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A. D. Forbes T. S. Griggs

School of Mathematics and Statistics
The Open University
Walton Hall, Milton Keynes MK7 6AA
UNITED KINGDOM
anthony.d.forbes@gmail.com terry.griggs@open.ac.uk

Abstract

The design spectrum has been determined for eleven of the 21 graphs with six vertices and nine edges. In this paper we completely solve the design spectrum problem for the remaining ten graphs.

1 Introduction

Throughout this paper all graphs are simple. Let G be a graph. If the edge set of a graph K can be partitioned into edge sets of graphs each isomorphic to G, we say that there exists a *decomposition* of K into G. In the case where K is the complete graph K_n we refer to the decomposition as a G-*design* of order n. The *design spectrum* of G is the set of non-negative integers n for which there exists a G-design of order n. For completeness, we remark that the empty set is a G-design of order 0 as well as 1; these trivial cases are usually assumed henceforth. A complete solution of the spectrum problem often seems to be difficult. However it has been achieved in many cases, especially amongst the smaller graphs. We refer the reader to the survey article of Adams, Bryant and Buchanan [2] and, for more up to date results, the Web site maintained by Bryant and McCourt [4]. If the graph G has v vertices, e edges, and if d is the greatest common divisor of the vertex degrees, then a G-design of order n can exist only if the following conditions hold:

$$\text{(i) } n \le 1 \text{ or } n \ge v, \quad \text{(ii) } n-1 \equiv 0 \pmod{d}, \quad \text{(iii) } n(n-1) \equiv 0 \pmod{2e}. \qquad (1)$$

Except where (i) of (1) applies, adding an isolated vertex to a graph does not affect its design spectrum.

The problem for small graphs has attracted attention. The design spectrum has been determined for (i) all graphs with at most five vertices, (ii) all graphs with six vertices and at most seven edges, (iii) all graphs with six vertices and eight edges, with two possible exceptions, and (iv) eleven of the graphs with six vertices and nine edges. See [2] and [4] for details and references. In Table 1 we list the 21 graphs

Table 1: The 21 graphs with 6 vertices and 9 edges

n_1	G161	H^9_{10}	{{6,2},{6,1},{5,2},{5,1},{4,2},{4,1},{3,2},{3,1},{2,1}}
n_2	G156		{{4,3},{4,2},{4,1},{5,2},{5,1},{6,1},{3,2},{3,1},{2,1}}
n_3	G162	H^9_9	{{3,4},{3,2},{3,1},{6,2},{6,1},{5,2},{5,1},{4,1},{2,1}}
n_4	G170	H^9_8	{{4,3},{4,2},{4,1},{6,2},{6,1},{5,2},{5,1},{3,2},{3,1}}
n_5	G164		{{4,3},{4,2},{4,1},{6,3},{6,1},{5,2},{5,1},{3,1},{2,1}}
n_6	G163	H^9_1	{{6,3},{6,2},{5,3},{5,1},{4,2},{4,1},{3,2},{3,1},{2,1}}
n_7	G166	H^9_7	{{4,2},{4,3},{4,1},{6,2},{6,1},{5,3},{5,1},{2,3},{2,1}}
n_8	G158	H^9_5	{{5,3},{5,2},{5,1},{4,3},{4,2},{4,1},{6,1},{3,1},{2,1}}
n_9	G157		{{4,3},{4,2},{4,1},{5,3},{5,2},{6,1},{3,2},{3,1},{2,1}}
n_{10}	G155		{{5,3},{5,2},{5,1},{4,3},{4,2},{4,1},{3,2},{3,1},{2,1}}
n_{11}	G159	H^9_6	{{5,2},{5,3},{5,1},{4,2},{4,3},{4,1},{6,1},{2,3},{2,1}}
n_{12}	G168		{{4,5},{4,2},{4,1},{3,5},{3,2},{3,1},{6,2},{6,1},{2,1}}
n_{13}	G173	H^9_4	{{5,3},{5,2},{5,1},{4,3},{4,2},{4,1},{6,2},{6,1},{3,1}}
n_{14}	G175	H^9_2	{{6,3},{6,2},{6,1},{5,3},{5,2},{5,1},{4,3},{4,2},{4,1}}
n_{15}	G165		{{4,3},{4,2},{4,1},{6,5},{6,1},{3,2},{3,1},{5,1},{2,1}}
n_{16}	G169		{{4,3},{4,2},{4,1},{6,5},{6,2},{3,2},{3,1},{5,1},{2,1}}
n_{17}	G167		{{4,6},{4,2},{4,1},{3,5},{3,2},{3,1},{6,2},{5,1},{2,1}}
n_{18}	G160		{{5,3},{5,2},{5,1},{4,6},{4,2},{4,1},{3,2},{3,1},{2,1}}
n_{19}	G172		{{5,3},{5,2},{5,1},{4,6},{4,2},{4,1},{3,2},{3,1},{6,1}}
n_{20}	G171		{{5,3},{5,2},{5,1},{4,2},{4,6},{4,1},{3,6},{3,1},{2,1}}
n_{21}	G174	H^9_3	{{6,4},{6,3},{6,2},{5,3},{5,2},{5,1},{4,2},{4,1},{3,1}}

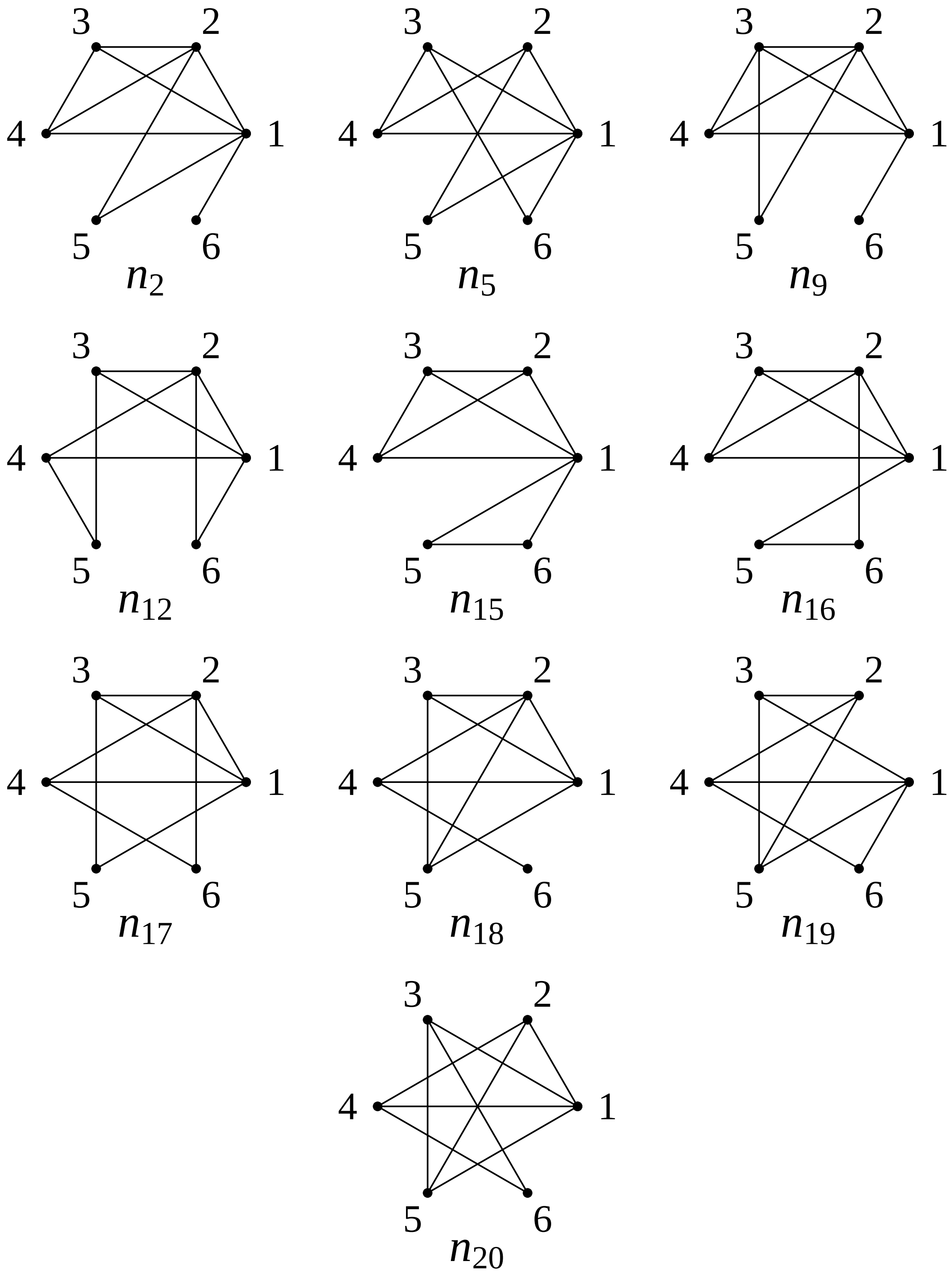
with six vertices and nine edges. The numbering in the first column corresponds to the ordering of the nine-edge graphs within the list of all 156 graphs of six vertices available at [14]. The second column identifies the graphs as they appear in *An Atlas of Graphs* by Read and Wilson [16]. In the third column we give the identities of the graphs as they appear in [4], where appropriate. The fourth column contains the edge sets, where the vertices have been labelled in non-increasing order of degree.

The design spectrum problem was solved for graph n_1 by Adams, Billington and Hoffman [1], for graph n_{14} ($K_{3,3}$) by Guy and Beineke [11], for graph n_6 by Mullin, Poplove and Zhu [15], and for graphs n_3, n_4, n_7, n_8, n_{11}, n_{13} and n_{21} by Kang, Zhao and Ma [12]. The necessary conditions (1) are sufficient except that there is no design of order 9 for n_1, n_3, n_4, n_8, n_{11}, n_{13}, and there is no design of order 10 for n_{14}. See also [4]. Graph n_{10} actually represents a K_5 with an edge removed plus an isolated vertex, and its spectrum is the same as that of its 5-vertex component ([6, 9, 13]). We now state our results.

Theorem 1.1 *Designs of order n exist for graphs n_2, n_5, n_9, n_{12}, n_{16}, n_{17}, n_{19} and n_{20} if and only if $n \equiv 0,\ 1 \pmod 9$ and $n \neq 9$.*

Theorem 1.2 *Designs of order n exist for graph n_{18} if and only if $n \equiv 0, 1 \pmod 9$ and $n \neq 9, 10$.*

Figure 1: Graphs with 6 vertices and 9 edges



Theorem 1.3 *Designs of order n exist for graph n_{15} if and only if $n \equiv 0, 1 \pmod 9$ and $n \neq 9, 10$.*

With these results, the design spectrum for graphs with six vertices and nine edges is completely solved.

Theorems 1.1 and 1.2 are proved in Section 4, and Theorem 1.3 in Section 5. For our computations and in the presentation of our results we represent the labelled graph n_i by a subscripted ordered 6-tuple $(z_1, z_2, \ldots, z_6)_i$, where $z_1 = 1$, $z_2 = 2$, $\ldots$, $z_6 = 6$ give the edge sets in Table 1 and the illustrations in Figure 1. For a graph G with 9 edges, the numbers of occurrences of G in a decomposition into G of the complete graph K_n, the complete r-partite graph K_{n^r} and the complete $(r+1)$-partite graph $K_{n^r m^1}$ are respectively

$$\frac{n(n-1)}{18}, \quad \frac{n^2 r(r-1)}{18} \quad \text{and} \quad \frac{nr(n(r-1)+2m)}{18}.$$

2 Non-existence results

Proposition 2.1 *A design of order 9 does not exist for graphs n_2, n_5, n_9, n_{12}, n_{15}, n_{16}, n_{17}, n_{18}, n_{19} and n_{20}.*

Proof These results are easily established by complete computer searches. However, it might be of interest to provide alternative proofs for some of the graphs. The complete graph K_9 is 8-regular and has 36 edges; so a design of order 9 consists of 4 graphs. In the following proofs we attempt to label the graphs of the design from the set $\{0, 1, \ldots, 8\}$ such that the edges of the four graphs partition the edges of a K_9 whose vertices are labelled with the same set.

For graphs n_5 and n_{15}, arrange the vertices so that they have degrees $(5, 3, 3, 3, 2, 2)$ in that order. Suppose there are a labels attached to vertices of degrees $\{5, 3\}$, b labels to vertices of degrees $\{3, 3, 2\}$ and c labels to vertices of degrees $\{2, 2, 2, 2\}$, exhausting all partitions of 8 into elements from $\{2, 3, 5\}$. Thus $a = 4$, $a + 2b = 12$, $b + 4c = 8$ and hence $b = 4$, $c = 1$. So, by symmetry and without loss of generality, we can label each graph $(*, *, *, *, *, 8)$, leaving labels $0, 1, \ldots, 7$ for the remaining vertices, which then form a decomposition of K_8 into a 5-vertex, 7-edge graph. The graph is identified in [2] as G_{19} in the case of n_5, or G_{16} in the case of n_{15}. But there is no G_{19} or G_{16} design of order 8, [2].

