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ASYMPTOTIC INFERENCE FOR UNIT ROOTS IN SPATIAL TRIANGULAR AUTOREGRESSION

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Asymptotic inference for unit roots in spatial triangular autoregression

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Abstract

A spatial autoregressive process with two parameters is investigated in the stable and unstable cases. It is shown that the limiting distribution of the least squares estimator of these parameters is normal and the rate of convergence is $n^{3/2}$ if one of the parameters equals zero and n otherwise.

1 Introduction

Consider the AR(1) time series model

$$X_k = \begin{cases} \alpha X_{k-1} + \varepsilon_k, & k \geqslant 1, \\ 0, & k = 0. \end{cases}$$

The least squares estimator $\widehat{\alpha}_n$ of α based on the observations $\{X_k : k = 1, \dots, n\}$ is

$$\widehat{\alpha}_n = \frac{\sum_{k=1}^n X_{k-1} X_k}{\sum_{k=1}^n X_{k-1}^2}.$$

It is well known that in the stable (or, in other words, asymptotically stationary) case when $|\alpha| < 1$, the sequence $(\widehat{\alpha}_n)_{n \geqslant 1}$ is asymptotically normal (see Mann and Wald

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[15] or Anderson [1]), namely,

$$n^{1/2}(\widehat{\alpha}_n - \alpha) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1-\alpha^2).$$

In the unstable (or, in other words, unit root) case when $\alpha = 1$, the sequence $(\widehat{\alpha}_n)_{n\geqslant 1}$ is not asymptotically normal but

$$n(\widehat{\alpha}_n - 1) \xrightarrow{\mathcal{D}} \frac{\int_0^1 W(t)\,\mathrm{d}W(t)}{\int_0^1 W^2(t)\,dt},$$

where $\{W(t) : t \in [0,1]\}$ denotes a standard Wiener process (see e.g. White [23], Phillips [19] or Chan and Wei [10]).

The analysis of spatial models is of interest in many different fields such as geography, geology, biology and agriculture. One can turn to Basu and Reinsel [4] for a discussion on these applications. These authors considered a special case of the so called unilateral AR model having the form

$$X_{k,\ell} = \sum_{i=0}^{p_1}\sum_{j=0}^{p_2} \alpha_{i,j} X_{k-i,\ell-j} + \varepsilon_{k,\ell}, \qquad \alpha_{0,0} = 0. \tag{1.1}$$

A special case of the above model is the so-called doubly geometric spatial autoregressive process

$$X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} - \alpha\beta X_{k-1,\ell-1} + \varepsilon_{k,\ell},$$

introduced by Martin [16]. This was the first spatial autoregressive model for which unstability has been studied. It is, in fact, the simplest spatial model, since the product structure $\varphi(x,y) = x^2 - \alpha x - \beta y + \alpha\beta = (x-\alpha)(y-\beta)$ of its characteristic polynomial ensures that it can be considered as some kind of combination of two autoregressive processes on the line, and several properties can be derived by the analogy of one–dimensional autoregressive processes. This model has been used by Jain [14] in the study of image processing, by Martin [17], Cullis and Gleeson [11], Basu and Reinsel [5] in agricultural trials and by Tjøstheim [21] in digital filtering.

In the stable case when $|\alpha| < 1$ and $|\beta| < 1$, asymptotic normality of several estimators $(\widehat{\alpha}_{m,n}, \widehat{\beta}_{m,n})$ of (α, β) based on the observations $\{X_{k,\ell} : 1 \leqslant k \leqslant m \text{ and } 1 \leqslant \ell \leqslant n\}$ has been shown (e.g. Tjøstheim [20, 22] or Basu and Reinsel [3, 4]), namely,

$$\sqrt{mn}\begin{pmatrix}\widehat{\alpha}_{m,n} - \alpha \\ \widehat{\beta}_{m,n} - \beta\end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma_{\alpha,\beta})$$

as $m, n \to \infty$ with $m/n \to \text{constant} > 0$ with some covariance matrix $\Sigma_{\alpha,\beta}$.

In the unstable case when $\alpha = \beta = 1$, in contrast to the AR(1) model, the sequence of Gauss–Newton estimators $(\widehat{\alpha}_{n,n}, \widehat{\beta}_{n,n})$ of (α, β) has been shown to be asymptotically normal (see Bhattacharyya, Khalil and Richardson [7] and Bhattacharyya, Richardson and Franklin [8]), namely,

$$n^{3/2}\begin{pmatrix}\widehat{\alpha}_{n,n}-\alpha\\ \widehat{\beta}_{n,n}-\beta\end{pmatrix}\stackrel{\mathcal{D}}{\longrightarrow}\mathcal{N}(0,\Sigma)$$

with some covariance matrix Σ. In the unstable case $\alpha=1$, $|\beta|<1$ the least squares estimator turns out to be asymptotically normal again (Bhattacharyya, Khalil and Richardson [7]).

Baran, Pap and Zuijlen [2] discussed a special case of the model (1.1), namely, when $p_1=p_2=1$, $\alpha_{0,1}=\alpha_{1,0}=:\alpha$ and $\alpha_{1,1}=0$, which is the simplest spatial model, that can not be reduced somehow to autoregressive models on the line. This model is stable in case $|\alpha|<1/2$ (see e.g. Whittle [24], Besag [9] or Basu and Reinsel [4]), and unstable if $|\alpha|=1/2$. In [2] the asymptotic normality of the least squares estimator of the unknown parameter α is proved both in stable and unstable cases.

In the present paper we study the asymptotic properties of a more complicated special case of the model (1.1) with $p_1=p_2=1$, $\alpha_{1,1}=0$, $\alpha_{1,0}=:\alpha$ and $\alpha_{0,1}=:\beta$.

Our zero start triangular spatial autoregressive process $\{X_{k,\ell}:k,\ell\in\mathbb{Z},\ k+\ell\geqslant 0\}$ is defined as

$$X_{k,\ell}=\begin{cases}\alpha X_{k-1,\ell}+\beta X_{k,\ell-1}+\varepsilon_{k,\ell}, & \text{for}\ \ k+\ell\geqslant 1,\\ 0, & \text{for}\ \ k+\ell=0.\end{cases}\tag{1.2}$$

This model is stable in case $|\alpha|+|\beta|<1$ (see again Whittle [24], Besag [9] and Basu and Reinsel [4]), and unstable if $|\alpha|+|\beta|=1$.

For a set $H\subset\{(k,\ell)\in\mathbb{Z}^2:k+\ell\geqslant 1\}$, the least squares estimator $(\widehat{\alpha}_H,\widehat{\beta}_H)$ of (α,β) based on the observations $\{X_{k,\ell}:(k,\ell)\in H\}$ can be obtained by minimizing the sum of squares

$$\sum_{(k,\ell)\in H}\big(X_{k,\ell}-\alpha X_{k-1,\ell}-\beta X_{k,\ell-1}\big)^2$$

with respect to α and β, and it has the form

$$\begin{pmatrix}\widehat{\alpha}_H\\ \widehat{\beta}_H\end{pmatrix}=\left(\sum_{(k,\ell)\in H}\begin{pmatrix}X_{k-1,\ell}^2 & X_{k-1,\ell}X_{k,\ell-1}\\ X_{k-1,\ell}X_{k,\ell-1} & X_{k,\ell-1}^2\end{pmatrix}\right)^{-1}\sum_{(k,\ell)\in H}\begin{pmatrix}X_{k-1,\ell}X_{k,\ell}\\ X_{k,\ell-1}X_{k,\ell}\end{pmatrix}.$$

For $k,\ell\in\mathbb{Z}$ with $k+\ell\geqslant 1$, consider the triangle

$$T_{k,\ell}:=\{(i,j)\in\mathbb{Z}^2:i+j\geqslant 1,\ \ i\leqslant k\ \ \text{and}\ \ j\leqslant\ell\}.$$

For simplicity, we shall write $T_n:=T_{n,n}$ for $n\in\mathbb{N}$.

Theorem 1.1 *Let* $\{\varepsilon_{k,\ell}:k,\ell\in\mathbb{Z},\ k+\ell\geqslant 1\}$ *be independent random variables with* $\mathsf{E}\,\varepsilon_{k,\ell}=0$, $\operatorname{Var}\varepsilon_{k,\ell}=1$ *and* $\sup\{\mathsf{E}\,\varepsilon_{k,\ell}^4:k,\ell\in\mathbb{Z},\ k+\ell\geqslant 1\}<\infty$. *Assume that the model* (1.2) *is satisfied.*

If $|\alpha| + |\beta| < 1$ *then*

$$(mn)^{1/2} \begin{pmatrix} \widehat{\alpha}_{T_{m,n}} - \alpha \\ \widehat{\beta}_{T_{m,n}} - \beta \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma_{\alpha,\beta}) \quad \text{as} \quad m, n \to \infty \quad \text{with} \quad m/n \to \text{constant} > 0,$$

where

$$\Sigma_{\alpha,\beta} := \frac{1}{2\sigma^2_{\alpha,\beta}(1 - \varrho^2_{\alpha,\beta})} \begin{pmatrix} 1 & -\varrho_{\alpha,\beta} \\ -\varrho_{\alpha,\beta} & 1 \end{pmatrix},$$

and

$$\sigma^2_{\alpha,\beta} := \big((1+\alpha+\beta)(1+\alpha-\beta)(1-\alpha+\beta)(1-\alpha-\beta)\big)^{-1/2},$$

$$\varrho_{\alpha,\beta} := \begin{cases} \dfrac{(1-\alpha^2-\beta^2)\sigma^2_{\alpha,\beta} - 1}{2\alpha\beta\sigma^2_{\alpha,\beta}}, & \text{if } \alpha\beta \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $|\alpha| + |\beta| = 1$ *and* $0 < |\alpha| < 1$ *then*

$$(mn)^{1/2} \begin{pmatrix} \widehat{\alpha}_{T_{m,n}} - \alpha \\ \widehat{\beta}_{T_{m,n}} - \beta \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, (2\varrho^2_{\alpha})^{-1}\overline{\Psi}_{\alpha,\beta}\big) \quad \text{as} \quad m, n \to \infty$$

with $m/n \to \text{constant} > 0$, *where* $\varrho^2_\alpha = \big(|\alpha|(1-|\alpha|)\big)^{-1}$ *and* $\overline{\Psi}_{\alpha,\beta}$ *denotes the adjoint matrix of*

$$\Psi_{\alpha,\beta} := \begin{pmatrix} 1 & \text{sign}(\alpha\beta) \\ \text{sign}(\alpha\beta) & 1 \end{pmatrix}.$$

If $|\alpha| + |\beta| = 1$ *and* $|\alpha| \in \{0, 1\}$ *then*

$$(mn)^{3/4} \begin{pmatrix} \widehat{\alpha}_{T_{m,n}} - \alpha \\ \widehat{\beta}_{T_{m,n}} - \beta \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma) \qquad \text{as} \quad m, n \to \infty \quad \text{with} \quad m/n \to \text{constant} > 0,$$

where

$$\Sigma := \frac{3}{4} I.$$

For the sake of simplicity, we carry out the proof only for $m = n$. The general case can be handled with slight modifications. We can write

$$\begin{pmatrix} \widehat{\alpha}_{T_n} - \alpha \\ \widehat{\beta}_{T_n} - \beta \end{pmatrix} = B_n^{-1} A_n$$

with

$$A_n := \sum_{(k,\ell)\in T_n} \begin{pmatrix} X_{k-1,\ell}\varepsilon_{k,\ell} \\ X_{k,\ell-1}\varepsilon_{k,\ell} \end{pmatrix}, \qquad B_n := \sum_{(k,\ell)\in T_n} \begin{pmatrix} X^2_{k-1,\ell} & X_{k-1,\ell}X_{k,\ell-1} \\ X_{k-1,\ell}X_{k,\ell-1} & X^2_{k,\ell-1} \end{pmatrix},$$

Concerning the asymptotic behavior of A_n and B_n we can prove the following propositions.

Proposition 1.2 *If* $|\alpha| + |\beta| < 1$ *then*

$$n^{-2} B_n \xrightarrow{\mathsf{L}_2} \Sigma_{\alpha,\beta}^{-1} \qquad \text{as} \quad n \to \infty.$$

If $|\alpha| + |\beta| = 1$ *and* $0 < |\alpha| < 1$ *then*

$$n^{-5/2} B_n \xrightarrow{\mathsf{L}_2} \sigma_\alpha^2 \Psi_{\alpha,\beta} \qquad \text{as} \quad n \to \infty,$$

where

$$\sigma_\alpha^2 := \frac{2^{9/2}}{15\sqrt{\pi|\alpha|(1-|\alpha|)}}.$$

If $|\alpha| + |\beta| = 1$ *and* $|\alpha| \in \{0, 1\}$ *then*

$$n^{-3} B_n \xrightarrow{\mathsf{L}_2} \Sigma^{-1} \qquad \text{as} \quad n \to \infty.$$

Proposition 1.3 *If* $|\alpha| + |\beta| < 1$ *then*

$$n^{-1} A_n \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \Sigma_{\alpha,\beta}^{-1}\right) \qquad \text{as} \quad n \to \infty.$$

If $|\alpha| + |\beta| = 1$ *and* $0 < |\alpha| < 1$ *then*

$$n^{-5/4} A_n \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_\alpha^2 \Psi_{\alpha,\beta}\right) \qquad \text{as} \quad n \to \infty.$$

If $|\alpha| + |\beta| = 1$ *and* $|\alpha| \in \{0, 1\}$ *then*

$$n^{-3/2} A_n \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \Sigma^{-1}\right) \qquad \text{as} \quad n \to \infty.$$

In cases $|\alpha| + |\beta| < 1$ and $|\alpha| + |\beta| = 1$, $|\alpha| \in \{0, 1\}$ these propositions obviously imply the corresponding statements of Theorem 1.1. In the third case we have $B_n^{-1} = \bar{B}_n / \det B_n$, where $\bar{B}_n$ denotes the adjoint matrix of B_n. Thus, the statement of Theorem 1.1 in case $|\alpha| + |\beta| = 1$ and $0 < |\alpha| < 1$ is a consequence of the following propositions.

Proposition 1.4 *If* $|\alpha| + |\beta| = 1$ *and* $0 < |\alpha| < 1$ *then*

$$n^{-9/2} \det B_n \xrightarrow{\mathsf{P}} 2\sigma_\alpha^2 \varrho_\alpha^2 \qquad \text{as} \quad n \to \infty.$$

Proposition 1.5 *If* $|\alpha| + |\beta| = 1$ *and* $0 < |\alpha| < 1$ *then*

$$n^{-7/2} \bar{B}_n A_n \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, 2\sigma_\alpha^4 \varrho_\alpha^2 \overline{\Psi}_{\alpha,\beta}\big) \qquad \text{as} \quad n \to \infty.$$

Corollary 1.6 *If* $|\alpha| + |\beta| \leqslant 1$ *then*

$$\left(\sum_{(k,\ell) \in T_n} \begin{pmatrix} X_{k-1,\ell}^2 & X_{k-1,\ell} X_{k,\ell-1} \\ X_{k-1,\ell} X_{k,\ell-1} & X_{k,\ell-1}^2 \end{pmatrix} \right)^{1/2} \begin{pmatrix} \widehat{\alpha}_{T_{m,n}} - \alpha \\ \widehat{\beta}_{T_{m,n}} - \beta \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, I)$$

as $m, n \to \infty$ *with* $m/n \to \text{constant} > 0$, *where we take the (uniquely defined) positive semidefinite square root.*

The aim of the following discussion is to show that it suffices to prove Propositions 1.2 – 1.5 for $\alpha \geqslant 0$ and $\beta \geqslant 0$ with $\alpha+\beta \leqslant 1$. First we note that the random variable $X_{k,\ell}$ can be expressed as a linear combination of the variables $\{\varepsilon_{i,j} : (i,j) \in T_{k,\ell}\}$, namely,

$$X_{k,\ell} = \sum_{(i,j)\in T_{k,\ell}} \binom{k+\ell-i-j}{k-i} \alpha^{k-i}\beta^{\ell-j}\varepsilon_{i,j} \tag{1.3}$$

for $k,\ell \in \mathbb{Z}$ with $k+\ell \geqslant 1$. Now put $\widetilde{\varepsilon}_{k,\ell} := (-1)^{k+\ell}\varepsilon_{k,\ell}$ for $k,\ell\in\mathbb{Z}$ with $k+\ell\geqslant 1$. Then $\{\widetilde{\varepsilon}_{k,\ell} : k,\ell\in\mathbb{Z},\ k+\ell\geqslant 1\}$ are independent random variables with $\mathsf{E}\,\widetilde{\varepsilon}_{k,\ell}=0$, $\operatorname{Var}\widetilde{\varepsilon}_{k,\ell}=1$ and $\sup\{\mathsf{E}\,\widetilde{\varepsilon}_{k,\ell}^4 : k,\ell\in\mathbb{Z},\ k+\ell\geqslant 0\}<\infty$. Consider the zero start triangular spatial AR process $\{\widetilde{X}_{k,\ell} : k,\ell\in\mathbb{Z},\ k+\ell\geqslant 0\}$ defined by

$$\widetilde{X}_{k,\ell} = \begin{cases} -\alpha\widetilde{X}_{k-1,\ell} - \beta\widetilde{X}_{k,\ell-1} + \widetilde{\varepsilon}_{k,\ell}, & \text{for } k+\ell\geqslant 1,\\ 0, & \text{for } k+\ell=0,\end{cases}$$

Then, by the representation (1.3),

$$\widetilde{X}_{k,\ell} = \sum_{(i,j)\in T_{k,\ell}} \binom{k+\ell-i-j}{k-i} (-\alpha)^{k-i}(-\beta)^{\ell-j}\widetilde{\varepsilon}_{i,j} = (-1)^{k+\ell}X_{k,\ell}$$

for $k,\ell\in\mathbb{Z}$ with $k+\ell\geqslant 0$. Hence,

$$\widetilde{A}_n := \sum_{(k,\ell)\in T_n} \begin{pmatrix} \widetilde{X}_{k-1,\ell}\widetilde{\varepsilon}_{k,\ell} \\ \widetilde{X}_{k,\ell-1}\widetilde{\varepsilon}_{k,\ell}\end{pmatrix} = -A_n,$$

$$\widetilde{B}_n := \sum_{(k,\ell)\in T_n} \begin{pmatrix} \widetilde{X}_{k-1,\ell}^2 & \widetilde{X}_{k-1,\ell}\widetilde{X}_{k,\ell-1} \\ \widetilde{X}_{k-1,\ell}\widetilde{X}_{k,\ell-1} & \widetilde{X}_{k,\ell-1}^2\end{pmatrix} = B_n.$$

Consequently, in order to prove Propositions 1.2 – 1.5 for $\alpha\leqslant 0$ and $\beta\leqslant 0$ with $\alpha+\beta\geqslant -1$ it suffices to prove them for $\alpha\geqslant 0$ and $\beta\geqslant 0$ with $\alpha+\beta\leqslant 1$.

