Bose-Einstein or HBT correlation signature of a second order QCD phase transition

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Abstract. For particles emerging from a second order QCD phase transition, we show that a recently introduced shape parameter of the Bose-Einstein correlation function, the Lévy index of stability equals to the correlation exponent - one of the critical exponents that characterize the behavior of the matter in the vicinity of the second order phase transition point. Hence the shape of the Bose-Einstein / HBT correlation functions, when measured as a function of bombarding energy and centrality in various heavy ion reactions, can be utilized to locate experimentally the second order phase transition and the critical end point of the first order phase transition line in QCD.

INTRODUCTION

The study fractal phenomena was initiated in high energy particle and nuclear physics by Bialas and Peschanski in ref. [1], with the motivation of searching for a second order phase transition by studying intermittency or the the power-law behavior of moments of the multiplicity distribution in narrowing bins of the momentum space, see refs. [2, 3] for excellent reviews on this topic.

The mathematical properties of Bose-Einstein correlation functions for Lévy stable sources were written up by three of us in refs. [4, 5], and are recapitulated in the next section. In ref. [6] we have added a physical interpretation and showed, that in case of jet physics, the fractal properties of QCD cascades can naturally be measured by the Lévy index of stability of the Bose-Einstein correlation functions. Our analytic results were similar in spirit to the numerical investigations of Wilk and collaborators in ref. [7]. Note that these correlations are frequently referred to as Hanbury Brown - Twiss or HBT correlations in the literature of heavy ion physics.

Bialas realized, that Bose-Einstein correlations and intermittency might be deeply connected [8], and considered a distribution of Gaussians where the radius parameter of the Gaussian has a power-law distribution, thus giving a way to the study fractals in coordinate space with the help of Bose-Einstein correlations. Brax and Peschanski were the first to introduce Lévy distributions, in momentum space, to multiparticle production in high energy physics [9]. They have suggested to use the measured value of the Lévy index of stability to signal quark gluon plasma production in heavy ion physics. Here we reconnect these seemingly different topics, and show how the excitation function of the shape parameter of the correlation function can be utilized to locate experimentally the critical end-point of QCD.

BOSE-EINSTEIN CORRELATIONS & LÉVY STABLE SOURCES

The two-particle Bose-Einstein correlation function is defined with the help of the two-particle and single-particle invariant momentum distributions as:

$$C_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{N_2(\mathbf{k}_1,\mathbf{k}_2)}{N_1(\mathbf{k}_1)\,N_1(\mathbf{k}_2)}. \tag{1}$$

If long-range correlations can be neglected or corrected for, and if the short-range correlations are dominated by Bose-Einstein correlations, this two-particle Bose-Einstein correlation function is related to the Fourier-transformed source distribution. For clarity, let us consider the case of a one-dimensional, factorized source,

$$S(x,k) = f(x)\,g(k). \tag{2}$$

In this case [4, 5], the Bose-Einstein correlation function is

$$C_2(k_1,k_2) = 1 + |\tilde{f}(q)|^2, \tag{3}$$

where the Fourier transformed source density (often referred to as the *characteristic function*) and the relative momentum are defined as

$$\tilde{f}(q) = \int \mathrm{d}x \exp(iqx)\, f(x), \qquad q = k_1 - k_2. \tag{4}$$

For the case of the jets decaying to jets to jets and so on, as well as at a second order phase transition, where fluctuations appear on all possible scales with a power-law tailed distribution, the final position of a particle is given by a large number of position shifts, hence the distribution of the final position x is obtained as a convolution,

$$x = \sum_{i=1}^{n} x_i, \qquad f(x) = \int \Pi_{i=1}^{n} \mathrm{d}x_i \Pi_{j=1}^{n} f_j(x_j)\, \delta(x - \sum_{k=1}^{n} x_k). \tag{5}$$

In the case of particle emission from QCD jets, that the fractal defining the particle emission is infrared stable: adding one more, very soft gluon does not change the resulting source distributions. A similar property holds for systems at a second order phase transition: the system becomes invariant under a renormalization group transformation. Bose-Einstein correlation functions for such particle emitting sources were evaluated recently by three of us, which we summarize here following refs. [4] and [5].

