|_{op}$ *is a norm.*

Proof. Let $A, B \in \mathcal{B}(X)$ and a be any scalar.

(i) We know that X is, in particular, a normed space. Thus, for every $x \in X$ we obtain $0 \leq \|Ax\|_X < \infty$ and $0 \leq \|x\|_X < \infty$. Thus, for $x \neq 0$ it holds $0 \leq \frac{\|Ax\|_X}{\|x\|_X} < \infty$ and we get that $0 \leq \|A\|_{op} < \infty$.

(ii) It is clear that $\|A\|_{op} = 0$ if and only if $A = 0$.

(iii) $\|aA\|_{op} = \sup\{\|(aA)x\|_X/\|x\|_X : x \neq 0\} = \sup\{|a|\|Ax\|_X/\|x\|_X : x \neq 0\} = |a|\|A\|_{op}$.

(iv)
$$\|A+B\|_{op} = \sup\left\{\frac{\|(A+B)x\|_X}{\|x\|_X} : x \neq 0\right\} = \sup\left\{\frac{\|Ax+Bx\|_X}{\|x\|_X} : x \neq 0\right\},$$
here $\|Ax+Bx\|_X \leq \|Ax\|_X + \|Bx\|_X$ because X is a normed space. Then,
$$\begin{aligned}\|A+B\|_{op} \leq& \sup\left\{\frac{\|Ax\|_X}{\|x\|_X} + \frac{\|Bx\|_X}{\|x\|_X} : x \neq 0\right\}\\ \leq& \sup\{\|Ax\|_X/\|x\|_X : x \neq 0\} + \sup\{\|Bx\|_X/\|x\|_X : x \neq 0\}\\ =& \|A\|_{op} + \|B\|_{op}.\end{aligned}$$
■

An immediate consequence of Definition 2.1.2 is that $\|Av\|_X \leq \|A\|_{op}\|v\|_X$ for each $v \in X$.

Proposition 2.1.9. *If $A, B \in \mathcal{B}(X)$, then $BA \in \mathcal{B}(X)$ and $\|BA\|_{op} \leq \|B\|_{op}\|A\|_{op}$. Therefore, the operator norm is compatible with the composition or multiplication of operators.*

Proof. BA is clearly linear and (being a composition of continuous mappings) continuous. For any $x \in X$, $\|BAx\|_X = \|B(Ax)\|_X \leq \|B\|_{op}\|Ax\|_X \leq \|B\|_{op}\|A\|_{op}\|x\|_X$. Hence, $\|BA\|_{op} \leq \|B\|_{op}\|A\|_{op}$. ■

With what we have done, we say that $(\mathcal{B}(X), \|\cdot\|_{op})$ is a normed space. Furthermore, $\mathcal{B}(X)$ is a normed algebra.

Theorem 2.1.10. *$(\mathcal{B}(X), \|\cdot\|_{op})$ is a Banach space.*

Proof. By Propositions 2.1.8 and 2.1.9 we know that $\mathcal{B}(X)$ is a normed space. Then, all that remains is checking the completeness.

Let $(B_k)_{k\in\mathbb{N}}$ be a Cauchy sequence on $\mathcal{B}(X)$. Then, for every $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that
$$m, n > N(\varepsilon) \Longrightarrow \|B_m - B_n\|_{op} < \varepsilon.$$
Let $x \in X$. Note that,
$$\frac{\|B_m(x) - B_n(x)\|_X}{\|x\|_X} = \left\|\frac{B_m(x) - B_n(x)}{x}\right\|_X \leq \left\|\frac{(B_m - B_n)(x)}{x}\right\|_X \leq \|B_m - B_n\|_{op} < \varepsilon.$$
Then, for each $x \in X$, $(B_k(x))_{k\in\mathbb{N}}$ is a Cauchy sequence on X. Since X is complete, the sequence has a limit and we denote it by $B(x)$.

Let $\alpha, \beta \in \mathbb{F}$ and $x_1, x_2 \in X$. We have:
$$\begin{aligned}B(\alpha x_1 + \beta x_2) &= \lim_{k\to\infty} B_k(\alpha x_1 + \beta x_2) = \lim_{k\to\infty} (B_k(\alpha x_1) + B_k(\beta x_2))\\ &= \lim_{k\to\infty} (\alpha B_k(x_1) + \beta B_k(x_2)) = \alpha B(x_1) + \beta B(x_2).\end{aligned}$$
Thus, B is linear.

We know that every Cauchy sequence is bounded, so we get that
$$\lim_{k\to\infty} \|B_k\|_{op} = \|\lim_{k\to\infty} B_k\|_{op} = \|B\|_{op} < \infty.$$

Finally, note that

$$\lim_{k\to\infty} \|B_k - B\|_{op} = \lim_{k\to\infty} \sup\left\{\frac{\|B_k(x) - Bx\|_X}{\|x\|_X} : x \neq 0\right\}$$
$$= \sup\left\{\lim_{k\to\infty} \frac{\|B_k(x) - Bx\|_X}{\|x\|_X}\right\} = 0.$$

Therefore, $B \in \mathcal{B}(X)$ and the sequence $(B_k)_{k\in\mathbb{N}}$ converges to B. ■

Definition 2.1.11 (Finite rank operator). *Let $K \in \mathcal{B}(X)$. An operator K is said to be of finite rank if its range,* $\operatorname{ran}(K)$*, is finite dimensional.*

We denote the set of the finite rank operators on X as $\mathcal{B}_{00}(X)$.

Definition 2.1.12 (Relatively compact set). *Let X be a Banach space and let $\Omega \subset X$. The subset Ω is called relatively compact if its closure is compact.*

Definition 2.1.13 (Relatively compact operator). *Let X be a Banach space, let $\Omega \subset X$, and let $K\colon \Omega \longrightarrow X$ be a bounded linear operator. K is said to be a relatively compact operator if for any bounded set $S \subset \Omega$, $K(S)$ is a relatively compact set on X.*

Definition 2.1.14 (Compact operator). *Let $K \in \mathcal{B}(X)$ and let $S \subset X$ be a bounded subset. We say that if $K(S)$ is a relatively compact subset of X, then K is compact.*

We denote the set of all compact operators on X as $\mathcal{B}_0(X)$. If we assume that X is a Hilbert space, then

$$\mathcal{B}_{00}(X) \subset \mathcal{B}_0(X).$$

This is a known result. Additionally, $\mathcal{B}_0(X) \subset \mathcal{B}(X)$ for any set X.

Erik Ivar Fredholm was born in Stockholm, Sweden, in 1866. Fredholm received the best education and he displayed his brilliance at an early age.

Fredholm's thesis treated a topic in the theory of partial differential equations, which had applications to the study of deformations of objects subjected to interior or exterior forces. Later, he acquired fame for his solution of Fredholm integral equation, which has wide applications in physics.

This work constitutes an achievement in functional analysis and spectral theory.

Compact operators originated in the study of integral equations. The theory of compact operators, known as the Riesz–Schauder theory, is a generalization of the classical Fredholm operator theory for integral equations, which has found many important applications in both ordinary and partial differential equations.

2.2 Infinite matrices

Infinite matrices can be regarded as a generalization of linear algebra to infinite dimensions. On the other hand, we appeal to infinite matrices because they have an important application to the theory of summability of divergent sequences and series, and the Heisenberg–Dirac theory of quantum mechanics.

At this point, we focus on infinite matrices and their spectral phenomena. According to this, we work with the Hilbert spaces $\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{N})$. For these spaces we consider the set $(e_j)_{j\in J}$ where $J=\mathbb{N}$ or $J=\mathbb{Z}$, and $\boldsymbol{e_j}$ is the vector whose jth entry is 1 and the remaining entries are zero. Clearly, $(e_j)_{j\in J}$ is an orthonormal basis for $\ell^2(J)$.

Theorem 2.2.1. *Let $(e_j)_{j\in J}$ be a basis. Then, an infinite matrix representation for an operator $A\in\mathcal{B}(\ell^2(J))$ is given by $a_{j,k}=\langle A\boldsymbol{e_k},\boldsymbol{e_j}\rangle$.*

Proof. For $J=\mathbb{Z}$. Let $(e_j)_{j\in\mathbb{Z}}$ be an orthonormal basis for $\ell^2(\mathbb{Z})$. Then, for $\boldsymbol{\alpha}\in\ell^2(\mathbb{Z})$ we have

$$A\boldsymbol{\alpha}=A\sum_{j\in\mathbb{Z}}\alpha_j\boldsymbol{e_j}=\sum_{j\in\mathbb{Z}}\alpha_jA\boldsymbol{e_j}=\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & A\boldsymbol{e_{-1}} & A\boldsymbol{e_0} & A\boldsymbol{e_1} & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix}\cdot\boldsymbol{\alpha}.$$

Note that, $A\boldsymbol{e_k}=\sum_{j\in\mathbb{Z}}\langle A\boldsymbol{e_k},\boldsymbol{e_j}\rangle\boldsymbol{e_j}$. Whence, if we say that T_A is $(a_{j,k})_{j,k\in\mathbb{Z}}$, we have that $A\boldsymbol{\alpha}=T_A\cdot\boldsymbol{\alpha}$.

The case $J=\mathbb{N}$ is similar. ■

Straightaway, we understand $\ell^2(J)$ as a space of infinite columns. The action of A on $\ell^2(J)$ can be described as a multiplication by the infinite matrix T_A. Let $a_{j,k}\equiv\langle A\boldsymbol{e_k},\boldsymbol{e_j}\rangle$.

In the case of $J=\mathbb{Z}$ we have the following matrix representation,

$$A\boldsymbol{x}=\left(\begin{array}{ccc|ccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{-2,-2} & a_{-2,-1} & a_{-2,0} & a_{-2,1} & \cdots \\ \cdots & a_{-1,-2} & a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots \\ \hline \cdots & a_{0,-2} & a_{0,-1} & a_{0,0} & a_{0,1} & \cdots \\ \cdots & a_{1,-2} & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right)\left(\begin{array}{c} \vdots \\ x_{-2} \\ x_{-1} \\ \hline x_0 \\ x_1 \\ \vdots \end{array}\right)=\left(\begin{array}{c} \vdots \\ y_{-2} \\ y_{-1} \\ \hline y_0 \\ y_1 \\ \vdots \end{array}\right)=\boldsymbol{y}$$

with $y_i=\sum_{k\in\mathbb{Z}}a_{j,k}x_k$. Similarly, when $J=\mathbb{N}$ we have the next one,

$$A\boldsymbol{x}=\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & 2_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}=\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix}=\boldsymbol{y}$$

with $y_i=\sum_{k\in\mathbb{N}}a_{j,k}x_k$.

Example 2.2.2. Let $S\colon\ell^2(\mathbb{N})\longrightarrow\ell^2(\mathbb{N})$ be the shift operator given by

$$S(\alpha_1,\alpha_2,\ldots)=(0,\alpha_1,\alpha_2,\ldots).$$

(i) Let $\boldsymbol{\alpha},\boldsymbol{\beta}\in\ell^2(\mathbb{N})$ and $a,b\in\mathbb{F}$. Then, we get

$$\begin{aligned} S(a\cdot\boldsymbol{\alpha}+b\cdot\boldsymbol{\beta}) =&(0,a\alpha_1+b\beta_1,a\alpha_2+b\beta_2,\ldots)=(0,a\alpha_1,a\alpha_2,\ldots)+(0,b\beta_1,b\beta_2\ldots)\\ =&a\cdot(0,\alpha_1,\alpha_2,\ldots)+b\cdot(0,\beta_1,\beta_2,\ldots)=a\cdot S\boldsymbol{\alpha}+b\cdot S\boldsymbol{\beta}. \end{aligned}$$

Therefore, S is linear.

(ii) Let $\boldsymbol{\alpha} \in \ell^2(\mathbb{N})$ with $\boldsymbol{\alpha} \neq \mathbf{0}$. We want to calculate $\|S\|_{op}$. Note that, it holds the following: $\|S\boldsymbol{\alpha}\|_2 = \|(0, \alpha_1, \alpha_2, \ldots)\|_2 = \|\boldsymbol{\alpha}\|_2$. Thus, $\|S\|_{op} = 1$ and we conclude that S is bounded. Thus, $S \in \mathcal{B}(\ell^2(\mathbb{N}))$.

(iii) Using the standard orthonormal basis for $\ell^2(\mathbb{N})$, we obtain

$$\langle S\boldsymbol{e_j}, \boldsymbol{e_i}\rangle = \langle \boldsymbol{e_{j+1}}, \boldsymbol{e_i}\rangle = \delta_{j+1,i}.$$

Therefore, the matrix corresponding to S and $(e_j)_{j\in\mathbb{N}}$ is the lower diagonal matrix

$$T_S = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Before continuing, we present a short review of the mathematical work of Israel Gohberg.

He was born in 1928. He began his mathematical studies around 1950. Later, he became head of the mathematics department and a full professor of the functional analysis course at the Mathematical Institute of the newly organized Moldavian Academy of Science. With two doctoral degrees and many honors which marked his career, he supervised more than 40 doctoral students.

The mathematical work of Gohberg is extensive and influential. His contributions belong to the field of analysis, operator theory and linear algebra. He was a leader in these research areas:

- Integral equations.
- Theory of non-selfadjoint operators.
- Spectral theory and factorization of matrices and operator functions.
- Inversion problems for structured matrices.

His papers and books are oriented to theory of Fredholm operators (perturbation, index and representations). Furthermore, he made a connection between the theory of commutative algebras and integral operators.

