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FORWARD INTEREST RATE CURVES IN DISCRETE TIME SETTINGS DRIVEN BY RANDOM FIELDS

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Forward interest rate curves in discrete time settings driven by random fields[1]

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Abstract

In this paper we study the term structure of forward interest rates in discrete time settings. We introduce a generalisation of the classical Heath-Jarrow-Morton type models. The forward rates corresponding to different time to maturity values will be equipped with different driving processes. In this way we use a discrete time random field to drive the forward rates instead of a single process. Since we are interested only in arbitrage free markets, we derive several no-arbitrage formulas and we also give examples for the structure of the driving field. We give sufficient conditions for the uniqueness of the no-arbitrage measure and finally present some examples.

Keywords. Forward interest rate, HJM model, no-arbitrage, equivalent martingale measure, martingale, random field, AR sheet, discrete time processes

1 Introduction

In this paper we study interest rate and bond pricing structures. In the literature one can find several approaches to the formulation of interest rate structures and based on them one can derive prices of bonds and other interest rate dependent financial assets. An overview on this subject is given e.g. in [7].

Our approach is based on an idea of Heath, Jarrow and Morton [4]. They constructed a continuous time model for the so-called forward rate structures and derived the bond prices from this structure as follows.

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Let $f(t,x)$ denote the instantaneous forward rate at time t with time to maturity x, where $x, t \in \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of the nonnegative real numbers. In the Heath-Jarrow-Morton (HJM) model the forward rates are assumed to follow the dynamics

$$df(t,x) = \alpha(t,x)\ dt + \sigma(t,x)\ dW(t), \tag{1}$$

where $\{W(t)\}_{t\in\mathbb{R}_+}$ is a standard Wiener process. In an integral form, we have

$$f(t,x) = f(0,x) + \int_0^t \alpha(u,x)\ du + \int_0^t \sigma(u,x)\ dW(u). \tag{2}$$

Having defined the forward rate dynamics, they proposed the following definition for the bond price. Denoting the price of a zero coupon bond at time t with maturity date s by $P(t,s)$, they defined the bond price by

$$P(t,s) = \exp\left\{-\int_0^{s-t} f(t,u)\ du\right\}, \qquad 0 \le t \le s. \tag{3}$$

One should emphasise that for any value $x \ge 0$ in (1), the forward rate process $\{f(t,x)\}_{t\in\mathbb{R}_+}$ is driven by the same Wiener process. Considering, for instance, the case where $\sigma(u,x)$ is deterministic, this means that the same 'shocks' have effect to all of the forward rates, which seems not to be very realistic. Therefore it is natural to generalise the model by introducing a random driving field instead of a single driving process. In this way forward rates with different time to maturity can be driven by different processes.

Such generalisation of the continuous time model has been proposed by Kennedy [6]. Later, Goldstein [2] and Santa-Clara and Sornette [9] studied such models. We can formulate the main idea as follows. Let $\{Z(t,s)\}_{t,s\in\mathbb{R}_+}$ be a random field and suppose that for each fixed $x \in \mathbb{R}_+$, the forward rate dynamics is given by

$$df(t,x) = \alpha(t,x)\ dt + \sigma(t,x)\ Z(dt,x), \tag{4}$$

where $\{Z(t,s)\}_{t\in\mathbb{R}_+}$ is a martingale for any $s \ge 0$. Writing (4) in an integral form, we have

$$f(t,x) = f(0,x) + \int_0^t \alpha(u,x)\ du + \int_0^t \sigma(u,x)\ Z(du,x). \tag{5}$$

We shall call a model like (1) classical in contrast to model (4).

The HJM model (see [4]) as well as the models studied in [6], [2] and [9] are continuous time models. It is natural to investigate also the discrete time analogue of such a model. One can find several papers on the discrete versions of the classical HJM models. Here we mention [3], [5] and [8].

In this paper, our main aim is to construct a discrete time forward interest rate model, where the forward rates corresponding to different time to maturity values are driven by different discrete time processes, that is, the forward rates are driven by a random field with discrete parameters. We emphasise that this generalisation is not simply leading to the K-factor models in a discrete setting. In the first part of the paper, in Section 2 and 3, we study classical discrete time HJM models. For calculational convenience, we mostly use a continuously compounding convention for the formulation of the bond price processes and for the discount factor process (see *Model A* in the forthcoming sections). We have to note here that results similar to that of Section 3 can be found in the literature, e.g. in [3]. However, we find it important to derive and present them in our way (which is slightly different) in order to get a comparable picture to our new model, which is introduced in Section 4.

Our second aim is to characterise no-arbitrage in the classical (Section 3) and in the new setup (Section 5), which has its consequences for the pricing problem. We will also investigate the uniqueness of the equivalent martingale measure. We give examples for random field models (Section 6) and study the consequences of no-arbitrage for them.

Based on the present study, we will be able to make a limiting transition in order to arrive at a continuous time model as suggested by [6], [2], [9] and to characterize no-arbitrage in that model. In this way we will also find the precise stochastic tools (e.g. stochastic integrals) needed for the continuous time limit models. Results in this direction together with results on parameter estimation and pricing problems will be published in our forthcoming papers (see e.g. [1]).

2 Classical HJM-type models

First we shall describe the type of financial market which is the subject of our study in this paper. The main purpose is to give and study a model for the zero coupon bonds with different maturity times. Like in the Heath-Jarrow-Morton type models, for this purpose one should introduce first the forward interest rate processes. Moreover, we need to construct models for the discount factor process of the market. This is needed for any pricing question in such a market and it is also important to emphasise that the no-arbitrage criterion can only be written by taking the discount factor into account.

Having given the definition of the forward rates, we show two possible ways to introduce the bond price processes and the discount factor. Both formulations have certain advantages and disadvantages in our problems. We call these approaches *Model A* and *Model B*.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_k\}_{k\in\mathbb{Z}_+}$, where $\mathbb{Z}_+$ denotes the set of nonnegative integers.

For what follows, $f(k, j)$ will denote the instantaneous forward rate at time k with time to maturity j, where $k \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_+$. We assume that the initial values $f(0, j)$, $j \in \mathbb{Z}_+$, are $\mathcal{F}_0$-measurable, since they are known at time 0. Next, we suppose that after time 0 the forward rates are given by the following equations:

$$f(k+1,j) = f(k,j) + \alpha(k,j) + \sigma(k,j)\left(S_{k+1} - S_k\right) \tag{6}$$

for $k, j \in \mathbb{Z}_+$, where $\{S_k\}_{k\in\mathbb{Z}_+}$ is a martingale with respect to the filtration $\{\mathcal{F}_k\}_{k\in\mathbb{Z}_+}$. For the increments of the process we use the notation $\Delta S_k = S_{k+1} - S_k$, $k \in \mathbb{Z}_+$. Furthermore, $\alpha(k,j)$ and $\sigma(k,j)$ are random variables which are supposed to be measurable with respect to $\mathcal{F}_k$ for all $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+$. Equivalently with (6), one can use the form

$$f(k,j) = f(0,j) + \sum_{i=0}^{k-1} \alpha(i,j) + \sum_{i=0}^{k-1} \sigma(i,j)\Delta S_i. \tag{7}$$

Now, it is natural to define the interest rate holding for the period $t = k$ to $t = k+1$ by

$$r(k) := f(k,0) \qquad \text{for all} \quad k \in \mathbb{Z}_+.$$

Model A. In this approach one can say that we formulate the returns of assets and also the discount factor using a continuous compounding convention, which leads in fact to a certain exponential form. In other words, the logreturns (the logarithm of the returns) are modeled directly and not the returns. This looks very much like the continuous formulation.