Next put $\widehat{\varepsilon}_{k,\ell} := (-1)^k\varepsilon_{k,\ell}$ for $k,\ell\in\mathbb{Z}$ with $k+\ell\geqslant 1$. Then $\{\widehat{\varepsilon}_{k,\ell} : k,\ell\in\mathbb{Z},\ k+\ell\geqslant 1\}$ are independent random variables with $\mathsf{E}\,\widehat{\varepsilon}_{k,\ell}=0$, $\operatorname{Var}\widehat{\varepsilon}_{k,\ell}=1$ and $\sup\{\mathsf{E}\,\widehat{\varepsilon}_{k,\ell}^4 : k,\ell\in\mathbb{Z},\ k+\ell\geqslant 0\}<\infty$. Consider the zero start triangular spatial AR process $\{\widehat{X}_{k,\ell} : k,\ell\in\mathbb{Z},\ k+\ell\geqslant 0\}$ defined by

$$\widehat{X}_{k,\ell} = \begin{cases} -\alpha\widehat{X}_{k-1,\ell} + \beta\widehat{X}_{k,\ell-1} + \widehat{\varepsilon}_{k,\ell}, & \text{for } k+\ell\geqslant 1,\\ 0, & \text{for } k+\ell=0.\end{cases}$$

Then, by the representation (1.3),

$$\widehat{X}_{k,\ell} = \sum_{(i,j)\in T_{k,\ell}} \binom{k+\ell-i-j}{k-i} (-\alpha)^{k-i}\beta^{\ell-j}\widehat{\varepsilon}_{i,j} = (-1)^{k}X_{k,\ell}$$

for $k, \ell \in \mathbb{Z}$ with $k + \ell \geqslant 0$. Hence,

$$\widehat{A}_n := \sum_{(k,\ell)\in T_n} \begin{pmatrix} \widehat{X}_{k-1,\ell}\widehat{\varepsilon}_{k,\ell} \\ \widehat{X}_{k,\ell-1}\widehat{\varepsilon}_{k,\ell} \end{pmatrix} = \sum_{(k,\ell)\in T_n} \begin{pmatrix} -X_{k-1,\ell}\varepsilon_{k,\ell} \\ X_{k,\ell-1}\varepsilon_{k,\ell} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A_n,$$

$$\begin{aligned} \widehat{B}_n &:= \sum_{(k,\ell)\in T_n} \begin{pmatrix} \widehat{X}_{k-1,\ell}^2 & \widehat{X}_{k-1,\ell}\widehat{X}_{k,\ell-1} \\ \widehat{X}_{k-1,\ell}\widehat{X}_{k,\ell-1} & \widehat{X}_{k,\ell-1}^2 \end{pmatrix} \\ &= \sum_{(k,\ell)\in T_n} \begin{pmatrix} X_{k-1,\ell}^2 & -X_{k-1,\ell}X_{k,\ell-1} \\ -X_{k-1,\ell}X_{k,\ell-1} & X_{k,\ell-1}^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} B_n \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Consequently, in order to prove Propositions 1.2 – 1.5 for $\alpha\beta \leqslant 0$ with $|\alpha|+|\beta| \leqslant 1$ it suffices to prove them for $\alpha \geqslant 0$ and $\beta \geqslant 0$ with $\alpha + \beta \leqslant 1$.

The proof of Propositions 1.2, 1.3, 1.4 and 1.5 are provided in Sections 3, 4, 5 and 6, respectively. Section 2 is devoted to the limiting behavior of the covariance structure of the random field $\{X_{k,\ell} : k, \ell \in \mathbb{Z},\ k + \ell \geqslant 0\}$.

2 Covariance structure

By the representation (1.3), we obtain that for all $k_1, \ell_1, k_2, \ell_2 \in \mathbb{Z}$ with $k_1 + \ell_1 \geqslant 0$ and $k_2 + \ell_2 \geqslant 0$, and for all $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} &\operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}) \\ &= \sum_{(i,j)\in T_{k_1,\ell_1}\cap T_{k_2,\ell_2}} \binom{k_1+\ell_1-i-j}{k_1-i}\binom{k_2+\ell_2-i-j}{k_2-i} \alpha^{k_1+k_2-2i}\beta^{\ell_1+\ell_2-2j}, \end{aligned} \tag{2.1}$$

where an empty sum is defined to be equal to 0.

Lemma 2.1 *If* $|\alpha| + |\beta| < 1$ *then*

$$|\operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \leqslant \frac{(|\alpha| + |\beta|)^{|k_1-k_2|+|\ell_1-\ell_2|}}{\big(1 - (|\alpha| + |\beta|)^2\big)^2}.$$

If $|\alpha| + |\beta| = 1$ *and* $0 < |\alpha| < 1$ *then*

$$|\operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \leqslant C(\alpha)\sqrt{k_1 + \ell_1 + k_2 + \ell_2}$$

with some constant $C(\alpha) > 0$.

If $|\alpha| + |\beta| = 1$ *and* $|\alpha| \in \{0, 1\}$ *then*

$$|\operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \leqslant k_1 + \ell_1 + k_2 + \ell_2.$$

Remark 2.2 In case $|\alpha| + |\beta| < 1$ one can derive the sharper estimate

$$|\operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \leqslant \sigma_{\alpha,\beta}^2 |\alpha|^{|k_1-k_2|}|\beta|^{|\ell_1-\ell_2|}$$

using the stationary solution of the equation (1.2) considering it on the whole lattice $\mathbb{Z}^2$ (see the proof of Proposition 2.3), but we do not need it.

Proof of Lemma 2.1. Suppose that $|\alpha| + |\beta| < 1$. Formula (2.1) implies

$$\begin{aligned}
|\mathrm{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| &\leqslant \sum_{(i,j)\in T_{k_1,\ell_1}\cap T_{k_2,\ell_2}} (|\alpha| + |\beta|)^{k_1+k_2+\ell_1+\ell_2-2i-2j} \\
&\leqslant (|\alpha| + |\beta|)^{|k_1-k_2|+|\ell_1-\ell_2|} \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}(|\alpha| + |\beta|)^{2(u+v)}.
\end{aligned}$$

We have

$$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}(|\alpha| + |\beta|)^{2(u+v)} = \Big(\sum_{w=0}^{\infty}(|\alpha| + |\beta|)^{2w}\Big)^2 = \frac{1}{\big(1-(|\alpha| + |\beta|)^2\big)^2},$$

hence we obtain the statement.

Now let $|\alpha| + |\beta| = 1$ and $0 < |\alpha| < 1$. If $k_1 + \ell_1 = 0$ or $k_2 + \ell_2 = 0$ then $\mathrm{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}) = 0$. By the monotonicity properties of the binomial coefficients and by Stirling's formula, for all $\alpha \in (0, 1)$ there exists a constant $c(\alpha) > 0$ such that

$$\binom{n}{k}\alpha^k(1-\alpha)^{n-k} \leqslant \frac{c(\alpha)}{\sqrt{n}}, \qquad \text{for } n = 1, 2, \ldots, \ 0 \leqslant k \leqslant n. \tag{2.2}$$

(This is also a consequence of the expansion in the Local Central Limit Theorem for Bernoulli random variables; see Petrov [18, Chapter VII, Theorem 6].) Hence, if $k_1 + \ell_1 \geqslant 1$ and $k_2 + \ell_2 \geqslant 1$ with $k_1 \leqslant k_2$ and $\ell_1 \leqslant \ell_2$ then by (2.1) and (2.2),

$$\begin{aligned}
&|\mathrm{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \\
&\quad = \left|\sum_{m=0}^{k_1+\ell_1-1}\sum_{u=0}^{m}\binom{m}{u}\binom{m+k_2-k_1+\ell_2-\ell_1}{u+k_2-k_1}\alpha^{k_2-k_1+2u}\beta^{\ell_2-\ell_1+2m-2u}\right| \\
&\quad \leqslant 1 + \sum_{m=1}^{k_1+\ell_1-1}\frac{c(|\alpha|)}{\sqrt{m+k_2-k_1+\ell_2-\ell_1}}\sum_{u=0}^{m}\binom{m}{u}|\alpha|^u(1-|\alpha|)^{m-u} \\
&\quad \leqslant 1 + c(|\alpha|)\int_0^{k_1+\ell_1}\frac{\mathrm{d}x}{\sqrt{x+k_2-k_1+\ell_2-\ell_1}} \\
&\quad = 1 + 2c(|\alpha|)\left(\sqrt{k_2+\ell_2} - \sqrt{k_2-k_1+\ell_2-\ell_1}\right) \\
&\quad \leqslant (1+2c(|\alpha|))\sqrt{k_1+k_2+\ell_1+\ell_2}.
\end{aligned}$$

If $k_1+\ell_1 \geqslant 1$ and $k_2+\ell_2 \geqslant 1$ with $k_1 \leqslant k_2$ and $\ell_1 \geqslant \ell_2$ then again by (2.1) and (2.2),

$$
\begin{aligned}
&|\operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})| \\
&\qquad = \left| \sum_{m=0}^{k_1+\ell_2-1} \sum_{u=0}^{m} \binom{m+\ell_1-\ell_2}{u} \binom{m+k_2-k_1}{u+k_2-k_1} \alpha^{k_2-k_1+2u} \beta^{\ell_1-\ell_2+2m-2u} \right| \\
&\qquad \leqslant 1 + \sum_{m=1}^{k_1+\ell_2-1} \frac{c(|\alpha|)}{\sqrt{m+\ell_1-\ell_2}} \sum_{u=0}^{m} \binom{m+k_2-k_1}{u+k_2-k_1} |\alpha|^{k_2-k_1+u} (1-|\alpha|)^{m-u} \\
&\qquad \leqslant 1 + c(|\alpha|) \int_0^{k_1+\ell_2} \frac{\mathrm{d}x}{\sqrt{x+\ell_1-\ell_2}} = 1 + 2c(|\alpha|) \left(\sqrt{k_1+\ell_1} - \sqrt{\ell_1-\ell_2} \right) \\
&\qquad \leqslant (1+2c(|\alpha|)) \sqrt{k_1+k_2+\ell_1+\ell_2}.
\end{aligned}
$$

The other cases follow by symmetry.

Next, let $\alpha = 1$ and $\beta = 0$. Then for $\ell_1 \neq \ell_2$ we have $\operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}) = 0$. Moreover $\operatorname{Cov}(X_{k_1,\ell}, X_{k_2,\ell}) = \ell + \min\{k_1; k_2\}$. The other cases can be handled similarly. □

For $n \in \mathbb{N}$, let us introduce the piecewise constant random fields

$$
\begin{aligned}
Y_{1,0}^{(n)}(s,t) &:= X_{[ns]+1,[nt]}, & Y_{0,1}^{(n)}(s,t) &:= X_{[ns],[nt]+1}, \\
Z_{1,0}^{(n)}(s,t) &:= n^{-1/4} X_{[ns]+1,[nt]}, & Z_{0,1}^{(n)}(s,t) &:= n^{-1/4} X_{[ns],[nt]+1}, \\
U_{1,0}^{(n)}(s,t) &:= n^{-1/2} X_{[ns]+1,[nt]}, & U_{0,1}^{(n)}(s,t) &:= n^{-1/2} X_{[ns],[nt]+1},
\end{aligned}
$$

for $s,t \in \mathbb{R}$ with $s+t \geqslant 0$.

Proposition 2.3 *Let* $s_1, t_1, s_2, t_2 \in \mathbb{R}$ *with* $s_1+t_1 > 0$, $s_2+t_2 > 0$.

If $|\alpha| + |\beta| < 1$ *then*

$$
\begin{pmatrix} \operatorname{Cov}(Y_{1,0}^{(n)}(s_1,t_1), Y_{1,0}^{(n)}(s_2,t_2)) & \operatorname{Cov}(Y_{1,0}^{(n)}(s_1,t_1), Y_{0,1}^{(n)}(s_2,t_2)) \\ \operatorname{Cov}(Y_{1,0}^{(n)}(s_2,t_2), Y_{0,1}^{(n)}(s_1,t_1)) & \operatorname{Cov}(Y_{0,1}^{(n)}(s_1,t_1), Y_{0,1}^{(n)}(s_2,t_2)) \end{pmatrix} \to y_{\alpha,\beta}(s_1,t_1,s_2,t_2)
$$

as $n \to \infty$, *where*

$$
y_{\alpha,\beta}(s_1,t_1,s_2,t_2) = \begin{cases} \frac{1}{2}\Sigma_{\alpha,\beta}^{-1} = \sigma_{\alpha,\beta}^2 \begin{pmatrix} 1 & \varrho_{\alpha,\beta} \\ \varrho_{\alpha,\beta} & 1 \end{pmatrix}, & \text{if } s_1 = s_2, \ t_1 = t_2, \\ 0, & \text{otherwise.} \end{cases}
$$

Moreover, if $s_1 \neq s_2$ *or* $t_1 \neq t_2$ *then the convergence to* 0 *has an exponential rate.*

If $0 < \alpha < 1$ *and* $\beta = 1 - \alpha$ *then*

$$
\begin{pmatrix} \operatorname{Cov}(Z_{1,0}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)) & \operatorname{Cov}(Z_{1,0}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)) \\ \operatorname{Cov}(Z_{1,0}^{(n)}(s_2,t_2), Z_{0,1}^{(n)}(s_1,t_1)) & \operatorname{Cov}(Z_{0,1}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)) \end{pmatrix} \to z_{\alpha}(s_1,t_1,s_2,t_2)\mathbf{1}
$$

as $n \to \infty$, *where* $\mathbf{1}$ *denotes the two-by-two matrix of ones and*

$$z_\alpha(s_1,t_1,s_2,t_2) = \begin{cases} \frac{\sqrt{s_1+s_2+t_1+t_2}-\sqrt{|s_1-s_2|+|t_1-t_2|}}{\sqrt{2\pi\alpha(1-\alpha)}} & \textit{if} \ \ (1-\alpha)(s_1-s_2)=\alpha(t_1-t_2), \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover, if $(1-\alpha)(s_1-s_2) \neq \alpha(t_1-t_2)$ *then the convergence to* 0 *has an exponential rate.*

If $\alpha \in \{0,1\}$ *and* $\beta = 1-\alpha$ *then*

$$\begin{pmatrix} \mathrm{Cov}(U^{(n)}_{1,0}(s_1,t_1),U^{(n)}_{1,0}(s_2,t_2)) & \mathrm{Cov}(U^{(n)}_{1,0}(s_1,t_1),U^{(n)}_{0,1}(s_2,t_2)) \\ \mathrm{Cov}(U^{(n)}_{1,0}(s_2,t_2),U^{(n)}_{0,1}(s_1,t_1)) & \mathrm{Cov}(U^{(n)}_{0,1}(s_1,t_1),U^{(n)}_{0,1}(s_2,t_2)) \end{pmatrix} \to u_\alpha(s_1,t_1,s_2,t_2) I$$

as $n \to \infty$, *where*

$$u_\alpha(s_1,t_1,s_2,t_2) = \begin{cases} \frac{s_1+s_2+t_1+t_2-|s_1-s_2|-|t_1-t_2|}{2} & \textit{if} \ \ (1-\alpha)(s_1-s_2)=\alpha(t_1-t_2), \\ 0, & \textit{otherwise.} \end{cases}$$

In the proof of Proposition 2.3 we make use of the following two theorems.

Theorem 2.4 *Let* $\xi_1,\dots,\xi_k$ *and* $\eta_1,\dots,\eta_\ell$ *be independent, identically distributed random variables such that*

$$\mathsf{P}(\xi_1=1)=\mathsf{P}(\eta_1=0)=\alpha, \qquad \mathsf{P}(\xi_1=0)=\mathsf{P}(\eta_1=1)=1-\alpha$$

with some $\alpha \in (0,1)$. *Let*

$$S^{(\alpha)}_k := \xi_1+\dots+\xi_k, \qquad S^{(1-\alpha)}_\ell := \eta_1+\dots+\eta_\ell, \qquad S_{k,\ell} := S^{(\alpha)}_k + S^{(1-\alpha)}_\ell \tag{2.3}$$

and

$$\overline{\alpha} := \frac{\alpha k + (1-\alpha)\ell}{k+\ell}.$$

Then

$$\mathsf{P}(S_{k,\ell} \geqslant (k+\ell)x) \leqslant \mathrm{e}^{-(k+\ell)I_{\overline{\alpha}}(x)}, \qquad \textit{for all} \ \ x > \overline{\alpha},$$
$$\mathsf{P}(S_{k,\ell} \leqslant (k+\ell)x) \leqslant \mathrm{e}^{-(k+\ell)I_{\overline{\alpha}}(x)}, \qquad \textit{for all} \ \ x < \overline{\alpha},$$

with some $I_{\overline{\alpha}}(x) > 0$, *which depends only on* $\overline{\alpha}$ *and* x *(and does not depend on* k,ℓ *).*

Proof. By Hoeffding's inequality for independent, not necessarily identically distributed Bernoulli random variables (see Hoeffding [12]), the statement holds with

$$I_{\overline{\alpha}}(x) = \begin{cases} x\log\frac{x}{\overline{\alpha}} + (1-x)\log\frac{1-x}{1-\overline{\alpha}}, & x \in [0,1], \\ \infty, & \text{otherwise.} \end{cases}$$

Moreover, for $x \neq \overline{\alpha}$ we have that $I_{\overline{\alpha}}(x) > I_{\overline{\alpha}}(\overline{\alpha}) = 0$. □

Theorem 2.5 *For some* $\alpha \in (0,1)$ *let* $S_{k,\ell}$ *be the random variable defined by* (2.3) *and let*

$$m_{k,\ell} := \mathsf{E} S_{k,\ell}, \qquad b_{k,\ell} := \mathrm{Var} S_{k,\ell}, \qquad x_{j,k,\ell} := (j - m_{k,\ell})/\sqrt{b_{k,\ell}}.$$

Then we have the following asymptotic expansion in the Local Central Limit Theorem:

$$\left| \mathsf{P}(S_{k,\ell} = j) - \frac{1}{\sqrt{2\pi b_{k,\ell}}} \exp\left\{-x_{j,k,\ell}^2/2\right\} \right| \leqslant \frac{C_\alpha}{k+\ell}$$

for all $k, \ell \geqslant 1$ *and* $j \in \{0, 1, \ldots, k+\ell\}$, *with a constant* $C_\alpha > 0$ *depending only on* α *(and not depending on* k, ℓ, j *).*

Proof. Using the inversion formula, we get that

$$\mathsf{P}(S_{k,\ell} = j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}tj} \mathsf{E} \mathrm{e}^{itS_{k,\ell}} \, \mathrm{d}t = \frac{1}{2\pi\sqrt{b_{k,\ell}}} \int_{-\pi\sqrt{b_{k,\ell}}}^{\pi\sqrt{b_{k,\ell}}} \mathrm{e}^{-\mathrm{i}tx_{j,k,\ell}} f_{k,\ell}(t) \, \mathrm{d}t,$$

where $f_{k,\ell}$ denotes the characteristic function of $(S_{k,\ell} - m_{k,\ell})/\sqrt{b_{k,\ell}}$. Again by the inversion formula,

$$\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx - t^2/2} \, \mathrm{d}t, \qquad x \in \mathbb{R}.$$

Hence,

$$\begin{aligned} \Delta_{j,k,\ell} &:= \mathsf{P}(S_{k,\ell} = j) - \frac{1}{\sqrt{2\pi b_{k,\ell}}} \exp\left\{-x_{j,k,\ell}^2/2\right\} \\ &= \frac{1}{2\pi\sqrt{b_{k,\ell}}} \left(\int_{-\pi\sqrt{b_{k,\ell}}}^{\pi\sqrt{b_{k,\ell}}} \mathrm{e}^{-\mathrm{i}tx_{j,k,\ell}} f_{k,\ell}(t) \, \mathrm{d}t - \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx_{j,k,\ell} - t^2/2} \, \mathrm{d}t \right). \end{aligned}$$

Consequently,

$$|\Delta_{j,k,\ell}| \leqslant \frac{1}{2\pi\sqrt{b_{k,\ell}}} \big(J_1(k,\ell) + J_2(k,\ell) + J_3(k,\ell) \big),$$

where

$$\begin{aligned} J_1(k,\ell) &:= \int\limits_{|t| \leqslant 1/(4L_{k,\ell})} \left| f_{k,\ell}(t) - \mathrm{e}^{-t^2/2} \right| \mathrm{d}t, \\ J_2(k,\ell) &:= \int\limits_{1/(4L_{k,\ell}) \leqslant |t| \leqslant \pi\sqrt{b_{k,\ell}}} |f_{k,\ell}(t)| \, \mathrm{d}t, \\ J_3(k,\ell) &:= \int\limits_{|t| \geqslant 1/(4L_{k,\ell})} \mathrm{e}^{-t^2/2} \, \mathrm{d}t, \end{aligned}$$

and

$$L_{k,\ell} := b_{k,\ell}^{-3/2} \left(\sum_{i=1}^{k} \mathsf{E}|\xi_i - \mathsf{E}\xi_i|^3 + \sum_{i=1}^{\ell} \mathsf{E}|\eta_i - \mathsf{E}\eta_i|^3 \right).$$

We have

$$\left| f_{k,\ell}(t) - \mathrm{e}^{-t^2/2} \right| \leqslant 16 L_{k,\ell} |t|^3 \mathrm{e}^{-t^2/3} \qquad \text{for} \quad |t| \leqslant 1/(4L_{k,\ell}),$$

see Petrov [18, Chapter V, Lemma 1]. Hence

$$J_1(k,\ell) \leqslant 16 A L_{k,\ell} \qquad \text{with} \quad A := \int_{-\infty}^{\infty} |t|^3 \mathrm{e}^{-t^2/3} \, \mathrm{d}t = 9.$$

It is easy to check that

$$b_{k,\ell} = (k+\ell)\alpha(1-\alpha), \qquad L_{k,\ell} = \frac{1}{4 c_\alpha \sqrt{b_{k,\ell}}} \quad \text{with} \quad c_\alpha := \frac{1}{4(1 - 2\alpha + 2\alpha^2)}.$$