Consider the graphs n_{12}, n_{16} and n_{17}. Arrange the vertices of these graphs so that they have degrees $(3, 3, 4, 4, 2, 2)$ in that order. Suppose there are a labels attached to vertices of degrees $\{4, 4\}$, b labels to vertices of degrees $\{4, 2, 2\}$, c labels to vertices of degrees $\{3, 3, 2\}$ and d labels to vertices of degrees $\{2, 2, 2, 2\}$, accounting for all partitions of 8 into elements from $\{2, 3, 4\}$. Thus $2a + b = 2c = 2b + c + 4d = 8$ and hence $c = 4$. We assign labels to the degree 3 vertices from $\{0, 1, 2, 3\}$. For n_{16}, observe that the vertices of degree 3 are adjacent. So without loss of generality we label the vertices $(0, 1, *, *, *, *)$, $(0, 2, *, *, *, *)$, $(1, 3, *, *, *, *)$, $(2, 3, *, *, *, *)$. Now there is no way to create pair $\{0, 3\}$. For n_{12} and n_{17}, observe that the vertices of degree 3 are not adjacent, nor are the vertices of degree 2. So without loss of generality we label the vertices either $(0,1,*,*,*,*)$, $(0,1,*,*,*,*)$, $(2,3,*,*,*,*)$, $(2,3,*,*,*,*)$, or $(0,1,*,*,*,*)$, $(0,2,*,*,*,*)$, $(1,3,*,*,*,*)$, $(2,3,*,*,*,*)$. In each case it is impossible to create each pair from $\{0,1,2,3\}$ exactly once.

Graph n_{18} has vertex degrees $(4,4,3,3,3,1)$. Suppose there are a labels attached to vertices of degrees $\{4,4\}$, b labels attached to vertices of degrees $\{4,3,1\}$, and c labels attached to vertices of degrees $\{3,3,1,1\}$, accounting for all partitions of 8 into at most four elements from $\{1,3,4\}$. Considering vertices of degrees 1 and 3, we have $4 = b+2c = 12$, a contradiction.

Graphs n_{19} and n_{20} have vertex degrees $(4,3,3,3,3,2)$. Suppose there are a labels attached to vertices of degrees $\{4,4\}$, b labels attached to vertices of degrees $\{4,2,2\}$, c labels attached to vertices of degrees $\{3,3,2\}$ and d labels attached to vertices of degrees $\{2,2,2,2\}$. Considering vertices of degrees 2, 3 and 4, we obtain $2b+c+4d = 4$, $2c = 16$, $2a+b = 4$, which is impossible.

For the two remaining graphs, n_2 and n_9, we rely on the computer searches. □

Proposition 2.2 *A design of order* 10 *does not exist for graphs* n_{15} *and* n_{18}.

Proof Five copies of the graph are required. In the following proofs we attempt to label the graphs of the design from the set $\{0,1,\dots,9\}$ such that the edges of the five graphs partition the edges of a K_{10} whose vertices are labelled with the same set.

In n_{15} the vertices of degrees 5 and 2 form a triangle. Each of the five labels that must be attached to vertices of degree 5 must also be attached to two vertices of degree 2. So the triangles would have to form a decomposition of K_5, a triangle design of order 5, which does not exist.

The vertices of n_{18} have degrees $(4,4,3,3,3,1)$. Suppose there are a labels attached to vertices of degrees $\{4,4,1\}$, b labels attached to vertices of degrees $\{4,3,1,1\}$, c labels attached to vertices of degrees $\{3,3,3\}$, and d labels attached to vertices of degrees $\{3,3,1,1,1\}$. Thus $a+2b+3d = 5$, $b+3c+2d = 15$, $2a+b = 10$. Hence $a = c = 5$, $b = d = 0$. Without loss of generality we assume labels $0,1,2,3,4$ are attached to vertices of degrees $4,4,1$, and we label the five graphs $(0,1,*,*,*,*)$, $(0,2,*,*,*,*)$, $(1,3,*,*,*,*)$, $(2,4,*,*,*,*)$, $(3,4,*,*,*,*)$. However there is now no way to create pair $\{0,3\}$.

Alternatively, it is feasible to obtain these results by computer searches. □

Proposition 2.3 *If a decomposition of* $K_{a,b,b,b}$ *into graph* n_{15} *exists, then* $b^2 \equiv 2ab \pmod 6$ *and* $b/2 \le a \le 5b/4$.

Proof The number of n_{15} graphs in the decomposition is $b(a+b)/3$. A single n_{15} graph must span all four parts of a 4-partite graph, with one or two vertices of total degree 5 in each of three parts and a single vertex of degree 3 in the fourth part. Let P be the part with a vertices and suppose there are u copies of n_{15} with 5 edges incident with vertices in P and v copies of n_{15} with 3 edges incident with vertices in P. Then, since there are a vertices of degree $3b$ in P, $3ab = 5u + 3v$. Also

$b(a+b)/3 = u+v$. Solving gives $u = b(2a-b)/2$, $v = b(5b-4a)/6$ from which the asserted congruence and inequalities follow. □

A consequence of Proposition 2.3 is that there are no decompositions into n_{15} of the 4-partite graphs $K_{3,3,3,3}$, $K_{9,9,9,9}$ and $K_{6,6,6,9}$. The lack of these useful decompositions might explain why the spectrum problem for n_{15} seems to be rather more difficult than for any of the other graphs.

3 The Main Construction

We use Wilson's construction involving group divisible designs. Recall that a K-GDD of type $g_1^{t_1} \dots g_r^{t_r}$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where V is a base set of cardinality $v = t_1 g_1 + \dots + t_r g_r$, $\mathcal{G}$ is a partition of V into t_i subsets of cardinality g_i, $i = 1, \dots, r$, called *groups* and $\mathcal{B}$ is a collection of subsets of cardinalities $k \in K$, called *blocks*, which collectively have the property that each pair of elements from different groups occurs in precisely one block but no pair of elements from the same group occurs at all. A $\{k\}$-GDD is also called a k-GDD. As is well known, if there exist $k-2$ MOLS of side q, then there exists a k-GDD of type q^k. So when q is a prime power there exists a q-GDD of type q^q and a $(q+1)$-GDD of type q^{q+1} (obtained from affine and projective planes of order q respectively).

Proposition 3.1 *Suppose there exist G-designs of orders* 18, 19, 27, 28, 36 *and* 37. *Suppose also there exist decompositions into G of $K_{6,6,6,6}$ and $K_{6,6,6,9}$. Then there exist G-designs of orders $9t$ and $9t+1$, $t \geq 0$, except possibly* 9, 10, 45, 46, 54, 55, 63, 64, 108, 109, 117 *and* 118.

Proof There exist 4-GDDs of types 3^{4t} and 3^{4t+1} for $t \geq 1$, [3], as well as 4-GDDs of types $3^{4t}6^1$ for $t \geq 2$ and $3^{4t+1}6^1$ for $t \geq 1$, [17]; see also [5, Theorem 4.8.2].

Let $e = 0$ or 1. Inflate each point of the GDD by a factor of 6, thus expanding the blocks to complete 4-partite graphs $K_{6,6,6,6}$. If $e = 1$, add an extra point, ∞. Overlay the inflated groups, plus ∞ when $e = 1$, with K_{18+e} or K_{36+e} as appropriate. This gives designs of orders

$72t + e$	for $t \geq 1$	(using the 4-GDD of type 3^{4t}),
$72t + 18 + e$	for $t \geq 1$	(using the 4-GDD of type 3^{4t+1}),
$72t + 36 + e$	for $t \geq 2$	(using the 4-GDD of type $3^{4t}6^1$),
$72t + 54 + e$	for $t \geq 1$	(using the 4-GDD of type $3^{4t+1}6^1$),

representing orders $18t + e$, $t \geq 0$, except $18 + e$, $36 + e$, $54 + e$, $108 + e$.

For the remaining residue classes modulo 72, inflate the points in one group of size 3 by a factor of 9 and all other points by a factor of 6, thus expanding the blocks to complete 4-partite graphs $K_{6,6,6,6}$ and $K_{6,6,6,9}$. If $e = 1$, add an extra point, ∞. Overlay the inflated groups, plus ∞ when $e = 1$, with K_{18+e} or K_{27+e} or K_{36+e} as appropriate. This gives designs of orders

$72t + 9 + e$	for $t \geq 1$	(using the 4-GDD of type 3^{4t}),
$72t + 27 + e$	for $t \geq 1$	(using the 4-GDD of type 3^{4t+1}),
$72t + 45 + e$	for $t \geq 2$	(using the 4-GDD of type $3^{4t}6^1$),
$72t + 63 + e$	for $t \geq 1$	(using the 4-GDD of type $3^{4t+1}6^1$),

representing orders $18t + 9 + e$, $t \geq 0$, except $9 + e$, $27 + e$, $45 + e$, $63 + e$, $117 + e$. □

4 Theorems 1.1 and 1.2

Lemma 4.1 *Designs of order* 10 *exist for graphs* n_2, n_5, n_9, n_{12}, n_{16}, n_{17}, n_{19} *and* n_{20}.
Designs of orders 18, 19, 27, 28, 36, 37, 45, 46 *and* 63 *exist for each of graphs* n_2, n_5, n_9, n_{12}, n_{16}, n_{17}, n_{18}, n_{19} *and* n_{20}.
Designs of orders 54 *and* 55 *exist for each of graphs* n_2, n_9, n_{16} *and* n_{18}.
A design of order 64 *exists for graph* n_{18}.

Proof The decompositions are presented in Appendix A. □

Lemma 4.2 *There exist decompositions into* n_2, n_5, n_9, n_{12}, n_{16}, n_{17}, n_{18}, n_{19} *and* n_{20} *of the complete* 4*-partite graphs* $K_{6,6,6,6}$, $K_{9,9,9,9}$ *and* $K_{6,6,6,9}$.
There exist decompositions into n_5, n_{12}, n_{17}, n_{19} *and* n_{20} *of the complete* 3*-partite graph* $K_{6,6,6}$.
There exist decompositions into n_2, n_5, n_9, n_{16}, n_{17}, n_{19} *and* n_{20} *of the complete* 4*-partite graph* $K_{3,3,3,3}$.
There exists a decomposition into n_{12} *of the complete* 4*-partite graph* $K_{6,6,6,3}$.
There exist decompositions into n_{12} *and* n_{18} *of the complete* 6*-partite graph* $K_{18,18,18,18.18,27}$.

Proof The decompositions are presented in Appendix A. □

Proof of Theorems 1.1 and 1.2
The graphs under consideration consist of n_2, n_5, n_9, n_{12}, n_{16}, n_{17}, n_{18}, n_{19} and n_{20}. By Lemmas 4.1 and 4.2, there exist for each of these graphs designs of orders 18, 19, 27, 28, 36 and 37 as well as decompositions of $K_{6,6,6,6}$ and $K_{6,6,6,9}$. So by Propositions 2.1, 2.2 and 3.1 it suffices to construct designs of orders 10, 45, 46, 54, 55, 63, 64, 108, 109, 117 and 118, with the exception of order 10 for graph n_{18}. Those designs that are not provided directly by Lemma 4.1 are constructed as follows.

Orders 54 and 55 for graphs n_5, n_{12}, n_{17}, n_{19} and n_{20}. Inflate a 3-GDD of type 3^3 by a factor of 6 so that the blocks become $K_{6,6,6}$ graphs. For order 55 add an extra point, ∞. Overlay each group with K_{18}, or overlay each group plus ∞ with K_{19}. Since decompositions of K_{18}, K_{19} and $K_{6,6,6}$ exist by Lemmas 4.1 and 4.2, the construction yields designs of orders 54 and 55.

Order 64 for graphs n_2, n_5, n_9, n_{16}, n_{17}, n_{19} and n_{20}. There exists a 4-GDD of type $3^5 6^1$, [17]; see also [7] and [5, Table 4.10]. Inflate each point by a factor of 3 so that the blocks become $K_{3,3,3,3}$ graphs. Add an extra point, ∞. Overlay each group plus ∞ with K_{10} or K_{19}. Since decompositions of K_{10}, K_{19} and $K_{3,3,3,3}$ exist by Lemmas 4.1 and 4.2, the construction yields a design of order 64.

Order 64 for graph n_{12}. Take a 4-GDD of type 3^4. Inflate the points in one group by a factor of 3 and all other points by 6, so that the blocks become $K_{6,6,6,3}$ graphs. Add an extra point, ∞. Overlay each group plus ∞ with K_{10} or K_{19}. Since decompositions into n_{12} of K_{10}, K_{19} and $K_{6,6,6,3}$ exist by Lemmas 4.1 and 4.2, the construction yields a design of order 64.