Once we know how an operator $A \in \mathcal{B}(\ell^2(J))$ can be represented by an infinite matrix, we ask if every infinite matrix defines a bounded operator on $\ell^2(J)$. For answering this question we establish that an infinite matrix $(a_{j,k})_{j,k\in J}$ generates a bounded operator on $\ell^2(J)$, i.e., an infinite matrix represents a bounded operator on $\ell^2(J)$ if there exists an operator $A \in \mathcal{B}(\ell^2(J))$ such that $\langle A\boldsymbol{e_k}, \boldsymbol{e_j}\rangle$ is valid for all $j, k \in J$. Thus, we set up the next definition which says under what conditions it holds that a matrix generates a bounded operator.

Definition 2.2.3. *The infinite matrix $A = (a_{j,k})_{j,k\in J}$ generates a bounded operator on $\ell^2(J)$ if and only if there exists a constant $M < \infty$ such that for every $\boldsymbol{x} \in \ell^2(J)$ the following conditions hold:*

(i) *The series $y_j \equiv \sum_{k\in J} a_{j,k}x_k$ converge for all $j \in J$.*

(ii) $\boldsymbol{y} \equiv (y_j)_{j\in J}$ *belongs to* $\ell^2(J)$.

(iii) $\|\boldsymbol{y}\|_2^2 \leq M\|\boldsymbol{x}\|_2^2$.

The smallest constant M for which (iii) is true equals the norm: $\|A\|_{op}$.

Definition 2.2.4 (Banach algebra)**.** *Let $\mathcal{A}$ be a complex Banach space. Then, if $\mathcal{A}$ is a complete normed algebra, we say that $\mathcal{A}$ is a Banach algebra.*

If a Banach algebra has a unit element, which is denoted by $e, 1$, or I, it is referred to as a unitary Banach algebra. This unit satisfies that $\|e\| = \|1\| = \|I\| = 1$.

Definition 2.2.5 (Involution)**.** *Let $\mathcal{A}$ be a Banach algebra. A mapping $*\colon \mathcal{A} \longrightarrow \mathcal{A}$ is called an involution if it satisfies the following:*

(i) $(x+y)^* = x^* + y^*$.

(ii) $(cx)^* = \bar{c}x^*$.

(iii) $(xy)^* = y^*x^*$.

(iv) $x^{**} = x$.

For all $x, y \in \mathcal{A}$ and $c \in \mathbb{C}$.

Definition 2.2.6 (C^*-algebra)**.** *Let $\mathcal{A}$ be a Banach algebra with an involution $*$. Then, $\mathcal{A}$ is a C^*-algebra if $\|x^*x\|_{\mathcal{A}} = \|x\|_{\mathcal{A}}^2$.*

Example 2.2.7. Let S be a set. The collection of all complex functions that are essentially bounded on S is $L^\infty(S)$ and it is a unitary Banach algebra with these operations:

(i) $(f+g)(x) = f(x) + g(x)$.

(ii) $(fg)(x) = f(x)g(x)$.

(iii) $(\alpha f)(x) = \alpha f(x)$.

The associated norm is $\|\cdot\|_\infty$. We show that given $f, g \in L^\infty(S)$, then $fg \in L^\infty(\mathcal{S})$.

Let $x \in S$. We get

$$\begin{aligned}
\|fg\|_\infty &= \inf\{M\colon \|(fg)(x)\| \leq M \quad \text{for } \mu\text{-a.e. } x \in S\} \\
&= \inf\{M\colon \|f(x)g(x)\| \leq M \quad \text{for } \mu\text{-a.e. } x \in S\} \\
&= \inf\{M\colon \|f(x)\|\|g(x)\| \leq M \quad \text{for } \mu\text{-a.e. } x \in S\} \\
&\leq \inf\{M_1\colon \|f(x)\| \leq M_1 \text{ for } \mu\text{-a.e. } x \in S\} \cdot \inf\{M_2\colon \|g(x)\| \leq M_2 \text{ for } \mu\text{-a.e. } x \in S\} \\
&= \|f\|_\infty \cdot \|g\|_\infty < \infty.
\end{aligned}$$

Thus, $fg \in L^\infty(S)$.

Example 2.2.8. Let $\mathcal{C}(K)$ be the space of continuous functions on a compact set K, with pointwise multiplication $(fg)(x) \equiv f(x)g(x)$, and unity the constant function 1. By Theorem 1.1.35 we know that $\mathcal{C}(K)$ is a Banach space. Thus, $\mathcal{C}(K)$ is an unitary Banach algebra.

Example 2.2.9. Let X be a Banach space. Let $\mathcal{B}(X)$ be the collection of all bounded linear operators on X and $\mathbb{I}_X$ be the identity operator. By Theorem 2.1.10 we get that $\mathcal{B}(X)$ is a Banach space. Therefore, $\mathcal{B}(X)$ is an unitary Banach algebra.

2.3 Spectrum

Proposition 2.3.1 (Uniqueness of multiplicative inverse)**.** *Let $\mathcal{A}$ be a Banach algebra with unity e. An element $a \in \mathcal{A}$ is said to be invertible on $\mathcal{A}$ if there exists an element $b \in \mathcal{A}$ such that $ab = ba = e$. Then, b is unique whenever it exists, in that case we denote it by a^{-1} and call it the inverse of a.*

Proof. Let $a \in \mathcal{A}$, such that $a \neq 0$. Suppose that there exist $b, b' \in \mathcal{A}$ with $b \neq b'$ and both are inverses of a. Then, we have $ab = ba = e$, and $ab' = b'a = e$. Now, note that

$$b = eb = (b'a)b = b'(ab) = b'e = b'.$$

Therefore, $b = b' = a^{-1}$, i.e., the multiplicative inverse of a exists and is unique. ∎

Definition 2.3.2 (Spectrum)**.** *Let A be a Banach algebra. The spectrum of an element $a \in \mathcal{A}$ is defined as the set*

$$\mathrm{sp}_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \colon a - \lambda e \quad \text{is not invertible in } \mathcal{A}\}.$$

Its complement $\mathbb{C} \setminus \mathrm{sp}_{\mathcal{A}}(a)$ is called the resolvent of a and we denote by $\rho_{\mathcal{A}}(a)$.

Example 2.3.3. Let $GL(n, \mathbb{C})$ be the general linear group of $n \times n$ invertible matrices over $\mathbb{C}$. The spectrum of a matrix consists of its eigenvalues.

Note that, it is possible to have different matrices with the same spectrum. The spectrum yields important information about a matrix, but need not characterize it uniquely.

Example 2.3.4. Let $\boldsymbol{x} \in \ell^\infty(\mathbb{N})$ given by $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}}$. The spectrum of the sequence $\boldsymbol{x}$ is $\mathrm{sp}_{\ell^\infty(\mathbb{N})}(\boldsymbol{x}) = \overline{\mathrm{Im}(\boldsymbol{x})}$. To see this note that the inverse of the sequence $\boldsymbol{x} - \lambda\mathbf{id} = (x_n - \lambda)_{n\in\mathbb{N}}$ is $\left(\frac{1}{x_n - \lambda}\right)_{n\in\mathbb{N}}$ and it is bounded if and only if there exists $c > 0$ such that $|x_n - \lambda| \geq c$ for each $n \in \mathbb{N}$.

Chapter 3

Laurent matrices

The french mathematician Pierre Alphonse Laurent found out that some complex functions admit an expansion into an infinite power series (positive and negative), this series is known as the Laurent series. We say that the Laurent series for a complex function a about a point c is given by

$$a(z) = \sum_{n\in\mathbb{Z}} a_n (z-c)^n.$$

It is important to mention that this kind of series takes into account the singularities of the function. In addition, the coefficients a_n are constants and can be defined by a line integral.

We will use these coefficients for the construction of an infinite matrix.

Similarly, let $(a_n)_{n=-\infty}^{\infty}$ be a sequence of complex numbers and A be the following infinite matrix

$$A \equiv \left(\begin{array}{ccc|cccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ \hline \cdots & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \cdots & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right).$$

We claim that A is a doubly-infinite matrix and is constant along their diagonals. Such matrices are known as Laurent matrices.

We can see A as an operator on $\mathcal{B}(\ell^2(\mathbb{Z}))$, thereby:

$$A\boldsymbol{x} = \left(\begin{array}{ccc|cccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ \cdots & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ \hline \cdots & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ \cdots & a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \cdots & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right) \left(\begin{array}{c} \vdots \\ x_{-2} \\ x_{-1} \\ \hline x_0 \\ x_1 \\ x_2 \\ \vdots \end{array}\right) = \left(\begin{array}{c} \vdots \\ \sum_{k=-\infty}^{\infty} a_{-2-k}x_k \\ \sum_{k=-\infty}^{\infty} a_{-1-k}x_k \\ \hline \sum_{k=-\infty}^{\infty} a_{-k}x_k \\ \sum_{k=-\infty}^{\infty} a_{1-k}x_k \\ \sum_{k=-\infty}^{\infty} a_{2-k}x_k \\ \vdots \end{array}\right).$$

In order to use the theory of bounded operators that are generated by infinite matrices, we are interested in the conditions under which a Laurent matrix generates a bounded operator on $\ell^2(\mathbb{Z})$.

The following theorem answers this question.

Theorem 3.0.5. *Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence. The associated Laurent matrix A generates a bounded operator on $\ell^2(\mathbb{Z})$ if and only if there is a function $a \in L^\infty(\mathbb{T})$ such that $(a_n)_{n\in\mathbb{Z}}$ is the sequence of the Fourier coefficients of a:*

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-in\theta}\, d\theta.$$

Consequently, we denote the Laurent matrix and the bounded operator generated by this matrix on $\ell^2(\mathbb{Z})$ by $L(a)$.

Otto Toeplitz, born at Breslau in 1881, studied mathematics in Breslau and Berlin. He preferred to consider mathematics as an art than as a science.

Toeplitz's mathematical interest was mainly in algebra. He liked to consider analysis as an algebra of an infinite number of variables. Most of his papers deal with problems of infinite matrices and the corresponding bilinear and quadratic forms. In 1911, he wrote *"Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen"* [11], Toeplitz considered the Laurent matrices and proved that the spectrum of the corresponding operator on $\ell^2(\mathbb{Z})$ is just the curve

$$\mathcal{R}(a) \equiv \Big\{ \sum_{n\in\mathbb{Z}} a_n t^n : t \in \mathbb{T} \Big\}.$$

In the same paper he established that the infinite matrix $(a_{j-k})_{j,k=0}^{\infty}$ induces a bounded operator on $\ell^2(\mathbb{Z}^+)$ if and only if the Laurent matrix generates a bounded operator on $\ell^2(\mathbb{Z})$. Because of this result the matrix $(a_n)_{n\in\mathbb{N}}$ is known as the Toeplitz matrix.

Definition 3.0.6 (Band matrix)**.** *Let $A = (a_{i,j})$ be a matrix. The subscripts of the entries of A are $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Let $k_1, k_2 > 0$ be integers. We say that A has lower bandwidth k_1 if $a_{i,j} = 0$ for $i > j + k_1$ and upper bandwidth k_2 if $a_{i,j} = 0$ for $j > i + k_2$. In any case, A is a band matrix.*

The bandwidth of the matrix A is $\max\{k_1, k_2\}$, in other words, there exists a constant $k \in \mathbb{N}$ such that $a_{i,j} = 0$ for $|i - j| > k$.

Example 3.0.7 (Tridiagonal matrices)**.** Let A be a matrix with $i, j = 1, 2, \ldots, n$. Consider $k_1 = k_2 = 1$, then we get that

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \ddots & & \vdots \\ 0 & a_{3,2} & a_{3,3} & a_{3,4} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

Example 3.0.8 (Hessenberg matrix)**.** A Hessenberg matrix is a special kind of square matrix because it is almost triangular.

Let A with $i, j = 1, 2, \dots, n$ be a square matrix and take $k_1 = 1$, and $k_2 = n - 1$. Then, we say that A is an upper Hessenberg matrix and has the following form

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

Additionally, if we take $k_1 = n - 1$, and $k_2 = 1$, then A is a lower Hessenberg matrix and we have,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \ddots & & \vdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

Definition 3.0.9 (Laurent polynomial)**.** *Let f be a function with the following form*

$$\sum_{k=-r}^{s} \alpha_k t^k, \quad \alpha_{-r} \neq 0 \quad ,and \quad \alpha_s \neq 0.$$

Thus, that linear combination of positive and negative powers is called a Laurent polynomial.

This polynomial is also called a trigonometric polynomial. The function a on $\mathbb{T}$, whose Fourier coefficients are just what we defined in Theorem 3.0.5, is referred to as the symbol of the Laurent matrix. Afterwards, we will see that in the case of Toeplitz band matrices, the symbol is a Laurent polynomial.

Definition 3.0.10 (Multiplication operator)**.** *Let $a \in L^\infty(\mathbb{T})$. Then, the multiplication operator*

$$M_a \colon L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$$

is given by $f \longmapsto af$.

Proposition 3.0.11. *Let $a \in L^\infty(\mathbb{T})$. Then, M_a is bounded and $\|M_a\|_{op} \leq \|a\|_\infty$.*

Proof. Let $f \in L^2(\mathbb{T})$. We have that $M_a(f) = af \in L^2(\mathbb{T})$. Notice that,

$$\|M_a(f)\|_2^2 = \oint_{\mathbb{T}} |a(t)f(t)|^2 \, d\mu \leq \oint_{\mathbb{T}} \|a\|_\infty^2 |f(t)|^2 \, d\mu = \|a\|_\infty^2 \|f\|_2^2 < \infty.$$

It means that, $\frac{\|M_a(f)\|_2^2}{\|f\|_2^2} \leq \|a\|_\infty^2$.