Hence

$$J_1(k,\ell) \leqslant \frac{144}{c_\alpha \sqrt{b_{k,\ell}}}. \tag{2.4}$$

Moreover $1/(4L_{k,\ell}) = c_\alpha \sqrt{b_{k,\ell}}$ implies

$$J_2(k,\ell) = \sqrt{b_{k,\ell}} \int\limits_{c_p \leqslant |t| \leqslant \pi} |f_{k,\ell}(u\sqrt{b_{k,\ell}})| \, \mathrm{d}u.$$

Clearly $g_{k,\ell}(u) := f_{k,\ell}(u\sqrt{b_{k,\ell}})$, $u \in \mathbb{R}$, is the characteristic function of $S_{k,\ell} - m_{k,\ell}$, hence

$$g_{k,\ell}(u) = \mathrm{e}^{-\mathrm{i}u m_{k,\ell}} h_1(u)^k h_2(u)^\ell,$$

where h_1 and h_2 denote the characteristic functions of ξ_1 and η_1, respectively. Hence

$$J_2(k,\ell) = \sqrt{b_{k,\ell}} \int\limits_{c_p \leqslant |t| \leqslant \pi} |h_1(u)|^k |h_2(u)|^\ell \, \mathrm{d}u.$$

We have

$$h_1(u) = 1 - \alpha + \alpha \mathrm{e}^{\mathrm{i}u}, \qquad h_2(u) = p + (1-p)\mathrm{e}^{\mathrm{i}u},$$

hence

$$|h_1(u)|^2 = |h_2(u)|^2 = 1 - 2\alpha(1-\alpha)(1 - \cos u).$$

Applying the inequality $1 - x \leqslant \mathrm{e}^{-x}$, $x \in \mathbb{R}$, we obtain

$$J_2(k,\ell) \leqslant \sqrt{b_{k,\ell}} \int\limits_{c_\alpha \leqslant |t| \leqslant \pi} \exp\big\{ -(k+\ell)\alpha(1-\alpha)(1-\cos u) \big\} \, \mathrm{d}u,$$

thus

$$J_2(k,\ell) \leqslant 2\pi \sqrt{b_{k,\ell}} \exp\big\{ -b_{k,\ell}(1 - \cos c_\alpha) \big\}. \tag{2.5}$$

Furthermore

$$J_3(k,\ell) = \int\limits_{|t|\geqslant c_\alpha\sqrt{b_{k,\ell}}} \mathrm{e}^{-t^2/2}\,\mathrm{d}t \leqslant \frac{1}{c_\alpha\sqrt{b_{k,\ell}}} \int\limits_{|t|\geqslant c_\alpha\sqrt{b_{k,\ell}}} |t|\mathrm{e}^{-t^2/2}\,\mathrm{d}t = \frac{2}{c_\alpha\sqrt{b_{k,\ell}}}\mathrm{e}^{-c_\alpha^2 b_{k,\ell}/2},$$

thus

$$J_3(k,\ell) \leqslant \frac{2}{c_\alpha\sqrt{b_{k,\ell}}}. \tag{2.6}$$

Collecting the estimates (2.4), (2.5) and (2.6) we conclude

$$b_{k,\ell}|\Delta_{j,k,\ell}| \leqslant \frac{19}{\pi c_\alpha} + b_{k,\ell}\exp\big\{-b_{k,\ell}(1-\cos c_\alpha)\big\}.$$

Consequently, we obtain the statement with

$$C_\alpha = \frac{1}{\alpha(1-\alpha)}\left(\frac{19}{\pi c_\alpha} + \frac{1}{(1-\cos c_\alpha)\mathrm{e}}\right),$$

since $\sup_{x\geqslant 0} x\mathrm{e}^{-x} = \mathrm{e}^{-1}$ and $b_{k,\ell} = (k+\ell)\alpha(1-\alpha)$. □

Proof of Proposition 2.3. Let $|\alpha|+|\beta|<1$. By Lemma 2.1,

$$|\mathrm{Cov}(Y_{1,0}^{(n)}(s_1,t_1), Y_{1,0}^{(n)}(s_2,t_2))| \leqslant \big(1-(|\alpha|+|\beta|)^2\big)^{-2}(|\alpha|+|\beta|)^{|[ns_1]-[ns_2]|+|[nt_1]-[nt_2]|}.$$

If $s_1\neq s_2$ then, for sufficiently large n, we have $\big|[ns_1]-[ns_2]\big| \geqslant n|s_1-s_2|/2$, since $\lim_{n\to\infty} n^{-1}\big|[ns_1]-[ns_2]\big| = |s_1-s_2|>0$. Consequently, $(|\alpha|+|\beta|)^{|[ns_1]-[ns_2]|}\to 0$, which implies that $\mathrm{Cov}(Y_{1,0}^{(n)}(s_1,t_1), Y_{1,0}^{(n)}(s_2,t_2))\to 0$. By symmetry, this also holds if $t_1\neq t_2$.

In case $s_1=s_2$, $t_1=t_2$, by formula (2.1),

$$\begin{aligned}\mathrm{Cov}\big(Y_{1,0}^{(n)}(s_1,t_1), Y_{1,0}^{(n)}(s_2,t_2)\big) &= \sum_{(i,j)\in T_{[ns_1]+1,[nt_1]}} \binom{[ns_1]+[nt_1]+1-i-j}{[ns_1]+1-i}^2 \alpha^{2[ns_1]+2-2i}\beta^{2[nt_1]-2j}\\ &= \sum_{\substack{u,v\in\mathbb{Z}_+,\\ u+v\leqslant[ns_1]+[nt_1]}} \binom{u+v}{u}^2\alpha^{2u}\beta^{2v}.\end{aligned}$$

Clearly $s_1+t_1>0$ implies that, for sufficiently large n, we have $[ns_1]+[nt_1]\geqslant n(s_1+t_1)/2$, since $\lim_{n\to\infty} n^{-1}([ns_1]+[nt_1]) = s_1+t_1>0$. Consequently, $[ns_1]+[nt_1]\to\infty$ as $n\to\infty$. Hence

$$\lim_{n\to\infty}\mathrm{Cov}(Y_{1,0}^{(n)}(s_1,t_1), Y_{1,0}^{(n)}(s_2,t_2)) = \sum_{u=0}^\infty\sum_{v=0}^\infty \binom{u+v}{u}^2\alpha^{2u}\beta^{2v}.$$

The aim of the following discussion is to show that the sum of this double series equals $\sigma^2_{\alpha,\beta}$. Let us consider the equation

$$X^*_{k,\ell} = \alpha X^*_{k-1,\ell} + \beta X^*_{k,\ell-1} + \varepsilon^*_{k,\ell}, \qquad \text{for } k,\ell\in\mathbb{Z}, \tag{2.7}$$

where $\{\varepsilon^*_{k,\ell} : k, \ell \in \mathbb{Z}\}$ are independent random variables with $\mathsf{E}\,\varepsilon^*_{k,\ell} = 0$, $\operatorname{Var}\varepsilon^*_{k,\ell} = 1$. Since $|\alpha| + |\beta| < 1$, it has a (P–a. s. unique) weakly stationary solution, for which we have the L^2–convergent infinite moving average representation

$$X^*_{k,\ell} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{u+v}{u}\alpha^u\beta^v\varepsilon^*_{k-u,\ell-v}. \tag{2.8}$$

(See Tjøstheim [20, Lemma 5.1].) This implies

$$\operatorname{Var}X^*_{k,\ell} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{u+v}{u}^2\alpha^{2u}\beta^{2v}.$$

Basu and Reinsel [4] proved that $\operatorname{Var}X^*_{k,\ell} = \sigma^2_{\alpha,\beta}$, hence we obtain the statement for $s_1 = s_2$, $t_1 = t_2$, and we finished the proof of the statement for $\operatorname{Cov}(Y^{(n)}_{1,0}(s_1,t_1), Y^{(n)}_{1,0}(s_2,t_2))$. The statement for $\operatorname{Cov}(Y^{(n)}_{0,1}(s_1,t_1), Y^{(n)}_{0,1}(s_2,t_2))$ follows by symmetry.

Next we investigate $\operatorname{Cov}\big(Y^{(n)}_{1,0}(s_1,t_1), Y^{(n)}_{0,1}(s_2,t_2)\big)$. By Lemma 2.1,

$$|\operatorname{Cov}(Y^{(n)}_{1,0}(s_1,t_1), Y^{(n)}_{0,1}(s_2,t_2))| \leqslant \frac{(|\alpha|+|\beta|)^{|[ns_1]-[ns_2]+1|+|[nt_1]-[nt_2]-1|}}{\big(1-(|\alpha|+|\beta|)^2\big)^2}.$$

If $s_1 \neq s_2$ or $t_1 \neq t_2$ then $\operatorname{Cov}(Y^{(n)}_{1,0}(s_1,t_1), Y^{(n)}_{0,1}(s_2,t_2)) \to 0$ can be proved as in the earlier case.

In case $s_1 = s_2$, $t_1 = t_2$, by formula (2.1),

$$\begin{aligned}
&\operatorname{Cov}(Y^{(n)}_{1,0}(s_1,t_1), Y^{(n)}_{0,1}(s_2,t_2)) \\
&= \sum_{(i,j)\in T_{[ns_1],[nt_1]}} \binom{[ns_1]+[nt_1]+1-i-j}{[ns_1]+1-i}\binom{[ns_1]+[nt_1]+1-i-j}{[ns_1]-i}\alpha^{2[ns_1]+1-2i}\beta^{2[nt_1]+1-2j} \\
&= \sum_{\substack{u,v\in\mathbb{Z}_+,\\ u+v\leqslant[ns_1]+[nt_1]-1}} \binom{u+v+1}{u+1}\binom{u+v+1}{u}\alpha^{2u+1}\beta^{2v+1}.
\end{aligned}$$

Considering again the stationary random field (2.8), we have

$$\operatorname{Cov}(X^*_{k,\ell-1}, X^*_{k-1,\ell}) = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{u+v+1}{u+1}\binom{u+v+1}{u}\alpha^{2u+1}\beta^{2v+1}.$$

Basu and Reinsel [4] proved that $\operatorname{Cov}(X^*_{k,\ell-1}, X^*_{k-1,\ell}) = \varrho_{\alpha\beta}\sigma^2_{\alpha,\beta}$. In fact, this can be proved directly by multiplying the equation (2.7) by $X^*_{k,\ell-1}$, $X^*_{k,\ell}$ and $X^*_{k-1,\ell}$, taking expectation of these three equations (using the weak stationarity of this random field), and solving this system of linear equations. Clearly $s_1 + t_1 > 0$ implies that $[ns_1] + [nt_1] \to \infty$ as $n \to \infty$, hence we obtain $\operatorname{Cov}(Y^{(n)}_{1,0}(s_1,t_1), Y^{(n)}_{0,1}(s_2,t_2)) \to$

$\varrho_{\alpha\beta}\sigma^2_{\alpha,\beta}$, and we finished the proof of the statement for $\operatorname{Cov}(Y^{(n)}_{1,0}(s_1,t_1), Y^{(n)}_{0,1}(s_2,t_2))$. The statement for $\operatorname{Cov}(Y^{(n)}_{0,1}(s_1,t_1), Y^{(n)}_{1,0}(s_2,t_2))$ follows by symmetry.

Now let $0<\alpha<1$ and $\beta = 1-\alpha$. By formula (2.1),

$$\operatorname{Cov}\big(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{1,0}(s_2,t_2)\big) = \frac{1}{\sqrt{n}} \sum_{(i,j)\in T_{[ns_1]\wedge[ns_2]+1,[nt_1]\wedge[nt_2]}} a_{i,j}(s_1,t_1,s_2,t_2), \quad (2.9)$$

where

$$\begin{aligned} a_{i,j}(s_1,t_1,s_2,t_2) := &\binom{[ns_1]+[nt_1]+1-i-j}{[ns_1]+1-i}\binom{[ns_2]+[nt_2]+1-i-j}{[ns_2]+1-i} \\ &\times \alpha^{[ns_1]+[ns_2]+2-2i}(1-\alpha)^{[nt_1]+[nt_2]-2j}. \end{aligned}$$

For $k,\ell\in\mathbb{N}$ let $S^{(\alpha)}_k$, $S^{(1-\alpha)}_\ell$ and $S_{k,\ell}$ be the random variables defined by (2.3). Then

$$a_{i,j}(s_1,t_1,s_2,t_2) = \mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]+1-i-j} = [ns_1]+1-i\big)\mathsf{P}\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]+1-i-j} = [nt_2]-j\big).$$

Moreover,

$$\operatorname{Cov}\big(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{1,0}(s_2,t_2)\big) = \frac{1}{\sqrt{n}} \sum_{m=1}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]+1} b_{m,n}(s_1,t_1,s_2,t_2), \quad (2.10)$$

where

$$\begin{aligned} b_{m,n}(s_1,t_1,s_2,t_2) :&= \sum_{i=m-[nt_1]\wedge[nt_2]}^{[ns_1]\wedge[ns_2]+1} a_{i,m-i}(s_1,t_1,s_2,t_2) \\ &= \mathsf{P}\big(S_{[ns_1]+[nt_1]+1-m,[ns_2]+[nt_2]+1-m} = [ns_1]+[nt_2]+1-m\big). \end{aligned}$$

We want to apply Theorem 2.4 for the terms of this sum. Let

$$k := [ns_1]+[nt_1]+1-m, \qquad \ell := [ns_2]+[nt_2]+1-m,$$
$$j := [ns_1]+[nt_2]+1-m, \qquad \overline{\alpha} := \frac{k\alpha+\ell(1-\alpha)}{k+\ell}.$$

Then $b_{m,n}(s_1,t_1,s_2,t_2) \leqslant \mathsf{P}(S_{k,\ell}=j)$. Consider

$$\begin{aligned} \gamma_{j,k,\ell,m} := \frac{j}{k+\ell} - \overline{\alpha} &= \frac{(1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])}{[ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m} \\ &\to \frac{(1-\alpha)(s_1-s_2)-\alpha(t_1-t_2)}{s_1+s_2+t_1+t_2} =: \gamma. \end{aligned}$$

If $(1-\alpha)(s_1-s_2) > \alpha(t_1-t_2)$ then $\gamma>0$, thus for sufficiently large $n\in\mathbb{N}$, we have

$$\gamma_{j,k,\ell,m} \geqslant \frac{\gamma}{2} > 0$$

for all $m \in \{1, \ldots, [ns_1] \wedge [ns_2] + [nt_1] \wedge [nt_2] + 1\}$. Hence

$$\begin{aligned} b_{m,n}(s_1, t_1, s_2, t_2) &\leqslant \mathsf{P}(S_{k,\ell} = j) = \mathsf{P}\big(S_{k,\ell} = (k+\ell)(\overline{\alpha} + \gamma_{j,k,\ell,m})\big) \\ &\leqslant \mathsf{P}\big(S_{k,\ell} \geqslant (k+\ell)(\overline{\alpha} + \gamma/2)\big) \end{aligned}$$

for sufficiently large $n \in \mathbb{N}$ and for all $m \in \{1, \ldots, [ns_1] \wedge [ns_2] + [nt_1] \wedge [nt_2] + 1\}$. Clearly $|s_1 - s_2| + |t_1 - t_2| > 0$ implies that

$$\begin{aligned} k + \ell = [ns_1] + [ns_2] + [nt_1] + [nt_2] + 2 - 2m &\geqslant \big|[ns_1] - [ns_2]\big| + \big|[nt_1] - [nt_2]\big| \\ &\geqslant (|s_1 - s_2| + |t_1 - t_2|)n/2 \end{aligned}$$

for sufficiently large $n \in \mathbb{N}$ and for all $m \in \{1, \ldots, [ns_1] \wedge [ns_2] + [nt_1] \wedge [nt_2] + 1\}$. Applying Theorem 2.4, we obtain

$$\mathsf{P}\big(S_{k,\ell} \geqslant (k+\ell)(\overline{\alpha} + \gamma/2)\big) \leqslant \exp\big\{ - n(|s_1 - s_2| + |t_1 - t_2|) I_{\overline{\alpha}}(\overline{\alpha} + \gamma/2)/2 \big\}.$$

Since $\gamma > 0$ implies $I(\overline{\alpha} + \gamma/2) > 0$, by (2.10) we obtain the statement for $(1-\alpha)(s_1 - s_2) > \alpha(t_1 - t_2)$. If $(1-\alpha)(s_1 - s_2) < \alpha(t_1 - t_2)$ then $\gamma < 0$, and for sufficiently large $n \in \mathbb{N}$, we have

$$\gamma_{j,k,\ell} \leqslant \frac{\gamma}{2} < 0$$

for all $m \in \{1, \ldots, [ns_1] \wedge [ns_2] + [nt_1] \wedge [nt_2] + 1\}$. Thus we conclude

$$\begin{aligned} b_{m,n}(s_1, t_1, s_2, t_2) &\leqslant \mathsf{P}(S_{k,\ell} = j) = \mathsf{P}\big(S_{k,\ell} = (k+\ell)(\overline{\alpha} + \gamma_{j,k,\ell})\big) \\ &\leqslant \mathsf{P}\big(S_{k,\ell} \leqslant (k+\ell)(\overline{\alpha} + \gamma/2)\big) \end{aligned}$$

for sufficiently large $n \in \mathbb{N}$ and for all $m \in \{1, \ldots, [ns_1] \wedge [ns_2] + [nt_1] \wedge [nt_2] + 1\}$. Applying Theorem 2.4, we obtain

$$\mathsf{P}\big(S_{k,\ell} \leqslant (k+\ell)(\overline{\alpha} + \gamma/2)\big) \leqslant \exp\big\{ - n(|s_1 - s_2| + |t_1 - t_2|) I_{\overline{\alpha}}(\overline{\alpha} + \gamma/2)/2 \big\}.$$

Now $\gamma < 0$ implies $I(\overline{\alpha} + \gamma/2) > 0$, and we finished the proof of the statement in case $(1-\alpha)(s_1 - s_2) \neq \alpha(t_1 - t_2)$.