It is assumed in the market that there is a stochastic discount factor process, say $\{M(k)\}_{k\in\mathbb{Z}_+}$, which is the key process in order to price the financial assets in the market. First, set $M(0) := 1$ and next we suppose that

$$M(k+1) = M(k)\exp\left\{-r(k) + \phi(k)\Delta S_k\right\}, \qquad k \in \mathbb{Z}_+, \tag{8}$$

where $\phi(k)$ is an $\mathcal{F}_k$-measurable random variable for all $k \in \mathbb{Z}_+$. Thus one can write

$$\log M(k+1) = \log M(k) - r(k) + \phi(k)\Delta S_k, \qquad k \in \mathbb{Z}_+, \tag{9}$$

or, alternatively,

$$M(k) = \exp\left\{-\sum_{i=0}^{k-1} r(i) + \sum_{i=0}^{k-1} \phi(i)\Delta S_i\right\} \qquad k \in \mathbb{Z}_+.$$

Let $P(k,\ell)$ denote the price of a zero coupon bond at time k with maturity ℓ for all $0 \le k \le \ell$. Hence we put $P(k,k) := 1$ and in general we define

$$P(k,\ell+1) = P(k,\ell)\exp\left\{-f(k,\ell-k)\right\}, \qquad 0 \le k \le \ell, \tag{10}$$

or, to put it in another way,

$$\log P(k,k+j+1) = \log P(k,k+j) - f(k,j), \qquad k,j \in \mathbb{Z}_+. \tag{11}$$

Thus, one has

$$P(k,\ell) = \exp\left\{ - \sum_{j=0}^{\ell-k-1} f(k,j) \right\}, \qquad 0 \le k \le \ell.$$

Model B. In discrete time settings it is also common to write the returns without the continuous compounding convention used in *Model A*, and so we can make them simpler. For instance, consider option theory. In the classical continuous time option pricing problems the stock price process is of an 'exponential type', since it is a geometric Brownian motion. However, in the discrete time settings we use the simple binomial or binary tree models, where the asset price is multiplied by a factor $(1+\rho)$ to get the new asset price.

In such type of formulation we can construct models for the discount factor and the bond process as follows.

Let $M(0) := 1$ again and define

$$M(k+1) = \frac{M(k)}{1 + r(k) - \phi(k)\Delta S_k}, \qquad k \in \mathbb{Z}_+,$$

that is,

$$M(k) = \frac{1}{\prod_{i=0}^{k-1}(1 + r(i) - \phi(i)\Delta S_i)}, \qquad k \in \mathbb{Z}_+.$$

In a similar way, the price of the zero coupon bond is defined by $P(k,k) := 1$, $k \in \mathbb{Z}_+$ and

$$P(k,\ell+1) = \frac{P(k,\ell)}{1 + f(k,\ell-k)}, \qquad 0 \le k \le \ell,$$

and hence we can write

$$P(k,\ell) = \frac{1}{\prod_{j=0}^{\ell-k-1}(1 + f(k,j))}, \qquad 0 \le k \le \ell.$$

3 No-arbitrage criteria in the classical model

The most important property one requires to make the model realistic is the no-arbitrage condition of the market.

Definition 1 *We say that the market satisfies the no-arbitrage criterion if there exists a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ which is equivalent with measure $\mathbb{P}$ such that for each $\ell \in \mathbb{Z}_+$ the discounted value process of the bond $\{P(k,\ell)M(k)\}_{0 \le k \le \ell}$ is a $\mathbb{P}^*$-martingale. Such a measure $\mathbb{P}^*$ will be called equivalent martingale measure.*

In the following, we shall write simply *a.s.* instead of $\mathbb{P}$-*a.s.* or $\mathbb{P}^*$-*a.s.*, if $\mathbb{P}^*$ is an equivalent martingale measure, since due to their equivalence the two notions are the same.

We mention here that one might find some equivalent formulations with this definition in the literature. In the following we present different forms of the condition at issue.

Proposition 1 *Suppose that $\mathbb{P}^*$ is an equivalent measure with $\mathbb{P}$ in* Model A. *Then $\mathbb{P}^*$ is an equivalent martingale measure if and only if we have for all $0 \le k < l$*

$$\begin{aligned} &\mathbb{E}^*\left(\exp\left\{\left(\phi(k) - \sum_{j=0}^{l-k-2}\sigma(k,j)\right)\Delta S_k\right\}\middle|\mathcal{F}_k\right) \\ &\quad = \exp\left\{r(k) - f(k,l-k-1) + \sum_{j=0}^{l-k-2}\alpha(k,j)\right\}, \qquad a.s., \end{aligned} \tag{12}$$

where $\mathbb{E}^*$ indicates expectation taken with respect to $\mathbb{P}^*$.

If the increment ΔS_k is $\mathbb{P}^*$-independent of $\mathcal{F}_k$ ($k \in \mathbb{Z}_+$) then the no-arbitrage condition (12) is equivalent with

$$G_k^*\left(\phi(k) - \sum_{j=0}^{\ell-k-2}\sigma(k,j)\right) = \exp\left\{r(k) - f(k,\ell-k-1) + \sum_{j=0}^{\ell-k-2}\alpha(k,j)\right\} \quad a.s. \tag{13}$$

for all $0 \le k < \ell$, where G_k^* is the moment generating function of ΔS_k with respect to the measure $\mathbb{P}^*$.

Proof. First note that

$$\frac{P(k+1,\ell)}{P(k,\ell)} = \exp\left\{f(k,\ell-k-1) - \sum_{j=0}^{\ell-k-2}\alpha(k,j) - \Delta S_k\sum_{j=0}^{\ell-k-2}\sigma(k,j)\right\},$$

$0 \le k < \ell$. Now, write

$$P(k+1,\ell)M(k+1) = P(k,\ell)M(k)A(k,\ell),$$

where

$$A(k,l) = \exp\left\{-r(k) + f(k,\ell-k-1) - \sum_{j=0}^{\ell-k-2}\alpha(k,j) + \left(\phi(k) - \sum_{j=0}^{\ell-k-2}\sigma(k,j)\right)\Delta S_k\right\}.$$

Hence, the no-arbitrage condition is equivalent with

$$\mathbb{E}^*(A(k,\ell)|\mathcal{F}_k) = 1 \qquad \text{a.s. for all } 0 \le k < \ell. \tag{14}$$

There remains the left hand side of (14) to be calculated:

$$\mathbb{E}^*(A(k,\ell)|\mathcal{F}_k) = \exp\left\{-r(k) + f(k,\ell-k-1) - \sum_{j=0}^{\ell-k-2} \alpha(k,j)\right\} \mathbb{E}^*\left(\exp\left\{c(k,\ell)\Delta S_k\right\}|\mathcal{F}_k\right),$$

where $c(k,l) = \phi(k) - \sum_{j=0}^{\ell-k-2}\sigma(k,j)$. Hence we obtain (12).