Then,

$$\|M_a\|_{op} = \sup\left\{\frac{\|M_a(f)\|_2}{\|f\|_2} : \|f\|_2 \neq 0\right\} = \sup\left\{\frac{\|af\|_2}{\|f\|_2} : \|f\|_2 \neq 0\right\}$$
$$\leq \sup\left\{\frac{\|a\|_\infty \|f\|_2}{\|f\|_2} : \|f\|_2 \neq 0\right\} = \|a\|_\infty. \qquad \blacksquare$$

The next proposition was taken from [5]. It will provide us the reverse inequality, $\|M_a\|_{op} \geq \|a\|_\infty$. To be clear in the proof of the following result, we define $\chi_j(t) \equiv t^j$ for $t \in \mathbb{T}$ and $j \in \mathbb{Z}$.

Proposition 3.0.12. *Let $A \in \mathcal{B}(L^2(\mathbb{T}))$. Suppose that there is a sequence $(a_n)_{n\in\mathbb{Z}}$ of complex numbers such that $\langle A\chi_j, \chi_k\rangle = a_{k-j}$. Then, there is a function a belonging to $L^\infty(\mathbb{T})$ such that $A = M_a$ and $(a_n)_{n\in\mathbb{Z}}$ is the Fourier coefficient sequence of a. Moreover, $\|M_a\|_{op} = \|a\|_\infty$.*

Proof. By Theorem 1.1.47 we get that $(L^2(\mathbb{T}), \|\cdot\|_2)$ is a Banach space, by Theorem 2.1.10 we have that $(\mathcal{B}(L^2(\mathbb{T})), \|\cdot\|_{op})$ is a Banach space. Additionally, we can conclude that $\mathcal{B}(L^2(\mathbb{T}))$ is a Banach algebra.

Consider $A \in \mathcal{B}(L^2(\mathbb{T}))$ and let $a \equiv A\chi_0$ be a function that belongs to $L^2(\mathbb{T})$. Then, the n-th Fourier coefficient of a is given by $\langle a, \chi_n\rangle = \langle A\chi_0, \chi_n\rangle = a_n$. Now, if $f \in L^\infty(\mathbb{T})$, then the functions Af and af belong to $L^2(\mathbb{T})$.

We claim that,

$$Af = af \quad \text{for each } f \in L^\infty(\mathbb{T}).$$

Foremost, take the Fourier coefficient sequence of f, $(f_n)_{n\in\mathbb{Z}}$. Some calculations give us that the j-th Fourier coefficient of af is $\sum_{k\in\mathbb{Z}} a_{j-k} f_k$ which we denote as

$$[af]_j = \sum_{k\in\mathbb{Z}} a_{j-k} f_k.$$

Carrying on, by Theorem 1.2.6 we already know that $\left\| f - \sum_{k=-N}^{N} f_k t^k \right\|_2^2 \xrightarrow[N\to\infty]{} 0$, i.e., the series converges to f in this norm.

Note that, we can represent Af as follows,

$$Af = \sum_{k\in\mathbb{Z}} f_k \cdot A\chi_k.$$

Thus,

$$\langle Af, \chi_j\rangle = \left\langle \sum_{k\in\mathbb{Z}} f_k \cdot A\chi_k, \chi_j \right\rangle = \sum_{k\in\mathbb{Z}} f_k \cdot \langle A\chi_k, \chi_j\rangle = \sum_{k\in\mathbb{Z}} f_k a_{j-k} = [Af]_j, \quad \forall j \in \mathbb{Z}.$$

Therefore, $[af]_j = [Af]_j$ for every $j \in \mathbb{Z}$ which implies that the functions Af and af are equal.

We now want to see that $a \in L^\infty(\mathbb{T})$. We establish the set E which is defined by $E \equiv \{t \in \mathbb{T} \colon |a(t)| > \|A\|_{op}\}$ with $E \subseteq \mathbb{T}$ and $\mu(E) > 0$. Denoting χ_E as the characteristic function of E and using the previous result, we obtain:

$$\begin{aligned}
\|A\chi_E\|_2^2 = \|a\chi_E\|_2^2 &= \int_E |a(t)|^2 \, dt = \oint_{\mathbb{T}} |a(t)|^2 \chi_E \, dt \\
&= \|a\|_2^2 \cdot \|\chi_E\|_2^2 \\
&> \|A\|_{op}^2 \cdot \|\chi_E\|_2^2.
\end{aligned}$$

However, this is impossible because $\|A\|_{op}$ is the supremum of the quotient $\frac{\|A\chi_E\|_2}{\|\chi_E\|_2}$ and so $|a(t)| \leq \|A\|_{op}$ μ-a.e. on $\mathbb{T}$. Hence, $a \in L^\infty(\mathbb{T})$ and taking the infimum

$$\inf\{M \colon |a(t)| \leq M \quad \text{for } \mu\text{-a.e. } t \in \mathbb{T}\} \leq \|A\|_{op},$$

it can be concluded that $\|a\|_\infty \leq \|A\|_{op}$.

Because of $Af = af$ the operators A and M_a coincide on a dense subset of $L^2(\mathbb{T})$ and both operators are bounded, it follows that $A = M_a$. Finally, $\|M_a\|_{op} = \|a\|_\infty$ is proved. ■

Theorem 3.0.13. *$\sqrt{2\pi}\|\Phi f\|_2 = \|f\|_2$ for each $f \in L^2(\mathbb{T})$.*

Proof. (i) Let $(\alpha_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. By Definition 1.1.51 we have that $\sum_{k\in\mathbb{Z}} |\alpha_k|^2 < \infty$. Now, we define $\sum_{k\in\mathbb{Z}} \alpha_k t^k \equiv f(t)$, by Parseval's identity we obtain that $f \in L^2(\mathbb{T})$, and $\Phi f = (a_k)_{k\in\mathbb{Z}}$.

(ii) We want to see that $2\pi\|\Phi f\|_2^2 = \|f\|_2^2$. By Parseval's identity we get,

$$2\pi\|\Phi f\|_2^2 = 2\pi\|(f_k)_{k\in\mathbb{Z}}\|_2^2 = 2\pi \sum_{k\in\mathbb{Z}} |f_k|^2 = \|f\|_2^2.$$ ■

Now, we can understand the properties of Laurent matrices by knowing that these are the matrix representations of multiplication operators on $L^2(\mathbb{T})$.

From now on, we will use b to refer the function in $L^2(\mathbb{T})$ or the Fourier coefficients in $\ell^2(\mathbb{Z})$.

Proposition 3.0.14. *Let $a \in L^\infty(\mathbb{T})$. Then, $L(a) = \Phi M_a \Phi^{-1}$.*

Proof. Let $b(t) = \sum_{n\in\mathbb{Z}} b_n t^n \in L^2(\mathbb{T})$. Then, we easily get

$$[\Phi M_a \Phi^{-1}](b) = \Phi M_a(b) = \Phi ab = (ab)_{n\in\mathbb{Z}}.$$

To calculate the Fourier coefficients of ab we will need the following observation:

$$\begin{aligned}
a(t)b(t) &= \Bigg(\sum_{j\in\mathbb{Z}} a_j t^j\Bigg)\Bigg(\sum_{n\in\mathbb{Z}} b_n t^n\Bigg) \\
&= \cdots + \Bigg(\sum_{k=-\infty}^{\infty} a_k b_{-2-k}\Bigg)t^{-2} + \Bigg(\sum_{k=-\infty}^{\infty} a_k b_{-1-k}\Bigg)t^{-1} + \cdots \\
&= \sum_{n=-\infty}^{\infty} \Bigg(\sum_{k=-\infty}^{\infty} a_k b_{n-k}\Bigg)t^n.
\end{aligned}$$

Therefore, the n-th Fourier coefficient of ab is given by $[ab]_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k}$. Additionally, we conclude that $(ab)_{n\in\mathbb{Z}} = L(a)b$. ■

Proposition 3.0.15. *Let $a, b \in L^\infty(\mathbb{T})$. Then, $L(a)L(b) = L(ab)$ and $\|L(a)\|_{op} = \|a\|_\infty$.*

Proof. By Proposition 3.0.14 we know that,

$$\begin{aligned} L(a)L(b) =& (\Phi M_a \Phi^{-1})(\Phi M_b \Phi^{-1}) = \Phi M_a (\Phi^{-1}\Phi) M_b \Phi^{-1} = \Phi M_a M_b \Phi^{-1} \\ =& \Phi M_{ab} \Phi^{-1} = L(ab). \end{aligned}$$

Further, by Propositions 2.1.9 and 3.0.12 we have,

$$\begin{aligned} \|L(a)\|_{op} &= \|\Phi M_a \Phi^{-1}\|_{op} \\ &= \|\Phi\|_{op} \cdot \|M_a\|_{op} \cdot \|\Phi^{-1}\|_{op} \\ &= \|\Phi\|_{op} \cdot \|\Phi^{-1}\|_{op} \cdot \|M_a\|_{op} \\ &= \|M_a\|_{op} = \|a\|_\infty. \end{aligned}$$

■

When dealing with linear operators on a Hilbert space we may approximate them (with respect to the $L^\infty(\mathbb{T})$ norm) with finite matrices by choosing an orthonormal basis of the space. For multiplication operators on $L^2(\mathbb{T})$, we study how well the eigenvalues of those matrices approximate the spectrum of each of these operators, which (as we will see) is the essential range of the symbol.

If the symbol a belongs to $L^\infty(\mathbb{T})$, then it is not a function but an equivalence class of functions. This means that to compute the spectrum, it is required the understanding of this element. Then, the notion of the range of $a \in L^\infty(\mathbb{T})$ must be approached with some care.

Definition 3.0.16 (Essential range)**.** *Let $a \in L^\infty(\mathbb{T})$. We denote by $\mathcal{R}(a)$ the essential range of a as an element of $L^\infty(\mathbb{T})$. Likewise, we may define $\mathcal{R}(a)$ as the spectrum of the multiplication operator M_a on $L^2(\mathbb{T})$.*

We can use that $a \in L^\infty(\mathbb{T})$ to define a measure m_a on the σ-algebra of Borel sets in $\mathbb{C}$ thereby:

$$m_a(S) \equiv \mu\{t \in \mathbb{T} \colon a(t) \in S\}, \quad S \subseteq \mathbb{C}.$$

This measure only depends on the equivalence class of a as an element of $L^\infty(\mathbb{T})$. Besides, if μ is a finite measure, then so is m_a. However, in the case that μ is only σ-finite, m_a need not be σ-finite or could be infinite. In all cases m_a is a countably additive measure defined on the Borel σ-algebra of $\mathbb{C}$.

Now, let G be the union of all open subsets of $\mathbb{C}$ having m_a-measure zero. It implies that $m_a(G) = 0$. Note that, G is the largest open set of m_a-measure zero and if we consider $F = \mathbb{C} \setminus G$ which is a closed set that satisfies:

A complex number λ belongs to F if and only if for every $\varepsilon > 0$ it holds

$$\mu\{t \in \mathbb{T} \colon |a(t) - \lambda| < \varepsilon\} > 0.$$

Moreover, every point of the complement of F has a neighborhood of m_a-measure zero. Therefore, the essential range of a is the nonempty and compact set:

$$\mathcal{R}(a) = \Big\{\lambda \in \mathbb{C} \colon \mu\{t \in \mathbb{T} \colon |a(t) - \lambda| < \varepsilon\} > 0 \; \forall \varepsilon > 0\Big\}.$$

Theorem 3.0.17. *Let $a \in L^\infty(\mathbb{T})$. Then, it holds*

(i) *If $0 \notin \mathcal{R}(a)$, then the inverse of $L(a)$ is $L(a^{-1})$.*

(ii) $\mathrm{sp}_{\ell^2(\mathbb{Z})} L(a) = \mathcal{R}(a)$.

Proof. Let $a \in L^\infty(\mathbb{T})$.

(i) Assuming that $0 \notin \mathcal{R}(a)$, it is possible to consider the multiplicative inverse of a, i.e., $a^{-1} = \frac{1}{a}$. Propositions 3.0.14 and 3.0.15 give us $L(a) = \Phi M_a \Phi^{-1}$, $L(a^{-1}) = \Phi M_{a^{-1}} \Phi^{-1}$, and $L(a)L(a^{-1}) = \Phi M_{aa^{-1}} \Phi^{-1}$. Thus, $M_{aa^{-1}} = \mathbb{I}$ and we can deduce that $L(a)L(a^{-1}) = \mathbb{I}$.

(ii) Take $\lambda \in \mathrm{sp}_{\ell^2(\mathbb{Z})} L(a)$. Then, $L(a) - \lambda\mathbb{I}$ is not invertible. Note that, $L(a) - \lambda\mathbb{I} = L(a - \lambda)$. By item (i) of the proof we get that $L^{-1}(a) = L(\frac{1}{a})$, this means that

$$\frac{1}{a} \in L^\infty(\mathbb{T}) \iff 0 \notin \mathcal{R}(a).$$

Therefore, $L(a - \lambda)$ is not invertible if and only if $\lambda \in \mathcal{R}(a)$. ∎

3.1 Examples of symbols

In this section we are going to include some examples of symbol classes that allow us to discuss their representations.

1. (Band matrices). Suppose that the band matrix $L(a)$ is given by $a_n = 0$ if $|n| > 2$. Then, the symbol a is a trigonometric polynomial:

$$a(t) = \sum_{n=-2}^{2} a_n t^n \quad \text{with} \quad t = e^{i\theta}.$$

The matrix $L(a)$ is given as follows

$$\left(\begin{array}{cccc|cccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 1 & 2 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 2 & 1 & 0 & 0 & \cdots \\
\hline
\cdots & 0 & 0 & 1 & 2 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 2 & 1 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 2 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right).$$

The corresponding symbol is $a(e^{i\theta}) = 2 + 2\cos\theta$. Besides, Theorem 3.0.17 permits us to say that the spectrum of the operator is the line segment $[0, 4]$.