Next consider the case $(1-\alpha)(s_1 - s_2) = \alpha(t_1 - t_2) \geqslant 0$. Then $[ns_1] \geqslant [ns_2]$ and $[nt_1] \geqslant [nt_2]$, hence

$$\operatorname{Cov}\big(Z_{1,0}^{(n)}(s_1, t_1), Z_{1,0}^{(n)}(s_2, t_2)\big) = \frac{1}{\sqrt{n}} \sum_{m=1}^{[ns_2]+[nt_2]+1} b_{m,n}(s_1, t_1, s_2, t_2),$$

where

$$\begin{aligned} b_{m,n}(s_1, t_1, s_2, t_2) &:= \sum_{i=m-[nt_2]}^{[ns_2]+1} a_{i,m-i}(s_1, t_1, s_2, t_2) \\ &= \mathsf{P}\big(S_{[ns_1]+[nt_1]+1-m,[ns_2]+[nt_2]+1-m} = [ns_1] + [nt_2] + 1 - m\big) = \mathsf{P}(S_{k,\ell} = j) \end{aligned}$$

with the earlier notations. We are going to apply Theorem 2.5. We have $j - k\alpha - \ell(1-\alpha) = (1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])$, hence

$$|b_{n,m}(s_1,t_1,s_2,t_2) - \widetilde{b}_{n,m}(s_1,t_1,s_2,t_2)| \leqslant \frac{C_\alpha}{[ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m}, \quad (2.11)$$

if $[ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m > 0$, where

$$\widetilde{b}_{n,m}(s_1,t_1,s_2,t_2) := \frac{\exp\left\{-\frac{\big((1-\alpha)([ns_1]-[ns_2])-\alpha([nt_1]-[nt_2])\big)^2}{2\alpha(1-\alpha)([ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m)}\right\}}{\sqrt{2\pi\alpha(1-\alpha)([ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m)}}.$$

In case $(1-\alpha)(s_1-s_2)=\alpha(t_1-t_2)>0$ we have $[ns_1]-[ns_2]+[nt_1]-[nt_2]>0$ for all sufficiently large $n \in \mathbb{N}$, hence (2.11) holds for all $m \in \{1,\ldots,[ns_2]+[nt_2]+1\}$. In case $(1-\alpha)(s_1-s_2)=\alpha(t_1-t_2)=0$ we have $[ns_1]-[ns_2]+[nt_1]-[nt_2]=0$, thus (2.11) holds for all $m \in \{1,\ldots,[ns_2]+[nt_2]\}$. The term with $m = [ns_2]+[nt_2]+1$ can be omitted since $n^{-1/2} b_{n,[ns_2]+[nt_2]+1}(s_1,t_1,s_2,t_2) \leqslant n^{-1/2} \to 0$. Obviously, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{m=1}^{[ns_2]+[nt_2]} \frac{C_\alpha}{[ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m} \\
&\leqslant \frac{C_\alpha}{\sqrt{n}} \int_1^{[ns_2]+[nt_2]+1} \frac{\mathrm{d}x}{[ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2x} \\
&\leqslant \frac{C_\alpha}{2\sqrt{n}} \log([ns_1]+[ns_2]+[nt_1]+[nt_2]) \to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

hence we may replace $b_{n,m}(s_1,t_1,s_2,t_2)$ with $\widetilde{b}_{n,m}(s_1,t_1,s_2,t_2)$. Furthermore,

$$\begin{aligned}
&\big|(1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])\big| \\
&= \big|(1-\alpha)([ns_1]-ns_1) - (1-\alpha)([ns_2]-ns_2) - \alpha([nt_1]-nt_1) + \alpha([nt_2]-nt_2)\big| \leqslant 1.
\end{aligned}$$

Using this inequality and that, for sufficiently large n, we have $[ns_1]-[ns_2]+[nt_1]-[nt_2] \geqslant n(s_1-s_2+t_1-t_2)/2$ if $(1-\alpha)(s_1-s_2)=\alpha(t_1-t_2)>0$, and applying the inequality $1-\mathrm{e}^{-x} \leqslant x$, $x \in \mathbb{R}$, it is easy to show that

$$\left|\widetilde{b}_{n,m}(s_1,t_1,s_2,t_2) - \frac{1}{\sqrt{2\pi\alpha(1-\alpha)\big(n(s_1+s_2+t_1+t_2)+2-2m\big)}}\right| \leqslant \frac{c_\alpha(s_1,t_1,s_2,t_2)}{n^{3/2}}$$

for sufficiently large n and for $m \in \{1,\ldots,[ns_2]+[nt_2]\}$, where $c_\alpha(s_1,t_1,s_2,t_2)$ is a constant depending only on $\alpha, s_1, t_1, s_2, t_2$ (not depending on n, m). Thus, it suffices to determine the limit of

$$\frac{1}{\sqrt{n}} \sum_{m=1}^{[ns_2]+[nt_2]} \frac{1}{\sqrt{2\pi\alpha(1-\alpha)\big(n(s_1+s_2+t_1+t_2)+2-2m\big)}} \qquad \text{as } n \to \infty,$$

which clearly equals

$$\int_0^{s_2+t_2} \frac{\mathrm{d}x}{\sqrt{2\pi\alpha(1-\alpha)(s_1+s_2+t_1+t_2-2x)}} = \frac{\sqrt{s_1+s_2+t_1+t_2} - \sqrt{|s_1-s_2|+|t_1-t_2|}}{\sqrt{2\pi\alpha(1-\alpha)}}.$$

The case $(1-\alpha)(s_1-s_2)=\alpha(t_1-t_2)<0$ can be handled similarly, hence we finished the proof of the statement for $\operatorname{Cov}(Z_{1,0}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2))$.

By formula (2.1),

$$\operatorname{Cov}\big(Z_{1,0}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)\big) = \frac{1}{\sqrt{n}} \sum_{(i,j)\in T_{([ns_1]+1)\wedge[ns_2],\,[nt_1]\wedge([nt_2]+1)}} a_{i,j}(s_1,t_1,s_2,t_2), \quad (2.12)$$

where now

$$\begin{aligned} a_{i,j}(s_1,t_1,s_2,t_2) &:= \binom{[ns_1]+[nt_1]+1-i-j}{[ns_1]+1-i}\binom{[ns_2]+[nt_2]+1-i-j}{[ns_2]-i} \\ &\qquad \times \alpha^{[ns_1]+[ns_2]+1-2i}(1-\alpha)^{[nt_1]+[nt_2]+1-2j} \\ &= \mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]+1-i-j} = [ns_1]+1-i\big)\mathsf{P}\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]+1-i-j} = [nt_2]+1-j\big). \end{aligned}$$

Moreover,

$$\operatorname{Cov}\big(Z_{1,0}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)\big) = \frac{1}{\sqrt{n}} \sum_{m=1}^{([ns_1]+1)\wedge[ns_2]+[nt_1]\wedge([nt_2]+1)} b_{m,n}(s_1,t_1,s_2,t_2), \quad (2.13)$$

where now

$$\begin{aligned} b_{m,n}(s_1,t_1,s_2,t_2) &:= \sum_{i=m-[nt_1]\wedge([nt_2]+1)}^{([ns_1]+1)\wedge[ns_2]} a_{i,m-i}(s_1,t_1,s_2,t_2) \\ &\leqslant \mathsf{P}\big(S_{[ns_1]+[nt_1]+1-m,[ns_2]+[nt_2]+1-m} = [ns_1]+[nt_2]+2-m\big). \end{aligned}$$

We want to apply Theorem 2.4 for the terms of this sum. Let

$$\begin{aligned} k &:= [ns_1]+[nt_1]+1-m, & \ell &:= [ns_2]+[nt_2]+1-m, \\ j &:= [ns_1]+[nt_2]+2-m, & \overline{\alpha} &:= \frac{k\alpha+\ell(1-\alpha)}{k+\ell}. \end{aligned}$$

Then $b_{m,n}(s_1,t_1,s_2,t_2) \leqslant \mathsf{P}(S_{k,\ell}=j)$. Consider

$$\begin{aligned} \gamma_{j,k,\ell,m} := \frac{j}{k+\ell} - \overline{\alpha} &= \frac{1+(1-\alpha)([ns_1]-[ns_2])-\alpha([nt_1]-[nt_2])}{[ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m} \\ &\to \frac{(1-\alpha)(s_1-s_2)-\alpha(t_1-t_2)}{s_1+s_2+t_1+t_2} =: \gamma. \end{aligned}$$

If $(1-\alpha)(s_1-s_2)\neq\alpha(t_1-t_2)$ then we obtain the statement in the same way as in the earlier case.

Consider the case $(1-\alpha)(s_1-s_2)=\alpha(t_1-t_2)>0$. Then

$$\operatorname{Cov}\big(Z_{1,0}^{(n)}(s_1,t_1),Z_{0,1}^{(n)}(s_2,t_2)\big)=\frac{1}{\sqrt{n}}\sum_{m=1}^{[ns_2]+[nt_2]+1}b_{m,n}(s_1,t_1,s_2,t_2),$$

for sufficiently large $n\in\mathbb{N}$, where now

$$\begin{aligned}b_{m,n}(s_1,t_1,s_2,t_2)&:=\sum_{i=m-[nt_2]-1}^{[ns_2]}a_{i,m-i}(s_1,t_1,s_2,t_2)\\&=\mathsf{P}\big(S_{[ns_1]+[nt_1]+1-m,[ns_2]+[nt_2]+1-m}=[ns_1]+[nt_2]+2-m\big)=\mathsf{P}(S_{k,\ell}=j)\end{aligned}$$

with the earlier notations.

We are going to apply Theorem 2.5. We have $j-k\alpha-\ell(1-\alpha)=(1-\alpha)([ns_1]-[ns_2])-\alpha([nt_1]-[nt_2])$, hence

$$|b_{n,m}(s_1,t_1,s_2,t_2)-\widetilde{b}_{n,m}(s_1,t_1,s_2,t_2)|\leqslant\frac{C_\alpha}{[ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m},\quad(2.14)$$

if $[ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m>0$, where

$$\widetilde{b}_{n,m}(s_1,t_1,s_2,t_2):=\frac{\exp\left\{-\frac{\big((1-\alpha)([ns_1]-[ns_2])-\alpha([nt_1]-[nt_2])+1\big)^2}{2\alpha(1-\alpha)([ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m)}\right\}}{\sqrt{2\pi\alpha(1-\alpha)([ns_1]+[ns_2]+[nt_1]+[nt_2]+2-2m)}}.$$

Using similar methods as before one can prove that $\widetilde{b}_{n,m}(s_1,t_1,s_2,t_2)$ has the same approximation as the one defined by (2.11), hence

$$\lim_{n\to\infty}\operatorname{Cov}\big(Z_{1,0}^{(n)}(s_1,t_1),Z_{0,1}^{(n)}(s_2,t_2)\big)=\lim_{n\to\infty}\operatorname{Cov}\big(Z_{1,0}^{(n)}(s_1,t_1),Z_{1,0}^{(n)}(s_2,t_2)\big).$$

The other two cases can be handled in a similar way.

Now, let $\alpha=1$ and $\beta=0$, i.e. $X_{k,\ell}=X_{k-1,\ell}+\varepsilon_{k,\ell}$ for $k+\ell\geqslant 0$. Thus, if $t_1\neq t_2$ we have

$$\operatorname{Cov}\big(U_{1,0}^{(n)}(s_1,t_1),U_{1,0}^{(n)}(s_2,t_2)\big)=\frac{1}{n}\operatorname{Cov}\big(X_{[ns_1]+1,[nt_1]},X_{[ns_2]+1,[nt_2]}\big)=0,$$

for sufficiently large n, while for $t_1=t_2$

$$\operatorname{Cov}\big(U_{1,0}^{(n)}(s_1,t_1),U_{1,0}^{(n)}(s_2,t_2)\big)=\frac{1}{n}\big([nt_1]+1+\min\{[ns_1],[ns_2]\}\big)\to t_1+\min\{s_1,s_2\}$$

as $n\to\infty$. It is easy to see that $\operatorname{Cov}\big(U_{0,1}^{(n)}(s_1,t_1),U_{0,1}^{(n)}(s_2,t_2)\big)$ has the same limit as $n\to\infty$, while

$$\lim_{n\to\infty}\operatorname{Cov}\big(U_{1,0}^{(n)}(s_1,t_1),U_{0,1}^{(n)}(s_2,t_2)\big)=\lim_{n\to\infty}\operatorname{Cov}\big(U_{0,1}^{(n)}(s_1,t_1),U_{1,0}^{(n)}(s_2,t_2)\big)=0.$$

The case $\alpha=0$ and $\beta=1$ can be handled in a similar way. □

In the case of $0<\alpha<1$ and $\beta=1-\alpha$ we can also estimate the difference of the covariances.

Proposition 2.6 *If* $0 < \alpha < 1$ *and* $\beta = 1-\alpha$ *then there exists a constant* $K_\alpha > 0$ *such that*

$$\left|\operatorname{Cov}(Z_{i,j}^{(n)}(s_1,t_1), Z_{j,i}^{(n)}(s_2,t_2)) - \operatorname{Cov}(Z_{i,j}^{(n)}(s_1,t_1), Z_{i,j}^{(n)}(s_2,t_2))\right| \leqslant K_\alpha n^{-1/2}$$

for all $n \in \mathbb{N}$, $s_1, t_1, s_2, t_2 \in \mathbb{R}$ *with* $s_1+t_1 > 0$, $s_2+t_2 > 0$ *and* $(i,j) \in \{(0,1),(1,0)\}$.

In the proof of Proposition 2.6 we make use the following theorem.

Theorem 2.7 *Let* $\alpha \in (0,1)$ *and for all* $k,\ell \geqslant 1$ *and* $0 \leqslant j \leqslant k+\ell-1$ *let* $S_{k,\ell}$, $b_{k,\ell}$, *and* $x_{j,k,\ell}$ *be as in Theorem 2.5 and*

$$\widetilde{\Delta}_{j,k,\ell} := \Big(\mathsf{P}(S_{k,\ell} = j+1) - \mathsf{P}(S_{k,\ell} = j)\Big) - \frac{1}{\sqrt{2\pi b_{k,\ell}}}\Big(\exp\left\{-x_{j+1,k,\ell}^2/2\right\} - \exp\left\{-x_{j,k,\ell}^2/2\right\}\Big).$$

Then there exists a constant $C_\alpha > 0$ *depending only on* α *(and not on* k, ℓ *and* j*) such that*

$$|\widetilde{\Delta}_{j,r,\ell}| \leqslant \frac{C_\alpha}{(k+\ell)^{3/2}}.$$

Proof. The proof is similar to that of Theorem 2.5. Using the inversion formula, we get

$$\mathsf{P}(S_{k,\ell} = j+1) - \mathsf{P}(S_{k,\ell} = j) = \frac{1}{2\pi\sqrt{b_{k,\ell}}} \int_{-\pi\sqrt{b_{k,\ell}}}^{\pi\sqrt{b_{k,\ell}}} \mathrm{e}^{-\mathrm{i}tx_{j,k,\ell}} \left(\mathrm{e}^{-\mathrm{i}t/\sqrt{b_{k,\ell}}} - 1\right) f_{k,\ell}(t)\,\mathrm{d}t,$$

where $f_{k,\ell}$ denotes the characteristic function of $(S_{k,\ell} - m_{k,\ell})/\sqrt{b_{k,\ell}}$. As

$$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx - t^2/2}\,\mathrm{d}t, \qquad x \in \mathbb{R},$$

we obtain

$$\begin{aligned}\widetilde{\Delta}_{j,k,\ell} = \frac{1}{2\pi\sqrt{b_{k,\ell}}}\Bigg(&\int_{-\pi\sqrt{b_{k,\ell}}}^{\pi\sqrt{b_{k,\ell}}} \mathrm{e}^{-\mathrm{i}tx_{j,k,\ell}} \left(\mathrm{e}^{-\mathrm{i}t/\sqrt{b_{k,\ell}}} - 1\right) f_{k,\ell}(t)\,\mathrm{d}t \\ &- \int_{-\infty}^{\infty} \left(\mathrm{e}^{-\mathrm{i}t/\sqrt{b_{k,\ell}}} - 1\right)\mathrm{e}^{-\mathrm{i}tx_{j,k,\ell} - t^2/2}\,\mathrm{d}t\Bigg).\end{aligned}$$

Consequently,

$$|\widetilde{\Delta}_{j,k,\ell}| \leqslant \frac{1}{2\pi\sqrt{b_{k,\ell}}}\big(\widetilde{J}_1(k,\ell) + \widetilde{J}_2(k,\ell) + \widetilde{J}_3(k,\ell)\big),$$

where

$$\widetilde{J}_1(k,\ell) := \int\limits_{|t|\leqslant 1/(4L_{k,\ell})} \left|\mathrm{e}^{-\mathrm{i}t/\sqrt{b_{k,\ell}}} - 1\right| \left|f_{k,\ell}(t) - \mathrm{e}^{-t^2/2}\right| \mathrm{d}t,$$

$$\widetilde{J}_2(k,\ell) := \int\limits_{1/(4L_{k,\ell})\leqslant |t|\leqslant \pi\sqrt{b_{k,\ell}}} 2|f_{k,\ell}(t)|\,\mathrm{d}t,$$

$$\widetilde{J}_3(k,\ell) := \int\limits_{|t|\geqslant 1/(4L_{k,\ell})} 2\mathrm{e}^{-t^2/2}\,\mathrm{d}t,$$

and $L_{k,\ell}$ is the same as in the proof of Theorem 2.5, i.e.

$$L_{k,\ell} = \frac{1}{4c_\alpha\sqrt{b_{k,\ell}}},$$

where $b_{k,\ell} = (k+\ell)\alpha(1-\alpha)$ and $c_\alpha := 1/(4(1-2\alpha+2\alpha^2))$.

Using the results obtained in the proof of Theorem 2.5 it is obvious that $\widetilde{J}_2(k,\ell)$ and $\widetilde{J}_3(k,\ell)$ converge to zero in exponential order as $k+\ell\to\infty$.

Applying again

$$\left|f_{k,\ell}(t) - \mathrm{e}^{-t^2/2}\right| \leqslant 16L_{k,\ell}|t|^3\mathrm{e}^{-t^2/3} \qquad \text{for } |t|\leqslant 1/(4L_{k,\ell}),$$

(see Petrov [18, Chapter V, Lemma 1]) and that $|1-\mathrm{e}^{-\mathrm{i}x}|\leqslant |x|$ for all $x\in\mathbb{R}$, we obtain

$$\widetilde{J}_1(k,\ell) \leqslant \frac{\widetilde{C}_\alpha}{b_{k,\ell}}, \qquad \text{with } \widetilde{C}_\alpha := \frac{4}{c_\alpha}\int_{-\infty}^{\infty} |t|^4\mathrm{e}^{-t^2/3}\,\mathrm{d}t$$

which completes the proof. □

Corollary 2.8 *Let* $\alpha\in(0,1)$. *There exists a constant* $C_\alpha>0$ *such that for all* $k,\ell\geqslant 1$ *and* $0\leqslant j\leqslant k+\ell-1$ *we have*

$$\left|\mathsf{P}(S_{k,\ell}=j+1) - \mathsf{P}(S_{k,\ell}=j)\right| \leqslant \frac{C_\alpha}{k+\ell}.$$

Proof of Proposition 2.6. Without loss of generality we can assume that $(i,j)=(1,0)$. Let

$$w^{(n)}(s_1,t_1,s_2,t_2) := \sqrt{n}\Big(\mathrm{Cov}\big(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{0,1}(s_2,t_2)\big) - \mathrm{Cov}\big(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{1,0}(s_2,t_2)\big)\Big).$$

Assume that $(1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])\geqslant 0$ and consider first the case $[ns_1]\geqslant[ns_2]$ and $[nt_1]\leqslant[nt_2]$. From the definition of the random fields $Z^{(n)}_{1,0}$ and

$Z_{0,1}^{(n)}$ using similar arguments as in the proof of Proposition 2.3 we have

$$\begin{aligned}
w^{(n)}(s_1,t_1,s_2,t_2) &= \sum_{m=1}^{[ns_2]+[nt_1]+1} \Big(\mathsf{P}\big(S_{[ns_1]+[nt_1]+1-m,[ns_2]+[nt_2]+1-m} = [ns_1]+[nt_2]+2-m\big) \\
&\qquad - \mathsf{P}\big(S_{[ns_1]+[nt_1]+1-m,[ns_2]+[nt_2]+1-m} = [ns_1]+[nt_2]+1-m\big)\Big) \\
&= \widehat{w}^{(n)}(s_1,t_1,s_2,t_2) - \alpha^{[ns_1]-[ns_2]}(1-\alpha)^{[nt_2]-[nt_1]}, \qquad (2.15)
\end{aligned}$$

where

$$\begin{aligned}
&\widehat{w}^{(n)}(s_1,t_1,s_2,t_2) \\
&\quad := \sum_{m=1}^{[ns_2]+[nt_1]} \Big(\mathsf{P}\big(S_{[ns_1]-[ns_2]+m,[nt_2]-[nt_1]+m} = [ns_1]-[ns_2]+[nt_2]-[nt_1]+1+m\big) \\
&\qquad\qquad - \mathsf{P}\big(S_{[ns_1]-[ns_2]+m,[nt_2]-[nt_1]+m} = [ns_1]-[ns_2]+[nt_2]-[nt_1]+m\big)\Big).
\end{aligned}$$

If we apply Theorem 2.7 to approximate the terms of $\widehat{w}^{(n)}(s_1,t_1,s_2,t_2)$ then the error of approximation is

$$\sum_{m=1}^{[ns_2]+[nt_1]} \frac{C_\alpha}{([ns_1]-[ns_2]+[nt_2]-[nt_1]+2m)^{3/2}} \leqslant C_\alpha \sum_{m=1}^{\infty} \frac{1}{m^{3/2}} < \infty, \qquad (2.16)$$

while for the normal approximation $\widetilde{w}^{(n)}(s_1,t_1,s_2,t_2)$ of $\widehat{w}^{(n)}(s_1,t_1,s_2,t_2)$ we have

$$\begin{aligned}
|\widetilde{w}^{(n)}(s_1,t_1,s_2,t_2)| &\leqslant \sum_{m=1}^{[ns_2]+[nt_1]} \frac{2\big((1-\alpha)([ns_1]-[ns_2])-\alpha([nt_1]-[nt_2])\big)+1}{\pi^{1/2}\big(2\alpha(1-\alpha)([ns_1]-[ns_2]+[nt_2]-[nt_1]+2m)\big)^{3/2}} \\
&\qquad \times \exp\left\{-\frac{\big((1-\alpha)([ns_1]-[ns_2])-\alpha([nt_1]-[nt_2])\big)^2}{2\alpha(1-\alpha)([ns_1]-[ns_2]+[nt_2]-[nt_1]+2m)}\right\}.
\end{aligned}$$

If $(1-\alpha)([ns_1]-[ns_2])-\alpha([nt_1]-[nt_2]) \leqslant 1$ then

$$|\widetilde{w}^{(n)}(s_1,t_1,s_2,t_2)| \leqslant \frac{3}{\alpha^{3/2}(1-\alpha)^{3/2}} \sum_{m=1}^{\infty} \frac{1}{m^{3/2}} < \infty. \qquad (2.17)$$

Otherwise, we have

$$
\begin{aligned}
|\widetilde{w}^{(n)}(s_1,t_1,s_2,t_2)| \leqslant 3 \sum_{m=1}^{[ns_2]+[nt_1]} & \frac{(1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])}{\pi^{1/2}(2\alpha)^{3/2}(1-\alpha)^{3/2}([ns_1]-[ns_2]+[nt_2]-[nt_1]+2m)^{3/2}} \\
& \times \exp\left\{ -\frac{\big((1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])\big)^2}{2\alpha(1-\alpha)([ns_1]-[ns_2]+[nt_2]-[nt_1]+2m)} \right\} \\
\leqslant \frac{3^{5/2}}{\pi^{1/2}2^{3/2}\big((1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])\big)^3} \\
+ 3 \int_1^{[ns_2]+[nt_1]} & \frac{(1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])}{\pi^{1/2}(2\alpha)^{3/2}(1-\alpha)^{3/2}([ns_1]-[ns_2]+[nt_2]-[nt_1]+2x)^{3/2}} \\
& \times \exp\left\{ -\frac{\big((1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])\big)^2}{2\alpha(1-\alpha)([ns_1]-[ns_2]+[nt_2]-[nt_1]+2x)} \right\} \mathrm{d}x \\
\leqslant 4 + \frac{1}{\alpha(1-\alpha)} \widetilde{\Phi} & \left(\frac{(1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2])}{\big(2\alpha(1-\alpha)([ns_1]-[ns_2]+[nt_2]-[nt_1]+2)\big)^{1/2}} \right) \leqslant \frac{4}{\alpha(1-\alpha)},
\end{aligned}
\tag{2.18}
$$

where

$$
\widetilde{\Phi}(x) := \frac{2}{\pi^{1/2}} \int_0^x \mathrm{e}^{-t^2/2} \mathrm{d}t, \qquad x > 0.
$$

Thus, in the case under consideration (2.15)–(2.18) imply the statement of the proposition.