Orders 108 and 109 for all nine graphs. Inflate a 4-GDD of type 3^4 by a factor of 9 so that the blocks become $K_{9,9,9,9}$ graphs. For order 109, add an extra point. Overlay the groups with K_{27} or K_{28}. Decompositions of K_{27}, K_{28} and $K_{9,9,9,9}$ exist by Lemmas 4.1 and 4.2.

Orders 117 and 118 for graphs n_2, n_5, n_9, n_{16}, n_{17}, n_{19} and n_{20}. Take a 4-GDD of type $6^5 9^1$, [10]; see also [5, Theorem 4.9.4]. Inflate the points by a factor of 3 so that the blocks become $K_{3,3,3,3}$ graphs. For order 118, add an extra point, ∞. For 117, overlay each group with K_{18} or K_{27}. For 118, overlay each group plus ∞ with K_{19} or K_{28}. Decompositions of K_{18}, K_{19}, K_{27}, K_{28} and $K_{3,3,3,3}$ exist by Lemmas 4.1 and 4.2.

Orders 117 and 118 for graphs n_{12} and n_{18}. These are constructed from the trivial 6-GDD of type 1^6, where the points of one group are inflated by 27 and all other points by 18 so that the blocks become 6-partite graphs $K_{18,18,18,18,18,27}$. Overlay the groups with K_{18} and K_{27} for order 117. Overlay the groups plus an extra point with K_{19} and K_{28} for order 118. Decompositions of K_{18}, K_{19}, K_{27}, K_{28} and $K_{18,18,18,18,18,27}$ exist by Lemmas 4.1 and 4.2. □

5 Theorem 1.3

Lemma 5.1 *Designs of orders* 18, 19, 27, 28, 36, 37, 45, 46, 54, 55, 63, 64, 81 *and* 82 *exist for graph* n_{15}.

Proof The decompositions are presented in Appendix A. □

Lemma 5.2 *There exist decompositions into* n_{15} *of the complete multipartite graphs* $K_{6,6,6,6}$, $K_{6,6,6,3}$, $K_{3,3,3,3,3}$, $K_{18,18,18,18,27}$, $K_{18,18,18,18,18,27}$ *and* $K_{3,3,3,3,3,3,3}$.

Proof The decompositions are presented in Appendix A. □

Proof of Theorem 1.3
There exist 4-GDDs of types 6^t for $t \geq 5$, $6^t 3^1$ for $t \geq 4$, $6^t 9^1$ for $t \geq 4$ and $6^t 15^1$ for $t \geq 6$, [10, 18]; see also [5, Theorem 4.9.4]. Inflate each point in the groups of sizes 9 and 15 by a factor of 3 and all other points by 6 thus expanding the blocks to complete 4-partite graphs $K_{6,6,6,6}$ and $K_{6,6,6,3}$ for which decompositions exist by Lemma 5.2.

Let $e = 0$ or 1. Take the inflated 4-GDDs of types 6^t, $6^t 3^1$, $6^t 9^1$ and $6^t 15^1$. Add an extra point, ∞, if $e = 1$. Overlay the groups, together with ∞ if $e = 1$, with K_{18+e} K_{27+e}, K_{36+e} or K_{45+e} as appropriate, noting that these decompositions are available by Lemma 5.1. This construction gives designs of orders

$36t + e$	for $t \geq 5$	(using the 4-GDD of type 6^t),
$36t + 45 + e$	for $t \geq 6$	(using the 4-GDD of type $6^t 15^1$),
$36t + 18 + e$	for $t \geq 4$	(using the 4-GDD of type $6^t 3^1$),
$36t + 27 + e$	for $t \geq 4$	(using the 4-GDD of type $6^t 9^1$),

representing orders $9t + e$ for $e = 0$, 1 and $t \geq 0$ except {36, 72, 108, 144}, {9, 45, 81, 117, 153, 189, 225}, {18, 54, 90, 126}, {27, 63, 99, 135}, {37, 73, 109, 145}, {10, 46, 82, 118, 154, 190, 226}, {19, 55, 91, 127} and {28, 64, 100, 136} in residue classes 0, 9, 18, 27, 1, 10, 19 and 28 modulo 36 respectively. The missing values are handled as follows.

Orders 9 and 10 are excluded by Proposition 2.1, and orders 18, 19, 27, 28, 36, 37, 45, 46, 54, 55, 63, 64, 81 and 82 are given by Lemma 5.1.

Below, we give only brief details by merely specifying the ingredients for Wilson's construction, namely the complete graphs, the complete multipartite graphs, the group divisible designs and, unless it is clear, how the points of the GDDs are inflated. The decompositions of the graphs into n_{15} exist by Lemmas 5.1 and 5.2.

Order 72 is constructed from K_{18}, $K_{6,6,6,6}$ and a 4-GDD of type 3^4.

Order 108 is constructed from K_{18}, $K_{6,6,6,6}$, $K_{6,6,6,3}$ and a 4-GDD of type $3^5 6^1$. Inflate the points in the group of size 6 by 3, all others by 6.

Order 144 is constructed from K_{18}, $K_{6,6,6,6}$ and a 4-GDD of type 3^8.

Order 117 is constructed from K_{18}, K_{27}, $K_{18,18,18,18,18,27}$ and a 6-GDD of type 1^6.

Order 153 is constructed from K_{18}, K_{27}, $K_{6,6,6,6}$, $K_{6,6,6,3}$ and a 4-GDD of type $3^7 9^1$. Inflate the points in the group of size 9 by 3, all others by 6.

Order 189 is constructed from K_{27}, $K_{3,3,3,3,3,3,3}$ and a 7-GDD of type 9^7 created from an affine plane of order 9 by removing two groups.

Order 225 is constructed from K_{45}, $K_{3,3,3,3,3}$ and a 5-GDD of type 15^5 ([8]; see also [5, Theorem 4.16]).

Order 90 is constructed from K_{18}, $K_{6,6,6,6}$ and a 4-GDD of type 3^5.

Order 126 is constructed from K_{18}, K_{36}, $K_{6,6,6,6}$ and a 4-GDD of type $3^5 6^1$.

Order 99 is constructed from K_{18}, K_{27}, $K_{18,18,18,18,27}$ and a 5-GDD of type 1^5.

Order 135 is constructed from K_{27}, $K_{3,3,3,3,3}$ and a 5-GDD of type 9^5 created from an affine plane of order 9 by removing four groups.

Order n, $n = 73$, 109, 145, 118, 154, 190, 226, 91, 127, 100, 136, is constructed in a similar manner to order $n - 1$. In each case we add an extra point and use the appropriate decompositions of K_{9t+1}. □

6 Concluding Remarks

We wish to thank a referee for alerting us to the relatively recent paper of Wei and Ge, [18], a result of which asserting the existence of a 4-GDD of type $6^7 15^1$ allowed a small improvement to our proof of Theorem 1.3.

With four exceptions all decompositions in Appendix A were obtained by a special computer program written in the C language. The designs where existence could not be decided by this program are of orders 18, 54, 64 and 81 for graph n_{15}. In these cases we had to adopt alternative methods.

We are of the opinion that the existence of an n_{15} design of order 18 is surprising. It was obtained from the partial Steiner triple system of order 18 with 17 blocks,

$$\begin{aligned}\mathcal{B} \;=\; &\{\{0,1,2\},\{0,3,4\},\{1,3,5\},\{2,3,6\},\{1,4,7\},\{2,4,8\},\\ &\{2,5,9\},\{4,5,10\},\{6,8,10\},\{7,8,9\},\{0,9,10\},\{11,7,10\},\\ &\{12,6,9\},\{13,3,7\},\{14,1,6\},\{15,0,8\},\{16,17,5\}\}.\end{aligned}$$

The leave of $\mathcal{B}$, a graph with 18 vertices and 102 edges, admits a decomposition into 17 tetrahedra:

$$\begin{aligned}\mathcal{D} \;=\; &\{\{5,6,7,15\},\{2,7,12,16\},\{0,7,14,17\},\{0,6,13,16\},\\ &\{0,5,11,12\},\{5,8,13,14\},\{4,6,11,17\},\{2,11,14,15\},\\ &\{2,10,13,17\},\{4,9,14,16\},\{1,9,11,13\},\{3,9,15,17\},\\ &\{4,12,13,15\},\{3,8,11,16\},\{1,8,12,17\},\{3,10,12,14\},\{1,10,15,16\}\}.\end{aligned}$$

We conjecture that $\mathcal{B}$ is up to isomorphism the only PSTS(18) with this property. To construct an n_{15} design of order 18 it suffices to pair off the triples in $\mathcal{B}$ with the quadruples in $\mathcal{D}$ that such that each pair $\{B,D\}$, $B \in \mathcal{B}$, $D \in \mathcal{D}$, has non-empty intersection.

In a similar manner we obtained a decomposition of K_{64} into n_{15} starting with a suitable PSTS(64) with 224 blocks, and of K_{81} starting from a PSTS(81) with 360 blocks. In each case we were able to exploit a non-trivial automorphism.

For the decomposition of K_{54}, we start with a partial Steiner system of order 54, PS(2, 4, 54), with 153 blocks and where no point has even degree. Denote the block set of this system by $\mathcal{S}$. Then $\mathcal{S}$ is obtained by expanding

$$\begin{aligned}&\{\{16,38,1,32\},\{49,34,9,1\},\{45,30,53,47\},\{13,0,32,42\},\{39,46,32,6\},\\ &\{18,42,46,49\},\{27,3,6,1\},\{24,43,32,44\},\{25,3,46,37\},\{52,11,28,26\},\\ &\{27,15,32,2\},\{16,34,51,3\},\{10,29,37,7\},\{2,3,17,29\},\{26,30,15,19\},\\ &\{48,5,16,0\},\{37,47,27,11\}\}\end{aligned}$$

to 153 blocks by the mapping $x \mapsto x+6 \pmod{54}$. We extend $\mathcal{S}$ by six blocks to $\mathcal{S}^* = \mathcal{S} \cup \mathcal{C}$, where $\mathcal{C}$ is the configuration,

$$\begin{aligned}\mathcal{C} \;=\; &\{\{2,7,23,20\},\{24,23,25,42\},\{20,25,41,38\},\\ &\{42,41,43,6\},\{38,43,5,2\},\{6,5,7,24\}\}.\end{aligned}$$

The points of $\mathcal{C}$ are chosen so that no pair of $\mathcal{C}$ is present in $\mathcal{S}$, and it is clear that $\mathcal{S}^*$ also has no points of even degree. The leave of $\mathcal{S}^*$ admits a decomposition into 159 triangles and the n_{15} design of order 54 is constructed from an appropriate complete matching of the blocks of $\mathcal{S}^*$ with these triangles.

Finally, observe that we have also solved the design spectrum problem for $K_4 \cup K_3$, the disjoint union of a tetrahedron and a triangle. The spectrum is the same as that of n_{15}. As alternatives to complete computer searches, the proofs for n_{15} in Propositions 2.1 and 2.2 are easily adapted for $K_4 \cup K_3$ to prove that designs of orders 9 and 10 do not exist. For order 9, denote by A and B the sets of labels attached to vertices of degrees $\{3, 3, 2\}$ and $\{2, 2, 2, 2\}$ respectively. Then $|A| = 8$ and $|B| = 1$. By removing the B label and all eight AB edges, we find that we require a $K_4 \cup K_2$ design of order 8, which does not exist, [2]. For order 10, denote by A and B the sets of labels that appear on vertices of degrees $\{3, 3, 3\}$ and $\{3, 2, 2, 2\}$ respectively. Then $|A| = |B| = 5$ and hence the B labels would need to form a triangle design of order 5, which does not exist. For the rest of the spectrum, the proof follows that of Theorem 1.3, and one way of obtaining the required decompositions into $K_4 \cup K_3$ from corresponding decompositions into n_{15} is as follows: (i) if necessary, obtain the full set of graphs by expanding the orbits, (ii) disassemble each graph into a tetrahedron and a triangle, (iii) find a complete matching of pairs of disjoint tetrahedra and triangles, and (iv) assemble the pairs to form graphs $K_4 \cup K_3$.

A The Decompositions

Proof of Lemma 4.1

$\boldsymbol{K_{10}}$ Let the vertex set be Z_{10}. The decompositions consist of
$(0, 1, 2, 7, 3, 4)_2$, $(0, 1, 3, 5, 2, 6)_5$, $(0, 1, 2, 5, 9, 4)_9$, $(0, 1, 2, 3, 6, 5)_{12}$,
$(0, 1, 2, 5, 6, 3)_{16}$, $(0, 1, 3, 2, 6, 7)_{17}$, $(0, 1, 2, 9, 7, 6)_{20}$
under the action of the mapping $x \mapsto x + 2 \pmod{10}$, and
$(0, 1, 2, 3, 4, 5)_{19}$, $(8, 2, 5, 6, 7, 0)_{19}$, $(3, 9, 4, 2, 7, 8)_{19}$, $(5, 8, 1, 4, 9, 6)_{19}$,
$(6, 0, 1, 9, 7, 3)_{19}$.