To see (13), note that $c(k,\ell)$ is measurable with respect to $\mathcal{F}_k$, and recall that ΔS_k is independent of $\mathcal{F}_k$. Hence

$$\mathbb{E}^*\left(\exp\left\{c(k,\ell)\Delta S_k\right\}|\mathcal{F}_k\right) = G_k^*(c(k,\ell)) \qquad \text{a.s.}$$

for $0 \le k < \ell$. □

Corollary 1 *If for all $k \ge 0$, the r.v. ΔS_k is $\mathbb{P}^*$-independent of $\mathcal{F}_k$ and standard normal with respect to an equivalent martingale measure $\mathbb{P}^*$ then we can write the no-arbitrage condition in* Model A *in the form*

$$f(k,m) = r(k) + \sum_{j=0}^{m-1}\alpha(k,j) - \frac{1}{2}\left[\phi(k) - \sum_{j=0}^{m-1}\sigma(k,j)\right]^2, \quad a.s.,\ k \ge 0,\ m \ge 0. \quad (15)$$

Moreover,

$$f(k,m) = f(0,m+k) + \sum_{i=0}^{k-1} a(i,k+m-i-1) + \sum_{i=0}^{k-1}\sigma(i,k+m-i-1)\Delta S_i, \qquad a.s. \quad (16)$$

for $k,m \in \mathbb{Z}_+$, where

$$a(i,\ell) = \sigma(i,\ell)\left[\sum_{j=0}^{\ell-1}\sigma(i,j) - \phi(i) + \frac{1}{2}\sigma(i,\ell)\right], \qquad \text{for } i,\ell \in \mathbb{Z}_+.$$

Proof. Indeed, due to the fact that $G_k^*(z) = \exp\{\frac{1}{2}z^2\}$ we have

$$r(k) - f(k,\ell-k-1) + \sum_{j=0}^{\ell-k-2}\alpha(k,j) = \frac{1}{2}\left[\phi(k) - \sum_{j=0}^{\ell-k-2}\sigma(k,j)\right]^2 \qquad \text{a.s.}$$

for $0 \le k < \ell$. Then, with $m = \ell - k - 1$ we obtain (15).

To derive formula (16) we use (15) to obtain for $i \ge 0$ and $\ell \ge 0$

$$f(i,\ell+1) - f(i,\ell) = \alpha(i,\ell) + \phi(i)\sigma(i,\ell) - \frac{1}{2}\left[\sum_{j=0}^{\ell}\sigma(i,j)\right]^2 + \frac{1}{2}\left[\sum_{j=0}^{\ell-1}\sigma(i,j)\right]^2$$

$$= \alpha(i,\ell) + \phi(i)\sigma(i,\ell) - \sigma(i,\ell)\sum_{j=0}^{\ell-1}\sigma(i,j) - \frac{1}{2}\sigma(i,\ell)^2.$$

Substitution of $\alpha(i,\ell)$ in this expression by using (6) leads to

$$\begin{aligned} f(i+1,\ell) - f(i,\ell+1) &= \sigma(i,\ell)\Delta S_i + \sigma(i,\ell)\left[\sum_{j=0}^{\ell-1}\sigma(i,j) - \phi(i) + \frac{1}{2}\sigma(i,\ell)\right] \\ &= \sigma(i,\ell)\Delta S_i + a(i,\ell) \end{aligned} \tag{17}$$

and hence to (16). □

Fix a maturity time T and suppose that we are interested in the interest rate corresponding to the interval $[T, T+1]$. Before T, we do not know $r(T)$. If we are at time k then our 'prediction' for $r(T)$ is $f(k,m)$, where $m = T-k$. Thus, formula (16) explains how the first prediction $f(0,T)$ is modified period by period up to time k in order to arrive finally at the value $f(k,m)$.

Remark 1 So far we studied the no-arbitrage criterion only in *Model A*. We should mention, however, that similar no-arbitrage criteria can be given in *Model B* as well. Without proof we present the analogue of (12) in *Model B*: for $0 \le k < \ell$

$$\begin{aligned} \mathbb{E}^*\left(\frac{1}{(1+r(k)-\phi(k)\Delta S_k)\prod_{j=0}^{\ell-k-2}(1+f(k,j)+\alpha(k,j)+\sigma(k,j)\Delta S_k)}\middle|\mathcal{F}_k\right) \\ = \frac{1}{\prod_{j=0}^{\ell-k-1}(1+f(k,j))}, \quad \text{a.s.} \end{aligned} \tag{18}$$

Comparing (12) to (18) one can see that in *Model A* the verification of the no-arbitrage criteria seems to be easier than in *Model B*. For instance, we derived (13) in *Model A*, which is quite useful, since in most of the cases we will suppose to have independence between the increments and the σ-algebra generated by the past. Thus we have a formula written in terms of moment generating functions which might be easily calculated for certain probability laws. Unfortunately, in *Model B* we cannot derive a formula which would be as good for practical purposes as (13). One has to mention, however, that for other purposes one might find *Model B* more appropriate than *Model A*.

Proposition 2 *Consider* Model A. *Let us suppose that we have an equivalent martingale measure* $\mathbb{P}^*$ *such that the increment* ΔS_k *is* $\mathbb{P}^*$*-independent of* $\mathcal{F}_k$ *for* $k \in \mathbb{Z}_+$ *and it concentrates on* $\{1,-1\}$*, i.e.,*

$$p_k^* := \mathbb{P}^*(\Delta S_k = 1) = 1 - \mathbb{P}^*(\Delta S_k = -1) \in (0,1).$$

(a) Then $\mathbb{P}^*$ *is a unique equivalent martingale measure. Moreover,* $\phi(k)$ *is deterministic for all* $k \in \mathbb{Z}_+$*.*

Assume, furthermore, that for $k, j \in \mathbb{Z}_+$ the $\alpha(k, j)$, $\sigma(k, j)$ and $f(0, j)$ are all deterministic and $\phi(k) \neq 0$.
(b) Then we have $\sigma(k, \ell) = \sigma(k)$, $k, \ell \in \mathbb{Z}_+$, and the values $\alpha(k, \ell)$, $k, \ell \geq 0$ are all uniquely determined in the model provided that $\{\sigma(k)\}_{k\in\mathbb{Z}_+}$, $\{f(0, j)\}_{j\in\mathbb{Z}_+}$'s and $\{\phi(k)\}_{k\in\mathbb{Z}_+}$ are given.