In the case $(1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2]) \geqslant 0$ and $[nt_1] > [nt_2]$ (which implies $[ns_1] > [ns_2]$) we have

$$
\begin{aligned}
w^{(n)}(s_1,t_1,s_2,t_2) = \widehat{w}^{(n)}(s_1,t_1,s_2,t_2) &+ \mathsf{P}\big(S^{(\alpha)}_{[ns_1]-[ns_2]+[nt_1]-[nt_2]} = [ns_1]-[ns_2]+1\big) \\
&- \mathsf{P}\big(S^{(\alpha)}_{[ns_1]-[ns_2]+[nt_1]-[nt_2]} = [ns_1]-[ns_2]\big),
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{w}^{(n)}(s_1,t_1,s_2,t_2) := \sum_{m=1}^{[ns_2]+[nt_2]} & \mathsf{P}\big(S_{[ns_1]-[ns_2]+[nt_1]-[nt_2]+m,m} = [ns_1]-[ns_2]+1+m\big) \\
& - \mathsf{P}\big(S_{[ns_1]-[ns_2]+[nt_1]-[nt_2]+m,m} = [ns_1]-[ns_2]+m\big),
\end{aligned}
$$

and the statement can be proved similarly to the previous case.

Case $(1-\alpha)([ns_1]-[ns_2]) - \alpha([nt_1]-[nt_2]) < 0$ follows by symmetry. □

In order to estimate covariances we make use the following lemma which is a generalization of Lemma 11 of [2].

Lemma 2.9 *Let* $\xi_1, \ldots, \xi_N$ *be independent random variables with* $\mathsf{E}\xi_i = 0$, $\mathsf{E}\xi_i^2 = 1$ *for all* $i = 1, \ldots, N$, *and* $M_4 := \max_{1 \leqslant i \leqslant N} \mathsf{E}\xi_i^4 < \infty$. *Let* $a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}, c_1, \ldots, c_{n_3}, d_1, \ldots, d_{n_4} \in \mathbb{R}$, $n_1, n_2, n_3, n_4 \leqslant N$ *and*

$$X := \sum_{i=1}^{n_1} a_i \xi_i, \quad Y := \sum_{j=1}^{n_2} b_j \xi_j, \quad Z := \sum_{i=1}^{n_3} c_i \xi_i, \quad W := \sum_{j=1}^{n_4} d_j \xi_j.$$

Then

$$\mathrm{Cov}(XY, ZW) = \sum_{i=1}^{n_1 \wedge n_2 \wedge n_3 \wedge n_4} (\mathsf{E}\xi_i^4 - 3)\, a_i b_i c_i d_i + \mathrm{Cov}(X, Z)\mathrm{Cov}(Y, W) + \mathrm{Cov}(X, W)\mathrm{Cov}(Y, Z).$$

Moreover, if $a_i, b_i, c_i, d_i \geqslant 0$ *then*

$$0 \leqslant \mathrm{Cov}(XY, ZW) \leqslant M_4 \mathrm{Cov}(X, Z)\mathrm{Cov}(Y, W) + M_4 \mathrm{Cov}(X, W)\mathrm{Cov}(Y, Z),$$

and

$$0 \leqslant \mathsf{E}XYZW \leqslant M_4\big(\mathsf{E}XZ\,\mathsf{E}YW + \mathsf{E}XW\,\mathsf{E}YZ + \mathsf{E}XY\,\mathsf{E}ZW\big).$$

Proof. We have

$$\mathrm{Cov}(XY, ZW) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j_1=1}^{n_3} \sum_{j_2=1}^{n_4} a_{i_1} b_{i_2} c_{j_1} d_{j_2} \mathrm{Cov}(\xi_{i_1}\xi_{i_2}, \xi_{j_1}\xi_{j_2}).$$

It is easy to check that

$$\mathsf{E}\xi_{i_1}\xi_{i_2}\xi_{j_1}\xi_{j_2} = \begin{cases} \mathsf{E}\xi_{i_1}^4, & \text{if } i_1 = i_2 = j_1 = j_2, \\ 1, & \text{if } i_1 = i_2 \neq j_1 = j_2 \text{ or } i_1 = j_1 \neq i_2 = j_2 \text{ or } i_1 = j_2 \neq i_2 = j_1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\mathrm{Cov}(\xi_{i_1}\xi_{i_2}, \xi_{j_1}\xi_{j_2}) = \begin{cases} \mathsf{E}\xi_{i_1}^4 - 1, & \text{if } i_1 = i_2 = j_1 = j_2, \\ 1, & \text{if } i_1 = j_1 \neq i_2 = j_2 \text{ or } i_1 = j_2 \neq i_2 = j_1, \\ 0, & \text{otherwise.} \end{cases}$$

Short calculation shows that

$$\mathrm{Cov}(XY, ZW) = \sum_{i=1}^{n_1 \wedge n_2 \wedge n_3 \wedge n_4} (\mathsf{E}\xi_i^4 - 3)\, a_i b_i c_i d_i + \mathrm{Cov}(X, Z)\mathrm{Cov}(Y, W) + \mathrm{Cov}(X, W)\mathrm{Cov}(Y, Z).$$

If the coefficients are non negative then obviously

$$0 \leqslant \mathrm{Cov}(XY, ZW) \leqslant (M_4 - 3)^+ \sum_{i=1}^{n_1 \wedge n_2 \wedge n_3 \wedge n_4} a_i b_i c_i d_i + \mathrm{Cov}(X, Z)\mathrm{Cov}(Y, W) + \mathrm{Cov}(X, W)\mathrm{Cov}(Y, Z).$$

Moreover,

$$\sum_{i=1}^{n_1\wedge n_2\wedge n_3\wedge n_4} a_ib_ic_id_i \leqslant \frac{1}{2}\Big(\sum_{i=1}^{n_1\wedge n_3} a_i^2c_i^2\Big)^{1/2}\Big(\sum_{i=1}^{n_1\wedge n_3} b_i^2d_i^2\Big)^{1/2} + \frac{1}{2}\Big(\sum_{i=1}^{n_1\wedge n_4} a_i^2d_i^2\Big)^{1/2}\Big(\sum_{i=1}^{n_2\wedge n_3} b_i^2c_i^2\Big)^{1/2} \leqslant \frac{1}{2}\big(\mathrm{Cov}(X,Z)\mathrm{Cov}(Y,W)+\mathrm{Cov}(X,W)\mathrm{Cov}(Y,Z)\big).$$

Hence,

$$\mathrm{Cov}(XY,ZW) \leqslant M_4\big(\mathrm{Cov}(X,Z)\mathrm{Cov}(Y,W)+\mathrm{Cov}(X,W)\mathrm{Cov}(Y,Z)\big),$$

since $M_4 \geqslant 1$ implies $(M_4-3)^+ + 2 \leqslant 2M_4$. Furthermore,

$$\mathsf{E}XYZW = \mathrm{Cov}(XY,ZW)+\mathsf{E}XY\,\mathsf{E}ZW \leqslant M_4\big(\mathsf{E}XZ\,\mathsf{E}YW+\mathsf{E}XW\,\mathsf{E}YZ\big)+\mathsf{E}XY\,\mathsf{E}ZW$$

which directly implies the second statement of the lemma. □

3 Proof of Proposition 1.2

According to the results of the Introduction, we may assume $\alpha \geqslant 0$ and $\beta \geqslant 0$. In this case $\Psi_{\alpha,\beta} = \mathbf{1}$. First we show that

$$n^{-2}\mathsf{E}B_n \to \Sigma_{\alpha,\beta}^{-1}, \qquad \text{if } \alpha+\beta<1; \tag{3.1}$$

$$n^{-5/2}\mathsf{E}B_n \to \sigma_\alpha^2\mathbf{1}, \qquad \text{if } \alpha+\beta=1 \text{ and } 0<\alpha<1; \tag{3.2}$$

$$n^{-3}\mathsf{E}B_n \to \Sigma^{-1}, \qquad \text{if } \alpha+\beta=1 \text{ and } \alpha\in\{0,1\}. \tag{3.3}$$

If $\alpha+\beta<1$ we have

$$\frac{1}{n^2}\mathsf{E}B_n = \frac{1}{n^2}\sum_{(k,\ell)\in T_n}\begin{pmatrix}\mathsf{E}X_{k-1,\ell}^2 & \mathsf{E}X_{k-1,\ell}X_{k,\ell-1}\\ \mathsf{E}X_{k-1,\ell}X_{k,\ell-1} & \mathsf{E}X_{k,\ell-1}^2\end{pmatrix}$$

$$= \frac{1}{n^2}\sum_{(k,\ell)\in T_n}\begin{pmatrix}\mathrm{Var}Y_{0,1}^{(n)}\big(\frac{k-1}{n},\frac{\ell-1}{n}\big) & \mathrm{Cov}\Big(Y_{0,1}^{(n)}\big(\frac{k-1}{n},\frac{\ell-1}{n}\big),Y_{1,0}^{(n)}\big(\frac{k-1}{n},\frac{\ell-1}{n}\big)\Big)\\ \mathrm{Cov}\Big(Y_{0,1}^{(n)}\big(\frac{k-1}{n},\frac{\ell-1}{n}\big),Y_{1,0}^{(n)}\big(\frac{k-1}{n},\frac{\ell-1}{n}\big)\Big) & \mathrm{Var}Y_{1,0}^{(n)}\big(\frac{k-1}{n},\frac{\ell-1}{n}\big)\end{pmatrix}$$

$$= \iint_T \begin{pmatrix}\mathrm{Var}Y_{0,1}^{(n)}(s,t) & \mathrm{Cov}\big(Y_{0,1}^{(n)}(s,t),Y_{1,0}^{(n)}(s,t)\big)\\ \mathrm{Cov}\big(Y_{0,1}^{(n)}(s,t),Y_{1,0}^{(n)}(s,t)\big) & \mathrm{Var}Y_{1,0}^{(n)}(s,t)\end{pmatrix}\mathrm{d}s\mathrm{d}t,$$

where $T := \{(s,t)\in\mathbb{R}^2 : s+t\geqslant 0,\ s\leqslant 1,\ t\leqslant 1\}$. By Lemma 2.1

$$\big|\mathrm{Cov}\big(Y_{q_1,q_2}^{(n)}(s,t),Y_{r_1,r_2}^{(n)}(s,t)\big)\big| \leqslant \big(1-(\alpha+\beta)\big)^{-2}, \qquad (q_1,q_2),(r_1,r_2)\in\{(1,0),(0,1)\},$$

hence the dominated convergence theorem and Proposition 2.3 imply

$$\lim_{n\to\infty}\frac{1}{n^2}\mathsf{E}B_n = \iint\limits_T \lim_{n\to\infty}\begin{pmatrix} \operatorname{Var}Y_{0,1}^{(n)}(s,t) & \operatorname{Cov}\big(Y_{0,1}^{(n)}(s,t), Y_{1,0}^{(n)}(s,t)\big) \\ \operatorname{Cov}\big(Y_{0,1}^{(n)}(s,t), Y_{1,0}^{(n)}(s,t)\big) & \operatorname{Var}Y_{1,0}^{(n)}(s,t) \end{pmatrix} \mathrm{d}s\mathrm{d}t$$

$$= \sigma_{\alpha,\beta}^2 \begin{pmatrix} 1 & \varrho_{\alpha,\beta} \\ \varrho_{\alpha,\beta} & 1 \end{pmatrix} \iint\limits_T \mathrm{d}s\mathrm{d}t = 2\sigma_{\alpha,\beta}^2 \begin{pmatrix} 1 & \varrho_{\alpha,\beta} \\ \varrho_{\alpha,\beta} & 1 \end{pmatrix} = \Sigma_{\alpha,\beta}^{-1}.$$

In case $\alpha+\beta=1,\ 0<\alpha<1$ we obtain in the same manner

$$\frac{1}{n^{5/2}}\mathsf{E}B_n = \iint\limits_T \begin{pmatrix} \operatorname{Var}Z_{0,1}^{(n)}(s,t) & \operatorname{Cov}\big(Z_{0,1}^{(n)}(s,t), Z_{1,0}^{(n)}(s,t)\big) \\ \operatorname{Cov}\big(Z_{0,1}^{(n)}(s,t), Z_{1,0}^{(n)}(s,t)\big) & \operatorname{Var}Z_{1,0}^{(n)}(s,t) \end{pmatrix} \mathrm{d}s\mathrm{d}t.$$

Again, by Lemma 2.1

$$\left|\operatorname{Cov}\big(Z_{q_1,q_2}^{(n)}(s,t), Z_{r_1,r_2}^{(n)}(s,t)\big)\right| \leqslant Cn^{-1/2}\big(2[ns]+2[nt]+2\big)^{1/2} \leqslant C(2s+2t+2)^{1/2},$$

$(q_1,q_2),(r_1,r_2)\in\{(1,0),(0,1)\}$. The function $(s,t)\mapsto C(2s+2t+2)^{1/2}$ is integrable on the triangle T, hence the dominated convergence theorem applies. By Proposition 2.3 we have

$$\lim_{n\to\infty}\frac{1}{n^{5/2}}\mathsf{E}B_n = \frac{1}{\sqrt{\pi\alpha(1-\alpha)}}\iint\limits_T \sqrt{s+t}\,\mathrm{d}s\mathrm{d}t\,\mathbf{1} = \frac{2^{9/2}}{15\sqrt{\pi\alpha(1-\alpha)}}\mathbf{1}.$$

At the end, if $\alpha+\beta=1$, and $\alpha\in\{0,1\}$ by Lemma 2.1 we have

$$\left|\operatorname{Cov}\big(U_{q_1,q_2}^{(n)}(s,t), U_{r_1,r_2}^{(n)}(s,t)\big)\right| \leqslant Cn^{-1}\big(2[ns]+2[nt]+2\big) \leqslant C(2s+2t+2),$$

$(q_1,q_2),(r_1,r_2)\in\{(1,0),(0,1)\}$. Hence, Proposition 2.3 and the dominated convergence theorem imply

$$\lim_{n\to\infty}\frac{1}{n^3}\mathsf{E}B_n = \iint\limits_T (s+t)\,\mathrm{d}s\mathrm{d}t\,I = \frac{4}{3}I.$$

Besides (3.1)–(3.3) to prove Proposition 2.3 we have to show

$$\frac{1}{n^\gamma}\operatorname{Var}\Big(\sum_{(k,\ell)\in T_n} X_{k-1,\ell}^2\Big) = \frac{1}{n^\gamma}\sum_{(k_1,\ell_1),(k_2,\ell_2)\in T_n}\operatorname{Cov}\big(X_{k_1-1,\ell_1}^2, X_{k_2-1,\ell_2}^2\big) \to 0, \tag{3.4}$$

$$\frac{1}{n^\gamma}\operatorname{Var}\Big(\sum_{(k,\ell)\in T_n} X_{k-1,\ell}X_{k,\ell-1}\Big) = \frac{1}{n^\gamma}\sum_{(k_1,\ell_1),(k_2,\ell_2)\in T_n}\operatorname{Cov}\big(X_{k_1-1,\ell_1}X_{k_1,\ell_1-1}, X_{k_2-1,\ell_2}X_{k_2,\ell_2-1}\big) \to 0, \tag{3.5}$$

$$\frac{1}{n^\gamma}\operatorname{Var}\Big(\sum_{(k,\ell)\in T_n} X_{k,\ell-1}^2\Big) = \frac{1}{n^\gamma}\sum_{(k_1,\ell_1),(k_2,\ell_2)\in T_n}\operatorname{Cov}\big(X_{k_1,\ell_1-1}^2, X_{k_2,\ell_2-1}^2\big) \to 0 \tag{3.6}$$

as $n \to \infty$, where

$$\gamma := \begin{cases} 4, & \text{if } \alpha+\beta<1; \\ 5, & \text{if } \alpha+\beta=1 \text{ and } 0<\alpha<1; \\ 6, & \text{if } \alpha+\beta=1 \text{ and } \alpha\in\{0,1\}. \end{cases} \tag{3.7}$$

Let $\alpha+\beta=1$ and $0<\alpha<1$. Using Lemma 2.9 we have

$$\frac{1}{n^5}\operatorname{Var}\Big(\sum_{(k,\ell)\in T_n} X_{k-1,\ell}^2\Big) \leqslant 2M_4 \iint_T\iint_T \operatorname{Cov}\big(Z_{0,1}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)\big)^2 \mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2,$$

$$\begin{aligned} \frac{1}{n^5}\operatorname{Var}\Big(\sum_{(k,\ell)\in T_n} X_{k-1,\ell}X_{k,\ell-1}\Big) \leqslant{}& M_4 \iint_T\iint_T \operatorname{Cov}\big(Z_{0,1}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)\big) \\ &\times \operatorname{Cov}\big(Z_{1,0}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)\big)\mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2 \\ &+ M_4 \iint_T\iint_T \operatorname{Cov}\big(Z_{0,1}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)\big) \\ &\times \operatorname{Cov}\big(Z_{1,0}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)\big)\mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2, \end{aligned}$$

$$\frac{1}{n^5}\operatorname{Var}\Big(\sum_{(k,\ell)\in T_n} X_{k,\ell-1}^2\Big) \leqslant 2M_4 \iint_T\iint_T \operatorname{Cov}\big(Z_{1,0}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)\big)^2 \mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2,$$

which by Lemma 2.1, Proposition 2.3 and by the dominated convergence theorem implies (3.4)–(3.6). In cases $\alpha+\beta<1$ and $\alpha+\beta=1$, $\alpha\in\{0,1\}$ (3.4)–(3.6) can be proved in a similar way. □

4 Proof of Proposition 1.3

First we show that $(A_n)_{n\geqslant 1}$ is a square integrable two dimensional martingale with respect to the filtration $(\mathcal{F}_n)_{n\geqslant 1}$, where $\mathcal{F}_n$ denotes the σ–algebra generated by the random variables $\{\varepsilon_{k,\ell} : (k,\ell)\in T_n\}$.