2. (Rational symbols). A rational function a belongs to $L^\infty(\mathbb{T})$ if and only if it does not have poles on $\mathbb{T}$. Such symbols define Laurent matrices whose entries decay as a geometric sequence. For example, for the following rational symbol

$$a(t) = 1 + \sum_{n=1}^{\infty} \alpha^n t^{-n} + \sum_{n=1}^{\infty} \beta^n t^n = 1 + \frac{\alpha}{t - \alpha} + \frac{\beta}{1/t - \beta}, \quad \text{for } t \in \mathbb{T},$$

the corresponding matrix $L(a)$ is

$$\left(\begin{array}{cccc|cccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \beta & 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \cdots \\ \cdots & \beta^2 & \beta & 1 & \alpha & \alpha^2 & \alpha^3 & \cdots \\ \hline \cdots & \beta^3 & \beta^2 & \beta & 1 & \alpha & \alpha^2 & \cdots \\ \cdots & \beta^4 & \beta^3 & \beta^2 & \beta & 1 & \alpha & \cdots \\ \cdots & \beta^5 & \beta^4 & \beta^3 & \beta^2 & \beta & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right), \quad \text{with} \quad |\alpha| < 1, |\beta| < 1.$$

3. We will discuss symbols in the Wiener algebra (see reference [3]), but first we need a definition.

   Definition 3.1.1 (Wiener algebra)**.** *The Wiener algebra $W \equiv W(\mathbb{T})$ is defined as the set of all functions $a\colon \mathbb{T} \longrightarrow \mathbb{C}$ with absolutely convergent Fourier series, that is, W is the colletion of all functions admitting the representation*

$$a(t) = \sum_{n=-\infty}^{\infty} a_n t^n, \quad \text{where } t \in \mathbb{T} \text{ and such that} \quad \textstyle\sum_{n\in\mathbb{Z}} |a_n| < \infty.$$

   The norm in the Wiener algebra, $\|\cdot\|_W$, is defined as follows

$$\|a\|_W \equiv \sum_{n\in\mathbb{Z}} |a_n|.$$

   It is well known that $(W, \|\cdot\|_W)$ is a Banach algebra with the pointwise algebraic operations.

   Remark 3.1.2. (i) Let $\mathcal{C}(\mathbb{T})$ be the set of all continuous functions on $\mathbb{T}$ with the maximum norm. Then, the space $(\mathcal{C}(\mathbb{T}), \|\cdot\|_\infty)$ is a Banach algebra and W is contained in it.

   - $W \subset \mathcal{C}(\mathbb{T})$. Let $a \in W$. It means that, $a(t) = \sum_{n\in\mathbb{Z}} a_n t^n$. For $N \in \mathbb{N}$, consider each function defined by $a^{(N)} \equiv \sum_{n=-N}^{N} a_n t^n \in \mathcal{C}(\mathbb{T})$. Then, using the maximum norm we obtain

$$\|a - a^{(N)}\|_\infty \xrightarrow[N\to\infty]{} 0.$$

     Therefore, by the uniform limit theorem we conclude that $a \in \mathcal{C}(\mathbb{T})$.

   (ii) Wiener's theorem says that if $a \in W$ and a has no zeros on $\mathbb{T}$, then $a^{-1} = \frac{1}{a} \in W$. Thus, by Theorem 3.0.17, the inverse of an invertible Laurent operator with a symbol on W is again a Laurent operator with a symbol on W.

4. (Continuous symbols). For a function $a \in W$ its image is $a(\mathbb{T})$ and coincides with the essential range of it. So, sp $L(a) = a(\mathbb{T})$.

   Looking at functions of $\mathcal{C}(\mathbb{T}) \setminus W$ can be difficult in terms of their Fourier coefficients. To understand the problem, we will study a special class of functions. Assume that

$(b_n)_{n=2}^{\infty}$ is a sequence of positive numbers converging monotonously to zero. Consider the series:

$$\sum_{n=2}^{\infty} b_n \sin n\theta = \sum_{n=2}^{\infty} \frac{b_n}{2i}(e^{in\theta} - e^{-in\theta}), \ e^{i\theta} \in \mathbb{T}. \tag{3.1.1}$$

The following result is well known:

(i) The series (3.1.1) is the Fourier series of a function in $\mathcal{C}(\mathbb{T})$ if and only if

$$b_n = o(1/n) \ \text{ as } \ n \to \infty.$$

(ii) The series (3.1.1) is the Fourier series of a function in $L^{\infty}(\mathbb{T})$ if and only if

$$b_n = O(1/n) \ \text{ as } \ n \to \infty.$$

Particularly, the symbol of the Laurent matrix induced by $(a_n)_{n\in\mathbb{Z}}$ with

$$a_{-1} = a_0 = a_1 = 0, \ \ a_n = \frac{1}{n \log |n|} \quad \text{for } |n| \geq 2$$

belongs to $\mathcal{C}(\mathbb{T}) \setminus W$; while the Laurent matrix defined by $(a_n)_{n\in\mathbb{Z}}$ with

$$a_{-1} = a_0 = a_1 = 0, \ \ a_n = \frac{\log |n|}{n} \quad \text{for } |n| \geq 2$$

does not generate a bounded operator on $\ell^2(\mathbb{Z})$.

5. (Piecewise continuous functions). A function $a \in L^{\infty}(\mathbb{T})$ is said to be piecewise continuous if for each $t \in \mathbb{T}$ the one-sided limits

$$a(t^+) \equiv \lim_{\varepsilon\to 0^+} a(e^{i(\theta+\varepsilon)}), \ \ a(t^-) \equiv \lim_{\varepsilon\to 0^-} a(e^{i(\theta+\varepsilon)})$$

exist. We denote by $\mathcal{PC} \equiv \mathcal{PC}(\mathbb{T})$ the set of all piecewise continuous functions on $\mathbb{T}$. It is well known that $\mathcal{PC}$ is a closed subalgebra of $L^{\infty}(\mathbb{T})$.

Functions belonging to $\mathcal{PC}$ have at most countably many jumps, i.e., for $a \in \mathcal{PC}$ we set up

$$\Lambda_a \equiv \{t \in \mathbb{T} \colon a(t^-) \neq a(t^+)\}.$$

This set is at most countable. In addition, for each $\delta > 0$ the set,

$$\{t \in \mathbb{T} \colon |a(t^+) - a(t^-)| > \delta\}$$

is finite. Given $a \in \mathcal{PC}$, we always assume that a is continuous on $\mathbb{T} \setminus \Lambda_a$. Therefore,

$$\mathcal{R}(a) = \bigcup_{t\in\mathbb{T}\setminus\Lambda_a} \{a(t)\} \cup \bigcup_{t\in\Lambda_a} \{a(t^-), a(t^+)\}. \tag{3.1.2}$$

By Theorem 3.0.17 we get that the Equation (3.1.2) is the spectrum of $L(a)$.

Chapter 4

Toeplitz matrices

Continuing with the study about bounded linear operators and invertibility, our main interest is to include a complete analysis of Toeplitz operators.

There are at least two reasons for the constant and rising interest in Toeplitz operators. Initially, Toeplitz operators have an important connection with a variety of problems in physics, probability theory, information, and control theory. Moreover, Toeplitz operators constitute one of the most important classes of non-selfadjoint operators and they are a fascinating example in topics such as operator theory and theory of Banach algebras.

Definition 4.0.3 (Toeplitz matrix)**.** *A Toeplitz matrix is defined by*

$$(a_{j-k})_{j,k=0}^{\infty} \equiv \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix is completely determined by its entries in the first row and first column, this is, by the sequence

$$(a_n)_{n\in\mathbb{Z}} = \{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}.$$

Previously, we described under which conditions a Laurent matrix generates a bounded linear operator on $\ell^2(\mathbb{Z})$. Now, we present a new theorem which associates a Toeplitz matrix with a bounded linear operator on $\ell^2(\mathbb{N})$.

Theorem 4.0.4. *Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence. The Toeplitz matrix $(a_{j-k})_{j,k=0}^{\infty}$ generates a bounded operator on $\ell^2(\mathbb{N})$ if and only if there is a function $a \in L^\infty(\mathbb{T})$ such that $(a_n)_{n\in\mathbb{Z}}$ is the sequence of the Fourier coefficients.*

This theorem was established by Otto Toeplitz in his paper *"Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen"* (1911, [11]). As we could see in Theorem 3.0.5, Toeplitz' paper actually deals with Laurent matrices. Nevertheless, the proof of Theorem 4.0.4 is a footnote of the paper and it is on this theorem that the name Toeplitz matrix was given.

In accordance with our work we only need the "if portion" of the theorem, which we can easily prove. Before describing the proof we introduce a definition.

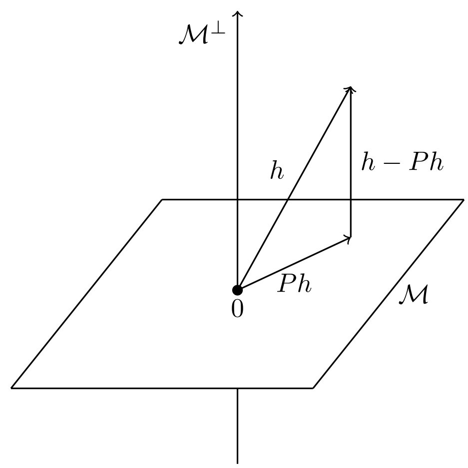


Figure 4.1: Orthogonal projection.

Definition 4.0.5 (Orthogonal projection)**.** *An orthogonal projection on a Hilbert space $\mathcal{H}$ is a linear map $P\colon \mathcal{H} \longrightarrow \mathcal{H}$ that satisfies*

$$P^2 = P \textit{ and } \langle Px, y\rangle = \langle x, Py\rangle, \text{ for every } x, y \in \mathcal{H}.$$

In the following theorem we recall some basic properties of projections. The proof can be found in every text of discrete mathematics (see e.g. [9]).

Theorem 4.0.6. *If $\mathcal{M}$ is a linear subspace of $\mathcal{H}$ and $h \in \mathcal{H}$. Let Ph be the only element of $\mathcal{M}$ such that $h - Ph \perp \mathcal{M}$. Then,*

(i) *P is a linear map onto $\mathcal{H}$.*

(ii) *$\|Ph\| \leq \|h\|$ for each $h \in \mathcal{H}$.*

(iii) *$P^2 = P$ where P^2 is the composition of P with itself.*

(iv) $\ker P = \mathcal{M}^\perp$ *and* $\operatorname{ran} P = \mathcal{M}$.

From the second item of the theorem it is possible to say that $\|P\|_{op} \leq 1$. Additionally, if e is a unitary vector in $\mathcal{M}$, then $Pe = e$. It implies that $\frac{\|Pe\|_{\mathcal{H}}}{\|e\|_{\mathcal{H}}} = 1$, i.e., $\|P\|_{op} = 1$.

Now, suppose that there is a function $a \in L^\infty(\mathbb{T})$. We identify $\ell^2(\mathbb{N})$ as a subspace of $\ell^2(\mathbb{Z})$ and denote by P the orthogonal projection of $\ell^2(\mathbb{Z})$ onto $\ell^2(\mathbb{N})$. Thus, the operator A given by an infinite Toeplitz matrix can be written as $PL(a)P$. This shows that A generates a bounded operator on $\ell^2(\mathbb{N})$ whenever $a \in L^\infty(\mathbb{T})$.

Moreover, notice that $\|A\|_{op} = \|PL(a)P\|_{op} \leq \|L(a)\|_{op} = \|a\|_\infty$.

The function $a \in L^\infty(\mathbb{T})$ is known as the symbol of the operator induced by this matrix on $\ell^2(\mathbb{N})$. Henceforth, we will denote the Toeplitz operator by $T(a)$.

The next theorems will help us to prove the important Theorem 4.0.9.

Theorem 4.0.7 (Baire's category)**.** *Let X be a nonempty complete metric space. Then, X cannot be written as a countable union of nowhere dense subsets. Hence, if X is the union of countably many closed sets $(A_k)_{k\in\mathbb{N}}$, then at least one of the A_k's must contain an open set.*

For a proof see [1].

Theorem 4.0.8 (Banach and Steinhaus)**.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of operators $A_n \in \mathcal{B}(X,Y)$ such that $(A_n x)_{n\in\mathbb{N}}$ is a convergent sequence on Y for each $x \in X$. Therefore, $\sup_{n\geq 1}\|A_n\|_{op} < \infty$, the operator A defined by $Ax \equiv \lim_{n\to\infty} A_n x$ belongs to $\mathcal{B}(X,Y)$, and $\|A\|_{op} \leq \liminf_{n\to\infty}\|A_n\|_{op}$.*

This theorem is also known as the uniform boundedness principle.

Proof. Let X be a Banach space, Y a normed space, and $(A_n)_{n\in\mathbb{N}} \in \mathcal{B}(X,Y)$ be a sequence of operators. Suppose that for every $x \in X$, it holds

$$\sup\{\|A_n x\|_Y : n \in \mathbb{N}\} < \infty.$$

For every $k \in \mathbb{N}$, we define the sets

$$X_k \equiv \{x \in X\colon \sup\{\|A_n x\|_Y : n \in \mathbb{N}\} \leq k\}.$$

Note that, the sets X_k's are closed and the fact that $(A_n)_{n\in\mathbb{N}}$ is a pointwise bounded sequence implies that

$$X = \bigcup_{k\in\mathbb{N}} X_k \neq \emptyset.$$

Baire's category Theorem 4.0.7 guarantees that one of these closed sets contains an open ball, so there exists k_0 such that,

$$\overline{B_\varepsilon(x_0)} \equiv \{x \in X\colon \|x - x_0\|_X \leq \varepsilon\} \subseteq X_{k_0}.$$

Thus, $\|A_n x\|_Y \leq k_0$ for any $x \in B_\varepsilon(x_0)$ and $n \in \mathbb{N}$.