Various forms of the Central Limit Theorem state, that under certain conditions, the distribution of the sum of large number of random variables converges (for $n \to \infty$) to a limit distribution. In case of “normal" elementary processes, that have finite means and variances, the limit distribution of their sum is a Gaussian. In case of random motion in a thermal medium, such position distribution corresponds to normal diffusion. However, near a second order phase transition point, fluctuations appear on all scales and the variance of the elementary process diverges, corresponding to the so-called anomalous diffusion. In this case, the system still may be invariant under convolution, and the shape of the limit distribution becomes independent from the number of elementary steps.

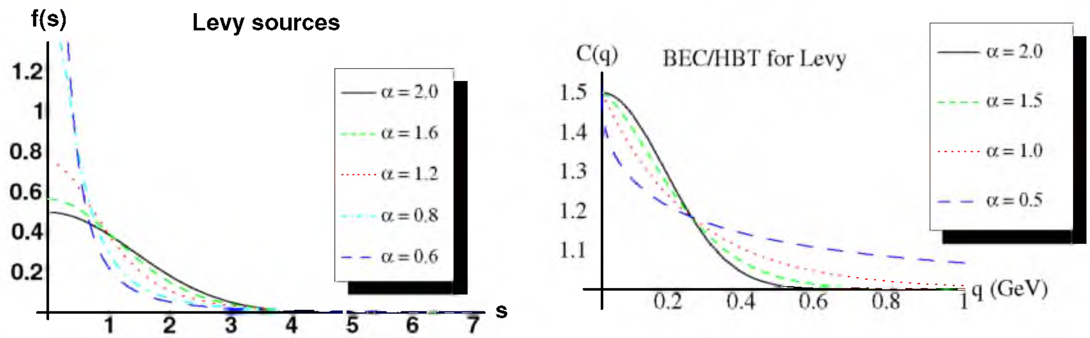

Figure 1. (left) Source functions for univariate symmetric Lévy laws, as a function of the dimensionless variable $s = r/R$, for various values of the Lévy index of stability, α. (right) Bose-Einstein correlation (or HBT) correlation functions for univariate symmetric Lévy laws, for a fixed scale parameter of $R = 0.8$ fm and various values of the Lévy index of stability, α.

Stable distributions are precisely those limit distributions that can occur in Generalized Central Limit theorems. Their study was begun by the mathematician Paul Lévy in the 1920's. The stable distributions can be given in terms of their characteristic functions, as the Fourier transform of a convolution is a product of the Fourier-transforms,

$$\tilde{f}(q) = \prod_{i=1}^{n} \tilde{f}_i(q), \tag{6}$$

and limit distributions appear when the convolution of one more elementary process does not change the shape of the limit distribution, but it results only in a modification of the parameters of the limit distribution. The characteristic function of univariate and symmetric stable distributions is

$$\tilde{f}(q) = \exp\left(iq\delta - |\gamma q|^{\alpha}\right), \tag{7}$$

where the support of the density function $f(x)$ is $(-\infty, \infty)$. Deep mathematical results imply that the index of stability, α, satisfies the inequality $0 < \alpha \leq 2$, so that the source distribution be always positive. These Lévy distributions are indeed stable under convolutions, in the sense of the following relations:

$$\tilde{f}_i(q) \;=\; \exp\left(iq\delta_i - |\gamma_i q|^{\alpha}\right), \qquad \prod_{i=1}^{n} \tilde{f}_i(q) = \exp\left(iq\delta - |\gamma q|^{\alpha}\right), \tag{8}$$

$$\gamma^{\alpha} \;=\; \sum_{i=1}^{n} \gamma_i^{\alpha}, \qquad \delta = \sum_{i=1}^{n} \delta_i. \tag{9}$$

Thus the Bose-Einstein correlation functions for uni-variate, symmetric stable distributions (after a core-halo correction, and a re-scaling) read as

$$C(q;\alpha) = 1 + \lambda \exp\left(-|qR|^{\alpha}\right). \tag{10}$$

Refs. [4] and [5] discuss further examples and details and generalize these results to three dimensional, hydrodynamically expanding, core-halo type sources, as well as to three-particle correlations.

THE ANOMALOUS DIMENSION OF QCD JETS AND BE/HBT

In QCD, jets emit jets that emit additional jets and so on. The resulting fractal structure of QCD jets was related to intermittency and power-law dependence of multiplicity moments on the bin-size in momentum space with the help of a beautiful geometric interpretation of the color dipole picture in refs. [10, 11, 12], and an infrared stable measure on the parton states, related to the hadronic multiplicity distribution. These ideas were developed further in refs. [13, 14, 15], [16] as well as in refs. [17, 18].