Proof. In this setup (13) can be written in the form for $0 \leq k < \ell$

$$p_k^* \exp\left\{\phi(k) - \sum_{j=0}^{\ell-k-2} \sigma(k, j)\right\} + (1 - p_k^*) \exp\left\{-\phi(k) + \sum_{j=0}^{\ell-k-2} \sigma(k, j)\right\}$$
$$= \exp\left\{r(k) - f(k, \ell - k - 1) + \sum_{j=0}^{\ell-k-2} \alpha(k, j)\right\}, \quad \text{a.s.} \tag{19}$$

Since (19) is linear in p_k^* it has a unique solution for p_k^*. Furthermore, (19) for $\ell = k + 1$ gives that $\phi(k)$ is deterministic for all $k \in \mathbb{Z}_+$.

Now, we turn to prove (b). By using (7) we can write (19) for $0 \leq k < \ell$ as

$$p_k^* \exp\left\{\phi(k) - \sum_{j=0}^{\ell-k-2} \sigma(k, j)\right\} + (1 - p_k^*) \exp\left\{-\phi(k) + \sum_{j=0}^{\ell-k-2} \sigma(k, j)\right\}$$
$$= \exp\left\{f(0, 0) - f(0, \ell - k - 1) + \sum_{i=0}^{k-1} \Big[\alpha(i, 0) - \alpha(i, \ell - k - 1)\Big]\right. \tag{20}$$
$$\left. + \sum_{i=0}^{k-1} \Big[\sigma(i, 0) - \sigma(i, \ell - k - 1)\Big] \Delta S_i + \sum_{j=0}^{\ell-k-2} \alpha(k, j)\right\}, \quad \text{a.s.}$$

The left hand-side of (20) is deterministic, so has to be the right hand-side. So,

$$\sum_{i=0}^{k-1} [\sigma(i, 0) - \sigma(i, \ell - k - 1)] \Delta S_i = 0 \qquad \text{a.s. for } 0 \leq k < \ell$$

and hence

$$\sigma(i, 0) - \sigma(i, j) = 0 \quad \text{for } i, j \in \mathbb{Z}_+.$$

It follows that we have a.s. for $0 \leq k < \ell$

$$p_k^* \exp\{\phi(k) - (\ell - k - 1)\sigma(k)\} + (1 - p_k^*) \exp\{-\phi(k) + (\ell - k - 1)\sigma(k)\}$$
$$= \exp\left\{f(0, 0) - f(0, \ell - k - 1) + \sum_{i=0}^{k-1} \Big[\alpha(i, 0) - \alpha(i, \ell - k - 1)\Big] + \sum_{j=0}^{\ell-k-2} \alpha(k, j)\right\}. \tag{21}$$

Equation (21) shows that under our assumptions the values $\alpha(k,\ell)$ cannot be chosen freely when we would like to set the parameters of the model (since the $\alpha(k,\ell)$'s are uniquely given in a recursive way from (21)). □

Remark 2 We saw in the model studied in statement (b) of Proposition 2 that the $\sigma(k,j)$'s do not depend on j, which means that the model is not as general as one would think. The reason for that, as it can clearly be seen from the proof, is the fact that there is only one driving process for the forward rates corresponding to different time to maturity values. This is a reason why in the next section of this paper we introduce a more general setup.

Remark 3 We mention that one could also consider the even simpler case where $\phi(k) = 0$ for all $k \in \mathbb{Z}_+$. Then the only difference is that one can choose the value $\alpha(k,0)$ freely from a certain interval, but for $k \in \mathbb{Z}_+$ and $\ell > 0$, the values $\alpha(k,\ell)$ are all uniquely determined again. Indeed, for $\ell = k+1$ both sides of (19) reduce to 1. Thus p_k^* should be determined by the aid of the next equation. Due to the monotonicity of the exponential function, for $\ell = k+2$ the equation (20) can be satisfied if and only if

$$\alpha(k,0) \in \big(-\sigma(k,0) - \delta_k \, , \, \sigma(k,0) - \delta_k \big),$$

where

$$\delta_k = f(0,0) + f(0,1) + \sum_{i=0}^{k-1} \big[\alpha(i,0) - \alpha(i,1)\big] + \sum_{i=0}^{k-1} \big[\sigma(i,0) - \sigma(i,1)\big] \Delta S_i.$$

Thus the assumption $\phi(k) = 0$ gives a little freedom at the choice of the values $\alpha(k,0)$.

Remark 4 Another important assertion could be the following. Suppose now, that the $\sigma(k,j)$'s, $f(0,j)$'s and $\phi(k)$'s $(k,j \in \mathbb{Z}_+)$ are all deterministic in the model. Then the following two statements are equivalent:

(1) $\alpha(k,\ell)$ is deterministic for all $k,\ell \in \mathbb{Z}_+$,

(2) $\sigma(k,\ell) = \sigma(k)$, $k,\ell \in \mathbb{Z}_+$.

This is trivial since we saw that (19) gives a system of equation that p_k^*, $k \in \mathbb{Z}_+$, should fulfill almost surely, that is, for almost every $\omega \in \Omega$.

4 A new model, based on random fields

Definitions and assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and suppose that $\{\mathcal{F}_k\}_{k\in\mathbb{Z}_+}$ is a filtration on it.

First, suppose that $\{S(k,\ell)\}_{k,\ell\in\mathbb{Z}_+}$ is a random field, i.e. $S(k,\ell)$ is a random variable for all $k,\ell\in\mathbb{Z}_+$. We will use the notation $\Delta_1 S(k,\ell) := S(k+1,\ell) - S(k,\ell)$. *We impose the following assumption on the driving process S:*

(A1) For each $\ell \in \mathbb{Z}_+$ the process $\{S(k,\ell)\}_{k\in\mathbb{Z}_+}$ is a square-integrable martingale with respect to $\{\mathcal{F}_k\}_{k\in\mathbb{Z}_+}$, that is, $S(k,\ell)$ is $\mathcal{F}_k$-measurable and

$$\mathbb{E}\left(\Delta_1 S(k,\ell)|\mathcal{F}_k\right) = 0 \qquad \text{a.s. for } k \geq 0.$$

We shall write

$$c(k,\ell_1,\ell_2) := \operatorname{cov}\left(\Delta_1 S(k,\ell_1)\ ,\ \Delta_1 S(k,\ell_2)\right)$$

and

$$\sigma_{k,\ell}^2 := c(k,\ell,\ell) = \operatorname{Var}\Delta_1 S(k,\ell).$$

Note that for practical purposes one may assume furthermore that $c(k,\ell_1,\ell_2)$ does not depend on k. This would mean that the covariance of the increments is independent of the time parameter.