Now, let $\tilde{x} \in X$ with $\|\tilde{x}\|_X \leq 1$. It satisfies that,

$$\begin{aligned}\|A_n\tilde{x}\|_Y &= \varepsilon^{-1}\cdot\|A_n(x_0+\varepsilon\tilde{x}) - A_n x_0\|_Y \\ &\leq \varepsilon^{-1}\cdot(\|A_n(x_0+\varepsilon\tilde{x})\|_Y + \|A_n x_0\|_Y) \\ &\leq \varepsilon^{-1}(2k_0).\end{aligned}$$

Taking the supremum over $\tilde{x}$ in the unit ball of X, it follows that

$$\|A_n\|_{op} = \sup\{\|A_n\tilde{x}\|_Y\colon \|\tilde{x}\|_X \leq 1\} \leq 2\varepsilon^{-1}k_0 < \infty.$$

Therefore, taking the limit when n goes to infinity we get $\|A\|_{op} < \infty$. ■

Theorem 4.0.9 (Norm of a Toeplitz operator)**.** *Let $a \in L^\infty(\mathbb{T})$. Then,*

$$\|T(a)\|_{op} = \|a\|_\infty.$$

Proof. We already know that $\|A\|_{op} \leq \|a\|_\infty$, it implies that $\|T(a)\|_{op} \leq \|a\|_\infty$. We want to prove the reverse inequality. To accomplish that, for each n we will define the following operators

$$(S_n\boldsymbol{x})_k = \begin{cases} 0, & \text{if } k < -n, \\ x_k, & \text{if } k \geq -n. \end{cases}$$

It is apparent that $S_n \xrightarrow[n\to\infty]{} \mathbb{I}$ strongly on $\ell^2(\mathbb{Z})$. This allows us to say that $S_nL(a)S_n$ converges strongly to $L(a)$ and $\|S_nL(a)S_n\|_{op} = \|T(a)\|_{op}$.

By Theorem 4.0.8 with $X = Y = \ell^2(\mathbb{N})$, we have that

$$\|L(a)\|_{op} \leq \liminf_{n\to\infty} \|S_nL(a)S_n\|_{op} = \|T(a)\|_{op}.$$

Thus, by Proposition 3.0.15 we immediately have $\|T(a)\|_{op} \geq \|a\|_\infty$. ∎

4.1 Spectrum of a Toeplitz operator

The determination of the spectra of Toeplitz operators is a much difficult problem, since we must examine the connection between Fredholmness and invertibility. In a first approach we say that the study of Fredholmness and invertibility for singular integral operators over the unit circle (thus over smooth curves) is equivalent to the study of the corresponding problems for Toeplitz operators.

We will encompass Coburn's Lemma and some of the theory behind it, i.e., Calkin algebra, essential spectrum, and Hardy spaces.

Definition 4.1.1 (Fredholm)**.** *Let X be a Banach space and $A \in \mathcal{B}(X)$. For this operator we define its image or range and kernel,*

$$\operatorname{Ker} A \equiv \{x \in X \colon Ax = 0\} \quad \textit{and} \quad \operatorname{Im} A \equiv \{Ax \colon x \in X\}.$$

Then, the operator A is said to be Fredholm if Im A *is a closed subspace of X and the two quantities*

$$\alpha(A) \equiv \dim \operatorname{Ker} A \quad \textit{and} \quad \beta(A) \equiv \dim(X/\operatorname{Im} A)$$

are finite. The space $X/\operatorname{Im} A$ is referred to as the cokernel of A and denoted as Coker A.

Other definition of Fredholm: An operator $A \in \mathcal{B}(X)$ is said to be Fredholm if it is invertible module compact operators, i.e., if there exists an operator $B \in \mathcal{B}(X)$ such that $AB - \mathbb{I}$ and $BA - \mathbb{I}$ are compact.

Definition 4.1.2 (Index)**.** *If A is Fredholm, then the index of A is defined as*

$$\operatorname{Ind} A \equiv \alpha(A) - \beta(A).$$

4.2 The connection between Fredholmness and invertibility

John Williams Calkin (1909-1964) was an american mathematician. He was awarded his MA in 1934 and his PhD in 1937 by Harvard University. His PhD dissertation was *"Applications* *of the theory of Hilbert space to partial differential equations; the self-adjoint transformations in Hilbert space associated with a formal partial differential operator of the second order and elliptic type".*

Today, Calkin is mostly remembered by the algebras bearing his name. The relevant work dates back to 1941 when he published his paper *"Two-sided ideals and congruences in the ring of bounded operators in Hilbert spaces"* [6].

Definition 4.2.1 (Calkin algebra)**.** *Let $\mathcal{B}_0(X)$ denote the set of all compact operators on a Banach space X. So, $\mathcal{B}_0(X)$ is a closed two-sided ideal of the Banach space $\mathcal{B}(X)$. The algebra $\mathcal{B}(X)/\mathcal{B}_0(X)$ is known as the Calkin algebra of X.*

Remark 4.2.2. Let X be a Banach space. Then, it is possible to prove that an operator $A \in \mathcal{B}(X)$ is Fredholm if and only if the coset $A+\mathcal{B}_0(X)$ is invertible in the quotient algebra $\mathcal{B}(X)/\mathcal{B}_0(X)$.

Definition 4.2.3 (Essential spectrum)**.** *Let X be a Banach space and $A \in \mathcal{B}(X)$. The essential spectrum of A, $\operatorname{sp}_{\text{ess}} A$, is the spectrum of $A + \mathcal{B}_0(X)$ in $\mathcal{B}(X)/\mathcal{B}_0(X)$, that is,*

$$\operatorname{sp}_{\text{ess}} A \equiv \{\lambda \in \mathbb{C} \colon A - \lambda\mathbb{I} \quad \textit{is not Fredholm on } X\}.$$

Proposition 4.2.4. *If $A \in \mathcal{B}(X)$, then $\operatorname{sp}_{\text{ess}} A \subset \operatorname{sp} A$.*

Proof. Firstly, we make an observation. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces.

$$\text{If } A \in \mathcal{B}(\mathcal{H}), B \in \mathcal{B}(\mathcal{K}), \text{ and } T \in \mathcal{B}_0(\mathcal{H},\mathcal{K}) \text{ then } TA, BT \in \mathcal{B}_0(\mathcal{H},\mathcal{K}).$$

Assume that $\lambda \notin \operatorname{sp} A$. Thereby, $A - \lambda\mathbb{I}$ is invertible and we call B its inverse. By the observation we made it holds that,

$$\begin{aligned}((A-\lambda\mathbb{I}) + \mathcal{B}_0(X))(B + \mathcal{B}_0(X)) &= (A-\lambda\mathbb{I})B + (A-\lambda\mathbb{I})\mathcal{B}_0(X) + \mathcal{B}_0(X)B + \mathcal{B}_0(X)\mathcal{B}_0(X)\\ &= \mathbb{I} + \mathcal{B}_0(X).\end{aligned}$$

Thus, $A - \lambda\mathbb{I}$ is invertible in $\mathcal{B}(X)/\mathcal{B}_0(X)$, i.e., $\lambda \notin \operatorname{sp}_{\text{ess}} A$. ■

The theory of Hardy spaces originated in the context of complex function theory and Fourier analysis in the beginning of twentieth century. The classical Hardy space $H^p(X)$, where $0 < p < \infty$, consists of holomorphic functions f defined on the unit disc or upper half plane.

They were introduced by Frigyes Riesz (1923), but received that name due to the paper *"The mean value of the modulus of an analytic function"* [7] written by Godfrey Harold Hardy in 1915. In real analysis these spaces are certain spaces of distributions on the real line, which are boundary values of the holomorphic functions of the complex Hardy spaces and are related to $L^p(X)$ spaces.

The reason why we will work on Hardy spaces is that we can define the logarithm function for functions belonging to $H^2(\mathbb{T})$

Definition 4.2.5 (Hardy spaces)**.** *Let $(f_n)_{n\in\mathbb{Z}}$ be the sequence of the Fourier coefficients of f. On $L^2(\mathbb{T})$ the closed subspaces*

$$H^2(\mathbb{T}) \equiv \{f \in L^2(\mathbb{T}) \colon f_n = 0 \text{ for } n < 0\} \quad \textit{and} \quad H^2_-(\mathbb{T}) \equiv \{f \in L^2(\mathbb{T}) \colon f_n = 0 \text{ for } n \geq 0\}$$

are called Hardy spaces of $L^2(\mathbb{T})$.

It is clear that $L^2(\mathbb{T})$ decomposes into the orthogonal sum

$$L^2(\mathbb{T}) = H^2(\mathbb{T}) \oplus H^2_-(\mathbb{T}).$$

Let P stand for the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. It is possible to prove that

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{in\theta} \right\}_{n=0}^{\infty}$$

is an orthonormal basis of $H^2(\mathbb{T})$.

Let $a \in L^\infty(\mathbb{T})$. The matrix representation of the operator $P\colon H^2(\mathbb{T}) \longrightarrow H^2(\mathbb{T})$ given by $f \longmapsto P(af)$ is the Toeplitz matrix $T(a)$. Note that, the operator P is the compression $PM(a)P$ of the multiplication operator $M(a)$ to $H^2(\mathbb{T})$.

The following theorem is a result of the brothers Frigyes Riesz and Marcel Riesz on analytic measures. In addition, we will need it to prove the important Theorem 4.2.7.

Theorem 4.2.6 (F. and M. Riesz)**.** *A function in $H^2(\mathbb{T})$ vanishes either almost everywhere or almost nowhere on $\mathbb{T}$.*

Theorem 4.2.7 (Coburn's Lemma)**.** *Let $a \in L^\infty(\mathbb{T})$ and suppose a does not vanish identically. Then, $T(a)$ has a trivial kernel on $\ell^2(\mathbb{N})$ or its image is dense in $\ell^2(\mathbb{N})$. In particular, $T(a)$ is invertible if and only if $T(a)$ is Fredholm of index zero:*

$$\operatorname{sp}\, T(a) = \operatorname*{sp}_{\text{ess}} T(a) \cup \{\lambda \in \mathbb{C} \setminus \operatorname*{sp}_{\text{ess}} T(a)\colon \operatorname{Ind}(T(a) - \lambda \mathbb{I}) \neq 0\}. \tag{4.2.1}$$

Proof. Let $a \in L^\infty(\mathbb{T})$. Assume that $T(a)$ has a nontrivial kernel and the image of $T(a)$ is not dense in $\ell^2(\mathbb{N})$. First, we want to interpret the adjoint operator of $T(a)$. Let $x \in \operatorname{Ker} T(a)$ with $x \neq 0$, we get that $T(a)x = 0$. Taking conjugate in both sides we obtain $\overline{T(a)}\bar{x} = \overline{0}$ with $\bar{x} \neq 0$. What this is telling us is that $T(\bar{a})$ has nontrivial kernel. Besides, $\bar{a}(t) \equiv \overline{a(t)}$ for each $t \in \mathbb{T}$.

Now, there are nonzero functions $f_+ \in H^2(\mathbb{T})$ and $g_+ \in H^2(\mathbb{T})$ with $f_+ \neq 0$ and $g_+ \neq 0$ such that $T(a)f_+ = 0$ and $T(\bar{a})g_+ = 0$.

Notice that, $Paf_+ = 0$ and this implies that $f_- \equiv af_+ \in H^2_-(\mathbb{T})$. Also, $P\bar{a}g_+ = 0$ and it means that $g_- \equiv \bar{a}g_+ \in H^2_-(\mathbb{T})$. Theorem 4.2.6 gives us that, $f_+ \neq 0$ and $g_+ \neq 0$ μ-a.e. on $\mathbb{T}$. So,

$$\bar{g}_- f_+ = \overline{\bar{a}g_+} f_+ = a\bar{g}_+ f_+ = af_+\bar{g}_+ = f_-\bar{g}_+ \equiv \varphi.$$

By Hölder's inequality 1.1.42, $\varphi \in L^1(\mathbb{T})$. Furthermore, we know the following:

$$\bar{g}_- \in H^2(\mathbb{T}) \Rightarrow \bar{g}_- f_+ \in H^2(\mathbb{T}) \quad \text{and} \quad \bar{g}_+ \in H^2_-(\mathbb{T}) \Rightarrow f_-\bar{g}_+ \in H^2_-(\mathbb{T}).$$

Thus, we get $\varphi_n = (\bar{g}_- f_+)_n = 0$ for $n \geq 0$ and $\varphi_n = (f_-\bar{g}_+)_n = 0$ for $n \leq 0$, i.e., $\varphi = 0$. Since $f_+ \neq 0$ and $g_+ \neq 0$ μ-a.e. on $\mathbb{T}$, we deduce that $g_- = 0$ μ-a.e. on $\mathbb{T}$, and as we established that $g_- = \bar{a}g_+$ it indicates that $a = 0$. However, this is a contradiction and shows that $T(a)$ has a trivial kernel or a dense range.

On the other hand, suppose that $T(a)$ is Fredholm of index zero. It means that,

$$\operatorname{Ind} T(a) = \dim \operatorname{Ker} T(a) - \dim(\ell^2(\mathbb{N}) / \operatorname{Im} T(a)) = 0.$$

Then, by what we already did there are two options: $\operatorname{Ker} T(a) = \{0\}$ or $\overline{\operatorname{Im} T(a)} = \ell^2(\mathbb{N})$. This tells us that $a(t) \neq 0$ for every $t \in \mathbb{T}$. Thus, $T(a)$ is invertible.

If $T(a)$ were not invertible, we would have two cases: $\dim \operatorname{Ker} T(a) > 0$ and $\dim(\ell^2(\mathbb{N})/T(a)) = 0$, or $\dim \operatorname{Ker} T(a) = 0$ and $\dim(\ell^2(\mathbb{N})/T(a)) > 0$. In both cases we have that $\operatorname{Ind} T(a) \neq 0$. Therefore, $T(a)$ is not Fredholm.