In order to do this we give a useful decomposition of $A_n - A_{n-1}$, where $A_0 := (0,0)^\top$. Let $A_n^{(i)}$, $i=1,2$ denote the components of A_n. By representation (1.3),

$$\begin{aligned} A_n^{(1)} - A_{n-1}^{(1)} &= \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in T_{k-1,\ell}} \binom{k+\ell-1-i-j}{k-1-i}\alpha^{k-1-i}\beta^{\ell-j}\varepsilon_{i,j}, \\ A_n^{(2)} - A_{n-1}^{(2)} &= \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in T_{k,\ell-1}} \binom{k+\ell-1-i-j}{k-1-i}\alpha^{k-i}\beta^{\ell-1-j}\varepsilon_{i,j}. \end{aligned}$$

Collecting first the terms containing only $\varepsilon_{i,j}$ with $(i,j)\in T_n\setminus T_{n-1}$, and then the rest, we obtain the decomposition

$$A_n - A_{n-1} = A_{n,1} + \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell} A_{n,2,k,\ell}, \tag{4.1}$$

where $A_{n,1} = \left(A_{n,1}^{(1)}, A_{n,1}^{(2)}\right)^\top$ and $A_{n,2,k,\ell} = \left(\widetilde{A}_{n,2,k-1,\ell}, \widetilde{A}_{n,2,k,\ell-1}\right)^\top$ with

$$A_{n,1}^{(1)} := \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in T_{k-1,\ell}\setminus T_{n-1}} \binom{k+\ell-1-i-j}{k-1-i} \alpha^{k-1-i}\beta^{\ell-j}\varepsilon_{i,j}$$

$$= \sum_{k=-n+2}^{m} \sum_{i=-n+1}^{k-1} \alpha^{k-1-i}\varepsilon_{k,n}\varepsilon_{i,n}, \tag{4.2}$$

$$A_{n,1}^{(2)} := \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in T_{k,\ell-1}\setminus T_{n-1}} \binom{k+\ell-1-i-j}{k-i} \alpha^{k-i}\beta^{\ell-1-j}\varepsilon_{i,j}$$

$$= \sum_{\ell=-n+2}^{n} \sum_{j=-n+1}^{\ell-1} \beta^{\ell-1-j}\varepsilon_{n,\ell}\varepsilon_{n,j}, \tag{4.3}$$

$$\widetilde{A}_{n,2,k,\ell} := \sum_{(i,j)\in T_{k,\ell}\cap T_{n-1}} \binom{k+\ell-i-j}{k-i} \alpha^{k-i}\beta^{\ell-j}\varepsilon_{i,j}. \tag{4.4}$$

The components of $A_{n,1}$ are quadratic forms of the variables $\{\varepsilon_{i,j} : (i,j) \in T_n \setminus T_{n-1}\}$, hence $A_{n,1}$ is independent of $\mathcal{F}_{n-1}$. Besides this the terms $\widetilde{A}_{n,2,k,\ell}$ are linear combinations of the variables $\{\varepsilon_{i,j} : (i,j) \in T_{n-1}\}$, thus the vectors $A_{n,2,k,\ell}$ are measurable with respect to $\mathcal{F}_{n-1}$. Consequently,

$$\mathsf{E}(A_n - A_{n-1} \mid \mathcal{F}_{n-1}) = \mathsf{E}A_{n,1} + \sum_{(k,\ell)\in T_n\setminus T_{n-1}} A_{n,2,k,\ell}\mathsf{E}(\varepsilon_{k,\ell} \mid \mathcal{F}_{n-1}) = 0.$$

Hence $(A_n)_{n\geqslant 1}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_n)_{n\geqslant 1}$.

By the Martingale Central Limit Theorem (see [13]), in order to prove the statement of Proposition 1.3, it suffices to show that the conditional variances of the martingale differences converge in probability and to verify the conditional Lindeberg condition. To be precise, the statement is a consequence of the following two propositions, where $\mathbb{1}_H$ denotes the indicator function of a set H.

Proposition 4.1

$$n^{-2}\sum_{m=1}^{n} \mathsf{E}\left((A_m - A_{m-1})(A_m - A_{m-1})^\top \,\middle|\, \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \Sigma_{\alpha,\beta}^{-1},$$

$$\text{if } |\alpha| + |\beta| < 1;$$

$$n^{-5/2}\sum_{m=1}^{n} \mathsf{E}\left((A_m - A_{m-1})(A_m - A_{m-1})^\top \,\middle|\, \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \sigma_\alpha^2 \Psi_{\alpha,\beta},$$

$$\text{if } |\alpha|+|\beta|=1 \text{ and } 0<|\alpha|<1;$$

$$n^{-3}\sum_{m=1}^{n} \mathsf{E}\left((A_m - A_{m-1})(A_m - A_{m-1})^\top \,\middle|\, \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \Sigma^{-1},$$

$$\text{if } |\alpha|+|\beta|=1 \text{ and } |\alpha|\in\{0,1\},$$

as $n \to \infty$.

Proposition 4.2 *For all* $\delta > 0$,

$$n^{-\gamma/2} \sum_{m=1}^{n} \mathsf{E}\left(\|A_m - A_{m-1}\|^2 \mathbb{1}_{\{\|A_m - A_{m-1}\| \geqslant \delta n^{\gamma/4}\}} \,\Big|\, \mathcal{F}_{m-1} \right) \xrightarrow{\mathsf{P}} 0$$

as $n \to \infty$, *where* γ *is defined by* (3.7), *i.e.*

$$\gamma := \begin{cases} 4, & \text{if } |\alpha| + |\beta| < 1; \\ 5, & \text{if } |\alpha| + |\beta| = 1 \text{ and } 0 < |\alpha| < 1; \\ 6, & \text{if } |\alpha| + |\beta| = 1 \text{ and } |\alpha| \in \{0, 1\}. \end{cases}$$

Proof of Proposition 4.1. Let $U_m := \mathsf{E}\big((A_m - A_{m-1})(A_m - A_{m-1})^\top \,\big|\, \mathcal{F}_{m-1}\big)$. Again, without loss of generality we may assume $\alpha \geqslant 0$ and $\beta \geqslant 0$. First we show, that

$$n^{-2} \sum_{m=1}^{n} \mathsf{E} U_m \to \Sigma_{\alpha,\beta}^{-1}, \qquad \text{if } \alpha + \beta < 1; \tag{4.5}$$

$$n^{-5/2} \sum_{m=1}^{n} \mathsf{E} U_m \to \sigma_\alpha^2 \mathbf{1}, \qquad \text{if } \alpha + \beta = 1 \text{ and } 0 < \alpha < 1; \tag{4.6}$$

$$n^{-3} \sum_{m=1}^{n} \mathsf{E} U_m \to \Sigma^{-1}, \qquad \text{if } \alpha + \beta = 1 \text{ and } \alpha \in \{0, 1\}, \tag{4.7}$$

as $n \to \infty$. We have

$$A_m - A_{m-1} = \sum_{(k,\ell) \in T_m \setminus T_{m-1}} \begin{pmatrix} X_{k-1,\ell}\varepsilon_{k,\ell} \\ X_{k,\ell-1}\varepsilon_{k,\ell} \end{pmatrix}$$

and by representation (1.3) and independence of the $\varepsilon_{i,j}$, the terms in the summation have zero mean and they are mutually uncorrelated. Since $(X_{k-1,\ell}, X_{k,\ell-1})^\top$ and $\varepsilon_{k,\ell}$ are also independent, we obtain

$$\begin{aligned} \mathsf{E}(A_m - A_{m-1})(A_m - A_{m-1})^\top &= \sum_{(k,\ell) \in T_m \setminus T_{m-1}} \mathsf{E} \begin{pmatrix} X_{k-1,\ell}^2 & X_{k-1,\ell}X_{k,\ell-1} \\ X_{k-1,\ell}X_{k,\ell-1} & X_{k,\ell-1}^2 \end{pmatrix} \mathsf{E}\varepsilon_{k,\ell} \\ &= \sum_{(k,\ell) \in T_m \setminus T_{m-1}} \mathsf{E} \begin{pmatrix} X_{k-1,\ell}^2 & X_{k-1,\ell}X_{k,\ell-1} \\ X_{k-1,\ell}X_{k,\ell-1} & X_{k,\ell-1}^2 \end{pmatrix} = \mathsf{E}B_m - \mathsf{E}B_{m-1}, \end{aligned}$$

where B_0 equals the two-by-two matrix containing only zeros. Consequently, (4.5), (4.6) and (4.7) follow from (3.1), (3.2) and (3.3), respectively.

By the decomposition (4.1) and by the measurability of $A_{m,2,k,\ell}$ with respect to $\mathcal{F}_{m-1}$ one can derive that

$$\begin{aligned} U_m &= \mathsf{E}(A_{m,1}A_{m,1}^\top \,|\, \mathcal{F}_{m-1}) + \sum_{(k,\ell)\in T_m\setminus T_{m-1}} \mathsf{E}(A_{m,1}\varepsilon_{k,\ell} \,|\, \mathcal{F}_{m-1}) A_{m,2,k,\ell}^\top \\ &\quad + \sum_{(k,\ell)\in T_m\setminus T_{m-1}} A_{m,2,k,\ell}^\top \mathsf{E}(A_{m,1}\varepsilon_{k,\ell} \,|\, \mathcal{F}_{m-1}) \\ &\quad + \sum_{(k_1,\ell_1)\in T_m\setminus T_{m-1}} \sum_{(k_2,\ell_2)\in T_m\setminus T_{m-1}} A_{m,2,k_1,\ell_1} A_{m,2,k_2,\ell_2}^\top \mathsf{E}(\varepsilon_{k_1,\ell_1}\varepsilon_{k_2,\ell_2} \,|\, \mathcal{F}_{m-1}). \end{aligned}$$

By the independence of $A_{m,1}$ and $\{\varepsilon_{k,\ell} : (k,\ell) \in T_m \setminus T_{m-1}\}$ from $\mathcal{F}_{m-1}$, and by $\mathsf{E}(A_{m,1}\varepsilon_{k,\ell}) = (0,0)^\top$, one obtains

$$U_m = \mathsf{E}A_{m,1}A_{m,1}^\top + \sum_{(k,\ell)\in T_m\setminus T_{m-1}} A_{m,2,k,\ell}A_{m,2,k,\ell}^\top. \tag{4.8}$$

This means that to complete the proof of the proposition we have to show

$$\frac{1}{n^\gamma}\mathrm{Var}\Big(\sum_{m=1}^n \sum_{(k,\ell)\in T_m\setminus T_{m-1}} \widetilde{A}_{m,2,k-1,\ell}^2\Big) \to 0, \tag{4.9}$$

$$\frac{1}{n^\gamma}\mathrm{Var}\Big(\sum_{m=1}^n \sum_{(k,\ell)\in T_m\setminus T_{m-1}} \widetilde{A}_{m,2,k-1,\ell}\widetilde{A}_{m,2,k,\ell-1}\Big) \to 0, \tag{4.10}$$

$$\frac{1}{n^\gamma}\mathrm{Var}\Big(\sum_{m=1}^n \sum_{(k,\ell)\in T_m\setminus T_{m-1}} \widetilde{A}_{m,2,k,\ell-1}^2\Big) \to 0 \tag{4.11}$$

as $n\to\infty$, where γ is defined by (3.7).

$$\begin{aligned} &\mathrm{Var}\Big(\sum_{m=1}^n \sum_{(k,\ell)\in T_m\setminus T_{m-1}} \widetilde{A}_{m,2,k,\ell}^2\Big) \\ &\qquad = \sum_{m_1=1}^n \sum_{(k_1,\ell_1)\in T_{m_1}\setminus T_{m_1-1}} \sum_{m_2=1}^n \sum_{(k_2,\ell_2)\in T_{m_2}\setminus T_{m_2-1}} \mathrm{Cov}(\widetilde{A}_{m_1,2,k_1,\ell_1}^2, \widetilde{A}_{m_2,2,k_2,\ell_2}^2). \end{aligned}$$

By Lemma 2.9 $\mathrm{Cov}\big(\widetilde{A}_{m_1,2,k_1,\ell_1}^2, \widetilde{A}_{m_2,2,k_2\ell_2}^2\big) \leqslant 2M_4\mathrm{Cov}\big(\widetilde{A}_{m_1,2,k_1,\ell_1}, \widetilde{A}_{m_2,2,k_2,\ell_2}\big)^2$. Moreover,

$$\begin{aligned} &\mathrm{Cov}\big(A_{m_1,2,k_1,\ell_1}, A_{m_2,2,k_2,\ell_2}\big) \\ &= \sum_{(i,j)\in T_{k_1-1,\ell_1}\cap T_{m_1-1}\cap T_{k_2-1,\ell_2}\cap T_{m_2-1}} \binom{k_1+\ell_1-i-j}{k_1-i}\binom{k_2+\ell_2-i-j}{k_2-i}\alpha^{k_1+k_2-2i}\beta^{\ell_1+\ell_2-2j}, \end{aligned}$$

so representation (1.3) implies $\operatorname{Cov}\big(A_{m_1,2,k_1-1,\ell_1}, A_{m_2,2,k_2-1,\ell_2}\big) \leqslant \operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})$. Furthermore,

$$\sum_{m_1=1}^{n} \sum_{(k_1,\ell_1)\in T_{m_1}\setminus T_{m_1-1}} \sum_{m_2=1}^{n} \sum_{(k_2,\ell_2)\in T_{m_2}\setminus T_{m_2-1}} \operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})^2 = \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} \operatorname{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})^2$$

In a similar way one can show

$$\operatorname{Var}\bigg(\sum_{m=1}^{n} \sum_{(k,\ell)\in T_m\setminus T_{m-1}} \widetilde{A}_{m,2,k-1,\ell}^2 \widetilde{A}_{m,2,k,\ell-1}^2\bigg) \leqslant M_4 \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} \Big(\operatorname{Cov}(X_{k_1-1,\ell_1}, X_{k_2-1,\ell_2})\operatorname{Cov}(X_{k_1,\ell_1-1}, X_{k_2,\ell_2-1}) + \operatorname{Cov}(X_{k_1-1,\ell_1}, X_{k_2,\ell_2-1})\operatorname{Cov}(X_{k_1,\ell_1-1}, X_{k_2-1,\ell_2})\Big).$$

Hence, (4.9), (4.10) and (4.11) can be derived in a similar way as (3.4), (3.5) and (3.6), respectively. □

Proof of Proposition 4.2. As before, we consider only the case $\alpha \geqslant 0$ and $\beta \geqslant 0$. We have

$$\mathbb{1}_{\{\|A_m - A_{m-1}\| \geqslant \delta n^{\gamma/4}\}} \leqslant \delta^{-2} n^{-\gamma/2} \|A_m - A_{m-1}\|^2,$$

hence it suffices to show that

$$n^{-\gamma} \sum_{m=1}^{n} \mathsf{E}\left(\|A_m - A_{m-1}\|^4 \,\middle|\, \mathcal{F}_{m-1}\right) \stackrel{\mathsf{P}}{\longrightarrow} 0 \qquad \text{as} \quad n \to \infty, \tag{4.12}$$

where γ is defined by (3.7). By the decomposition (4.1) of $A_m - A_{m-1}$ and by the inequality $(x+y)^4 \leqslant 2^3(x^4 + y^4)$ for $x, y \in \mathbb{R}$,

$$\|A_m - A_{m-1}\|^4 \leqslant 2^3 \|A_{m,1}\|^4 + 2^3 \bigg\| \sum_{(k,\ell)\in T_m\setminus T_{m-1}} \varepsilon_{k,\ell} A_{m,2,k,\ell} \bigg\|^4.$$

By the independence of $A_{m,1}$ and $\mathcal{F}_{m-1}$, we have $\mathsf{E}\big(\|A_{m,1}\|^4 \,\big|\, \mathcal{F}_{m-1}\big) = \mathsf{E}\|A_{m,1}\|^4$. Applying the measurability of $A_{m,2,k,\ell}$ with respect to $\mathcal{F}_{m-1}$, we obtain that

$$\mathsf{E}\left(\bigg\| \sum_{(k,\ell)\in T_m\setminus T_{m-1}} \varepsilon_{k,\ell} A_{m,2,k,\ell} \bigg\|^4 \,\middle|\, \mathcal{F}_{m-1}\right) \leqslant \big((M_4 - 3)^+ + 3\big) \left(\sum_{(k,\ell)\in T_m\setminus T_{m-1}} \|A_{m,2,k,\ell}\|^2\right)^2.$$

Hence, in order to prove (4.12), it suffices to show that

$$\lim_{n\to\infty} n^{-\gamma}\sum_{m=1}^{n} \mathsf{E}\|A_{m,1}\|^4 = 0, \tag{4.13}$$

$$\lim_{n\to\infty} n^{-\gamma}\sum_{m=1}^{n} \mathsf{E}\left(\sum_{(k,\ell)\in T_m\setminus T_{m-1}} \|A_{m,2,k,\ell}\|^2\right)^2 = 0. \tag{4.14}$$

It is easy to see that using (4.2) and (4.3) we obtain

$$\|A_{m,1}\|^4 \leqslant 2^3\left(\sum_{k=-m+2}^{m}\sum_{i=-m+1}^{k-1}\alpha^{k-1-i}\varepsilon_{k,m}\varepsilon_{i,m}\right)^4 + 2^3\left(\sum_{\ell=-m+2}^{m}\sum_{j=-m+1}^{\ell-1}\beta^{\ell-1-j}\varepsilon_{m,\ell}\varepsilon_{m,j}\right)^4.$$

If $0<\alpha<1$ then by Lemma 12 of [2] we have $\mathsf{E}\|A_{m,1}\|^4 = O(m^2)$, while for $\alpha=1$ a short calculation shows that $\mathsf{E}\|A_{m,1}\|^4 = O(m^4)$ as $n\to\infty$. Hence, (4.13) is satisfied for all possible values of γ.

Furthermore, we have

$$\begin{aligned}&\mathsf{E}\left(\sum_{(k,\ell)\in T_m\setminus T_{m-1}} \|A_{m,2,k,\ell}\|^2\right)^2\\ &= \sum_{(k_1,\ell_1)\in T_m\setminus T_{m-1}}\ \sum_{(k_2,\ell_2)\in T_m\setminus T_{m-1}} \mathsf{E}\Big((\widetilde{A}^2_{m,2,k_1-1,\ell_1}+\widetilde{A}^2_{m,2,k_1,\ell_1-1})(\widetilde{A}^2_{m,2,k_2-1,\ell_2}+\widetilde{A}^2_{m,2,k_2,\ell_2-1})\Big).\end{aligned}$$

From Lemma 2.9 follows

$$\mathsf{E}(\widetilde{A}^2_{m,2,k_1,\ell_1}\widetilde{A}^2_{m,2,k_2,\ell_2}) \leqslant 3M_4\mathsf{E}\widetilde{A}^2_{m,2,k_1,\ell_1}\mathsf{E}\widetilde{A}^2_{m,2,k_2,\ell_2},$$

while using (4.4) and representation (1.3), one can see

$$\mathsf{E}\widetilde{A}^2_{m,2,k,\ell} \leqslant \operatorname{Var}X_{k,\ell}.$$

By Lemma 2.1 there exists a constant $C(\alpha,\beta)$ such that

$$\operatorname{Var}X_{k,\ell} \leqslant \begin{cases} C(\alpha,\beta), & \text{if } \alpha+\beta<1;\\ C(\alpha,\beta)\sqrt{k+\ell}, & \text{if } \alpha+\beta=1 \text{ and } 0<\alpha<1;\\ C(\alpha,\beta)(k+\ell), & \text{if } \alpha+\beta=1 \text{ and } \alpha\in\{0,1\}.\end{cases}$$

Now, it is easy to see that

$$\mathsf{E}\left(\sum_{(k,\ell)\in T_m\setminus T_{m-1}} \|A_{m,2,k,\ell}\|^2\right)^2 \leqslant 12M_4C(\alpha,\beta)m^{\gamma-4}(4m-1)^2,$$

which implies (4.14). □

5 Proof of Proposition 1.4

In what follows we will assume that $0 < \alpha < 1$ and $\beta = 1 - \alpha$. Consider the following expression of $\det B_n$

$$\det B_n = \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} W_{k_1,\ell_1,k_2,\ell_2},$$

where

$$W_{k_1,\ell_1,k_2,\ell_2} = X^2_{k_1,\ell_1-1} X^2_{k_2-1,\ell_2} - X_{k_1-1,\ell_1} X_{k_1,\ell_1-1} X_{k_2-1,\ell_2} X_{k_2,\ell_2-1}.$$

Using representation (1.3) of $X_{k,\ell}$ from Lemma 2.9 we obtain.