Now, we define the instantaneous forward rate $f(k,j)$ at time k with time to maturity j as follows:

$$f(k+1,j) = f(k,j) + \alpha(k,j) + \sigma(k,j)\Delta_1 S(k,j), \tag{22}$$

where $k\in\mathbb{Z}_+$, $j\in\mathbb{Z}_+$. One can write equivalently

$$f(k,j) = f(0,j) + \sum_{i=0}^{k-1}\alpha(i,j) + \sum_{i=0}^{k-1}\sigma(i,j)\Delta_1 S(i,j). \tag{23}$$

In (22) and (23) the $\alpha(k,j)$'s and $\sigma(k,j)$'s are all random variables for $k\in\mathbb{Z}_+$, $j\in\mathbb{Z}_+$. We shall suppose that for all $j\in\mathbb{Z}_+$, the processes $\{\alpha(k,j)\}_{k\in\mathbb{Z}_+}$ and $\{\sigma(k,j)\}_{k\in\mathbb{Z}_+}$ are adapted to the filtration $\{\mathcal{F}_k\}_{k\in\mathbb{Z}_+}$, i.e., $\alpha(k,j)$ and $\sigma(k,j)$ are $\mathcal{F}_k$-measurable.

Thus we have a model where the forward interest rate value $f(k,j)$ can be considered to be announced at time k since $f(k,j)$ is measurable with respect to $\mathcal{F}_k$.

As in the classical model, the interest rate at time k —holding for the period $t=k$ to $t=k+1$— is defined by

$$r(k) = f(k,0) \qquad \text{for} \quad k\in\mathbb{Z}_+.$$

Again one can build up two approaches for the construction of the discount factors and the bond price processes.

Model A. The stochastic discount factor process $\{M(k)\}_{k\in\mathbb{Z}_+}$ of the market is supposed to have the following dynamics: $M(0) := 1$ and

$$M(k+1) = M(k)\exp\left\{-r(k) + \sum_{j=0}^{\infty}\phi(k,j)\Delta_1 S(k,j)\right\}, \qquad k\in\mathbb{Z}_+, \tag{24}$$

where $\phi(k,j)$ is an $\mathcal{F}_k$-measurable random variable for $k,j \geq 0$, and we assume that $\sum_{j=0}^{\infty} \phi(k,j)\Delta_1 S(k,j)$ exists in L_2-sense. A natural way of discounting would be to take the defining equation (24) such that $\phi(k,j) = 0$ for all $k,j \in \mathbb{Z}_+$, that is, the discounting would be done only with the interest rate values, as it is often the case in the literature. However, (24) allows the discount factors to be also modified at time k by each of the shocks corresponding to time k. Similar discount processes were considered in [9].

The condition on the L_2-convergence is taken in order to guarantee the sum in (24) to be well defined. Here we mention that to guarantee the L_2-convergence one can find some sufficient conditions. For instance, consider the case where $\Delta_1 S(k,j)$ is independent of $\mathcal{F}_k$ for $k,j \in \mathbb{Z}_+$. Then the condition

$$\sum_{j=0}^{\infty} \sigma_{k,j} \sqrt{\mathbb{E}\phi(k,j)^2} < \infty, \qquad k \in \mathbb{Z}_+, \tag{25}$$

is sufficient for the L_2 convergence of the series at issue. Indeed, take $0 \leq m \leq n$. Then by the independence and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{j=m}^{n} \phi(k,j)\Delta_1 S(k,j)\right)^2 &= \left|\sum_{j_1=m}^{n}\sum_{j_2=m}^{n} \mathbb{E}\phi(k,j_1)\phi(k,j_2)\mathbb{E}\Delta_1 S(k,j_1)\Delta_1 S(k,j_2)\right| \\
&\leq \sum_{j_1=m}^{n}\sum_{j_2=m}^{n} \sqrt{\mathbb{E}\phi(k,j_1)^2\mathbb{E}\phi(k,j_2)^2\sigma_{k,j_1}^2\sigma_{k,j_2}^2} \\
&= \left(\sum_{j=m}^{n} \sigma_{k,j}\sqrt{\mathbb{E}\phi(k,j)^2}\right)^2 \longrightarrow 0 \quad \text{as } m \to \infty.
\end{aligned}$$

The definition of the price of a zero coupon bond $P(k,\ell)$ at time k with maturity ℓ for all $0 \leq k \leq \ell$ remains the same, i.e., $P(k,k) := 1$ and

$$P(k,\ell+1) = P(k,\ell)\exp\{-f(k,\ell-k)\}, \quad \text{if} \quad 0 \leq k \leq \ell. \tag{26}$$

For the further calculations we find it useful to derive the quotient $\frac{P(k+1,\ell)}{P(k,\ell)}$. It is easy to see that

$$\begin{aligned}
\frac{P(k+1,\ell)}{P(k,\ell)} &= \exp\left\{ f(k,\ell-k-1) - \sum_{j=0}^{\ell-k-2} \left(f(k+1,j) - f(k,j)\right)\right\} \\
&= \exp\left\{ f(k,\ell-k-1) - \sum_{j=0}^{\ell-k-2} \alpha(k,j) - \sum_{j=0}^{\ell-k-2} \sigma(k,j)\Delta_1 S(k,j)\right\}.
\end{aligned}$$

Model B. As in the classical case, one could take an alternative way to construct the discount and bond price processes as follows.

Set $M(0) := 1$ again and define

$$M(k+1) = \frac{M(k)}{1 + r(k) - \sum_{j=0}^{\infty} \phi(k,j)\Delta S(k,j)}, \qquad k > 0.$$

In a similar way, the price of the zero coupon bond is defined by $P(k,k) := 1$, $k \in \mathbb{Z}_+$ and

$$P(k,\ell+1) = \frac{P(k,\ell)}{1 + f(k,\ell-k)}, \qquad 0 \le k \le \ell.$$

5 No-arbitrage criteria in case of random fields

In this section we will give results which are analogous to the results given in Section 3.

Theorem 1 *Suppose that* $\mathbb{P}^*$ *is an equivalent measure with* $\mathbb{P}$ *in* Model A. *Then* $\mathbb{P}^*$ *is an equivalent martingale measure if and only if we have a.s. for all* $0 \le k < \ell$

$$\mathbb{E}^*\left(\exp\left\{\sum_{j=0}^{\infty} \psi_\ell(k,j)\Delta_1 S(k,j)\right\}\middle|\mathcal{F}_k\right) = \exp\left\{r(k) - f(k,\ell-k-1) + \sum_{j=0}^{\ell-k-2} \alpha(k,j)\right\}, \tag{27}$$

where

$$\psi_\ell(k,j) := \begin{cases} \phi(k,j) - \sigma(k,j), & \text{if } \; 0 \le j \le \ell-k-2 \\ \phi(k,j), & \text{if } \; \ell-k-1 \le j. \end{cases}$$

If, furthermore, $\phi(k,j) = 0$ *for* $j > N$*, where* $N \in \mathbb{Z}_+$ *is fixed, and for each* $k \in \mathbb{Z}_+$ *the increments* $\Delta_1 S(k,j)$*,* $j \in \mathbb{Z}_+$*, are all* $\mathbb{P}^*$*-independent of* $\mathcal{F}_k$ *then the no-arbitrage condition (27) can be written as*

$$\begin{aligned} &G^*_{k,N\vee(\ell-k-2)}(\psi_\ell(k,0),\ldots,\psi_\ell(k,N\vee(\ell-k-2))) \\ &\qquad = \exp\left\{r(k) - f(k,\ell-k-1) + \sum_{j=0}^{\ell-k-2} \alpha(k,j)\right\}, \end{aligned} \tag{28}$$

where $G^*_{k,i}$ *is the joint moment generating function of* $\Delta_1 S(k,0),\ldots,\Delta_1 S(k,i)$ *with respect to measure* $\mathbb{P}^*$.