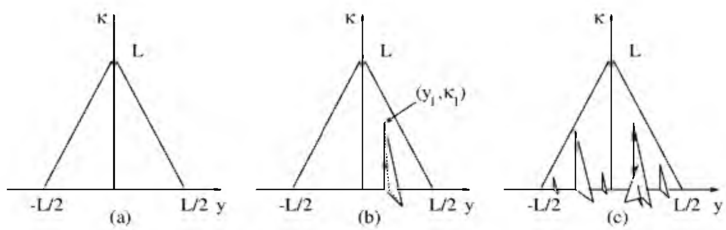

Figure 2. The phase-space of QCD jets in the (y, κ) plane, where $\kappa = \log(k_t^2)$. (a) The phase space available for a gluon emitted by a high energy $q\bar{q}$ system is a triangular region in the (y, κ) plane. (b) If one gluon is emitted at (y_1, κ_1), the phase space for a second (softer) gluon is given by the area of this folded surface. (c) The total gluonic phase space can be described by this multifaceted fractal surface [10, 11, 12].

A high energy $q\bar{q}$ system radiates gluons according to the dipole formula

$$dn = \frac{3\alpha_s}{4\pi^2} \frac{dk_\perp^2}{k_\perp^2} dy d\phi, \tag{11}$$

hence the phase-space for the emission of a gluon is given by the relation

$$|y| \leq \frac{1}{2} \ln(s/k_\perp^2), \tag{12}$$

which corresponds to the triangular region in a $(y, \ln k_\perp^2)$ diagram as shown in Fig. 2 (a). If two gluons are emitted, then the distribution of the hardest gluon is described by eq. (11). The distribution of the second, softer, gluon corresponds to two dipoles, the first is stretched between the quark and the first gluon, and the second between the first gluon and the anti-quark. The phase-space available for the second gluon corresponds to the folded surface in Fig. 2 (b), with the constraint $k_{\perp,2}^2 < k_{\perp,1}^2$, as the first gluon is assumed to be the hardest one. This procedure can be generalized so that the emission of a third, still softer gluon corresponds to radiation from three color dipoles, with n gluons emitted already the emission of the $n+1$-th gluon is given by a chain of $n+1$ dipoles. Thus, with many gluons, the gluonic phase space can be represented by a multi-faceted surface as illustrated in Fig. 2 (c). Each gluon adds a fold to the surface, which increases the phase-space for softer gluons. (Note, that in this process the recoils are neglected, as is normal in leading log approximation). Due to its iterative nature, the process generates a Koch-type fractal curve at the base-line. The length of this base-line of the partonic structure on Figure 2 (c) is proportional to the particle multiplicity. This curve is longer, when studied with higher resolution: it is a fractal curve, embedded into the four-dimensional energy-momentum space, characterized by the fractal dimension

$$d_f = 1 + \sqrt{\frac{3\alpha_s}{2\pi}}, \tag{13}$$

or one plus the anomalous dimension of QCD [10, 11, 12]. With the help of the Lund string fragmentation picture, this fractal in momentum space is mapped into a fractal in coordinate space, and the constant of conversion is the hadronic string tension, $\kappa \approx 1$ GeV/fm. This mapping does not change the fractal properties of the curve.

A walk, where the length of the steps is given by a Lévy distribution, and the direction of the steps is random, corresponds to a fractal curve, in physical terms it can be interpreted as the path of a test particle performing a generalized Brownian motion. This motion is referred to as anomalous diffusion and the probability that the test particle diffuses to distances r greater than a certain value of $|s|$ is given by $P(r > |s|) \propto |s|^{-\alpha}$. This relation is valid for anomalous diffusion in any dimensions. Thus the Lévy index of stability α is the fractal dimension of the trajectory of the corresponding anomalous diffusion [19]. When we apply this result to QCD, there are two key considerations.