$\boldsymbol{K_{18}}$ Let the vertex set be $Z_{17} \cup \{\infty\}$. The decompositions consist of
$(0, 1, 3, 11, 5, \infty)_2$, $(0, 1, 3, 8, 7, \infty)_9$, $(0, 1, 3, 5, 11, \infty)_{18}$
under the action of the mapping $x \mapsto x + 1 \pmod{17}$, $\infty \mapsto \infty$. For the other six graphs, with vertex set Z_{18} the decompositions consist of
$(14, 12, 1, 16, 2, 4)_5$, $(2, 0, 7, 4, 8, 10)_5$, $(8, 6, 13, 10, 14, 16)_5$,
$(16, 2, 6, 11, 9, 7)_5$, $(4, 8, 12, 17, 15, 13)_5$, $(10, 14, 0, 5, 3, 1)_5$,
$(10, 15, 11, 12, 9, 4)_5$, $(16, 3, 17, 0, 15, 10)_5$, $(4, 9, 5, 6, 3, 16)_5$,
$(0, 12, 11, 7, 9, 14)_5$, $(6, 0, 17, 13, 15, 2)_5$, $(12, 6, 5, 1, 3, 8)_5$,
$(8, 3, 1, 7, 11, 9)_5$, $(3, 1, 5, 2, 17, 13)_5$, $(5, 7, 11, 17, 9, 15)_5$,
$(13, 11, 15, 1, 9, 2)_5$, $(14, 7, 17, 15, 13, 9)_5$,
$(13, 6, 10, 14, 16, 15)_{12}$, $(1, 12, 16, 2, 4, 3)_{12}$, $(7, 0, 4, 8, 10, 9)_{12}$,
$(15, 0, 14, 3, 8, 10)_{12}$, $(3, 6, 2, 9, 14, 16)_{12}$, $(9, 12, 8, 15, 2, 4)_{12}$,
$(9, 16, 2, 5, 13, 11)_{12}$, $(15, 4, 8, 11, 1, 17)_{12}$, $(3, 10, 14, 17, 7, 5)_{12}$,
$(3, 13, 7, 11, 6, 4)_{12}$, $(9, 1, 13, 17, 12, 10)_{12}$, $(15, 7, 1, 5, 0, 16)_{12}$,
$(4, 1, 5, 6, 12, 14)_{12}$, $(5, 17, 2, 8, 11, 6)_{12}$, $(7, 10, 2, 11, 0, 12)_{12}$,
$(11, 14, 12, 17, 0, 5)_{12}$, $(13, 16, 0, 8, 6, 17)_{12}$,
$(5, 12, 7, 6, 1, 2)_{16}$, $(11, 0, 13, 12, 7, 8)_{16}$, $(17, 6, 1, 0, 13, 14)_{16}$,
$(7, 13, 9, 2, 10, 6)_{16}$, $(13, 1, 15, 8, 16, 12)_{16}$, $(1, 7, 3, 14, 4, 0)_{16}$,

$(10,13,3,5,9,4)_{16}$, $(16,1,9,11,15,10)_{16}$, $(4,7,15,17,3,16)_{16}$,
$(9,12,3,8,5,15)_{16}$, $(15,0,9,14,11,3)_{16}$, $(3,6,15,2,17,9)_{16}$,
$(10,0,16,2,8,5)_{16}$, $(2,11,8,14,5,4)_{16}$, $(4,12,10,14,2,17)_{16}$,
$(5,17,14,16,11,10)_{16}$, $(6,8,4,16,11,17)_{16}$,
$(4,9,10,11,14,2)_{17}$, $(10,15,16,17,2,8)_{17}$, $(16,3,4,5,8,14)_{17}$,
$(3,7,12,9,2,6)_{17}$, $(9,13,0,15,8,12)_{17}$, $(15,1,6,3,14,0)_{17}$,
$(3,10,6,8,13,7)_{17}$, $(9,16,12,14,1,13)_{17}$, $(15,4,0,2,7,1)_{17}$,
$(2,14,0,7,5,11)_{17}$, $(8,2,6,13,11,17)_{17}$, $(14,8,12,1,17,5)_{17}$,
$(5,12,11,4,15,6)_{17}$, $(6,17,0,5,16,9)_{17}$, $(7,13,4,5,17,10)_{17}$,
$(10,11,0,1,12,13)_{17}$, $(16,17,1,11,7,3)_{17}$,
$(10,9,16,12,11,7)_{19}$, $(16,15,4,0,17,13)_{19}$, $(4,3,10,6,5,1)_{19}$,
$(10,4,9,2,13,0)_{19}$, $(16,10,15,8,1,6)_{19}$, $(4,16,3,14,7,12)_{19}$,
$(9,6,5,0,7,8)_{19}$, $(15,12,11,6,13,14)_{19}$, $(3,0,17,12,1,2)_{19}$,
$(3,7,15,0,8,11)_{19}$, $(9,13,3,6,14,17)_{19}$, $(15,1,9,12,2,5)_{19}$,
$(11,0,5,4,14,8)_{19}$, $(1,17,5,7,13,11)_{19}$, $(8,16,2,12,5,17)_{19}$,
$(14,13,2,8,7,1)_{19}$, $(17,6,2,10,11,14)_{19}$,
$(2,14,10,15,6,8)_{20}$, $(8,2,16,3,12,14)_{20}$, $(14,8,4,9,0,2)_{20}$,
$(7,0,3,9,10,5)_{20}$, $(13,6,9,15,16,11)_{20}$, $(1,12,15,3,4,17)_{20}$,
$(10,4,15,13,5,3)_{20}$, $(16,10,3,1,11,9)_{20}$, $(4,16,9,7,17,15)_{20}$,
$(4,8,3,11,6,0)_{20}$, $(10,14,9,17,12,6)_{20}$, $(16,2,15,5,0,12)_{20}$,
$(11,2,5,1,7,8)_{20}$, $(0,5,12,6,13,11)_{20}$, $(1,5,13,17,14,2)_{20}$,
$(7,6,17,1,12,0)_{20}$, $(13,8,11,7,17,14)_{20}$.

$\boldsymbol{K_{19}}$ Let the vertex set be Z_{19}. The decompositions consist of
$(0,1,3,7,9,5)_2$, $(0,1,2,5,7,10)_5$, $(0,1,3,7,11,5)_9$,
$(0,1,3,5,13,7)_{12}$, $(0,1,3,8,4,10)_{16}$, $(0,1,3,5,9,12)_{17}$,
$(0,1,3,5,9,12)_{18}$, $(0,1,2,4,8,9)_{19}$, $(0,1,2,4,7,12)_{20}$
under the action of the mapping $x \mapsto x+1 \pmod{19}$.

$\boldsymbol{K_{27}}$ Let the vertex set be $Z_{26} \cup \{\infty\}$. The decompositions consist of
$(0,1,2,5,4,6)_2$, $(0,7,13,18,16,12)_2$, $(1,8,17,19,\infty,13)_2$,
$(0,1,2,5,4,8)_5$, $(0,7,9,14,13,17)_5$, $(1,12,13,2,\infty,3)_5$,
$(0,1,2,5,6,7)_9$, $(0,6,15,18,\infty,10)_9$, $(0,11,13,19,1,17)_9$,
$(0,1,2,3,6,5)_{12}$, $(0,6,13,14,4,21)_{12}$, $(0,11,17,19,3,\infty)_{12}$,
$(11,21,22,17,\infty,2)_{16}$, $(0,1,3,16,2,9)_{16}$, $(0,5,14,22,20,17)_{16}$,
$(0,18,2,9,21,\infty)_{17}$, $(0,1,3,4,14,11)_{17}$, $(0,25,5,13,20,17)_{17}$,
$(0,1,2,3,6,7)_{19}$, $(0,1,8,12,17,23)_{19}$, $(1,6,7,22,19,\infty)_{19}$,
$(20,12,4,19,21,\infty)_{20}$, $(0,2,3,6,13,21)_{20}$, $(1,3,22,7,8,19)_{20}$
under the action of the mapping $x \mapsto x+2 \pmod{26}$, $\infty \mapsto \infty$. For the remaining graph, the lack of a vertex of degree 2 makes the previous method impossible. With vertex set Z_{27} the decomposition consists of
$(3,2,19,11,7,13)_{18}$, $(6,5,22,14,10,16)_{18}$, $(9,8,25,17,13,19)_{18}$,
$(2,23,18,0,16,14)_{18}$, $(5,26,21,3,19,17)_{18}$, $(8,2,24,6,22,20)_{18}$,
$(17,1,20,5,0,4)_{18}$, $(20,4,23,8,3,7)_{18}$, $(23,7,26,11,6,10)_{18}$,
$(0,22,3,12,9,19)_{18}$, $(3,25,6,15,12,22)_{18}$, $(0,24,10,6,19,18)_{18}$,
$(1,25,4,7,22,0)_{18}$

under the action of the mapping $x \mapsto x + 9 \pmod{27}$.

$\boldsymbol{K_{28}}$ Let the vertex set be Z_{28}. The decompositions consist of

$(0, 23, 3, 24, 1, 19)_2$, $(14, 11, 18, 13, 24, 1)_2$, $(11, 10, 0, 22, 2, 6)_2$,
$(19, 1, 7, 5, 9, 23)_2$, $(1, 10, 8, 24, 13, 4)_2$, $(2, 24, 4, 13, 9, 15)_2$,
$(0, 9, 5, 13, 22, 3)_5$, $(25, 0, 11, 15, 4, 26)_5$, $(2, 25, 7, 18, 13, 12)_5$,
$(9, 3, 18, 15, 11, 19)_5$, $(12, 0, 26, 1, 11, 19)_5$, $(0, 6, 10, 2, 8, 19)_5$,
$(0, 6, 5, 14, 1, 2)_9$, $(4, 17, 25, 15, 3, 23)_9$, $(22, 0, 9, 3, 26, 12)_9$,
$(8, 24, 4, 3, 21, 9)_9$, $(9, 6, 19, 2, 18, 21)_9$, $(3, 7, 10, 15, 20, 1)_9$,
$(0, 2, 15, 22, 26, 17)_{12}$, $(17, 21, 10, 19, 24, 8)_{12}$, $(21, 16, 13, 20, 3, 26)_{12}$,
$(26, 7, 1, 3, 10, 25)_{12}$, $(2, 24, 14, 27, 15, 4)_{12}$, $(0, 7, 21, 27, 5, 16)_{12}$,
$(0, 8, 7, 21, 22, 4)_{16}$, $(0, 12, 10, 17, 11, 21)_{16}$, $(23, 22, 3, 8, 10, 14)_{16}$,
$(19, 0, 2, 25, 7, 3)_{16}$, $(7, 14, 5, 2, 1, 25)_{16}$, $(1, 14, 11, 13, 9, 8)_{16}$,
$(0, 15, 13, 9, 14, 7)_{17}$, $(24, 3, 2, 20, 4, 15)_{17}$, $(12, 0, 1, 5, 15, 2)_{17}$,
$(19, 1, 6, 9, 22, 25)_{17}$, $(11, 6, 15, 12, 22, 2)_{17}$, $(1, 14, 4, 22, 23, 5)_{17}$,
$(0, 19, 3, 23, 5, 12)_{18}$, $(19, 13, 6, 25, 18, 0)_{18}$, $(21, 25, 10, 4, 12, 11)_{18}$,
$(24, 4, 14, 8, 10, 23)_{18}$, $(2, 7, 10, 5, 8, 4)_{18}$, $(2, 11, 1, 19, 21, 26)_{18}$,
$(0, 8, 12, 6, 17, 25)_{19}$, $(1, 0, 22, 18, 8, 19)_{19}$, $(15, 24, 11, 7, 9, 10)_{19}$,
$(2, 5, 15, 10, 17, 14)_{19}$, $(17, 12, 13, 3, 14, 9)_{19}$, $(2, 16, 11, 19, 23, 20)_{19}$,
$(0, 1, 12, 18, 5, 11)_{20}$, $(16, 11, 8, 19, 14, 17)_{20}$, $(19, 2, 5, 7, 15, 6)_{20}$,
$(25, 22, 15, 3, 17, 24)_{20}$, $(4, 15, 14, 0, 21, 2)_{20}$, $(0, 13, 14, 25, 22, 10)_{20}$,

under the action of the mapping $x \mapsto x + 4 \pmod{28}$.