Proof. First note that

$$P(k+1,\ell)M(k+1) = P(k,\ell)M(k)A(k,\ell), \qquad 0 \le k < l,$$

where

$$A(k,\ell) = \exp\Bigg\{ -r(k) + f(k,\ell-k-1) - \sum_{j=0}^{\ell-k-2} \alpha(k,j) - \sum_{j=0}^{\ell-k-2} \sigma(k,j)\Delta_1 S(k,j) + \sum_{j=0}^{\infty} \phi(k,j)\Delta_1 S(k,j)\Bigg\}.$$

Now, the process $\{P(k,\ell)M(k)\}_{0\le k\le \ell}$ is a martingale if and only if

$$\mathbb{E}\left(A(k,\ell)|\mathcal{F}_k\right) = 1 \quad \text{a.s. for} \quad 0 \le k < \ell.$$

It only remains to be mentioned that $B(k,\ell) = \exp\left\{f(k,\ell-k-1) - \sum_{j=0}^{\ell-k-2}\alpha(k,j) - r(k)\right\}$ is measurable with respect to $\mathcal{F}_k$. Thus we get (27).

Next, (28) is also immediate in case of the independence of the increment $\Delta_1 S(k,j)$ $(j \in \mathbb{Z}_+)$ of $\mathcal{F}_k$. □

We mention that the assumption $\phi(k,j) = 0$ for $j > N$ was crucial in the derivation of the left hand-side of (28). By letting $N = \infty$ one would have further difficulties to calculate the conditional expectation in (27) in order to arrive at a simple formula like (28).

Corollary 2 *Assume that $\phi(k,j) = 0$ for $j > N$, where $N \in \mathbb{Z}_+$ is fixed. If the random vector $(\Delta_1 S(k,0), \Delta_1 S(k,1), \ldots, \Delta_1 S(k,j))$ is normally distributed with respect to an equivalent martingale measure $\mathbb{P}^*$ and $\mathbb{P}^*$-independent of $\mathcal{F}_k$ for all $k \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_+$ then the no-arbitrage criterion in* Model A *implies*

$$f(k,m) - \sum_{j=0}^{m-1} \alpha(k,j) - r(k) + \frac{1}{2}\sum_{j=0}^{N\vee(m-1)} \psi_{m+k+1}(k,j)^2\sigma_{k,j}^2 + \sum_{j_1=0}^{N\vee(m-1)}\sum_{j_2=j_1+1}^{N\vee(m-1)} \psi_{m+k+1}(k,j_1)\psi_{m+k-1}(k,j_2)c(k,j_1,j_2) = 0 \quad a.s. \tag{29}$$

for $k, m \in \mathbb{Z}_+$.
Furthermore,

$$f(k,m) = f(0,m+k) + \sum_{i=0}^{k-1} a(i,m+k-i-1) + \sum_{i=0}^{k-1}\sigma(i,m+k-i-1)\Delta_1 S(i,m+k-i-1), \tag{30}$$

where

$$a(i,\ell) = \sigma(i,\ell)\left[-\sum_{j=0}^{N\vee(\ell-1)}\phi(i,j)c(i,j,\ell) + \sum_{j=0}^{\ell-1}\sigma(i,j)c(i,j,\ell) + \frac{1}{2}\sigma_{i,\ell}^2\sigma(i,\ell)\right] \quad i,\ell \in \mathbb{Z}_+.$$

Proof. For $x_0, \dots, x_i \in \mathbb{R}$, the random variable $\zeta(x_0, \dots, x_i) = \sum_{j=0}^{i} x_j \Delta_1 S(k,j)$ is Gaussian with mean zero and variance

$$\sigma^2_{\zeta(x_0,\dots,x_i)} = \mathbb{E}\left(\sum_{j=0}^{i} x_j \Delta_1 S(k,j)\right)^2 = \sum_{j=0}^{i} x_j^2 \mathbb{E}\Delta_1 S(k,j)^2$$
$$+ 2\sum_{j_1=0}^{i} \sum_{j_2=j_1+1}^{i} x_{j_1} x_{j_2} \mathbb{E}\Delta_1 S(k,j_1)\Delta_1 S(k,j_2),$$

from which (29) follows directly by setting $m = \ell - k - 1$ in (28), since $G^*_{k,i} = \exp\left\{\frac{1}{2}\sigma^2_{\zeta(x_0,\dots,x_i)}\right\}$.

Now we turn to the derivation of (30). For this, we start by writing (29) in the form

$$f(k,m) = r(k) + \sum_{j=0}^{m-1} \alpha(k,j)$$
$$- \frac{1}{2}\left[\sum_{j=0}^{N} \phi(k,j)^2 \sigma^2_{k,j} - 2\sum_{j=0}^{N\wedge(m-1)} \phi(k,j)\sigma(k,j)\sigma^2_{k,j} + \sum_{j=0}^{m-1} \sigma(k,j)^2\sigma^2_{k,j}\right]$$
$$- \left[\sum_{j_1=0}^{N} \sum_{j_2=j_1+1}^{N} \phi(k,j_1)\phi(k,j_2)c(k,j_1,j_2) - \sum_{j_1=0}^{m-1} \sum_{j_2=j_1+1}^{N} \sigma(k,j_1)\phi(k,j_2)c(k,j_1,j_2)\right.$$
$$\left. + \sum_{j_1=0}^{m-2} \sum_{j_2=j_1+1}^{m-1} \sigma(k,j_1)\sigma(k,j_2)c(k,j_1,j_2) - \sum_{j_1=0}^{N\wedge(m-2)} \sum_{j_2=j_1+1}^{m-1} \phi(k,j_1)\sigma(k,j_2)c(k,j_1,j_2)\right].$$