Equation (4.2.1) suggests us that if we suppose $T(a)$ is not invertible, we will have that $T(a)$ is not invertible on the Calkin algebra, i.e., is not Fredholm or $\operatorname{Ind} T(a) \neq 0$. Hence, this equality of sets is true. ■

We finish this chapter with an example of the spectrum of a specific function. Take $a\colon \mathbb{T} \longrightarrow \mathbb{C}$ given by $t \longmapsto -it^{-1} - 0.2 + 0.7it^4$. Then, we have

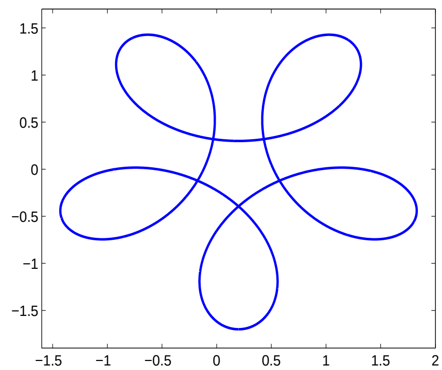


Figure 4.2: $\mathrm{sp}_{\mathrm{ess}}\, T(a)$.

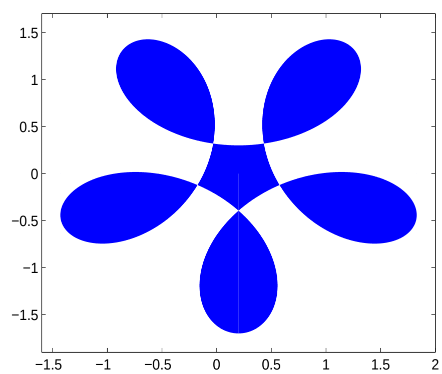


Figure 4.3: $\mathrm{sp}\, T(a)$.

Chapter 5

A review of "Asymptotic spectra of dense Toeplitz matrices are unstable"

In this last chapter we want to use our preliminary study, mathematical tools, and important results that we have obtained for understanding the paper *"Asymptotic spectra of dense Toeplitz matrices are unstable"* [2] written by Albrecht Böttcher and Sergei M. Grudsky and published in 2003.

Nowadays, there are many techniques developed for the asymptotic analysis of the eigenvalue distribution of Toeplitz matrices. What we will do is to present asymptotic estimates for the eigenvalues of a bounded operator on $\ell^2(\mathbb{N})$. In addition, the methods from the theory of Banach spaces may be handled to obtain quantitative estimates.

The increasing development in this theory answers eigenvalue problems of mechanics and mathematical physics.

The paper deals with the limiting set of the eigenvalues of the finite truncations of an infinite Toeplitz matrix whose symbol is continuous but not rational. Furthermore, we will consider a sequence of symbols $(a^{(n)})_{n\in\mathbb{N}}$ that converges uniformly to the symbol a. Also, we will find the limiting sets of this sequence and of a. Using the Hausdorff metric, we will exhibit that $\Lambda(a^{(n)})$ does not converge to $\Lambda(a)$. It declares us the limiting set is unstable with respect to small perturbations of the symbol in the uniform norm.

5.1 Introduction

Consider a function $a \in L^\infty(\mathbb{T})$ and let $(a_k)_{k\in\mathbb{Z}}$ be the sequence of its Fourier coefficients (see 1.2.3). We denote by $T(a)$ and $T_n(a)$ the infinite and $n \times n$ Toeplitz matrices with the symbol a, these are defined by

$$T(a) = (a_{j-k})_{j,k=0}^{\infty}, \quad T_n(a) = (a_{j-k})_{j,k=0}^{n-1}.$$

By Theorem 4.0.4 we know that $T(a)$ induces a bounded linear operator on $\ell^2(\mathbb{N})$.

In chapter 3 we saw examples of symbol classes, so we are familiar with the geometric representation that a continuous or piecewise continuous symbol a has. Thus, we have two cases:

1. If $a \in \mathcal{C}(\mathbb{T})$, then sp $T(a)$ is the union of the range of a and all the points in the plane encircled by $a(\mathbb{T})$ with nonzero winding number, i.e., the total number of times that the curve travels counterclockwise around the point.

2. If $a \in \mathcal{PC}(\mathbb{T})$, we denote by $a^{\#}(\mathbb{T})$ the continuous curve that arises from the essential range of a by filling in a line segment between the two endpoints of each jump of a. Then, sp $T(a)$ consists of $a^{\#}(\mathbb{T})$ and all points in the plane with nonzero winding number with respect to $a^{\#}(\mathbb{T})$.

We say that the spectrum of a Toeplitz operator $T(a)$ is continuous on $\mathcal{C}(\mathbb{T})$ and $\mathcal{PC}(\mathbb{T})$, if for every sequence $(a^{(n)})_{n\in\mathbb{N}}$ in $\mathcal{C}(\mathbb{T})$ or $\mathcal{PC}(\mathbb{T})$ converging uniformly to a, then

$$\lim_{n\to\infty} \text{sp } T(a^{(n)}) = \text{sp } T(a).$$

It means that, sp $T(a^{(n)})$ converges to sp $T(a)$ in the Hausdorff metric (see 1.1.2).

Let $(T_n(a))_{n\in\mathbb{N}}$ be the sequence of $n \times n$ Toeplitz matrices, where sp $T_n(a)$ is the set of eigenvalues of the truncated Toeplitz matrix $T_n(a)$. The asymptotic spectrum of $T(a)$ is given by the set

$$\Lambda(a) \equiv \limsup_{n\to\infty} \text{sp } T_n(a).$$

That is, the limiting set $\Lambda(a)$ is defined as follows

$$\lambda \in \Lambda(a) \Longleftrightarrow n_1 < n_2 < n_3 < \cdots \text{ and } \lambda_{n_k} \in \text{sp } T_{n_k}(a) \text{ such that } \lambda_{n_k} \longrightarrow \lambda.$$

Theorem 4.2.7 helps us to find the spectrum of a Toeplitz operator sp $T(a)$ when a is a rational symbol. In general the determination of the asymptotic spectrum $\Lambda(a)$ is a difficult task.

Example 5.1.1. Let $a(t) = \sum_{k=-r}^{s} a_k t^k$ be a Laurent polynomial, it implies that $T(a)$ is a band matrix. Denote by $z_1(\lambda), z_2(\lambda), z_3(\lambda), \ldots, z_{r+s}(\lambda)$ the zeros of the polynomial $z^r(a(z) - \lambda) = (\sum_{k=0}^{r+s} a_{k-r} z^k) - \lambda z^r$. We have labeled the zeros of the polynomial in such a way that $|z_1(\lambda)| \leq |z_2(\lambda)| \leq \cdots \leq |z_{r+s}(\lambda)|$.

In 1960, Palle Schmidt and Frank Spitzer proved, in their paper *"The Toeplitz matrices of an arbitrary Laurent polynomial"* (1960, [10]), that $\Lambda(a) = \{\lambda \in \mathbb{C} \colon |z_r(\lambda)| = |z_{r+1}(\lambda)|\}$. Additionally, they showed that the limiting set $\Lambda(a)$ is either a singleton or the union of finitely many analytic arcs.

Gàbor Szegő was a hungarian mathematician who worked mainly in function theory, classical orthogonal polynomials, isoperimetric inequalities, orthogonal polynomials on the unitary circle, and Toeplitz forms. This work led him to prove a number of limit theorems, among them the famous Szegő's first limit theorem and the strong Szegő limit theorem.

Given certain classes of continuous or piecewise continuous symbols a, he described the asymptotic eigenvalue distribution of $T_n(a)$ by the formulas of the Szegő type.

Let a be a continuous or piecewise continuous symbol and $\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_n^{(n)}$ be the eigenvalues of $T_n(a)$, then for every continuous function $\varphi\colon \mathbb{C} \to \mathbb{R}$ with compact support it holds,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j^{(n)}) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(a(e^{i\theta}))\, d\theta.$$

Since we are interested in the eigenvalue distribution of Toeplitz matrices, there are a few eigenvalues that are located at a far distance respect to the other eigenvalues. However, these can be omitted. Within our study, it is important to characterize them as isolated points of the overall shape of sp $T_n(a)$. Moreover, this formula is measuring the degree of dispersion of the eigenvalues with respect to the mean value of their distribution.

In order to find $\Lambda(a)$ there are two alternatives that have been tested. The first works only for rational symbols and what we do is to compute sp $T_n(a)$ for some large values of n and hope that we will get a good approximation for $\Lambda(a)$. Referring to large values of n we might state $n = 2^{13}$ as a maximum value for a regular computer and $n = 2^{20}$ as a maximum value for the fastest computer in the world, but these spectra are used in statistical physics with required matrix orders of 10^{20}. Thus, we discard this option.

The second alternative is to approximate a by a Laurent polynomial $a^{(n)}$, attempting that $\Lambda(a^{(n)})$ is close to $\Lambda(a)$. This procedure fails for piecewise continuous symbols because they can never be approximated uniformly by Laurent polynomials; and in this work we will show that it can fail for continuous symbols also, which is curious because a continuous function can be uniformly approximated by Laurent polynomials. Whence, we have the following result: the asymptotic spectrum $\Lambda(\cdot)$ is discontinuous on $\mathcal{C}(\mathbb{T})$.

5.2 Main result

In this section we will present the main result, that is Theorem 5.2.3, but first we need Lemmas 5.2.1 and 5.2.2. These lemmas and some calculations will help us to prove the theorem. We mention that some proofs were not in the paper, we had to construct them.

Consider the following symbol

$$a(t) = \frac{33 - (t + t^2)(1 - t^2)^{3/4}}{t}, \quad t \in \mathbb{T}. \tag{5.2.1}$$

Note that, the only singularity of this function occurs at $t = 1$ and at this point the derivative of a has a jump discontinuity. Then, a belongs to $\mathcal{PC}^\infty(\mathbb{T})$ but not to $\mathcal{C}^\infty(\mathbb{T})$. Besides, $T(a)$ is a lower Hessenberg matrix (see 3.0.8), the Fourier series of a is convergent with $a_k = 0$ for $k \le -2$. We define the Laurent polynomial $a^{(n)} \equiv P_n a$, where $P_n a$ (see 4.0.5) denotes the n-th partial sum of the Fourier series.

The american mathematician Harold Widom has researched in the areas of integral equations and operator theory, in particular the study of Toeplitz and Wiener–Hopf operators, and the asymptotic behavior of the spectra for various classes of operators. Widom and collaborators have used ideas from operator theory to obtain new results in random matrix theory and limiting distributions of the largest and smallest eigenvalues of random matrices.

In particular, Widom in his paper *"Eigenvalue distribution of non-selfadjoint Toeplitz matrices and the asymptotics of Toeplitz determinants in the case of nonvanishing index"*

(1990, [12]) gives results which show that the eigenvalues of $T_n(a)$ for the symbol (5.2.1) cluster along the its range $a(\mathbb{T})$.

For the symbol (5.2.1), Figures 5.1 and 5.2 show that the eigenvalues of $T_{128}(a)$ and $T_{256}(a)$ cluster along the range of a. Besides, we plot the eigenvalues of $T_{128}(P_n a)$ for $n = 4, 6, 8, 12$. These eigenvalue distributions exhibit that the set $\Lambda(P_n a)$ grows like a rampant tree and does not converge to $\Lambda(a)$ whenever n approaches to ∞.

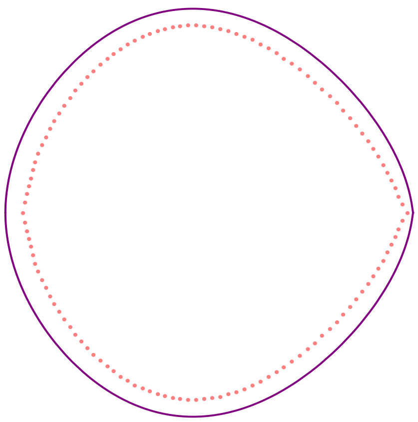


Figure 5.1: Range of $a(\mathbb{T})$ (purple) and the eigenvalues of $T_{128}(a)$ (pink).

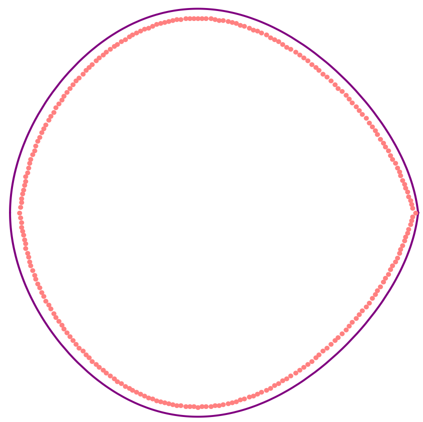


Figure 5.2: Range of $a(\mathbb{T})$ (purple) and the eigenvalues of $T_{256}(a)$ (pink).

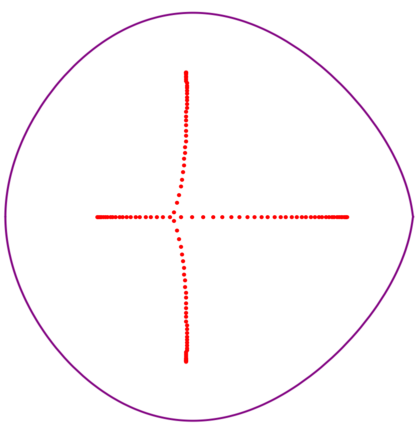


Figure 5.3: Range of $a(\mathbb{T})$ (purple) and the eigenvalues of $T_{128}(P_4 a)$ (red).