$\boldsymbol{K_{36}}$ Let the vertex set be $Z_{35} \cup \{\infty\}$. The decompositions consist of

$(0, 1, 3, 7, 11, 16)_2$, $(0, 8, 17, 22, 20, \infty)_2$,
$(0, 1, 3, 7, 12, 13)_9$, $(0, 10, 15, 27, 29, \infty)_9$,
$(0, 1, 3, 5, 11, 19)_{18}$, $(0, 6, 15, 18, 22, \infty)_{18}$

under the action of the mapping $x \mapsto x + 1 \pmod{35}$, $\infty \mapsto \infty$, and

$(\infty, 1, 4, 20, 27, 8)_5$, $(4, 24, 31, 25, 12, 3)_5$, $(21, 7, 22, 3, 2, 10)_5$,
$(24, 29, 18, 7, 3, 30)_5$, $(9, 20, 28, 0, 7, 26)_5$, $(2, 34, 30, 27, 6, 8)_5$,
$(30, 0, 34, 31, 33, 21)_5$, $(33, 18, 32, 21, 28, 12)_5$, $(0, 8, 27, 25, 17, 21)_5$,
$(1, 31, 34, 23, 11, 24)_5$,
$(0, 26, \infty, 18, 3, 23)_{12}$, $(10, 22, 12, 4, 23, 25)_{12}$, $(23, 30, 25, 20, 6, 9)_{12}$,
$(28, 11, 21, 15, 22, 29)_{12}$, $(8, 14, 3, 12, 1, 7)_{12}$, $(0, 8, 17, 34, 12, 4)_{12}$,
$(3, 16, 28, 22, 7, 32)_{12}$, $(15, 21, 24, 26, 4, 7)_{12}$, $(1, 9, 32, 34, 10, 16)_{12}$,
$(2, 34, 11, 29, 10, \infty)_{12}$,
$(9, 22, 18, 28, \infty, 33)_{16}$, $(31, 5, \infty, 7, 32, 1)_{16}$, $(7, 9, 26, 17, 19, 33)_{16}$,
$(7, 2, 15, 30, 1, 23)_{16}$, $(33, 17, 32, 0, 26, 14)_{16}$, $(21, 23, 0, 6, 19, 5)_{16}$,
$(2, 9, 16, 19, 28, 29)_{16}$, $(28, 16, 8, 15, 34, 21)_{16}$, $(0, 9, 4, 10, 8, 13)_{16}$,
$(0, 16, 5, 24, 3, 6)_{16}$,
$(31, 18, \infty, 33, 19, 14)_{17}$, $(31, 25, 14, 2, 34, 19)_{17}$, $(34, 32, 1, 2, 22, 3)_{17}$,
$(17, 7, 26, 0, 16, 14)_{17}$, $(1, 31, 28, 0, 15, 11)_{17}$, $(18, 10, 1, 0, 9, 30)_{17}$,
$(15, 17, 13, 3, 18, 9)_{17}$, $(18, 28, 4, 2, 11, 17)_{17}$, $(0, 9, 34, 22, 4, 25)_{17}$,
$(0, 16, 27, 28, \infty, 29)_{17}$,
$(23, 28, 7, 21, 20, \infty)_{19}$, $(31, \infty, 9, 17, 25, 16)_{19}$, $(12, 23, 22, 19, 27, 24)_{19}$,
$(28, 23, 3, 9, 5, 0)_{19}$, $(3, 20, 6, 32, 25, 16)_{19}$, $(9, 4, 15, 29, 5, 6)_{19}$,

$(0,9,11,17,7,15)_{19}$, $(19,34,26,8,2,32)_{19}$, $(1,2,11,10,28,7)_{19}$,
$(4,26,21,3,25,33)_{19}$,
$(17,26,8,0,16,\infty)_{20}$, $(16,\infty,24,27,4,11)_{20}$, $(10,14,5,9,24,27)_{20}$,
$(17,21,32,14,4,33)_{20}$, $(31,17,1,10,13,0)_{20}$, $(1,21,30,5,23,32)_{20}$,
$(26,33,32,15,24,9)_{20}$, $(3,33,32,4,22,2)_{20}$, $(0,8,32,33,20,18)_{20}$,
$(0,13,3,9,24,6)_{20}$
under the action of the mapping $x \mapsto x+5 \pmod{35}$, $\infty \mapsto \infty$.

$\boldsymbol{K_{37}}$ Let the vertex set be Z_{37}. The decompositions consist of
$(0,1,3,7,10,12)_2$, $(0,5,16,24,20,14)_2$,
$(0,1,2,5,7,13)_5$, $(0,8,9,25,22,19)_5$,
$(0,1,3,7,11,13)_9$, $(0,9,14,25,29,18)_9$,
$(0,1,3,5,11,10)_{12}$, $(0,7,18,21,6,20)_{12}$,
$(0,1,3,7,8,18)_{16}$, $(0,9,14,25,13,28)_{16}$,
$(0,1,3,5,9,16)_{17}$, $(0,7,23,19,10,27)_{17}$,
$(0,1,3,5,10,18)_{18}$, $(0,6,14,21,25,1)_{18}$,
$(0,1,2,4,7,13)_{19}$, $(0,2,14,21,22,10)_{19}$,
$(0,1,2,4,7,13)_{20}$, $(0,8,14,20,27,35)_{20}$
under the action of the mapping $x \mapsto x+1 \pmod{37}$.

$\boldsymbol{K_{45}}$ Let the vertex set be $Z_{44} \cup \{\infty\}$. The decompositions consist of
$(0,\infty,9,18,39,17)_2$, $(7,29,16,2,19,22)_2$, $(9,38,4,12,11,2)_2$,
$(5,17,3,31,1,6)_2$, $(37,17,6,14,16,31)_2$, $(13,32,21,31,20,19)_2$,
$(10,0,19,4,29,38)_2$, $(42,20,40,43,18,37)_2$, $(40,3,7,10,27,12)_2$,
$(2,35,14,27,8,6)_2$,
$(11,16,14,7,\infty,4)_5$, $(22,28,20,16,21,5)_5$, $(42,24,37,16,2,18)_5$,
$(31,32,16,41,8,43)_5$, $(21,16,1,17,3,13)_5$, $(36,20,26,33,39,34)_5$,
$(8,25,19,38,22,41)_5$, $(13,43,15,42,19,23)_5$, $(13,11,2,34,27,\infty)_5$,
$(2,18,37,31,7,11)_5$,
$(\infty,41,42,15,37,24)_9$, $(10,31,39,11,12,22)_9$, $(37,9,19,22,15,36)_9$,
$(15,8,20,1,29,36)_9$, $(18,7,9,42,40,22)_9$, $(19,6,31,16,32,25)_9$,
$(43,21,13,38,1,41)_9$, $(1,4,28,12,30,40)_9$, $(29,0,22,6,30,19)_9$,
$(2,4,8,39,18,5)_9$,
$(\infty,33,16,23,18,34)_{12}$, $(33,22,25,13,7,5)_{12}$, $(18,33,24,7,43,3)_{12}$,
$(8,6,14,26,35,19)_{12}$, $(34,2,30,9,8,20)_{12}$, $(10,20,7,35,11,41)_{12}$,
$(9,20,15,30,31,40)_{12}$, $(13,16,43,9,41,8)_{12}$, $(8,43,31,36,21,40)_{12}$,
$(33,38,8,11,12,31)_{12}$,
$(\infty,6,29,43,20,14)_{16}$, $(37,17,21,43,25,19)_{16}$, $(29,20,39,18,15,30)_{16}$,
$(32,37,2,16,8,30)_{16}$, $(2,5,3,34,19,4)_{16}$, $(18,37,27,0,34,8)_{16}$,
$(5,36,23,24,32,28)_{16}$, $(26,13,24,30,6,5)_{16}$, $(2,7,12,35,41,0)_{16}$,
$(3,15,12,23,22,19)_{16}$,
$(\infty,10,19,36,29,30)_{17}$, $(27,17,35,6,15,2)_{17}$, $(24,29,10,33,37,21)_{17}$,
$(13,22,14,6,37,7)_{17}$, $(7,40,37,23,20,32)_{17}$, $(4,24,8,38,37,27)_{17}$,
$(8,6,23,18,10,11)_{17}$, $(42,20,39,1,16,8)_{17}$, $(1,6,0,31,7,37)_{17}$,
$(3,7,5,8,21,29)_{17}$,
$(\infty,37,7,2,4,41)_{18}$, $(21,25,6,37,31,40)_{18}$, $(21,1,39,23,28,34)_{18}$,

$(36, 7, 31, 10, 22, 30)_{18}$, $(32, 31, 15, 14, 10, 12)_{18}$, $(11, 19, 7, 29, 32, 4)_{18}$,
$(17, 34, 6, 3, 38, 5)_{18}$, $(7, 16, 14, 0, 6, 32)_{18}$, $(24, 37, 28, 32, 34, 17)_{18}$,
$(1, 38, 0, 37, 24, 6)_{18}$,
$(\infty, 23, 18, 43, 33, 12)_{19}$, $(0, 40, 9, 18, 34, 3)_{19}$, $(43, 20, 31, 5, 25, 41)_{19}$,
$(15, 35, 7, 26, 38, 40)_{19}$, $(2, 22, 6, 39, 1, 9)_{19}$, $(3, 37, 2, 4, 40, 43)_{19}$,
$(37, 25, 41, 5, 7, 12)_{19}$, $(43, 34, 21, 1, 20, 28)_{19}$, $(8, 31, 6, 40, 4, 16)_{19}$,
$(2, 41, 14, 20, 38, 36)_{19}$,
$(\infty, 32, 11, 5, 18, 28)_{20}$, $(0, 14, 24, 25, 37, 13)_{20}$, $(12, 19, 35, 17, 31, 18)_{20}$,
$(20, 11, 31, 30, 22, 33)_{20}$, $(27, 11, 28, 41, 40, 31)_{20}$, $(3, 39, 8, 42, 26, 36)_{20}$,
$(0, 42, 4, 8, 26, 5)_{20}$, $(22, 10, 7, 14, 41, 29)_{20}$, $(35, 11, 34, 17, 29, 10)_{20}$,
$(1, 10, 21, 16, 37, 25)_{20}$

under the action of the mapping $x \mapsto x + 4 \pmod{44}$, $\infty \mapsto \infty$.

$\boldsymbol{K_{46}}$ Let the vertex set be Z_{46}. The decompositions consist of

$(0, 22, 19, 23, 36, 39)_2$, $(33, 7, 39, 41, 0, 38)_2$, $(19, 1, 38, 34, 44, 3)_2$,
$(42, 7, 8, 24, 36, 4)_2$, $(0, 15, 5, 26, 37, 2)_2$,
$(0, 35, 24, 8, 31, 3)_5$, $(42, 17, 41, 3, 11, 8)_5$, $(1, 37, 17, 35, 28, 6)_5$,
$(15, 44, 38, 2, 37, 35)_5$, $(0, 14, 18, 20, 5, 1)_5$,
$(0, 17, 40, 28, 38, 20)_9$, $(30, 34, 35, 33, 21, 38)_9$, $(9, 19, 15, 24, 41, 40)_9$,
$(3, 31, 22, 19, 38, 36)_9$, $(0, 11, 36, 19, 4, 22)_9$,
$(0, 3, 44, 35, 18, 1)_{12}$, $(42, 25, 41, 18, 31, 32)_{12}$, $(12, 43, 0, 37, 9, 23)_{12}$,
$(13, 32, 26, 40, 8, 36)_{12}$, $(2, 39, 43, 17, 5, 18)_{12}$,
$(0, 16, 37, 33, 25, 44)_{16}$, $(16, 36, 31, 38, 19, 35)_{16}$, $(23, 25, 31, 43, 16, 39)_{16}$,
$(37, 28, 40, 36, 24, 38)_{16}$, $(1, 25, 36, 42, 16, 35)_{16}$,
$(0, 40, 17, 8, 1, 6)_{17}$, $(42, 33, 6, 24, 27, 13)_{17}$, $(23, 30, 17, 26, 5, 10)_{17}$,
$(11, 1, 32, 6, 8, 5)_{17}$, $(1, 9, 36, 33, 3, 26)_{17}$,
$(0, 40, 38, 37, 9, 32)_{18}$, $(32, 12, 31, 35, 42, 27)_{18}$, $(2, 34, 35, 6, 29, 13)_{18}$,
$(35, 25, 13, 14, 39, 38)_{18}$, $(17, 4, 33, 38, 35, 31)_{18}$,
$(0, 30, 40, 16, 28, 29)_{19}$, $(24, 12, 16, 35, 15, 39)_{19}$, $(10, 42, 3, 31, 37, 7)_{19}$,
$(32, 9, 11, 17, 37, 8)_{19}$, $(1, 40, 11, 14, 41, 33)_{19}$,
$(0, 9, 17, 24, 33, 22)_{20}$, $(24, 8, 1, 27, 37, 2)_{20}$, $(15, 19, 35, 21, 1, 30)_{20}$,
$(44, 24, 4, 5, 12, 43)_{20}$, $(0, 11, 43, 36, 42, 8)_{20}$

under the action of the mapping $x \mapsto x + 2 \pmod{46}$.