Hence for $i \geq 0$, $\ell \geq 0$ we have

$$f(i,\ell+1) - f(i,\ell) = \alpha(i,\ell) + \mathbb{1}_{\{N \geq \ell\}}\sigma^2_{i,\ell}\phi(i,\ell)\sigma(i,\ell)$$
$$- \frac{1}{2}\sigma(i,\ell)^2\sigma^2_{i,\ell} + \sum_{j_2=\ell+1}^{N} \sigma(i,\ell)\phi(i,j_2)c(i,\ell,j_2)$$
$$- \sum_{j_1=0}^{\ell-1} \sigma(i,j_1)\sigma(i,\ell)c(i,j_1,\ell) + \sum_{j_1=0}^{N\wedge(\ell-1)} \phi(i,j_1)\sigma(i,\ell)c(i,j_1,\ell).$$

Substitution of $\alpha(i,\ell)$ in this expression by using (22) leads to

$$f(i+1,\ell)-f(i,\ell+1)$$

$$=\sigma(i,\ell)\left[\Delta_1 S(i,\ell)-\sum_{j=0}^{N\vee(\ell-1)}\phi(i,j)c(i,\ell,j)+\sum_{j=0}^{\ell-1}\sigma(i,j)c(i,\ell,j)+\frac{1}{2}\sigma_{i,\ell}^2\sigma(i,\ell)\right]$$

$$=\sigma(i,\ell)\Delta_1 S(i,\ell)+a(i,\ell),$$

and hence to (30). □

6 Examples for the driving process

In the following examples we shall suppose that $\{\eta(i,j)\}_{i,j\in\mathbb{Z}_+}$ form a white noise system, i.e., let $\eta(i,j)$ be i.i.d. random variables with mean zero and variance 1 for $i,j\in\mathbb{Z}_+$. We shall, furthermore, define $\mathcal{F}_k := \sigma\left(\eta(i,j)\mid i\le k,\ j\in\mathbb{Z}_+\right)$.

Example 1 Define the driving process as a partial sum of the $\eta(k,\ell)$'s, that is,

$$S(k,\ell) := \sum_{i=0}^{k}\sum_{j=0}^{\ell}\eta(i,j) \qquad k,\ell\in\mathbb{Z}_+.$$

It gives

$$S(k+1,\ell+1)=S(k,\ell+1)+S(k+1,\ell)-S(k,\ell)+\eta(k+1,\ell+1).$$

For each $\ell\in\mathbb{Z}_+$, the independence of the $\eta(i,j)$'s together with $\mathbb{E}\eta(i,j)=0$ imply that $\{S(k,\ell)\}_{k\in\mathbb{Z}_+}$ is a martingale with respect to $\{\mathcal{F}_k\}_{k\in\mathbb{Z}_+}$ for each $\ell\in\mathbb{Z}_+$. Furthermore, we have

$$\begin{aligned}\operatorname{cov}(\Delta_1 S(k,\ell_1),\Delta_1 S(k,\ell_2)) &= \sum_{j_1=0}^{\ell_1}\sum_{j_2=0}^{\ell_2}\mathbb{E}\eta(k+1,j_1)\eta(k+1,j_2)\\ &= \sum_{j=0}^{\ell_1\wedge\ell_2}\mathbb{E}\left(\eta(k+1,j)\right)^2=\ell_1\wedge\ell_2+1 := c(\ell_1,\ell_2).\end{aligned}$$

Hence this driving process fulfills the required assumptions, furthermore, the covariance function c is independent of the time parameter k.

Example 2 (AR model) Fix a constant $\rho\in\mathbb{R}$. We define the driving process by

$$S(k,\ell)=\sum_{i=0}^{k}\sum_{j=0}^{\ell}\rho^{\ell-j}\eta(i,j), \qquad k,\ell\in\mathbb{Z}_+.$$

Hence, in this case we have

$$S(k+1,\ell+1) = S(k,\ell+1) + \rho S(k+1,\ell) - \rho S(k,\ell) + \eta(k+1,\ell+1)$$

for $k,\ell \in \mathbb{Z}_+$. Then one can write

$$\Delta_1 S(k,\ell+1) = \rho \Delta_1 S(k,\ell) + \eta(k+1,\ell+1),$$

which means that $\{\Delta_1 S(k,\ell)\}_{\ell\in\mathbb{Z}_+}$ is an autoregression process (AR(1)) with coefficient ρ.

For this, we have

$$\begin{aligned}\operatorname{cov}(\Delta_1 S(i,\ell_1), \Delta_1 S(i,\ell_2)) &= \sum_{j_1=0}^{\ell_1}\sum_{j_2=0}^{\ell_2} \rho^{\ell_1+\ell_2-j_1-j_2} \mathbb{E}\eta(i+1,j_1)\eta(i+1,j_2) \\ &= \frac{\rho^{\ell_1+\ell_2+2} - \rho^{|\ell_1-\ell_2|}}{\rho^2-1} \qquad \text{for} \quad \rho \neq \pm 1.\end{aligned}$$

Note that we have again a covariance function that does not depend on the time parameter k.

For $\rho = 1$ we have the model studied in Example 1. For $\rho = -1$, one can easily derive $\operatorname{cov}(\Delta_1 S(i,\ell_1), \Delta_1 S(i,\ell_2)) = (-1)^{\ell_1+\ell_2}(\ell_1 \wedge \ell_2 + 1)$.

Finally we mention that by the choice $\rho = 0$ we obtain $\Delta_1 S(k,j) = \eta(k+1,j)$, $k,j \in \mathbb{Z}_+$. In this case the process $\{S(k,j)\}_{k\in\mathbb{Z}_+}$ is a discrete random walk. Moreover, $S(k,j_1)$ and $S(k,j_2)$ are evolving independently for $j_1 \neq j_2$ and hence this setup is not very realistic.

In the next proposition we shall consider a simple case of the AR model.

Proposition 3 (No-arbitrage in the AR model) *Let us assume that the forward rates are driven by an autoregression field presented in Example 2 such that $\eta(i,j)$ concentrates on the set $\{1,-1\}$ with*

$$p^*_{i,j} := \mathbb{P}^*(\eta(i,j)=1) = 1 - \mathbb{P}^*(\eta(i,j)=-1) \in (0,1),$$

where $\mathbb{P}^*$ is an equivalent martingale measure. Let $\rho \neq 0$, and $\sigma(k,j) \neq 0$ a.s., $k,j \in \mathbb{Z}_+$. Suppose that $\phi(k,j) = 0$ for $j > N$, $k \in \mathbb{Z}_+$ with some $N \in \mathbb{N}$.

Then the only equivalent martingale measure is $\mathbb{P}^*$.