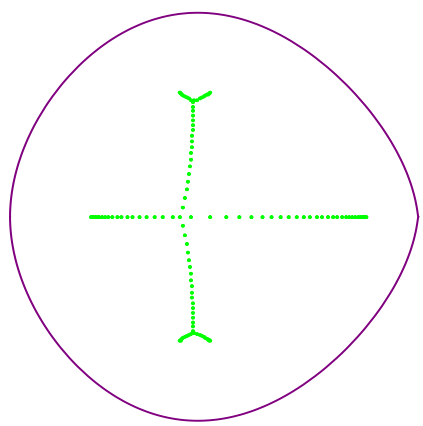


Figure 5.4: Range of $a(\mathbb{T})$ (purple) and the eigenvalues of $T_{128}(P_6 a)$ (green).

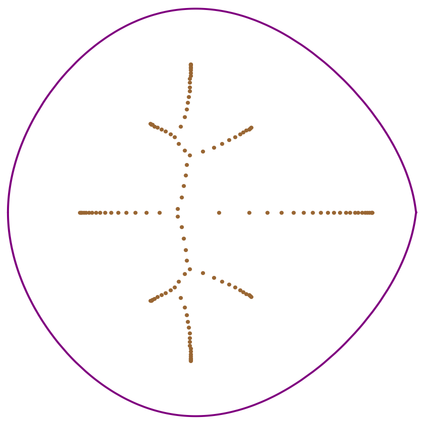


Figure 5.5: Range of $a(\mathbb{T})$ (purple) and the eigenvalues of $T_{128}(P_8 a)$ (brown).

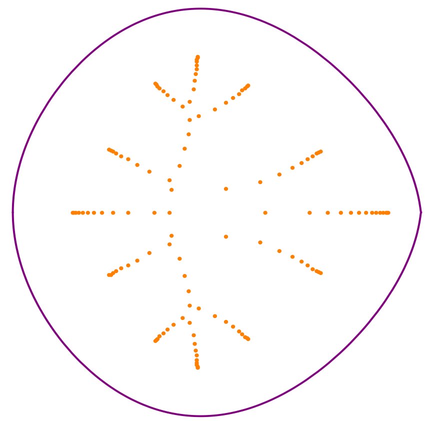


Figure 5.6: Range of $a(\mathbb{T})$ (purple) and the eigenvalues of $T_{128}(P_{12} a)$ (orange).

We will start with a computation. Take $\alpha = \frac{3}{4}$. For $t \in \mathbb{T}$, by the binomial theorem we have

$$(1-t^2)^\alpha = \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} t^{2n}. \tag{5.2.2}$$

The Weierstrass product formula is well known and it states that

$$\prod_{k=1}^{\infty} \left(1 - \frac{\alpha}{k}\right) e^{\alpha/k} = \frac{e^{\gamma\alpha}}{\Gamma(1-\alpha)}, \tag{5.2.3}$$

where γ is the Euler-Mascheroni constant.

Let us now treat about the connection between the harmonic series and γ. The earliest recorded appearance of the harmonic series seems to be in the 14-th century by the french mathematician Nicole Oresme $(1323, 1382)$. He knew how to add harmonic and geometric progressions as well as infinite geometric series, and he was the first to prove that the harmonic series diverges. Now, consider the n-th partial sum of the harmonic series called the harmonic number H_n defined as,

$$H_n \equiv 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

His proof relies on grouping the terms in the series furnishes the following inequality, sometimes called Oresme's inequality,

$$H_{2^m} > 1 + \frac{m}{2}, \quad m \geq 0. \tag{5.2.4}$$

A careful analysis of Oresme's inequality (5.2.4) shows that H_n increases at the same rate as the logarithm of n, this implies that the harmonic series has a logarithmic property.

Furthermore, the difference between H_n and $\log(n+1)$ decreases as n increases and eventually converges to the Euler's constant γ as n tends to infinity. Thus,

$$\gamma = \lim_{n\to\infty} (H_n - \log(n+1)). \tag{5.2.5}$$

It implies that, $H_n = \log(n+1) + \gamma + o(1)$ whenever $n \to \infty$.

Therefore, a calculation gives us,

$$
\begin{aligned}
(-1)^n \binom{\alpha}{n} &= \frac{1}{n!}\left((-1)^n \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\right) \\
&= \frac{1}{n!}\left((-1)^n \alpha(-1)(1-\alpha)(-2)(1-\frac{\alpha}{2})(-3)(1-\frac{\alpha}{3})\cdots(-(n-1))(1-\frac{\alpha}{n-1})\right) \\
&= -\alpha\frac{(n-1)!}{n!}\prod_{k=1}^{n-1}\left(1-\frac{\alpha}{k}\right) = -\frac{\alpha}{n}\prod_{k=1}^{n-1}\left(1-\frac{\alpha}{k}\right) \\
&= -\frac{\alpha}{n}\Big(\prod_{k=1}^{n-1}\left(1-\frac{\alpha}{k}\right)e^{\alpha/k}\Big)\Big(\prod_{k=1}^{n-1} e^{-\alpha/k}\Big) = -\frac{\alpha}{n}\Big(\prod_{k=1}^{n-1}\left(1-\frac{\alpha}{k}\right)e^{\alpha/k}\Big)e^{-\alpha H_{n-1}}.
\end{aligned}
\tag{5.2.6}
$$

Inserting Equations (5.2.3) and (5.2.5) in Equation (5.2.6) we obtain,

$$
(-1)^n \binom{\alpha}{n} = -\frac{\alpha}{n}\frac{e^{\gamma\alpha}}{\Gamma(1-\alpha)} e^{-\alpha(\log n + \gamma + o(1))}.
$$

If we use the Maclaurin series of $e^{o(1)}$, we get that

$$
(-1)^n \binom{\alpha}{n} = -\frac{\alpha e^{\gamma\alpha}}{n\Gamma(1-\alpha)} n^{-\alpha} e^{-\alpha\gamma} e^{o(1)} = -\frac{\alpha}{\Gamma(1-\alpha)}\frac{1}{n^{1+\alpha}}(1+o(1)). \tag{5.2.7}
$$

Now, if we consider the infinite sum

$$
\sum_{n=0}^{\infty}(-1)^n \binom{\alpha}{n} = -\frac{\alpha}{\Gamma(1-\alpha)}\sum_{n=0}^{\infty}\frac{1}{n^{1+\alpha}}(1+o(1)).
$$

In addition, by the n^p criterion for series with $p = \frac{7}{4}$ it is clear that the series converges. Moreover, it illustrates that the Fourier series of the function defined by (5.2.2) is absolutely convergent.

The next step is to deal with the algebra $H^\infty(\mathbb{T}) = \{h \in L^\infty(\mathbb{T})\colon h_n = 0 \text{ for } n < 0\}$. Let $a(t) \equiv t^{-1}(h_0 - \varphi(t))$ where $\varphi(t) = (t+t^2)(1-t^2)^{3/4}$ for $t \in \mathbb{T}$. We will exhibit h explicitly and prove that it belongs to $H^\infty(\mathbb{T})$. Likewise, we can take $h_0 > 0$ large enough such that $h_0 - \varphi$ is invertible in $H^\infty(\mathbb{T})$.

Lemma 5.2.1. *Let $\lambda \in \mathbb{D}$ and define $r_\lambda(\theta) \equiv \lambda(e^{i\theta}-1) + \varphi(e^{i\theta})$. Then, r_λ and its first two derivatives r'_λ, r''_λ satisfy the estimates*

$$
|r_\lambda^{(k)}(\theta)| \le C_k|\theta|^{\alpha-k}, \quad k = 0,1,2,
$$

for $0 < |\theta| < \pi$, where $C_k \in (0,\infty)$ are constants independent of λ and θ. Moreover, $C_0 \le 10$.

Proof. Recall that $\alpha = \frac{3}{4}$ and $\lambda \in \mathbb{D}$. Then, the Maclaurin series of $e^{i\theta}$ is $e^{i\theta} = \sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!}$. Thus, if we insert this series in the function r_λ and expand it, we obtain

$$
\begin{aligned}
r_\lambda(\theta) &= \delta_0\theta^{3/4} + \delta_1\theta + \delta_2\theta^{7/4} + O(\theta^2) \\
&= \theta^{3/4}(\delta_0 + \delta_1\theta^{1/4} + \delta_2\theta + O(\theta^{5/4})),
\end{aligned}
\tag{5.2.8}
$$

where, for example $\delta_0 = -2((-1)^{5/8}2^{3/4}), \delta_1 = i\lambda$, and $\delta_2 = \frac{3(-1)^{1/8}}{2^{1/4}} + 3(-1)^{1/8}2^{3/4}$. That is, $\delta_i \in \mathbb{C}$ for $i \in \mathbb{N}$.

The first derivative of r_λ is

$$\begin{aligned} r'_\lambda(\theta) &= \frac{3}{4}\delta_0\theta^{-1/4} + \delta_1 + \frac{7}{4}\delta_2\theta^{3/4} + O(\theta) \\ &= \theta^{-1/4}\left(\frac{3}{4}\delta_0 + \delta_1\theta^{1/4} + \frac{7}{4}\delta_2\theta + O(\theta^{5/4})\right). \end{aligned}$$

Lastly, the second derivative of r_λ is

$$\begin{aligned} r''_\lambda(\theta) &= -\frac{3}{16}\delta_0\theta^{-5/4} + \frac{21}{16}\delta_2\theta^{-1/4} + O(1) \\ &= \theta^{-5/4}\left(-\frac{3}{16}\delta_0 + \frac{21}{16}\delta_2\theta + O(\theta^{5/4})\right). \end{aligned}$$

Now, for r_λ we have the following calculation:

$$\begin{aligned} |r_\lambda(\theta)| &\leq |\theta|^{3/4}(|\delta_0| + |\delta_1|\cdot|\theta|^{1/4} + |\delta_2|\cdot|\theta| + O(|\theta|^{5/4})) \\ &\leq |\theta|^{3/4}C_0 = |\theta|^{\alpha}C_0, \end{aligned}$$

where C_0 is some positive constant.

We can ensure that the mentioned series are convergent because they come from the Maclaurin series of $e^{i\theta}$ and we already know that this series is convergent.

Handling with the first derivative we get this second calculation:

$$\begin{aligned} |r'_\lambda(\theta)| &\leq |\theta|^{-1/4}\left(\frac{3}{4}|\delta_0| + |\delta_1|\cdot|\theta|^{1/4} + \frac{7}{4}|\delta_2|\cdot|\theta| + O(|\theta|^{5/4})\right) \\ &\leq |\theta|^{-1/4}C_1 = |\theta|^{\alpha-1}C_1, \end{aligned}$$

where C_1 is some positive constant.

Finally, using the second derivative of r_λ we have:

$$\begin{aligned} |r''_\lambda(\theta)| &\leq |\theta|^{-5/4}\left(\frac{3}{16}|\delta_0| + \frac{21}{16}|\delta_2|\cdot|\theta| + O(|\theta|^{5/4})\right) \\ &\leq |\theta|^{-5/4}C_2 = |\theta|^{\alpha-2}C_2, \end{aligned}$$

where C_2 is some positive constant.

Therefore, $|r_\lambda^{(k)}(\theta)| \leq C_k|\theta|^{\alpha-k}$ for $k = 0, 1, 2$ as we desired.

Additionally, a simple calculation shows that $C_0 \leq 10$. ∎

Lemma 5.2.2. *Let $\lambda \in \mathbb{D}$ and define $h(t) \equiv h_0 - \lambda t - \varphi(t)$ for $t \in \mathbb{T}$. If $h_0 \geq 33$, then h is invertible in $H^\infty(\mathbb{T})$ and there is an n_0 such that the n-th Fourier coefficient $(h^{-1})_n$ of h^{-1} is nonzero for all $n \geq n_0$.*

Proof. Recall that $\alpha = 3/4$ and $\varphi(t) = (t+t^2)(1-t^2)^{\alpha}$. Using the function r_λ from Lemma 5.2.1 we obtain,

$$\begin{aligned} h(e^{i\theta}) &= h_0 - \lambda e^{i\theta} - \varphi(e^{i\theta}) = h_0 - \lambda - \lambda(e^{i\theta}-1) - \varphi(e^{i\theta}) \\ &= h_0 - \lambda - r_\lambda(\theta) = (h_0-\lambda)\left(1 - \frac{r_\lambda(\theta)}{h_0-\lambda}\right). \end{aligned}$$

By assumption $\lambda \in \mathbb{D}$ and $h_0 \geq 33$, so $|h_0 - \lambda| > 32$ and $\|r_\lambda\|_\infty = 1\cdot 2 + 1 \cdot 2 \cdot 2^{3/4} = 2(1+2^{3/4}) < 2(1+2) = 6 < 32$. Then, h is invertible in $H^\infty(\mathbb{T})$. Besides, we will find the multiplicative inverse of h and denote it as h^{-1}. Note that,

$$\frac{1}{h(e^{i\theta})} = \frac{1}{h_0-\lambda}\cdot\frac{1}{1-\frac{r_\lambda(\theta)}{h_0-\lambda}} = \frac{1}{h_0-\lambda}\cdot\sum_{j=0}^{\infty}\left(\frac{r_\lambda(\theta)}{h_0-\lambda}\right)^j, \quad \text{when } \left|\frac{r_\lambda(\theta)}{h_0-\lambda}\right| < 1. \tag{5.2.9}$$

From Equation (5.2.9) we have an explicit form of h^{-1}:

$$h^{-1}(e^{i\theta}) = \frac{1}{h_0-\lambda}\left[1 + \frac{r_\lambda(\theta)}{h_0-\lambda} + s_\lambda(\theta)\right],$$

with $s_\lambda(\theta) \equiv \sum_{j=2}^{\infty}\left(\frac{r_\lambda(\theta)}{h_0-\lambda}\right)^j$.