First, if gluon radiation is neglected, the $q\overline{q}$ system hadronizes as a 1+1 dimensional hadronic string, which has no fractal structure. If the gluon emission is switched on, the emission of gluon n from one of the n dipoles corresponds to a step of an anomalous diffusion in the plane transverse to the given dipole. Hence the anomalous dimension of QCD equals to the Lévy index of stability of this anomalous diffusion,

$$\sqrt{\frac{3\alpha_s}{2\pi}} = \alpha_{\text{Lévy}}. \tag{14}$$

Second, data on Bose-Einstein correlations are often determined in terms of the invariant momentum difference $Q_{\text{inv}} = \sqrt{-(p_1 - p_2)^2}$. Bose-Einstein correlation functions that depend on this invariant momentum difference can be obtained within the framework of the so-called τ-model. This model assumes a broad proper-time distribution, $H(\tau)$ and very strong correlations between coordinate and momentum in all directions, $x^\mu/\tau \propto p^\mu/m_{(t)}$. Hence $(x_1 - x_2)(p_1 - p_2) \propto \tau Q^2_{\text{inv}}$, see refs. [20, 21] for details. In this case, the Bose-Einstein correlation function measures the Fourier-transformed proper-time distribution $\tilde{H}$ in the following, unusual manner:

$$C_2(Q_{\text{inv}}) \simeq 1 + \lambda Re \tilde{H}^2 \left(\frac{Q^2_{\text{inv}}}{2m_{(t)}} \right), \tag{15}$$

where $m_{(t)}$ stands for the (transverse) mass of the pair for (two)- or more jet events. From this relation it follows, that $\alpha_{\text{BEC}} = 2\alpha_{\text{Lévy}}$. Thus we find the following relationship between the running QCD coupling constant α_s and the exponent of an invariant relative momentum dependent Bose-Einstein correlation function α_{BEC}:

$$\alpha_s = \frac{\pi}{6}\alpha^2_{\text{BEC}}. \tag{16}$$

In ref. [6] we have compared this leading log result to NA22 and UA1 correlation data of refs. [22],[23] and found a reasonable agreement with these data.

BOSE-EINSTEIN CORRELATIONS AT A SECOND ORDER QCD PHASE TRANSITION

The main motivation behind the experimental and theoretical program of high energy physics is to study the phase diagram of hot and dense hadronic matter. According to recent lattice QCD calculations at finite temperature and baryon density, there exist a line of first order phase transitions that separates the hadronic and the quark-gluon plasma (QGP) state. This line of the first order phase transitions ends at the critical end-point (CEP), where the transition from hadron gas to QGP becomes a second order phase transition. Recent lattice QCD calculations located [24] this CEP at $T_E = 162 \pm 2$ MeV and $\mu_E = 360 \pm 40$ MeV. Below these baryochemical values, the transition from a hadron gas to a QGP becomes a cross-over, and at vanishing net baryon density the critical temperature becomes $T_c = 164 \pm 2$ MeV (the errors are statistical only). In this calculation, the quark masses were already at the physical value, but the continuum extrapolation was missing. S. Katz presented improvements at the Quark Matter 2005 conference [25], using physical quark masses and working towards the continuum extrapolation. He reported $T_c = 189 \pm 8$ MeV for the critical temperature at $\mu_B = 0$.

At the CEP, the second order phase transition is characterized by the fixed point of the renormalization group transformations. In a quark-gluon plasma, the vacuum expectation value of the quark condensate $c = \langle \overline{q} q \rangle$ vanishes , while in the hadronic phase, this vacuum expectation value becomes non-zero. The correlation function of the order parameter is defined as $\rho(R) = \langle c(r+R)c(r)\rangle - \langle c \rangle^2$ and measures the spatial correlation between the pions. At the CEP, this correlation function decays as

$$\rho(R) \propto R^{-(d-2+\eta)}, \tag{17}$$

a power-law. The parameter η is called as the exponent of the correlation function.

For Lévy stable sources, corresponding to an anomalous diffusion with large fluctuations in coordinate space, the correlation between the initial and actual positions decays also as a power-law, where the exponent is given by the Lévy index of stability α as

$$\rho(R) \propto R^{-(1+\alpha)}. \tag{18}$$

As we are considering a QCD phase transition in a $d = 3$ three-dimensional coordinate space we find that the correlation exponent equals to the Lévy index of stability, $\alpha = \eta$. Stephanov, Rajagopal and Shuryak pointed out [26], that the universality class of the second order QCD phase transition is that of the 3d Ising model. For this universality class, the correlation exponent has been determined by Rieger [27] as

$$\alpha(\text{Lévy}) = \eta(\text{3d Ising}) = 0.50 \pm 0.05. \tag{19}$$

Fig. 1 indicates that the change in the shape of the correlation function is rather significant, if α decreases from its Gaussian value of 2 to 0.5, its characteristic value at the 2nd order QCD phase transition point. Fig. 3 illustrates how this shape parameter of the Bose-Einstein correlation function may depend on the relative temperature near the critical point. Hence studying the *shape* parameter of the two-particle Bose-Einstein or HBT correlation functions as a function of the bombarding energy or the centrality of the heavy ion collisions, a previously unknown tool is obtained to determine if the pions are emitted from the neighborhood of the critical end point of the QCD phase diagram.