$\boldsymbol{K_{54}}$ Let the vertex set be $Z_{53} \cup \{\infty\}$. The decompositions consist of

$(0, 25, 46, 47, 43, \infty)_2$, $(0, 2, 5, 13, 16, 19)_2$, $(0, 24, 9, 36, 4, 23)_2$,
$(19, 52, 46, 2, 7, \infty)_9$, $(33, 37, 8, 26, 38, 14)_9$, $(46, 36, 51, 14, 38, 5)_9$,
$(25, 33, 13, 4, 51, \infty)_{18}$, $(0, 1, 3, 5, 10, 16)_{18}$, $(0, 6, 22, 19, 36, 44)_{18}$

under the action of the mapping $x \mapsto x + 1 \pmod{53}$, $\infty \mapsto \infty$. With vertex set Z_{54} the decomposition into n_{16} consists of

$(6, 48, 42, 20, 49, 34)_{16}$, $(9, 51, 45, 23, 52, 37)_{16}$, $(12, 0, 48, 26, 1, 40)_{16}$,
$(15, 3, 51, 29, 4, 43)_{16}$, $(18, 6, 0, 32, 7, 46)_{16}$, $(21, 9, 3, 35, 10, 49)_{16}$,
$(48, 27, 1, 35, 29, 38)_{16}$, $(51, 30, 4, 38, 32, 41)_{16}$, $(0, 33, 7, 41, 35, 44)_{16}$,
$(3, 36, 10, 44, 38, 47)_{16}$, $(6, 39, 13, 47, 41, 50)_{16}$, $(9, 42, 16, 50, 44, 53)_{16}$,
$(4, 6, 37, 35, 46, 1)_{16}$, $(7, 9, 40, 38, 49, 4)_{16}$, $(10, 12, 43, 41, 52, 7)_{16}$,
$(13, 15, 46, 44, 1, 10)_{16}$, $(16, 18, 49, 47, 4, 13)_{16}$, $(19, 21, 52, 50, 7, 16)_{16}$,

$(51,13,10,14,36,19)_{16}$, $(0,16,13,17,39,22)_{16}$, $(3,19,16,20,42,25)_{16}$,
$(6,22,19,23,45,28)_{16}$, $(9,25,22,26,48,31)_{16}$, $(12,28,25,29,51,34)_{16}$,
$(3,28,41,4,13,38)_{16}$, $(6,31,44,7,16,41)_{16}$, $(9,34,47,10,19,44)_{16}$,
$(12,37,50,13,22,47)_{16}$, $(15,40,53,16,25,50)_{16}$, $(18,43,2,19,28,53)_{16}$,
$(15,49,11,14,39,44)_{16}$, $(18,52,14,17,42,47)_{16}$, $(21,1,17,20,45,50)_{16}$,
$(24,4,20,23,48,53)_{16}$, $(27,7,23,26,51,2)_{16}$, $(30,10,26,29,0,5)_{16}$,
$(31,20,13,5,9,18)_{16}$, $(34,23,16,8,12,21)_{16}$, $(37,26,19,11,15,24)_{16}$,
$(40,29,22,14,18,27)_{16}$, $(43,32,25,17,21,30)_{16}$, $(46,35,28,20,24,33)_{16}$,
$(23,30,33,53,2,34)_{16}$, $(26,33,36,2,5,37)_{16}$, $(29,36,39,5,8,40)_{16}$,
$(32,39,42,8,11,43)_{16}$, $(35,42,45,11,14,46)_{16}$, $(38,45,48,14,17,49)_{16}$,
$(46,19,27,0,32,5)_{16}$, $(52,25,33,6,38,11)_{16}$, $(2,8,14,50,38,26)_{16}$,
$(4,31,12,39,44,17)_{16}$, $(5,11,17,53,41,29)_{16}$
under the action of the mapping $x \mapsto x+18 \pmod{54}$.

K_{55} Let the vertex set be Z_{55}. The decompositions consist of
$(0,21,31,33,29,42)_2$, $(36,47,50,41,40,6)_2$, $(0,17,1,37,32,27)_2$,
$(0,11,2,3,53,12)_9$, $(29,43,3,48,25,8)_9$, $(0,17,24,49,44,16)_9$,
$(0,11,23,28,42,35)_{16}$, $(10,1,16,20,26,34)_{16}$, $(0,18,20,21,14,43)_{16}$,
$(0,13,50,38,9,2)_{18}$, $(43,35,14,12,15,1)_{18}$, $(0,10,3,16,43,14)_{18}$,
under the action of the mapping $x \mapsto x+1 \pmod{55}$.

K_{63} Let the vertex set be $Z_{62} \cup \{\infty\}$. The decompositions consist of
$(0,49,57,14,\infty,9)_2$, $(43,32,39,55,61,40)_2$, $(10,57,12,4,40,36)_2$,
$(25,57,1,36,15,24)_2$, $(45,17,4,19,46,23)_2$, $(58,18,42,14,55,6)_2$,
$(0,5,19,42,12,31)_2$,
$(49,56,52,10,\infty,38)_5$, $(50,46,41,20,3,53)_5$, $(2,36,43,37,1,47)_5$,
$(23,60,7,0,48,21)_5$, $(53,27,22,40,45,31)_5$, $(1,58,11,43,34,35)_5$,
$(1,6,20,12,46,30)_5$,
$(13,42,45,16,\infty,58)_9$, $(44,39,56,2,26,53)_9$, $(21,6,31,29,59,61)_9$,
$(46,19,31,42,5,24)_9$, $(59,21,40,39,42,55)_9$, $(37,32,16,26,50,53)_9$,
$(49,55,48,0,24,22)_9$,
$(19,22,38,49,\infty,30)_{12}$, $(47,49,11,48,36,43)_{12}$, $(17,56,25,50,22,27)_{12}$,
$(52,54,34,47,60,30)_{12}$, $(42,27,5,55,48,47)_{12}$, $(20,16,31,30,47,61)_{12}$,
$(22,31,13,54,27,43)_{12}$,
$(27,45,6,16,\infty,2)_{16}$, $(13,18,45,60,41,1)_{16}$, $(29,38,35,4,42,20)_{16}$,
$(39,40,2,4,51,46)_{16}$, $(1,61,25,9,15,44)_{16}$, $(6,9,20,52,47,5)_{16}$,
$(0,9,8,58,23,16)_{16}$,
$(25,56,14,40,\infty,38)_{17}$, $(30,53,11,57,39,52)_{17}$, $(57,55,45,22,20,10)_{17}$,
$(36,25,31,28,23,58)_{17}$, $(38,44,48,59,10,37)_{17}$, $(44,51,35,21,61,3)_{17}$,
$(0,22,41,36,3,35)_{17}$,
$(11,2,13,8,44,\infty)_{19}$, $(17,9,21,24,47,58)_{19}$, $(25,61,19,44,47,36)_{19}$,
$(55,31,18,16,36,3)_{19}$, $(47,23,31,22,24,6)_{19}$, $(36,29,24,58,38,26)_{19}$,
$(1,0,36,4,45,28)_{19}$,
$(26,4,50,17,3,\infty)_{20}$, $(60,39,31,12,21,1)_{20}$, $(18,20,29,10,25,26)_{20}$,
$(35,10,50,4,54,30)_{20}$, $(48,43,3,5,15,11)_{20}$, $(0,9,55,12,34,57)_{20}$,
$(1,15,17,14,37,44)_{20}$

under the action of the mapping $x \mapsto x+2 \pmod{62}$, $\infty \mapsto \infty$. With vertex set Z_{63} the decomposition into n_{18} consists of

$(53,15,17,24,14,12)_{18}$, $(56,18,20,27,17,15)_{18}$, $(59,21,23,30,20,18)_{18}$,
$(36,55,14,29,26,28)_{18}$, $(39,58,17,32,29,31)_{18}$, $(42,61,20,35,32,34)_{18}$,
$(51,37,24,5,57,58)_{18}$, $(54,40,27,8,60,61)_{18}$, $(57,43,30,11,0,1)_{18}$,
$(44,39,26,16,59,34)_{18}$, $(47,42,29,19,62,37)_{18}$, $(50,45,32,22,2,40)_{18}$,
$(14,52,58,56,61,48)_{18}$, $(17,55,61,59,1,51)_{18}$, $(20,58,1,62,4,54)_{18}$,
$(0,10,31,48,23,30)_{18}$, $(3,13,34,51,26,33)_{18}$, $(6,16,37,54,29,36)_{18}$,
$(11,27,25,31,30,48)_{18}$, $(14,30,28,34,33,51)_{18}$, $(17,33,31,37,36,54)_{18}$,
$(61,24,50,38,56,55)_{18}$, $(1,27,53,41,59,58)_{18}$, $(4,30,56,44,62,61)_{18}$,
$(41,34,45,32,6,21)_{18}$, $(44,37,48,35,9,24)_{18}$, $(47,40,51,38,12,27)_{18}$,
$(0,1,34,16,42,9)_{18}$, $(1,13,25,40,52,33)_{18}$, $(1,30,9,37,31,21)_{18}$,
$(6,7,40,22,48,37)_{18}$

under the action of the mapping $x \mapsto x+9 \pmod{63}$.

$\boldsymbol{K_{64}}$ Let the vertex set be $Z_{63} \cup \{\infty\}$. The decomposition consists of

$(35,37,\infty,24,6,12)_{18}$, $(38,40,\infty,27,9,15)_{18}$, $(41,43,\infty,30,12,18)_{18}$,
$(34,35,54,27,51,55)_{18}$, $(37,38,57,30,54,58)_{18}$, $(40,41,60,33,57,61)_{18}$,
$(8,44,45,5,3,35)_{18}$, $(11,47,48,8,6,38)_{18}$, $(14,50,51,11,9,41)_{18}$,
$(5,51,53,18,61,38)_{18}$, $(8,54,56,21,1,41)_{18}$, $(11,57,59,24,4,44)_{18}$,
$(55,13,43,19,18,5)_{18}$, $(58,16,46,22,21,8)_{18}$, $(61,19,49,25,24,11)_{18}$,
$(30,39,45,12,31,1)_{18}$, $(33,42,48,15,34,4)_{18}$, $(36,45,51,18,37,7)_{18}$,
$(59,21,55,17,14,23)_{18}$, $(62,24,58,20,17,26)_{18}$, $(2,27,61,23,20,29)_{18}$,
$(60,11,58,21,20,40)_{18}$, $(0,14,61,24,23,43)_{18}$, $(3,17,1,27,26,46)_{18}$,
$(50,22,40,13,18,61)_{18}$, $(53,25,43,16,21,1)_{18}$, $(56,28,46,19,24,4)_{18}$,
$(13,16,47,56,59,44)_{18}$, $(40,16,35,29,0,46)_{18}$, $(49,10,5,62,33,28)_{18}$,
$(1,25,20,14,48,37)_{18}$, $(1,44,32,4,61,35)_{18}$,

under the action of the mapping $x \mapsto x+9 \pmod{63}$, $\infty \mapsto \infty$. □

Proof of Lemma 4.2

$\boldsymbol{K_{6,6,6,6}}$ Let the vertex set be Z_{24} partitioned according to residue classes modulo 4. The decompositions consist of

$(0,1,3,10,6,11)_2$, $(0,1,2,11,7,5)_5$, $(0,1,7,10,12,2)_9$,
$(0,1,3,7,12,11)_{12}$, $(0,2,3,9,5,15)_{16}$, $(0,1,7,15,2,4)_{17}$,
$(0,1,3,6,10,17)_{18}$, $(0,1,2,10,7,13)_{19}$, $(0,1,2,6,11,9)_{20}$

under the action of the mapping $x \mapsto x+1 \pmod{24}$.