$$\mathsf{E} W_{k_1,\ell_1,k_2,\ell_2} = A^{(1)}_{k_1,\ell_1,k_2,\ell_2} + A^{(2)}_{k_1,\ell_1,k_2,\ell_2} + A^{(3)}_{k_1,\ell_1,k_2,\ell_2} + A^{(4)}_{k_1,\ell_1,k_2,\ell_2}, \tag{5.1}$$

where using notation (2.3)

$$\begin{aligned}
A^{(1)}_{k_1,\ell_1,k_2,\ell_2} = & \sum_{(i,j)\in T_{k_1\wedge(k_2-1),(\ell_1-1)\wedge\ell_2}} \big(\mathsf{E}\varepsilon^4_{i,j} - 3\big) \mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j} = k_1 - i\big)^2 \mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j} = \ell_2 - j\big)^2 \\
& - \sum_{(i,j)\in T_{k_1\wedge k_2-1,\ell_1\wedge\ell_2-1}} \big(\mathsf{E}\varepsilon^4_{i,j} - 3\big) \mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j} = k_1 - i\big) \mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j} = k_1 - 1 - i\big) \\
& \qquad \times \mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j} = \ell_2 - j\big) \mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j} = \ell_2 - 1 - j\big), \\
A^{(2)}_{k_1,\ell_1,k_2,\ell_2} = & \operatorname{Cov}^2(X_{k_1,\ell_1-1}, X_{k_2-1,\ell_2}) - \operatorname{Cov}(X_{k_1,\ell_1-1}, X_{k_2-1,\ell_2}) \operatorname{Cov}(X_{k_1-1,\ell_1}, X_{k_2,\ell_2-1}), \\
A^{(3)}_{k_1,\ell_1,k_2,\ell_2} = & \operatorname{Cov}^2(X_{k_1,\ell_1-1}, X_{k_2-1,\ell_2}) - \operatorname{Cov}(X_{k_1,\ell_1-1}, X_{k_2,\ell_2-1}) \operatorname{Cov}(X_{k_1-1,\ell_1}, X_{k_2-1,\ell_2}), \\
A^{(4)}_{k_1,\ell_1,k_2,\ell_2} = & \operatorname{Var}(X_{k_1,\ell_1-1}) \operatorname{Var}(X_{k_2-1,\ell_2}) \\
& - \operatorname{Cov}(X_{k_1-1,\ell_1}, X_{k_1,\ell_1-1}) \operatorname{Cov}(X_{k_2-1,\ell_2}, X_{k_2,\ell_2-1}).
\end{aligned}$$

It is easy to see, that

$$n^{-9/2} \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} A^{(1)}_{k_1,\ell_1,k_2,\ell_2} = n^{-1/2} \iint_T \iint_T A^{(1)}_{[ns_1],[nt_1],[ns_2],[nt_2]} \mathrm{d}s_1 \mathrm{d}t_1 \mathrm{d}s_2 \mathrm{d}t_2. \tag{5.2}$$

Moreover,

$$\begin{aligned}
\big|A^{(1)}_{[ns_1],[nt_1],[ns_2],[nt_2]}\big| \leqslant |M_4 - 3| & \sum_{m=1}^{[ns_1]\wedge([ns_2]-1)+([nt_1]-1)\wedge[nt_2]} a_{m,n}(s_1, t_1, s_2, t_2) \\
& + |M_4 - 3| \sum_{m=1}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]-2} b_{m,n}(s_1, t_1, s_2, t_2),
\end{aligned}$$

where $M_4 := \sup\{\mathsf{E}\varepsilon_{k,\ell}^4 : k, \ell \in \mathbb{Z},\ k+\ell \geqslant 1\}$ and

$$\begin{aligned}
&a_{m,n}(s_1,t_1,s_2,t_2)\\
&\quad := \sum_{i=m-([nt_1]-1)\wedge[nt_2]}^{[ns_1]\wedge([ns_2]-1)} \mathsf{P}^2\big(S^{(\alpha)}_{[ns_1]+[nt_1]-1-m} = [ns_1]-i\big)\mathsf{P}^2\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]-1-m} = [nt_2]-m+i\big)\\
&\quad \leqslant \mathsf{P}^2\big(S_{[ns_1]+[nt_1]-1-m,[ns_2]+[nt_2]-1-m} = [ns_1]+[nt_2]-m\big),\\
&b_{m,n}(s_1,t_1,s_2,t_2)\\
&\quad := \sum_{i=m-[nt_1]\wedge[nt_2]+1}^{[ns_1]\wedge[ns_2]-1} \mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]-1-m} = [ns_1]-i\big)\mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]-1-m} = [ns_1]-1-i\big)\\
&\qquad\qquad \times \mathsf{P}\big(S^{1-\alpha}_{[ns_2]+[nt_2]-1-m} = [nt_2]-1-m+i\big)\mathsf{P}\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]-1-m} = [nt_2]-m+i\big)\\
&\quad \leqslant \mathsf{P}^2\big(S_{[ns_1]+[nt_1]-1-m,[ns_2]+[nt_2]-1-m} = [ns_1]+[nt_2]-1-m\big).
\end{aligned}$$

Theorem 2.5 implies that there exists a constant D_α such that

$$\mathsf{P}(S_{k,\ell} = j) \leqslant \frac{D_\alpha}{\sqrt{k+\ell}}$$

holds for all $k, \ell \geqslant 1$ and $j \in \{0, 1, \ldots, k+\ell\}$. Hence,

$$\begin{aligned}
\big|A^{(1)}_{[ns_1],[nt_1],[ns_2],[nt_2]}\big| &\leqslant 2|M_4-3| \sum_{m=1}^{[ns_1]\wedge([ns_2]-1)+([nt_1]-1)\wedge[nt_2]} \frac{D_\alpha^2}{[ns_1]+[ns_2]+[nt_1]+[nt_2]-2-2m}\\
&\leqslant 2|M_4-3|D_\alpha^2 \int_1^{[ns_1]\wedge([ns_2]-1)+([nt_1]-1)\wedge[nt_2]+1} \frac{\mathrm{d}x}{[ns_1]+[ns_2]+[nt_1]+[nt_2]-2-2x}\\
&\leqslant |M_4-3|D_\alpha^2 \log\big([ns_1]+[ns_2]+[nt_1]+[nt_2]\big). \qquad (5.3)
\end{aligned}$$

By the dominated convergence theorem (5.2) and (5.3) imply

$$\lim_{n\to\infty} n^{-9/2} \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} A^{(1)}_{k_1,\ell_1,k_2,\ell_2} = 0. \qquad (5.4)$$

What concerns the second term in (5.1), we have

$$\begin{aligned}
&n^{-9/2} \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} A^{(2)}_{k_1,\ell_1,k_2,\ell_2} = \iint_T \iint_T \operatorname{Cov}(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{0,1}(s_2,t_2))\\
&\quad \times \sqrt{n}\big(\operatorname{Cov}(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{0,1}(s_2,t_2)) - \operatorname{Cov}(Z^{(n)}_{0,1}(s_1,t_1), Z^{(n)}_{1,0}(s_2,t_2))\big)\mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2
\end{aligned}$$

By Proposition 2.3 we have

$$\lim_{n\to\infty} \operatorname{Cov}(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{0,1}(s_2,t_2)) = z_\alpha(s_1,t_1,s_2,t_2). \qquad (5.5)$$

Moreover, from Proposition 2.3 also follows, that if $(1-\alpha)(s_1-s_2)\neq\alpha(t_1-t_2)$ then both $\mathrm{Cov}(Z_{1,0}^{(n)}(s_1,t_1),Z_{0,1}^{(n)}(s_2,t_2))$ and $\mathrm{Cov}(Z_{0,1}^{(n)}(s_1,t_1),Z_{1,0}^{(n)}(s_2,t_2))$ converge to 0 in exponential order, as $n\to\infty$. Hence, if $(1-\alpha)(s_1-s_2)\neq\alpha(t_1-t_2)$ then

$$\lim_{n\to\infty}\sqrt{n}\big(\mathrm{Cov}(Z_{1,0}^{(n)}(s_1,t_1),Z_{0,1}^{(n)}(s_2,t_2))-\mathrm{Cov}(Z_{0,1}^{(n)}(s_1,t_1),Z_{1,0}^{(n)}(s_2,t_2))\big)=0. \quad (5.6)$$

By Lemma 2.1 and Proposition 2.6 we can apply dominated convergence theorem that together with (5.6) and (5.5) implies

$$\lim_{n\to\infty} n^{-9/2}\sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n} A^{(2)}_{k_1,\ell_1,k_2,\ell_2}=0. \quad (5.7)$$

Using similar arguments one can show that

$$\lim_{n\to\infty} n^{-9/2}\sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n} A^{(3)}_{k_1,\ell_1,k_2,\ell_2}=0. \quad (5.8)$$

At the end we have

$$\begin{aligned}
&n^{-9/2}\sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n} A^{(4)}_{k_1,\ell_1,k_2,\ell_2}\\
&\quad=\iint\limits_T \mathrm{Var}(Z_{1,0}^{(n)}(s,t))\mathrm{d}s\mathrm{d}t\iint\limits_T\sqrt{n}\big(\mathrm{Var}(Z_{0,1}^{(n)}(s,t))-\mathrm{Cov}(Z_{0,1}^{(n)}(s,t),Z_{1,0}^{(n)}(s,t))\big)\mathrm{d}s\mathrm{d}t\\
&\qquad+\iint\limits_T \mathrm{Cov}(Z_{0,1}^{(n)}(s_1,t_1),Z_{1,0}^{(n)}(s_2,t_2))\mathrm{d}s\mathrm{d}t\\
&\qquad\times\iint\limits_T\sqrt{n}\big(\mathrm{Var}(Z_{1,0}^{(n)}(s,t))-\mathrm{Cov}(Z_{0,1}^{(n)}(s,t),Z_{1,0}^{(n)}(s,t))\big)\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Proposition 2.3 and the dominated convergence theorem imply

$$\lim_{n\to\infty}\iint\limits_T\mathrm{Var}(Z_{1,0}^{(n)}(s,t))\mathrm{d}s\mathrm{d}t=\lim_{n\to\infty}\iint\limits_T\mathrm{Cov}(Z_{1,0}^{(n)}(s,t),Z_{0,1}^{(n)}(s,t))\mathrm{d}s\mathrm{d}t=\frac{2^{9/2}}{15\sqrt{\pi\alpha(1-\alpha)}}.$$

Short straightforward calculation shows that

$$\begin{aligned}
&\sqrt{n}\big(\mathrm{Var}(Z_{1,0}^{(n)}(s,t))-\mathrm{Cov}(Z_{0,1}^{(n)}(s,t),Z_{1,0}^{(n)}(s,t))\big)\\
&\quad=\sqrt{n}\big(\mathrm{Var}(Z_{0,1}^{(n)}(s,t))-\mathrm{Cov}(Z_{0,1}^{(n)}(s,t),Z_{1,0}^{(n)}(s,t))\big)\\
&\quad=1+\sum_{m=1}^{[ns]+[nt]}\Big(\mathsf{P}(S_{[ns]+[nt]+1-m,[ns]+[nt]+1-m}=[ns]+[nt]+1-m)\\
&\qquad-\mathsf{P}(S_{[ns]+[nt]+1-m,[ns]+[nt]+1-m}=[ns]+[nt]-m)\Big)=1+R_{[ns]+[nt]},
\end{aligned}$$

where

$$R_{[ns]+[nt]} := \sum_{m=1}^{[ns]+[nt]} \Big(\mathsf{P}(S_{m,m} = m) - \mathsf{P}(S_{m,m} = m-1) \Big).$$

Obviously, the limit of $R_{[ns]+[nt]}$, if it exists, does not depend on s and t, it is equal to

$$R_\infty := \sum_{m=1}^{\infty} \Big(\mathsf{P}(S_{m,m} = m) - \mathsf{P}(S_{m,m} = m-1) \Big). \tag{5.9}$$

First we show that the terms of the sum (5.9) are strictly positive.

$$\begin{aligned}
\mathsf{P}(S_{m,m} = m-1) &= \sum_{i=0}^{m-1} \mathsf{P}(S_m^{(\alpha)} = i)\mathsf{P}(S_m^{(1-\alpha)} = m-1-i) \\
&\leqslant \Big(\sum_{i=0}^{m-1} \mathsf{P}(S_m^{(\alpha)} = i)^2 \Big)^{1/2} \Big(\sum_{i=0}^{m-1} \mathsf{P}(S_m^{(1-\alpha)} = m-1-i)^2 \Big)^{1/2} \\
&\leqslant \Big(\sum_{i=0}^{m-1} \mathsf{P}(S_m^{(\alpha)} = i)^2 \Big)^{1/2} \Big(\sum_{i=1}^{m} \mathsf{P}(S_m^{(\alpha)} = i)^2 \Big)^{1/2} < \sum_{i=0}^{m} \mathsf{P}(S_m^{(\alpha)} = i)^2 \\
&= \sum_{i=0}^{m} \mathsf{P}(S_m^{(\alpha)} = i)\mathsf{P}(S_m^{(1-\alpha)} = m-i) = \mathsf{P}(S_{m,m} = m)
\end{aligned}$$

where the last inequality follows from $0 < \alpha < 1$. Now, let

$$\widetilde{R}_n := \sum_{m=1}^{n} \Big(\mathsf{P}(S_{m,m} = m) - \mathsf{P}(S_{m,m} = m-1) \Big).$$

For all $k \in \mathbb{N}$, we have

$$\big|\widetilde{R}_{n+k} - \widetilde{R}_n\big| = \sum_{m=n+1}^{n+k} \Big| \mathsf{P}(S_{m,m} = m) - \mathsf{P}(S_{m,m} = m-1) \Big|. \tag{5.10}$$

Let $\delta > 0$. If we apply Theorem 2.7 to approximate the terms of the sum in (5.10), the error of the approximation is

$$\sum_{m=n+1}^{n+k} \frac{C_\alpha}{(2m)^{3/2}} \leqslant \frac{C_\alpha}{2^{3/2}} \sum_{m=n+1}^{\infty} \frac{1}{m^{3/2}} \leqslant \frac{C_\alpha}{2^{3/2}} \int_n^\infty \frac{1}{x^{3/2}} \mathrm{d}x \leqslant \frac{C_\alpha}{\sqrt{2n}} < \frac{\delta}{2}, \tag{5.11}$$

if n is large enough. The approximation of $\big|\widetilde{R}_{n+k} - \widetilde{R}_n\big|$ equals

$$\begin{aligned}
&\sum_{m=n+1}^{n+k} \frac{1}{\sqrt{2\pi m\alpha(1-\alpha)}} \big| 1 - \mathrm{e}^{-1/(4m\alpha(1-\alpha))} \big| \qquad (5.12) \\
&\qquad \leqslant \frac{1}{\sqrt{\pi}(4\alpha(1-\alpha))^{3/2}} \mathrm{e}^{-1/(4n\alpha(1-\alpha))} \sum_{m=n+1}^{\infty} \frac{1}{m^{3/2}} \leqslant \frac{\delta}{2},
\end{aligned}$$

if n is large enough. Hence, (5.11) and (5.12) imply the absolute convergence of (5.9). Using the definition $\mathsf{P}(S_{0,0} = -1) := 0$ we have

$$\begin{aligned} 1 + R_\infty &= \sum_{m=0}^{\infty} \Big(\mathsf{P}\big(S_m^{(\alpha)} + S_m^{(1-\alpha)} = m\big) - \mathsf{P}\big(S_m^{(\alpha)} + S_m^{(1-\alpha)} = m-1\big) \Big) \\ &= \lim_{\beta \to 1-\alpha} \sum_{m=0}^{\infty} \Big(\mathsf{P}\big(S_m^{(\alpha)} + S_m^{(\beta)} = m\big) - \mathsf{P}\big(S_m^{(\alpha)} + S_m^{(\beta)} = m-1\big) \Big) \\ &= \lim_{\beta \to 1-\alpha} \big(F_4(1,1,1,1;\alpha^2,\beta^2) - \alpha\beta F_4(2,2,2,2;\alpha^2,\beta^2) \big) \end{aligned}$$

where $F_4(a,b,c,d;x,y)$ is a hipergeometric series of two parameters defined by

$$F_4(a,b,c,d;x,y) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (d)_n m! n!} x^m y^n, \qquad \sqrt{|x|} + \sqrt{|y|} \leqslant 1,$$

$a,b,c,d \in \mathbb{N}$ and $(a)_n := a(a+1)\dots(a+n-1)$ (see e.g. [6]). As

$$F_4\Big(a,b,a,b;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\Big) = \frac{(1-x)^b(1-y)^a}{1-xy}$$

we have

$$\begin{aligned} 1 + R_\infty = \lim_{\beta \to 1-\alpha} & \frac{\big(1 \mp \sigma_{\alpha,\beta}^{-2} + \alpha^2 - \beta^2\big)\big(1 \mp \sigma_{\alpha,\beta}^{-2} - \alpha^2 + \beta^2\big)}{4\alpha\beta\big(1 \mp \sigma_{\alpha,\beta}^{-2} - (\alpha+\beta)^2\big)\big(1 \mp \sigma_{\alpha,\beta}^{-2} - (\alpha-\beta)^2\big)} \\ & \times \Big(\big(1 \mp \sigma_{\alpha,\beta}^{-2} + \alpha^2 - \beta^2\big)\big(1 \mp \sigma_{\alpha,\beta}^{-2} - \alpha^2 + \beta^2\big) - 4\alpha\beta \Big) = \frac{1}{2}\varrho_\alpha^2 \end{aligned} \tag{5.13}$$

since $\sigma_{\alpha,\beta}^{-2} \to 0$ if $\beta \to 1-\alpha$. Since the right hand side of (5.13) can serve as a dominating function for $R_{[ns]+[nt]}$, from the dominated convergence theorem follows

$$\lim_{n\to\infty} \iint\limits_T \sqrt{n}\big(\operatorname{Var}(Z_{1,0}^{(n)}(s,t)) - \operatorname{Cov}(Z_{0,1}^{(n)}(s,t), Z_{1,0}^{(n)}(s,t))\big)\mathrm{d}s\mathrm{d}t = \varrho_\alpha^2, \tag{5.14}$$

which together with (5.4), (5.7) and (5.8) implies

$$\lim_{n\to\infty} n^{-9/2}\mathsf{E}\det B_n = 2\sigma_\alpha^2\varrho_\alpha^2.$$

Now, let us deal with the variance of $\det B_n$. Obviously,

$$\begin{aligned} \frac{1}{n^9}\operatorname{Var}\big(\det B_n\big) &= \frac{1}{n^9} \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} \sum_{(k_3,\ell_3)\in T_n} \sum_{(k_4,\ell_4)\in T_n} \operatorname{Cov}\big(W_{k_1,\ell_1,k_2,\ell_2}, W_{k_3,\ell_3,k_4,\ell_4}\big) \\ &= \iint\limits_T \iint\limits_T \iint\limits_T \iint\limits_T n\operatorname{Cov}\big(\widetilde{W}_n(s_1,t_1,s_2,t_2), \widetilde{W}_n(s_3,t_3,s_4,t_4)\big)\mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2\mathrm{d}s_3\mathrm{d}t_3\mathrm{d}s_4\mathrm{d}t_4, \end{aligned} \tag{5.15}$$

where

$$\widetilde{W}_n(s_1,t_1,s_2,t_2) := \big(Z_{1,0}^{(n)}(s_1,t_1)Z_{0,1}^{(n)}(s_2,t_2)\big)^2 - Z_{1,0}^{(n)}(s_1,t_1)Z_{0,1}^{(n)}(s_1,t_1)Z_{1,0}^{(n)}(s_2,t_2)Z_{0,1}^{(n)}(s_2,t_2).$$