Furthermore, based on an universality class argument, we have determined that the second order QCD phase transition at the critical end point will be signaled with the value of $\alpha = 0.5$, a very spiky Bose-Einstein correlation function indeed.

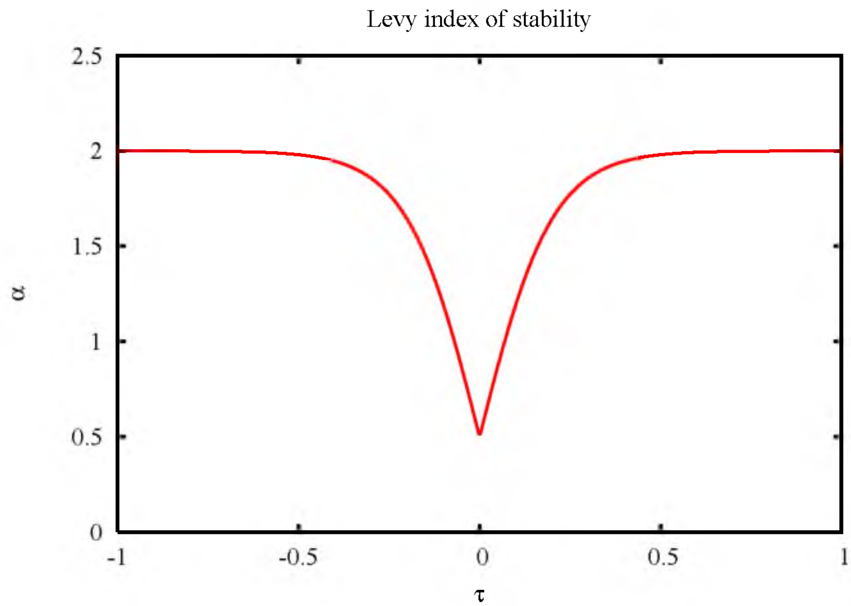

Figure 3. Illustration of the behavior of the Lévy index of stability of Bose-Einstein correlations as a function of the dimensionless temperature variable $\tau = (T - T_c)/T_c$ in the neighborhood of the critical endpoint of the 1st order phase transition line in QCD. At the critical endpoint, the phase transition becomes 2nd order and the Lévy index of stability decreases to the correlation exponent of QCD. As this transition has the same universality class as that of the 3d Ising model, one expects a decrease from the $\alpha \approx 2$ values that are characteristic to a Boltzmann gas and normal diffusion to $\alpha = 0.5$, corresponding to the correlation exponent of QCD at the critical endpoint. As shown in Fig. 1, such a change in the shape parameter makes the Bose-Einstein correlation functions much sharper than a simple Gaussian, so the spiky structure of the correlation function could be used to search for this point experimentally.

CONCLUSIONS

We have recapitulated earlier results that indicate, that the general shape of the Bose-Einstein or HBT correlation functions is a stretched exponential or Lévy stable form, where the Lévy index of stability becomes a new shape parameter of the correlation function with $0 < \alpha \leq 2$ and the popular Gaussian parameterization corresponds to the $\alpha = 2$ particular, special case. Then we have studied two physically interesting examples.

In case of particle emission from jets, we have recapitulated the connection between the stability index of the Bose-Einstein/HBT correlation functions and the running coupling constant of QCD.

We have also considered a scenario, when the power-law tail of a Lévy distribution of the particle emission *in the coordinate space* appears due to a second-order QCD phase transition. In this case, the Lévy index of stability of the Bose-Einstein or HBT correlation function was shown to be equal to the correlation exponent of QCD. This value is known to be 0.5 ± 0.05 from universality class considerations. Hence by measuring the excitation function of the Lévy index of stability (the shape parameter of the two-particle Bose-Einstein or HBT correlation functions), one can experimentally determine the bombarding energy and centrality range where a heavy ion collision hits the critical end point of QCD. Clearly, more work is necessary to check to what extent this interesting effect can be masked by the decays of various resonances, hydrodynamic expansion, and by the time evolution of the particle emitting source between the second order phase transition point and the freeze-out temperature.

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