$\boldsymbol{K_{9,9,9,9}}$ Let the vertex set be $\{0,1,\ldots,35\}$ partitioned into $\{3j+i : j=0,1,\ldots,8\}$, $i=0,1,2$, and $\{27,28,\ldots,35\}$. The decompositions consist of

$(0,1,5,27,8,10)_2$, $(0,13,2,32,29,33)_2$,
$(0,1,2,27,5,10)_5$, $(0,28,29,14,7,16)_5$,
$(0,1,5,27,12,13)_9$, $(0,2,10,30,35,32)_9$,
$(0,1,5,8,27,30)_{12}$, $(0,2,13,35,3,27)_{12}$,
$(0,1,5,27,11,30)_{16}$, $(0,32,8,25,7,20)_{16}$,
$(0,1,5,8,27,29)_{17}$, $(0,2,13,35,29,12)_{17}$,
$(0,1,5,8,27,18)_{18}$, $(0,13,2,28,34,8)_{18}$,
$(0,1,2,5,27,13)_{19}$, $(0,1,11,30,33,20)_{19}$,

$(0, 1, 2, 5, 27, 12)_{20}$, $(0, 8, 11, 28, 32, 24)_{20}$
under the action of the mapping $x \mapsto x+1 \pmod{27}$ for $x < 27$, $x \mapsto (x+1 \pmod 9)) + 27$ for $x \geq 27$.
$\boldsymbol{K_{6,6,6,9}}$ Let the vertex set be $\{0, 1, \ldots, 26\}$ partitioned into $\{3j+i : j = 0, 1, \ldots, 5\}$, $i = 0, 1, 2$, and $\{18, 19, \ldots, 26\}$. The decompositions consist of

$(0, 13, 5, 25, 18, 19)_2$, $(17, 13, 9, 22, 15, 25)_2$, $(10, 17, 3, 18, 20, 5)_2$,
$(1, 0, 17, 21, 23, 25)_2$, $(2, 9, 1, 26, 20, 18)_2$,
$(0, 16, 17, 19, 18, 23)_5$, $(2, 15, 10, 23, 4, 0)_5$, $(10, 6, 17, 18, 14, 24)_5$,
$(6, 7, 17, 22, 2, 21)_5$, $(1, 2, 6, 19, 20, 26)_5$,
$(0, 21, 11, 16, 10, 20)_9$, $(21, 8, 4, 3, 20, 15)_9$, $(15, 20, 1, 11, 12, 22)_9$,
$(19, 16, 14, 6, 20, 1)_9$, $(2, 0, 19, 13, 17, 18)_9$,
$(0, 1, 19, 22, 2, 20)_{12}$, $(8, 6, 22, 20, 16, 13)_{12}$, $(16, 12, 21, 17, 10, 2)_{12}$,
$(0, 21, 11, 13, 15, 17)_{12}$, $(1, 26, 9, 17, 24, 11)_{12}$,
$(0, 16, 24, 14, 23, 11)_{16}$, $(0, 17, 10, 19, 4, 24)_{16}$, $(18, 9, 4, 14, 15, 22)_{16}$,
$(23, 10, 3, 14, 4, 25)_{16}$, $(3, 5, 4, 20, 11, 19)_{16}$,
$(0, 17, 19, 21, 11, 13)_{17}$, $(17, 10, 12, 24, 1, 0)_{17}$, $(14, 4, 24, 23, 6, 17)_{17}$,
$(15, 23, 10, 11, 19, 9)_{17}$, $(0, 25, 4, 1, 20, 2)_{17}$,
$(0, 1, 19, 21, 5, 2)_{18}$, $(5, 6, 22, 23, 16, 0)_{18}$, $(12, 14, 18, 25, 1, 13)_{18}$,
$(0, 13, 11, 14, 18, 20)_{18}$, $(0, 20, 8, 4, 16, 23)_{18}$,
$(0, 1, 25, 20, 14, 11)_{19}$, $(14, 9, 24, 13, 4, 6)_{19}$, $(3, 7, 20, 24, 5, 2)_{19}$,
$(10, 5, 0, 19, 22, 9)_{19}$, $(2, 0, 16, 18, 23, 13)_{19}$,
$(0, 13, 23, 14, 2, 6)_{20}$, $(20, 14, 0, 1, 4, 19)_{20}$, $(24, 6, 17, 7, 13, 21)_{20}$,
$(8, 3, 15, 19, 18, 14)_{20}$, $(19, 4, 16, 2, 6, 20)_{20}$

under the action of the mapping $x \mapsto x+3 \pmod{18}$ for $x < 18$, $x \mapsto (x+3 \pmod 9)) + 18$ for $x \geq 18$.
$\boldsymbol{K_{6,6,6}}$ Let the vertex set be $\{0, 1, \ldots, 17\}$ partitioned into $\{2j+i : j = 0, 1, \ldots, 5\}$, $i = 0, 1$, and $\{12, 13, \ldots, 17\}$. The decompositions consist of

$(0, 12, 13, 3, 1, 5)_5$, $(0, 12, 1, 3, 17, 5)_{12}$, $(0, 3, 12, 14, 5, 4)_{17}$,
$(0, 2, 3, 12, 17, 5)_{19}$, $(0, 1, 3, 12, 16, 10)_{20}$

under the action of the mapping $x \mapsto x+1 \pmod{12}$ for $x < 12$, $x \mapsto (x+1 \pmod 6)) + 12$ for $x \geq 12$.
$\boldsymbol{K_{3,3,3,3}}$ Let the vertex set be Z_{12} partitioned according to residue class modulo 4. The decompositions consist of

$(0, 7, 5, 10, 6, 9)_2$, $(7, 2, 8, 9, 4, 1)_2$, $(3, 6, 4, 5, 9, 8)_2$,
$(2, 0, 1, 3, 11, 5)_2$, $(8, 11, 1, 6, 5, 10)_2$, $(10, 4, 9, 11, 1, 3)_2$,
$(0, 5, 7, 10, 6, 2)_5$, $(2, 9, 5, 4, 8, 3)_5$, $(3, 1, 10, 4, 0, 8)_5$,
$(6, 4, 9, 7, 11, 3)_5$, $(8, 1, 5, 7, 6, 11)_5$, $(11, 1, 9, 10, 2, 0)_5$,
$(0, 7, 9, 2, 6, 10)_9$, $(6, 8, 3, 1, 2, 4)_9$, $(8, 7, 10, 5, 1, 9)_9$,
$(1, 2, 4, 11, 5, 0)_9$, $(4, 9, 10, 3, 11, 7)_9$, $(11, 0, 5, 6, 3, 8)_9$,
$(0, 9, 2, 11, 3, 10)_{16}$, $(5, 11, 4, 10, 0, 6)_{16}$, $(2, 3, 8, 5, 4, 9)_{16}$,
$(1, 10, 0, 7, 11, 8)_{16}$, $(4, 6, 1, 3, 7, 5)_{16}$, $(7, 8, 6, 9, 2, 1)_{16}$
$(0, 10, 3, 7, 1, 5)_{17}$, $(7, 1, 6, 8, 9, 2)_{17}$, $(2, 3, 9, 5, 0, 8)_{17}$,
$(4, 11, 2, 1, 7, 10)_{17}$, $(5, 6, 0, 4, 11, 3)_{17}$, $(8, 9, 11, 10, 6, 4)_{17}$,
$(0, 5, 10, 6, 7, 3)_{19}$, $(3, 8, 2, 5, 1, 4)_{19}$, $(10, 9, 3, 4, 8, 1)_{19}$,

$(1, 8, 6, 11, 7, 0)_{19}$, $(4, 9, 2, 6, 7, 11)_{19}$, $(11, 0, 2, 9, 5, 10)_{19}$,
$(0, 2, 7, 9, 1, 4)_{20}$, $(4, 2, 10, 5, 3, 7)_{20}$, $(3, 1, 5, 6, 8, 0)_{20}$,
$(7, 6, 2, 9, 8, 11)_{20}$, $(10, 8, 0, 9, 11, 3)_{20}$, $(11, 4, 5, 1, 6, 10)_{20}$.

$\boldsymbol{K_{6,6,6,3}}$ Let the vertex set be $\{0, 1, \ldots, 20\}$ partitioned into $\{3j+i : j = 0, 1, \ldots, 5\}$, $i = 0, 1, 2$, and $\{18, 19, 20\}$. The decomposition consists of
$(0, 19, 10, 5, 15, 1)_{12}$, $(9, 11, 13, 18, 0, 10)_{12}$, $(9, 1, 14, 5, 6, 2)_{12}$,
$(10, 2, 18, 6, 13, 3)_{12}$, $(0, 16, 2, 11, 4, 20)_{12}$, $(1, 20, 3, 8, 19, 11)_{12}$
under the action of the mapping $x \mapsto x + 6 \pmod{18}$ for $x < 18$, $x \mapsto x$ for $x \geq 18$.

$\boldsymbol{K_{18,18,18,18,18,27}}$ Let the vertex set be $\{0, 1, \ldots, 116\}$ partitioned into $\{5j + i : j = 0, 1, \ldots, 17\}$, $i = 0, 1, 2, 3, 4$, and $\{90, 91, \ldots, 116\}$. The decompositions consist of
$(0, 9, 68, 88, 99, 78)_{12}$, $(13, 77, 102, 54, 3, 103)_{12}$, $(26, 103, 73, 33, 65, 9)_{12}$,
$(73, 40, 101, 86, 23, 76)_{12}$, $(25, 87, 101, 31, 27, 26)_{12}$, $(79, 3, 55, 91, 71, 90)_{12}$,
$(0, 19, 72, 95, 24, 102)_{12}$,
$(0, 81, 78, 77, 67, 103)_{18}$, $(42, 103, 16, 26, 84, 55)_{18}$, $(4, 62, 41, 114, 110, 6)_{18}$,
$(60, 109, 4, 11, 21, 30)_{18}$, $(58, 89, 1, 101, 104, 69)_{18}$, $(27, 20, 114, 63, 19, 0)_{18}$,
$(0, 38, 44, 99, 62, 7)_{18}$
under the action of the mapping $x \mapsto x + 1 \pmod{90}$ for $x < 90$, $x \mapsto (x - 90 + 3 \pmod{27}) + 90$ for $x \geq 90$. □

Proof of Lemma 5.1

$\boldsymbol{K_{18}}$ With vertex set Z_{18} the decomposition consists of
$(6, 5, 7, 15, 1, 14)_{15}$, $(16, 2, 7, 12, 5, 17)_{15}$, $(7, 0, 14, 17, 8, 9)_{15}$,
$(6, 0, 13, 16, 8, 10)_{15}$, $(5, 0, 11, 12, 4, 10)_{15}$, $(5, 8, 13, 14, 2, 9)_{15}$,
$(6, 4, 11, 17, 2, 3)_{15}$, $(2, 11, 14, 15, 0, 1)_{15}$, $(2, 10, 13, 17, 4, 8)_{15}$,
$(4, 9, 14, 16, 1, 7)_{15}$, $(1, 9, 11, 13, 3, 5)_{15}$, $(15, 3, 9, 17, 0, 8)_{15}$,
$(4, 12, 13, 15, 0, 3)_{15}$, $(3, 8, 11, 16, 7, 13)_{15}$, $(12, 1, 8, 17, 6, 9)_{15}$,
$(10, 3, 12, 14, 0, 9)_{15}$, $(10, 1, 15, 16, 7, 11)_{15}$.

$\boldsymbol{K_{19}}$ With vertex set Z_{19} the decomposition consists of
$(0, 1, 3, 8, 4, 10)_{15}$
under the action of the mapping $x \mapsto x + 1 \pmod{19}$.

$\boldsymbol{K_{27}}$ With vertex set $Z_{26} \cup \{\infty\}$ the decomposition consists of
$(0, 23, 1, 16, \infty, 19)_{15}$, $(20, 11, 25, 12, 0, 22)_{15}$, $(1, 6, 18, 21, 3, 11)_{15}$
under the action of the mapping $x \mapsto x + 2 \pmod{26}$, $\infty \mapsto \infty$.

$\boldsymbol{K_{28}}$ With vertex set Z_{28} the decomposition consists of
$(0, 5, 15, 4, 17, 3)_{15}$, $(22, 14, 10, 8, 13, 5)_{15}$, $(13, 6, 1, 19, 16, 15)_{15}$,
$(24, 15, 8, 6, 17, 4)_{15}$, $(25, 19, 22, 21, 10, 16)_{15}$, $(2, 3, 19, 23, 7, 12)_{15}$
under the action of the mapping $x \mapsto x + 4 \pmod{28}$.

$\boldsymbol{K_{36}}$ With vertex set $Z_{35} \cup \{\infty\}$ the decomposition consists of
$(\infty, 4, 26, 13, 12, 30)_{15}$, $(20, 0, 6, 31, 29, 27)_{15}$, $(1, 19, 16, 27, 10, 20)_{15}$,
$(4, 34, 11, 17, 23, 15)_{15}$, $(15, 18, 29, 19, 10, 2)_{15}$, $(20, 19, 4, 33, 21, 18)_{15}$,
$(24, 17, 16, 12, 30, 27)_{15}$, $(31, 1, 17, 13, 33, 29)_{15}$, $(0, 2, 12, 18, 23, 28)_{15}$,
$(3, 2, 17, 28, 11, 18)_{15}$
under the action of the mapping $x \mapsto x + 5 \pmod{35}$, $\infty \mapsto \infty$.

$\boldsymbol{K_{37}}$ With vertex set Z_{37} the decomposition consists of
$(0, 1, 3, 8, 9, 20)_{15}$, $(0, 4, 16, 22, 10, 23)_{15}$

under the action of the mapping $x \mapsto x+1 \pmod{37}$.

$\boldsymbol{K_{45}}$ With vertex set $Z_{44} \cup \{\infty\}$ the decomposition consists of

$(3,2,13,39,12,\infty)_{15}$, $(33,38,7,3,\infty,30)_{15}$, $(21,34,30,15,8,6)_{15}$, $(42,12,17,40,15,28)_{15}$, $(27,24,0,43,3,32)_{15}$, $(33,17,29,36,34,13)_{15}$, $(31,20,21,29,1,43)_{15}$, $(13,40,19,30,28,35)_{15}$, $(42,32,20,24,4,10)_{15}$, $(2,7,10,30,9,20)_{15}$

under the action of the mapping $x \mapsto x+4 \pmod{44}$, $\infty \mapsto \infty$.