Proof. Let $G^*_{i,j}$ denote the moment generating function of $\eta(i,j)$ taken with respect to $\mathbb{P}^*$. Clearly, $G^*_{i,j}(x) = p^*_{i,j}e^x + (1-p^*_{i,j})e^{-x}$. In this case (28) gives us the following system of equations:

$$\prod_{i=0}^{(\ell-k-2)\vee N} G^*_{k+1,i}\left(\sum_{j=i}^{(\ell-k-2)\vee N} \rho^{j-i}\psi_\ell(k,j)\right) = \exp\left\{r(k) - f(k,\ell-k-1) + \sum_{j=0}^{\ell-k-2}\alpha(k,j)\right\}, \tag{31}$$

for $0 \le k < \ell$. In this special case we have

$$\psi_\ell(k,j) = \begin{cases} \phi(k,j) - \sigma(k,j), & \text{if } 0 \le j \le (\ell-k-2) \wedge N, \\ -\sigma(k,j), & \text{if } N+1 \le j \le \ell-k-2, \\ \phi(k,j), & \text{if } \ell-k-1 \le j \le N, \\ 0, & \text{otherwise.} \end{cases}$$

In order to get a more detailed picture of this problem, below we rewrite (31) for two particular cases. Thus, for $\ell = k+1$, (31) leads to

$$\prod_{i=0}^{N} G^*_{k+1,i}\left(\sum_{j=i}^{N} \rho^{j-i}\phi(k,j)\right) = 1, \tag{32}$$

secondly, with $k+1 \le \ell \le k+N+2$ we obtain

$$\begin{aligned} &\prod_{i=0}^{N} G^*_{k+1,i}\left(\sum_{j=i}^{N} \rho^{j-i}\phi(k,j) - \sum_{j=i}^{\ell-k-2} \rho^{j-i}\sigma(k,j)\right) = \\ &\exp\left\{ r(k) - f(k,\ell-k-1) + \sum_{j=0}^{\ell-k-2} \alpha(k,j)\right\}. \end{aligned} \tag{33}$$

Consider now the case where $\phi(k,j) \neq 0$ for $k \in \mathbb{Z}_+$ and $0 \le j \le N$. Then we have $N+2$ equations from (33) to determine $p^*_{k+1,0}, \dots, p^*_{k+1,N}$. Taking the ratio of (33) for $\ell+1$ and ℓ we obtain

$$\begin{aligned} &\frac{\prod_{i=0}^{\ell-k-2} G^*_{k+1,i}\left(\sum_{j=i}^{N} \rho^{j-i}\phi(k,j) - \sum_{j=i}^{\ell-k-1} \rho^{j-i}\sigma(k,j)\right)}{\prod_{i=0}^{\ell-k-2} G^*_{k+1,i}\left(\sum_{j=i}^{N} \rho^{j-i}\phi(k,j) - \sum_{j=i}^{\ell-k-2} \rho^{j-i}\sigma(k,j)\right)} \\ &\qquad = \exp\left\{f(k,\ell-k-1) - f(k,\ell-k) + \alpha(k,\ell-k-1)\right\}. \end{aligned} \tag{34}$$

Now, (34) gives a condition that $p^*_{k+1,\ell-k-1}$ has to fulfill. (Note that only $\ell-k-1$ moment generating functions occur in (34), which correspond to $\eta(k+1,0)$, $\eta(k+1,1)$,...,$\eta(k+1,\ell-k-2)$.) Since the function

$$p \mapsto \frac{pe^a + (1-p)e^{-a}}{pe^b + (1-p)e^{-b}}, \qquad p \in (0,1),$$

is strictly monotone if $a \neq b$, thus it follows that $p^*_{k+1,\ell-k-1}$ is uniquely determined for $\ell = k+1, \dots, k+N+1$.

One can easily see that as we increase the value of ℓ in (31), such that $\ell > k+N+2$, in each step one more generating function, namely $G^*_{k+1,\ell-k-1}$ occurs on the left hand-side of (31). Thus, for any $\ell > k+N+2$, (31) gives the condition for $p^*_{k+1,\ell-k-1}$ and

$\alpha(k, \ell-k-2)$. From this, we can see the uniqueness of $p^*_{k+1,j}$ for $j \geq N+2$ as well. Thus, we have shown that $\mathbb{P}^*$ is unique.

□

Remark 5 Consider the autoregression model discussed in Proposition 3 and suppose that the assumptions of this proposition are valid. If, furthermore, we assume that $\{\sigma(k,j)\}_{k,j\in\mathbb{Z}_+}$ and $\{f(0,j)\}_{j\in\mathbb{Z}_+}$ are all deterministic, then $\alpha(k,j)$, $k > 0$, $j \in \mathbb{Z}_+$, cannot be deterministic. This can be seen from (31). Indeed, the left hand-side of (31) is deterministic, hence the right hand-side has to be deterministic. The latter is

$$r(k) - f(k,\ell-k-1) - \sum_{j=0}^{\ell-k-2} \alpha(k,j) = \sum_{i=0}^{k-1} \alpha(i,0) - \alpha(i,\ell-k-1)$$
$$+ \sum_{i=0}^{k-1} \left(\sigma(i,0)\eta(i+1,0) - \sigma(i,\ell-k-1) \sum_{j=0}^{\ell-k-1} \rho^{\ell-k-1-j}\eta(i+1,j) \right) - \sum_{j=0}^{\ell-k-2} \alpha(k,j). \tag{35}$$

Let us write $\alpha(k,j) = \beta(k,j) + m(k,j)$, $k,j \in \mathbb{Z}_+$, where $\mathbb{E}\alpha(k,j) = m(k,j)$. One can see from (34) that even if (35) has to be deterministic, and so $\beta(k,\ell-k-1)$ cannot be chosen freely, there is still a little freedom at the choice of the value $m(k,\ell-k-1)$. Namely, one could derive an interval such that choosing $m(k,\ell-k-1)$ from that interval, the solution $p^*_{k+1,\ell-k-1}$ would be in (0,1). Since the calculation of such intervals for each $\alpha(k,j)$ could be done only in a recursive way, it would be fairly complicated.

Remark 6 There are two possible ways one could build up a model we study now. One possibility is to suppose that we fix $\sigma(k,j)$, $\phi(k,j)$, $f(0,j)$ and $\alpha(k,j)$ first. Here we have to emphasise that, as we saw so far, $\alpha(k,j)$ cannot be chosen freely in order to guarantee the existence of an equivalent martingale measure. In fact, depending on the construction, sometimes $\alpha(k,j)$ is not chosen freely at all, sometimes only its shift parameter $m(k,j)$ could be chosen freely from a certain interval. Further difficulties might be caused by the fact that these intervals can only be calculated in a recursive way. Having set up $\alpha(k,j)$ as well, the next step would be to determine the equivalent martingale measure, since it is a key object at pricing problems. Therefore, even if this way would be natural, we have calculational difficulties in this case.

Another way to look at and to construct the model would be the following. Since we may have difficulties with the choice of $\alpha(k,j)$ (more precisely with the choice of $m(k,j)$), we fix $\sigma(k,j)$, $\phi(k,j)$, $f(0,j)$, and furthermore, we fix an equivalent martingale measure. Having done this, $\alpha(k,j)$ is already uniquely defined. That is why many authors choose this second way.

Remark 7 There are certainly further questions that one could ask in the setting we introduced. First, the problem of pricing of interest rate derivatives. Secondly, one can study the limiting connections between the discrete and continuous time models. A third interesting area would be the estimations of parameters in our models. The latter two problems are considered in [1].

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