We desire to find the n-th Fourier coefficient of h^{-1}. Then, given that the Fourier transform is linear (see Theorem 1.2.5) we obtain that each coefficient of h^{-1} is

$$(h^{-1})_n = (h_0-\lambda)^{-2}(r_\lambda)_n + (h_0-\lambda)^{-1}(s_\lambda)_n \text{ for } n \geq 1. \tag{5.2.10}$$

Using Equation (5.2.2) it holds that,

$$\varphi(t) = (t+t^2)(1-t^2)^{\alpha} = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(t^{2k+1}+t^{2k+2}).$$

Equation (5.2.7) implies that whenever n approaches to ∞, we have $(r_\lambda)_n = E_\alpha\frac{1}{n^{1+\alpha}}(1+o(1))$ where E_α is a nonzero constant independent of λ, it guarantees us that $(r_\lambda)_n \neq 0$ for sufficiently large n.

Now, we will prove that $(s_\lambda)_n = O(1/n^2)$ as $n \longrightarrow \infty$ showing that $(s_\lambda)_n$ is arbitrarily close to zero for enough large n.

Since $\frac{C_0|\theta|^{3/4}}{h_0-1} \leq \frac{10\pi^{3/4}}{32} < 1$, we can estimate s_λ as follows

$$\begin{aligned} |s_\lambda(\theta)| &\leq \sum_{j=2}^{\infty}\left|\frac{r_\lambda(\theta)}{h_0-1}\right|^j \leq \sum_{j=2}^{\infty}\left(\frac{C_0|\theta|^{3/4}}{h_0-1}\right)^j \quad \text{by Lemma 5.2.1,} \\ &\leq \frac{\left(\frac{C_0|\theta|^{3/4}}{h_0-1}\right)^2}{1-\frac{C_0|\theta|^{3/4}}{h_0-1}} \quad \text{by the criterion of geometric series,} \\ &= |\theta|^{2\alpha}\frac{\frac{C_0^2}{(h_0-1)^2}}{1-\frac{C_0|\theta|^{3/4}}{h_0-1}} \leq |\theta|^{2\alpha}D_0, \end{aligned}$$

where $D_0 = \frac{25}{8(32-10\pi^{3/4})}$.

We use Equation (5.2.8) in s_λ and get that:

$$s_\lambda(\theta) = \theta^{3/2}\left(\hat{\delta}_0 + \hat{\delta}_1\theta^{1/4} + O(\theta^{1/2})\right), \tag{5.2.11}$$

where, for example $\hat{\delta}_0 = -\left(1/128 + i/128\right)$ and $\hat{\delta}_1 = \frac{(-1)^{1/8}\lambda}{128\cdot 2^{1/4}}$.

From Equation (5.2.11), we find the first and second derivatives of s_λ

$$s'_\lambda(\theta) = \theta^{1/2}\left(\frac{3}{2}\hat{\delta}_0 + \frac{7}{4}\hat{\delta}_1\theta^{1/4} + O(\theta^{1/2})\right) \tag{5.2.12}$$

$$s''_\lambda(\theta) = \theta^{-1/2}\left(\frac{3}{4}\hat{\delta}_0 + \frac{21}{16}\hat{\delta}_1\theta^{1/4} + O(\theta^{1/2})\right). \tag{5.2.13}$$

Now, analysing Equation (5.2.12) we can estimate $|s'_\lambda|$:

$$|s'_\lambda(\theta)| \leq |\theta|^{1/2}\left(\frac{3}{2}|\hat{\delta}_0| + \frac{7}{4}|\hat{\delta}_1|\cdot|\theta|^{1/4} + O(|\theta|^{1/2})\right) \leq |\theta|^{1/2}D_1 = |\theta|^{2\alpha}D_1,$$

where D_1 is some positive constant.

Lastly, from Equation (5.2.13) the second derivative of s_λ will satisfy that

$$|s''_\lambda(\theta)| \leq |\theta|^{-1/2}\left(\frac{3}{2}|\hat{\delta}_0| + \frac{21}{16}|\hat{\delta}_1|\cdot|\theta|^{1/4} + O(|\theta|^{1/2})\right) \leq |\theta|^{-1/2}D_2 = |\theta|^{2\alpha-1}D_2,$$

where D_2 is some positive constant.

Finally, from Definition 1.2.3 we can calculate $(s_\lambda)_n$ by integrating by parts twice:

$$(s_\lambda)_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}s_\lambda(\theta)\,d\theta = \frac{1}{2\pi i^2 n^2}\int_{-\pi}^{\pi} e^{-in\theta}s''_\lambda(\theta)\,d\theta.$$

Then, it is possible to enclose each $(s_\lambda)_n$ by using the estimates we found,

$$|(s_\lambda)_n| \leq \frac{D_2}{2\pi n^2}\int_{-\pi}^{\pi}|\theta|^{2\alpha-2}\,d\theta = \frac{D_2\pi^{2\alpha-1}}{\pi(2\alpha-1)}\frac{1}{n^2}.$$

Therefore, we have seen that $(s_\lambda)_n = O(1/n^2)$. This result combined with our study of $(r_\lambda)_n$ and Equation (5.2.10) gives us that $(h^{-1})_n \neq 0$ for all sufficiently large n. ■

At this point, we have done the necessary work for proving our main result. Now, we present Theorem 5.2.3 that concerns the limiting set of a continuous symbol a in the Hausdorff metric.

Theorem 5.2.3. *There exist a family of functions $a^{(n)} \in \mathcal{C}(\mathbb{T})$ and $a \in \mathcal{C}(\mathbb{T})$ such that $\|a^{(n)} - a\|_\infty \xrightarrow[n\to\infty]{} 0$ but $\Lambda(a^{(n)})$ does not converge to $\Lambda(a)$ in the Hausdorff metric.*

Proof. Recall that $h_0 \geq 33, \alpha = 3/4$, and $\lambda \in \mathbb{D}$. For every $t \in \mathbb{T}$, let $a(t) = t^{-1}(h_0 - \varphi(t))$ where $\varphi(t) = (t+t^2)(1-t^2)^\alpha$.

Notice that,

$$a(t)-\lambda = t^{-1}(h_0-\varphi(t))-\lambda = t^{-1}(h_0-\varphi(t)-\lambda t) = t^{-1}h(t),$$

where $h \in L^\infty(\mathbb{T})$ is the function we defined in Lemma 5.2.2.

Now, consider the Toeplitz matrix $T_{n+1}(h)$. After calculating its Fourier coefficients, we construct its Toeplitz matrix which is a lower triangular matrix. Thus, $h_n = 0$ for $n > 0$ and we conclude that $h \in H^\infty(\mathbb{T})$.

Consider such matrix $T_{n+1}(h)$, we can delete its first row and last column. This procedure will gives us the Toeplitz matrix $T_n(a-\lambda)$. It is well known that the Cramer's rule gives an explicit formula for the solution of a system of linear equations and also it may be useful to find the inverse of a nonsingular matrix, so it is possible to identify the $(n+1,1)$ entry of $T_{n+1}^{-1}(h)$,

$$[T_{n+1}^{-1}(h)]_{n+1,1} = (-1)^{n+2}\frac{\det T_n(a-\lambda)}{\det T_{n+1}(h)}. \tag{5.2.14}$$

Since h is an invertible function in $H^\infty(\mathbb{T})$, we get that $T_{n+1}^{-1}(h) = T_{n+1}(h^{-1})$ (see 3.0.17), it implies that $[T_{n+1}^{-1}(h)]_{n+1,1} = (h^{-1})_n$. Clearly, $\det T_{n+1}(h) = h_0^{n+1}$.

From Equation (5.2.14) we obtain that $\det T_n(a-\lambda) = (-1)^n h_0^{n+1}(h^{-1})_n$, for $n \in \mathbb{N}$. By Lemma 5.2.2, $(h^{-1})_n$ is nonzero for n sufficiently large and every $\lambda \in \mathbb{D}$. Then, it holds too for $\det T_n(a-\lambda)$ and we state that,

$$\mathbb{D} \cap \Lambda(a) = \emptyset. \tag{5.2.15}$$

We shall prove that $0 \in \Lambda(P_{2m+1}a)$ for every $m \geq 2$, where $P_{2m+1}a$ denotes the $(2m+1)$-th partial sum of the Fourier series. Define the polynomial $q(P_{2m+1}a)$ as follows

$$q(P_{2m+1}a)(z) \equiv h_0 - \sum_{k=0}^{m}(-1)^k\binom{\alpha}{k}(z^{2k+1}+z^{2k+2}). \tag{5.2.16}$$

We can notice that this polynomial is particularly defined and previous results will be helpful.

We conveniently label its zeros like $z_1, z_2, \ldots, z_{2m+2}$ belonging to $\mathbb{C}$ such that $|z_1| \leq |z_2| \leq \cdots \leq |z_{2m+2}|$. In Example 5.1.1 we saw that $\Lambda(a) = \{\lambda \in \mathbb{C}\colon |z_{r+1}(\lambda)| = |z_{r+2}(\lambda)|\}$. We know that for this function, $r = 0$. Then, we have that $0 \in \Lambda(P_{2m+1}a)$ if and only if $|z_1| = |z_2|$.

From Equation (5.2.16) it holds that the polynomial $q(P_{2m+1}a)$ has real coefficients. It is well known that if a polynomial with real coefficients has complex roots, then they come in conjugate pairs. So, it is enough to show that $q(P_{2m+1}a)$ has no real zeros. Immediately, it follows the equality $|z_1| = |z_2|$. Thus, we can think that these two roots are the ones closest to zero.

If we expand the polynomial $q(P_{2m+1}a)$, we get

$$\begin{aligned} q(P_{2m+1}a)(z) &= h_0 - (z+z^2) + \alpha(z^3+z^4) + \frac{\alpha(1-\alpha)}{2!}(z^5+z^6) \\ &\quad + \cdots + \frac{\alpha(1-\alpha)(2-\alpha)\cdots(m-1-\alpha)}{m!}(z^{2m+1}+z^{2m+2}) \\ &= h_0 - (z-z^2) + \sum_{k=1}^{m} b_k(z^{2k+1}+z^{2k+2}), \end{aligned} \tag{5.2.17}$$

where $b_k = (-1)^{k+1}\binom{\alpha}{k} > 0$ for all $k \in \mathbb{N}$.

Assuming that $|z| \leq 1$, we realize that

$$\begin{aligned}\left|\sum_{k=0}^{m}(-1)^k\binom{\alpha}{k}(z^{2k+1}+z^{2k+2})\right| &\leq \sum_{k=0}^{m}\left|\binom{\alpha}{k}\right|(|z|^{2k+1}+|z|^{2k+2}) \leq 2\sum_{k=0}^{\infty}\left|\binom{\alpha}{k}\right| \\ &= 2+2\alpha+2\left(\frac{\alpha(1-\alpha)}{2!}+\frac{\alpha(1-\alpha)(2-\alpha)}{3!}+\cdots\right) \\ &= 2+2\alpha+2\left(\frac{\alpha}{2}(1-\alpha/1)+\frac{\alpha}{3}(1-\alpha/1)(1-\alpha/2)+\cdots\right) \\ &= 2+2\alpha+2\sum_{k=1}^{\infty}\frac{\alpha}{k+1}(1-\alpha/1)\cdots(1-\alpha/k).\end{aligned}$$

Additionally, note that $e^{-\alpha/k} > 1-\alpha/k$ for every k. Whence,

$$\begin{aligned}\left|\sum_{k=0}^{m}(-1)^k\binom{\alpha}{k}(z^{2k+1}+z^{2k+2})\right| &< 2+2\alpha+2\sum_{k=1}^{\infty}\frac{\alpha}{k+1}e^{-\alpha/1}\cdots e^{-\alpha/k} \\ &= 2+2\alpha+2\sum_{k=1}^{\infty}\frac{\alpha}{k+1}e^{-\alpha H_k} \\ &< 2+2\alpha+2\sum_{k=1}^{\infty}\frac{\alpha}{k+1}e^{-\alpha(\log(k+1)-\log 2)} \\ &< 2+2\alpha+2\sum_{k=1}^{\infty}\frac{\alpha}{(k+1)}\frac{2^{\alpha}}{(k+1)^{\alpha}} \\ &< 2+2+4\sum_{k=1}^{\infty}\frac{1}{(k+1)^{3/2}} < 12 < 32.\end{aligned}$$

Hence, we have seen that the difference $h_0 - \sum_{k=0}^{m}(-1)^k\binom{\alpha}{k}(z^{2k+1}+z^{2k+2})$ is not zero. It means that $z(P_{2m+1}a)$ has no zeros in $[-1,1]$. On the other hand, if z is a real number it holds the following:

(i) If we suppose that $|z| \geq \frac{1}{\sqrt{\alpha}}$, then $\alpha(z^3+z^4) \geq z+z^2$.

(ii) Since $b_k(z^{2k+1}+z^{2k+2}) > 0$ for each k and all $|z| > 1$, we have from Equation (5.2.17) that $q(P_{2m+1}a)(z) \neq 0$ for all $|z| \geq \frac{1}{\sqrt{\alpha}}$.

(iii) Since $b_k(z^{2k+1}+z^{2k+2}) > 0$ for every k. If we assume that $1 < |z| < \frac{1}{\sqrt{\alpha}}$, then $|z+z^2| < \frac{1}{\sqrt{\alpha}} + \frac{1}{\alpha} < \sqrt{2}+2 < 33$. Thus, it follows that $q(P_{2m+1}a)(z) \neq 0$ for all real z satisfying $1 < |z| < \frac{1}{\sqrt{\alpha}}$.

With these previous observations we finally get that, $|z_1| = |z_2|$ and therefore $0 \in \Lambda(P_{2m+1}a)$ for every m.

Finally, by Equation (5.2.15) and $0 \in \Lambda(P_{2m+1}a)$ we arrive at the conclusion that the Hausdorff distance from $\Lambda(a)$ to $\Lambda(P_{2m+1}a)$ will be at least one. Thus, $\Lambda(P_{2m+1}a)$ does not converge to $\Lambda(a)$ in the Hausdorff metric. ■

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