$\boldsymbol{K_{46}}$ With vertex set Z_{46} the decomposition consists of

$(0,42,6,29,13,21)_{15}$, $(42,22,7,10,44,13)_{15}$, $(27,26,15,31,36,18)_{15}$, $(19,1,40,21,25,26)_{15}$, $(39,20,36,44,3,17)_{15}$

under the action of the mapping $x \mapsto x+2 \pmod{46}$.

$\boldsymbol{K_{54}}$ With vertex set Z_{54} the decomposition consists of

$(32,1,16,38,7,52)_{15}$, $(22,7,38,44,28,32)_{15}$, $(13,28,44,50,47,51)_{15}$, $(34,1,9,49,33,42)_{15}$, $(7,1,15,40,11,25)_{15}$, $(21,7,13,46,37,53)_{15}$, $(47,30,45,53,15,38)_{15}$, $(51,5,36,53,24,33)_{15}$, $(5,3,11,42,10,15)_{15}$, $(32,0,13,42,48,51)_{15}$, $(38,6,19,48,0,3)_{15}$, $(44,0,12,25,11,21)_{15}$, $(39,6,32,46,34,35)_{15}$, $(12,38,45,52,3,4)_{15}$, $(4,18,44,51,46,48)_{15}$, $(46,18,42,49,23,52)_{15}$, $(48,1,24,52,3,21)_{15}$, $(30,0,4,7,5,12)_{15}$, $(27,1,3,6,5,50)_{15}$, $(12,7,9,33,46,47)_{15}$, $(13,15,18,39,12,26)_{15}$, $(32,24,43,44,29,30)_{15}$, $(38,30,49,50,28,35)_{15}$, $(2,1,36,44,22,31)_{15}$, $(46,3,25,37,34,38)_{15}$, $(31,9,43,52,35,48)_{15}$, $(49,4,15,37,11,13)_{15}$, $(26,11,28,52,17,31)_{15}$, $(17,4,32,34,24,38)_{15}$, $(38,10,23,40,29,36)_{15}$, $(27,2,15,32,22,24)_{15}$, $(8,21,33,38,6,16)_{15}$, $(14,27,39,44,22,47)_{15}$, $(16,3,34,51,4,11)_{15}$, $(22,3,9,40,17,42)_{15}$, $(9,15,28,46,44,53)_{15}$, $(29,7,10,37,0,53)_{15}$, $(16,13,35,43,18,36)_{15}$, $(41,19,22,49,11,34)_{15}$, $(29,2,3,17,26,42)_{15}$, $(8,9,23,35,4,52)_{15}$, $(41,14,15,29,8,44)_{15}$, $(26,15,19,30,1,46)_{15}$, $(36,21,25,32,27,28)_{15}$, $(27,31,38,42,0,9)_{15}$, $(0,5,16,48,10,35)_{15}$, $(0,6,11,22,1,14)_{15}$, $(17,6,12,28,14,50)_{15}$, $(37,11,27,47,17,19)_{15}$, $(53,17,33,43,22,23)_{15}$, $(5,23,39,49,1,18)_{15}$, $(23,2,7,20,32,37)_{15}$, $(25,23,24,42,5,9)_{15}$

under the action of the mapping $x \mapsto x+18 \pmod{54}$.

$\boldsymbol{K_{55}}$ With vertex set Z_{55} the decomposition consists of

$(0,42,4,30,31,16)_{15}$, $(23,0,1,3,2,9)_{15}$, $(0,8,18,27,5,11)_{15}$

under the action of the mapping $x \mapsto x+1 \pmod{55}$.

$\boldsymbol{K_{63}}$ With vertex set $Z_{62} \cup \{\infty\}$ the decomposition consists of

$(1,27,28,50,\infty,48)_{15}$, $(1,21,56,0,51,60)_{15}$, $(11,27,25,16,28,54)_{15}$, $(8,20,16,40,22,6)_{15}$, $(27,46,31,21,5,38)_{15}$, $(10,0,59,28,15,43)_{15}$, $(0,17,25,55,23,41)_{15}$

under the action of the mapping $x \mapsto x+2 \pmod{62}$, $\infty \mapsto \infty$.

$\boldsymbol{K_{64}}$ With vertex set $Z_{63} \cup \{\infty\}$ the decomposition consists of

$(1,0,8,18,38,53)_{15}$, $(1,3,4,10,39,41)_{15}$, $(1,2,5,37,6,33)_{15}$, $(0,7,10,23,5,51)_{15}$, $(1,15,17,20,7,12)_{15}$, $(22,1,30,47,0,20)_{15}$, $(0,31,42,55,11,36)_{15}$, $(24,1,35,\infty,59,4)_{15}$, $(1,42,43,49,50,59)_{15}$, $(1,31,44,52,16,25)_{15}$, $(1,34,48,62,26,60)_{15}$, $(0,21,56,61,12,62)_{15}$,

$(0, 33, 43, 58, 19, 41)_{15}$, $(0, 6, 25, 54, 28, 48)_{15}$, $(0, 4, 13, 35, 37, 49)_{15}$,
$(0, 16, 17, 50, 52, 60)_{15}$, $(0, 26, 32, 44, 2, 39)_{15}$, $(0, 29, 59, \infty, 24, 47)_{15}$,
$(14, 0, 30, 40, 16, 2)_{15}$, $(3, 25, 49, \infty, 15, 31)_{15}$, $(7, 3, 26, 40, 17, 43)_{15}$,
$(6, 2, 43, 50, 15, 62)_{15}$, $(2, 22, 25, 29, 35, 52)_{15}$, $(2, 13, 15, 49, 34, 51)_{15}$,
$(8, 2, 12, 31, 62, 4)_{15}$, $(5, 15, 23, 43, 16, 4)_{15}$, $(6, 3, 24, 30, 44, 5)_{15}$,
$(14, 3, 21, 51, 49, 4)_{15}$, $(11, 2, 33, 53, 19, 1)_{15}$, $(23, 2, 26, 57, 50, 4)_{15}$,
$(3, 2, 41, 47, 16, 34)_{15}$, $(3, 8, 23, 44, 57, 0)_{15}$.
under the action of the mapping $x \mapsto x + 9 \pmod{63}$, $\infty \mapsto \infty$.

$\boldsymbol{K_{81}}$ With vertex set Z_{81} the decomposition consists of
$(0, 1, 4, 54, 2, 40)_{15}$, $(0, 9, 19, 21, 58, 78)_{15}$, $(0, 6, 7, 45, 61, 64)_{15}$,
$(0, 14, 23, 69, 8, 25)_{15}$, $(0, 20, 22, 35, 13, 75)_{15}$, $(0, 29, 48, 62, 15, 39)_{15}$,
$(0, 26, 32, 80, 16, 50)_{15}$, $(0, 17, 33, 56, 18, 71)_{15}$, $(0, 44, 46, 55, 30, 59)_{15}$,
$(0, 47, 57, 70, 34, 79)_{15}$, $(0, 52, 66, 73, 37, 67)_{15}$, $(0, 24, 60, 74, 38, 51)_{15}$,
$(0, 65, 76, 77, 49, 68)_{15}$, $(2, 1, 22, 38, 8, 20)_{15}$, $(5, 1, 21, 47, 31, 2)_{15}$,
$(2, 11, 32, 68, 3, 52)_{15}$, $(26, 2, 30, 50, 62, 3)_{15}$, $(1, 20, 46, 74, 13, 40)_{15}$,
$(2, 16, 49, 66, 53, 79)_{15}$, $(48, 2, 70, 78, 1, 19)_{15}$, $(34, 2, 39, 43, 41, 1)_{15}$,
$(2, 75, 76, 80, 51, 62)_{15}$, $(3, 12, 44, 49, 15, 57)_{15}$, $(3, 5, 13, 39, 21, 77)_{15}$,
$(1, 12, 15, 78, 11, 51)_{15}$, $(1, 39, 60, 67, 33, 70)_{15}$, $(3, 51, 53, 61, 76, 78)_{15}$,
$(1, 25, 53, 66, 57, 68)_{15}$, $(1, 28, 62, 77, 65, 69)_{15}$, $(1, 7, 23, 26, 8, 42)_{15}$,
$(1, 43, 49, 58, 16, 17)_{15}$, $(7, 4, 44, 53, 61, 2)_{15}$, $(4, 25, 35, 49, 32, 50)_{15}$,
$(24, 1, 32, 52, 58, 2)_{15}$, $(6, 1, 59, 76, 44, 62)_{15}$, $(41, 4, 42, 62, 0, 3)_{15}$,
$(14, 4, 26, 33, 44, 1)_{15}$, $(4, 43, 60, 69, 16, 67)_{15}$, $(5, 16, 34, 59, 15, 44)_{15}$,
$(5, 7, 42, 69, 11, 0)_{15}$
under the action of the mapping $x \mapsto x + 9 \pmod{81}$.

$\boldsymbol{K_{82}}$ With vertex set Z_{82} the decomposition consists of
$(0, 51, 73, 64, 77, 74)_{15}$, $(44, 15, 60, 67, 55, 61)_{15}$, $(33, 7, 19, 35, 67, 57)_{15}$,
$(6, 12, 74, 62, 52, 35)_{15}$, $(65, 6, 69, 27, 20, 64)_{15}$, $(16, 70, 12, 43, 55, 35)_{15}$,
$(32, 73, 80, 65, 45, 10)_{15}$, $(49, 50, 52, 17, 3, 60)_{15}$, $(0, 5, 30, 72, 43, 61)_{15}$
under the action of the mapping $x \mapsto x + 2 \pmod{82}$. □

Proof of Lemma 5.2

$\boldsymbol{K_{6,6,6,6}}$ Let the vertex set be Z_{24} partitioned according to residue classes modulo 4. The decomposition consists of
$(0, 1, 3, 10, 5, 11)_{15}$
under the action of the mapping $x \mapsto x + 1 \pmod{24}$.

$\boldsymbol{K_{6,6,6,3}}$ Let the vertex set be $\{0, 1, \ldots, 20\}$ partitioned into $\{3j+i : j = 0, 1, \ldots, 5\}$, $i = 0, 1, 2$, and $\{18, 19, 20\}$. The decomposition consists of
$(0, 2, 16, 18, 7, 14)_{15}$, $(17, 1, 19, 0, 16, 15)_{15}$, $(10, 2, 3, 19, 15, 5)_{15}$,
$(6, 17, 10, 18, 11, 1)_{15}$, $(1, 5, 9, 18, 3, 14)_{15}$, $(2, 1, 15, 20, 4, 12)_{15}$
under the action of the mapping $x \mapsto x+6 \pmod{18}$ for $x < 18$, $x \mapsto (x+1 \pmod 3)+18$ for $x \geq 18$.

$\boldsymbol{K_{3,3,3,3,3}}$ Let the vertex set be Z_{15} partitioned according to residue class modulo 5. The decomposition consists of
$(0, 1, 3, 7, 2, 9)_{15}$, $(2, 1, 4, 8, 3, 14)_{15}$
under the action of the mapping $x \mapsto x + 3 \pmod{15}$.

$\boldsymbol{K_{18,18,18,18,27}}$ Let the vertex set be $\{0, 1, \ldots, 98\}$ partitioned into $\{4j + i : j = 0, 1, \ldots, 17\}$, $i = 0, 1, 2, 3$, and $\{72, 73, \ldots, 98\}$. The decomposition consists of

$(0, 83, 51, 34, 37, 93)_{15}$, $(32, 2, 63, 74, 93, 38)_{15}$, $(57, 35, 32, 73, 8, 91)_{15}$, $(75, 56, 63, 10, 16, 15)_{15}$, $(46, 37, 19, 32, 31, 73)_{15}$, $(0, 10, 39, 80, 2, 79)_{15}$

under the action of the mapping $x \mapsto x + 1 \pmod{72}$ for $x < 72$, $x \mapsto (x - 72 + 3 \pmod{27}) + 72$ for $x \geq 72$.

$\boldsymbol{K_{18,18,18,18,18,27}}$ Let the vertex set be $\{0, 1, \ldots, 116\}$ partitioned into $\{5j + i : j = 0, 1, \ldots, 17\}$, $i = 0, 1, 2, 3, 4$, and $\{90, 91, \ldots, 116\}$. The decomposition consists of

$(0, 59, 96, 88, 71, 28)_{15}$, $(20, 32, 64, 110, 71, 17)_{15}$, $(46, 70, 93, 54, 60, 103)_{15}$, $(43, 36, 115, 30, 65, 103)_{15}$, $(29, 8, 111, 81, 11, 2)_{15}$, $(76, 53, 110, 12, 19, 20)_{15}$, $(0, 11, 53, 116, 4, 103)_{15}$

under the action of the mapping $x \mapsto x + 1 \pmod{90}$ for $x < 90$, $x \mapsto (x - 90 + 3 \pmod{27}) + 90$ for $x \geq 90$.

$\boldsymbol{K_{3,3,3,3,3,3,3}}$ Let the vertex set be Z_{21} partitioned according to residue class modulo 7. The decomposition consists of

$(0, 1, 4, 16, 2, 10)_{15}$

under the action of the mapping $x \mapsto x + 1 \pmod{21}$. □

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