Short calculation shows that the right hand side of (5.15) can be rewritten as

$$\iint\limits_T \iint\limits_T \iint\limits_T \iint\limits_T n\Big(\Theta_n^{(1)}(s_1,t_1,s_2,t_2,s_3,t_3,s_4,t_4) + \Theta_n^{(2)}(s_1,t_1,s_2,t_2,s_3,t_3,s_4,t_4) + 2\Theta_n^{(3)}(s_1,t_1,s_2,t_2,s_3,t_3,s_4,t_4)\Big)\mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2\mathrm{d}s_3\mathrm{d}t_3\mathrm{d}s_4\mathrm{d}t_4, \tag{5.16}$$

where

$$\begin{aligned}
&\Theta_n^{(1)}(s_1,t_1,s_2,t_2,s_3,t_3,s_4,t_4)\\
&\quad := \mathrm{Cov}\Big(Z_{0,1}^{(n)}(s_1,t_1)Z_{1,0}^{(n)}(s_2,t_2)\big(Z_{0,1}^{(n)}(s_1,t_1) - Z_{1,0}^{(n)}(s_1,t_1)\big)\big(Z_{1,0}^{(n)}(s_2,t_2) - Z_{0,1}^{(n)}(s_2,t_2)\big),\\
&\qquad Z_{0,1}^{(n)}(s_3,t_3)Z_{1,0}^{(n)}(s_4,t_4)\big(Z_{0,1}^{(n)}(s_3,t_3) - Z_{1,0}^{(n)}(s_3,t_3)\big)\big(Z_{1,0}^{(n)}(s_4,t_4) - Z_{0,1}^{(n)}(s_4,t_4)\big)\Big),\\
&\Theta_n^{(2)}(s_1,t_1,s_2,t_2,s_3,t_3,s_4,t_4) := \mathrm{Cov}\Big(Z_{0,1}^{(n)}(s_1,t_1)Z_{1,0}^{(n)}(s_1,t_1)\big(Z_{1,0}^{(n)}(s_2,t_2) - Z_{0,1}^{(n)}(s_2,t_2)\big)^2,\\
&\qquad Z_{0,1}^{(n)}(s_3,t_3)Z_{1,0}^{(n)}(s_3,t_3)\big(Z_{1,0}^{(n)}(s_4,t_4) - Z_{0,1}^{(n)}(s_4,t_4)\big)^2\Big),\\
&\Theta_n^{(3)}(s_1,t_1,s_2,t_2,s_3,t_3,s_4,t_4) := \mathrm{Cov}\Big(Z_{0,1}^{(n)}(s_1,t_1)Z_{1,0}^{(n)}(s_1,t_1)\big(Z_{1,0}^{(n)}(s_2,t_2) - Z_{0,1}^{(n)}(s_2,t_2)\big)^2,\\
&\qquad Z_{0,1}^{(n)}(s_3,t_3)Z_{1,0}^{(n)}(s_4,t_4)\big(Z_{0,1}^{(n)}(s_3,t_3) - Z_{1,0}^{(n)}(s_3,t_3)\big)\big(Z_{1,0}^{(n)}(s_4,t_4) - Z_{0,1}^{(n)}(s_4,t_4)\big)\Big).
\end{aligned}$$

By representation (1.3) the components $n\Theta_n^{(q)}$, $q=1,2,3$, of the integrand in (5.16) are linear combinations of covariances of form

$$\mathrm{Cov}(\varepsilon_{i_1,j_1}\varepsilon_{i_2,j_2}\varepsilon_{i_3,j_3}\varepsilon_{i_4,j_4}, \varepsilon_{i_5,j_5}\varepsilon_{i_6,j_6}\varepsilon_{i_7,j_7}\varepsilon_{i_8,j_8}), \tag{5.17}$$

where the indices $(i_r,j_r)\in\mathbb{Z}^2$, $r=1,2,\dots,8$ run either on triangle $T_{[ns_q],[nt_q]-1}$ or on $T_{[ns_q]-1,[nt_q]}$, $q=\big[(r+1)/2\big]$. The coefficients of the linear combinations are products of $1/n$ and two terms of form $\mathsf{P}\big(S^\alpha_{[ns_{m_r}]+[nt_{m_r}]-1-i_r-j_r} = [ns_{m_r}]-1-i_r\big)$, two terms of form $\mathsf{P}\big(S^\alpha_{[ns_{m_r}]+[nt_{m_r}]-1-i_r-j_r} = [ns_{m_r}]-i_r\big)$ and four terms of form

$$\mathsf{P}\big(S^\alpha_{[ns_{m_r}]+[nt_{m_r}]-1-i_r-j_r} = [ns_{m_r}]-i_r\big) - \mathsf{P}\big(S^\alpha_{[ns_{m_r}]+[nt_{m_r}]-1-i_r-j_r} = [ns_{m_r}]-1-i_r\big) =: \widehat{\Delta}^{(n)}_{i_r,j_r}(s_{m_r},t_{m_r}),$$

where $\cup_{r=1}^8\{m_r\}=\{1,2,3,4\}$. Corollary 2.8 implies that there exists a constant $C_\alpha>0$ depending only on α such that

$$\Big|\widehat{\Delta}^{(n)}_{i_r,j_r}(s_{m_r},t_{m_r})\Big| \leqslant \frac{C_\alpha}{[ns_{m_r}]+[nt_{m_r}]-1-i_r-j_r}. \tag{5.18}$$

Covariances of form (5.17) are equal to zero if the index sets $\{(i_r, j_r) : r = 1,2,3,4\}$ and $\{(i_r, j_r) : r = 5,6,7,8\}$ are disjoint. Besides the nonempty intersection of these sets, to obtain nonzero covariances in (5.17) for each $u \in \{1,2,\ldots,8\}$ there should exist at least one $v \in \{1,2,\ldots,8\}$ such that $u \neq v$ and $(i_u, j_u) = (i_v, j_v)$. Consider first the case, when $\{1,2,\ldots,8\}$ is divided into two disjoint subsets $\{u_1,u_2,u_3,u_4\}$ and $\{v_1,v_2,v_3,v_4\}$, $(i_{u_r}, j_{u_r}) = (i_{v_r}, j_{v_r})$, $r = 1,2,3,4$ holds and no other index pairs are equal. This configuration yields the highest amount of terms when we express $\mathrm{Cov}\big(\widetilde{W}_n(s_1,t_1,s_2,t_2), \widetilde{W}_n(s_3,t_3,s_4,t_4)\big)$. Expression (5.16) shows that the sum of the corresponding terms of $n^{-9}\mathrm{Var}\big(\det B_n\big)$ can be rewritten as the sum of terms of form

$$\iint\limits_T \iint\limits_T \iint\limits_T \iint\limits_T \mathrm{Cov}\big(Z^{(n)}_{u_1,v_1}(s_{m_1},t_{m_1}) Z^{(n)}_{u_2,v_2}(s_{m_2},t_{m_2})\big)$$

$$\times \mathrm{Cov}\big(Z^{(n)}_{u_3,v_3}(s_{m_3},t_{m_3}) Z^{(n)}_{u_4,v_4}(s_{m_4},t_{m_4})\big)$$

$$\times\sqrt{n}\Big(\mathrm{Cov}\big(Z^{(n)}_{u_5,v_5}(s_{m_5},t_{m_5}) Z^{(n)}_{u_5,v_5}(s_{m_6},t_{m_6})\big) - \mathrm{Cov}\big(Z^{(n)}_{u_5,v_5}(s_{m_5},t_{m_5}) Z^{(n)}_{v_5,u_5}(s_{m_6},t_{m_6})\big)\Big)$$

$$\times\sqrt{n}\Big(\mathrm{Cov}\big(Z^{(n)}_{u_6,v_6}(s_{m_7},t_{m_7}) Z^{(n)}_{u_6,v_6}(s_{m_8},t_{m_8})\big) - \mathrm{Cov}\big(Z^{(n)}_{u_6,v_6}(s_{m_7},t_{m_7}) Z^{(n)}_{v_6,u_6}(s_{m_8},t_{m_8})\big)\Big)$$

$$\mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2\mathrm{d}s_3\mathrm{d}t_3\mathrm{d}s_4\mathrm{d}t_4,$$

where $\{m_r : r = 1,2,\ldots,8\} = \{1,2,3,4\}$ and $(u_r, v_r) \in \{(0,1),(1,0)\}$, $r = 1,2,\ldots,6$. Lemma 2.1, Propositions 2.3 and 2.6 and the dominated convergence theorem imply that these terms of $n^{-9}\mathrm{Var}\big(\det B_n\big)$ converge to 0 as $n \to \infty$.

The next case is when the set $\{1,2,\ldots,8\}$ is divided into three disjoint subsets $\{u_1,u_2,u_3\}$ and $\{v_1,v_2,v_3\}$ and $\{w_1,w_2\}$ and either

$$(i_{u_r}, j_{u_r}) = (i_{v_r}, j_{v_r}) = (i_{w_r}, j_{w_r}), \quad r = 1,2, \qquad \text{and} \qquad (i_{u_3}, j_{u_3}) = (i_{v_3}, j_{v_3})$$

or

$$(i_{u_r}, j_{u_r}) = (i_{v_r}, j_{v_r}), \quad r = 1,2, \quad \text{and} \quad (i_{u_3}, j_{u_3}) = (i_{v_3}, j_{v_3}) = (i_{w_1}, j_{w_1}) = (i_{w_1}, j_{w_2})$$

holds and no other index pairs are equal. Inequality (5.18) implies that expressions of form

$$\sum_{(i,j)\in T_{[ns_1]\wedge[ns_2]\wedge[ns_3]-1,[nt_1]\wedge[nt_2]\wedge[nt_3]-1}} \mathsf{P}\big(S^{\alpha}_{[ns_1]+[nt_1]-i-j} = [ns_1]-i\big)\widehat{\Delta}^{(n)}_{i,j}(s_2,t_2)\widehat{\Delta}^{(n)}_{i,j}(s_3,t_3)$$

are bounded uniformly in n and $(s_r, t_r) \in T$, $r = 1,2,3$. Hence, Propositions 2.3 and 2.6 and the dominated convergence theorem imply that the terms of $n^{-9}\mathrm{Var}\big(\det B_n\big)$ corresponding to this second case also converge to 0 as $n \to \infty$. The remaining terms can be handled in a similar way. □

6 Proof of Proposition 1.5

Similarly to Section 5 it is enough to consider the case $0 < \alpha < 1$ and $\beta = 1 - \alpha$. We have

$$n^{-7/2}\bar{B}_n A_n = \big(n^{-5/2}\bar{B}_n - \sigma^2_\alpha \bar{\mathbf{1}}\big)n^{-1}A_n + n^{-1}\sigma^2_\alpha \bar{\mathbf{1}} A_n,$$

and short straightforward calculation shows that

$$\big(n^{-5/2}\bar{B}_n - \sigma_\alpha^2 \mathbf{1}\big) n^{-1} A_n = C_n + D_n,$$

where

$$C_n := n^{-5/4}\,\mathrm{diag}(A_n) n^{-9/4} \bar{B}_n (1,1)^\top,$$
$$D_n := \Big(\frac{1}{n^{5/2}} \sum_{(k,\ell)\in T_n} X_{k-1,\ell}X_{k,\ell-1} - \sigma_\alpha^2\Big)\frac{1}{n} Q_n (1,-1)^\top.$$

Here $\mathrm{diag}(A_n)$ denotes the two-by-two diagonal matrix having A_n in its main diagonal and

$$Q_n := (1,-1)A_n = \sum_{(k,\ell)\in T_n} \big((X_{k-1,\ell} - X_{k,\ell-1})\varepsilon_{k,\ell}\big). \tag{6.1}$$

By Proposition 1.2

$$\frac{1}{n^{5/2}} \sum_{(k,\ell)\in T_n} X_{k-1,\ell}X_{k,\ell-1} - \sigma_\alpha^2 \stackrel{\mathsf{L}_2}{\longrightarrow} 0 \qquad \text{as } n\to\infty. \tag{6.2}$$

Representation (1.3) and independence of the error terms $\varepsilon_{k,\ell}$ imply $\mathsf{E}Q_n = 0$ and

$$\begin{aligned}\frac{1}{n^2}\mathsf{E}Q_n^2 &= \frac{1}{n^2}\sum_{(k,\ell)\in T_n} (X_{k-1,\ell} - X_{k,\ell-1})^2 \qquad (6.3)\\ &= \iint_T \sqrt{n}\Big(\mathrm{Var}\big(Z_{1,0}^{(n)}(s,t)\big) + \mathrm{Var}\big(Z_{1,0}^{(n)}(s,t)\big) - 2\mathrm{Cov}\big(Z_{0,1}^{(n)}(s,t), Z_{1,0}^{(n)}(s,t)\big)\Big)\mathrm{d}s\mathrm{d}t.\end{aligned}$$

According to Proposition 2.6 the last term is bounded which together with (6.2) implies $D_n \stackrel{\mathsf{P}}{\longrightarrow} (0,0)^\top$ as $n\to\infty$.

$$\begin{aligned}\mathsf{E}\Big(\frac{1}{n^{9/4}}\bar{B}_n(1,1)^\top\Big) &= \frac{1}{n^{9/4}}\sum_{(k,\ell)\in T_n} \mathsf{E}\begin{pmatrix} X_{k,\ell-1}^2 - X_{k-1,\ell}X_{k,\ell-1}\\ X_{k,\ell-1}^2 - X_{k-1,\ell}X_{k,\ell-1}\end{pmatrix}\\ &= \frac{1}{n^{1/4}}\iint_T \sqrt{n}\begin{pmatrix}\mathrm{Var}\big(Z_{1,0}^{(n)}(s,t)\big) - \mathrm{Cov}\big(Z_{0,1}^{(n)}(s,t), Z_{1,0}^{(n)}(s,t)\big)\\ \mathrm{Var}\big(Z_{0,1}^{(n)}(s,t)\big) - \mathrm{Cov}\big(Z_{0,1}^{(n)}(s,t), Z_{1,0}^{(n)}(s,t)\big)\end{pmatrix}\mathrm{d}s\mathrm{d}t \to \begin{pmatrix}0\\0\end{pmatrix}\end{aligned}$$

as $n\to\infty$. Furthermore, using Lemma 2.9 we obtain

$$\begin{aligned}\mathrm{Var}\Big(\sum_{(k,\ell)\in T_n}(X_{k,\ell-1}^2 - X_{k-1,\ell}X_{k,\ell-1})\Big) = \sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n} & B_{k_1,\ell_1,k_2,\ell_2}^{(1)} + 2B_{k_1,\ell_1,k_2,\ell_2}^{(2)}\\ & + B_{k_1,\ell_1,k_2,\ell_2}^{(3)} + B_{k_1,\ell_1,k_2,\ell_2}^{(4)},\end{aligned}$$

where

$$B^{(1)}_{k_1,\ell_1,k_2,\ell_2} = \sum_{(i,j)\in T_{k_1\wedge k_2,\ell_1\wedge\ell_2-1}} (\mathsf{E}\varepsilon^4_{i,j}-3)\mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j}=k_1-i\big)^2\mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j}=\ell_2-1-j\big)^2$$
$$- \sum_{(i,j)\in T_{k_1\wedge(k_2-1),\ell_1\wedge\ell_2-1}} (\mathsf{E}\varepsilon^4_{i,j}-3)\mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j}=\ell_2-j\big)\mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j}=\ell_2-1-j\big)$$
$$\times \mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j}=k_1-i\big)^2$$
$$- \sum_{(i,j)\in T_{(k_1-1)\wedge k_2,\ell_1\wedge\ell_2-1}} (\mathsf{E}\varepsilon^4_{i,j}-3)\mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j}=k_1-i\big)\mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j}=k_1-1-i\big)$$
$$\times \mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j}=\ell_2-1-j\big)^2$$
$$+ \sum_{(i,j)\in T_{k_1\wedge k_2-1,\ell_1\wedge\ell_2-1}} (\mathsf{E}\varepsilon^4_{i,j}-3)\mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j}=k_1-i\big)\mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j}=k_1-1-i\big)$$
$$\times \mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j}=\ell_2-j\big)\mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j}=\ell_2-1-j\big),$$
$$B^{(2)}_{k_1,\ell_1,k_2,\ell_2} = \mathrm{Cov}(X_{k_1,\ell_1-1},X_{k_2,\ell_2-1})\big(\mathrm{Cov}(X_{k_1,\ell_1-1},X_{k_2,\ell_2-1})-\mathrm{Cov}(X_{k_1,\ell_1-1},X_{k_2-1,\ell_2})\big),$$
$$B^{(3)}_{k_1,\ell_1,k_2,\ell_2} = \mathrm{Cov}(X_{k_1,\ell_1-1},X_{k_2,\ell_2-1})\big(\mathrm{Cov}(X_{k_1-1,\ell_1},X_{k_2-1,\ell_2})-\mathrm{Cov}(X_{k_1-1,\ell_1},X_{k_2,\ell_2-1})\big),$$
$$B^{(4)}_{k_1,\ell_1,k_2,\ell_2} = \mathrm{Cov}(X_{k_1-1,\ell_1},X_{k_2,\ell_2-1})\big(\mathrm{Cov}(X_{k_1,\ell_1-1},X_{k_2-1,\ell_2})-\mathrm{Cov}(X_{k_1,\ell_1-1},X_{k_2,\ell_2-1})\big).$$

Hence, using the same arguments as in the proof of Proposition 1.4 (see (5.4) and (5.7)) one can verify

$$\lim_{n\to\infty}\frac{1}{n^{9/2}}\mathrm{Var}\bigg(\sum_{(k,\ell)\in T_n}(X^2_{k,\ell-1}-X_{k-1,\ell}X_{k,\ell-1})\bigg)=0.$$

Naturally, the same holds for the second component of $n^{-9/4}\bar{B}_n(1,1)^\top$, which means

$$n^{-9/4}\bar{B}_n(1,1)^\top \xrightarrow{\mathsf{L}_2} (0,0)^\top \qquad \text{as } n\to\infty. \tag{6.4}$$

Proposition 1.3 and (6.4) imply $C_n \xrightarrow{\mathsf{P}} (0,0)^\top$ as $n\to\infty$, so to prove the asymptotic normality of $n^{-7/2}\bar{B}_nA_n$ it suffices to show the asymptotic normality of $n^{-1}\sigma^2_\alpha\bar{\mathbf{1}}A_n = n^{-1}Q_n(\sigma^2_\alpha,-\sigma^2_\alpha)^\top$.

As A_n is a two dimensional martingale with respect to filtration $(\mathcal{F}_n)_{n\geqslant 1}$,

$$Q_n-Q_{n-1}=A^{(1)}_{n,1}-A^{(2)}_{n,1}+\sum_{(k,\ell)\in T_n\setminus T_{n-1}}\varepsilon_{k,\ell}\big(\widetilde{A}_{n,2,k-1,\ell}-\widetilde{A}_{n,2,k,\ell-1}\big) \tag{6.5}$$

is a martingale difference with respect to the same filtration. This means that similarly to the proof of Proposition 1.3 we can apply the Martingale Central Limit Theorem and the statement of Proposition 1.5 follows from the propositions below.

Proposition 6.1

$$n^{-2}\sum_{m=1}^{n}\mathsf{E}\big((Q_m-Q_{m-1})^2\,\big|\,\mathcal{F}_{m-1}\big)\xrightarrow{\mathsf{P}} 2\varrho^2_\alpha \qquad \textit{as } n\to\infty.$$

Proposition 6.2 *For all* $\delta > 0$,

$$n^{-2} \sum_{m=1}^{n} \mathsf{E}\left((Q_m - Q_{m-1})^2 \mathbb{1}_{\{|Q_m - Q_{m-1}| \geqslant \delta n\}} \,\middle|\, \mathcal{F}_{m-1} \right) \xrightarrow{\mathsf{P}} 0 \qquad \text{as} \quad n \to \infty.$$

Proof of Proposition 6.1. The proof is very similar to that of Proposition 4.1. Let $V_m := \mathsf{E}\big((Q_m - Q_{m-1})^2 \,\big|\, \mathcal{F}_{m-1}\big)$. The statement of Proposition 6.1 will follow from

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{m=1}^{n} \mathsf{E} V_m = 2\varrho_\alpha^2, \tag{6.6}$$

$$\lim_{n\to\infty} \frac{1}{n^4} \operatorname{Var}\left(\sum_{m=1}^{n} V_m \right) = 0. \tag{6.7}$$

By the martingale property of Q_n we have

$$\sum_{m=1}^{n} \mathsf{E} V_m = \sum_{m=1}^{n} \mathsf{E}\Big(\mathsf{E}\big(Q_m^2 \,\big|\, \mathcal{F}_{m-1}\big) - Q_{m-1}^2 \Big) = \sum_{m=1}^{n} \big(\mathsf{E} Q_m^2 - \mathsf{E} Q_{m-1}^2 \big) = \mathsf{E} Q_n^2$$

which together with (6.3) and (5.14) implies (6.6).

Furthermore, representations (6.1) of Q_n and (4.8) of U_n imply

$$V_n = (1,-1) U_n U_n^\top (1,-1)^\top = \mathsf{E}\big(A_{n,1}^{(1)} - A_{n,1}^{(2)}\big)^2 + \sum_{(k,\ell)\in T_n \setminus T_{n-1}} \big(\widetilde{A}_{n,2,k-1,\ell} - \widetilde{A}_{n,2,k,\ell-1}\big)^2.$$

This means that to verify (6.7) one has to show

$$\lim_{n\to\infty} \frac{1}{n^4} \operatorname{Var}\left(\sum_{m=1}^{n} \sum_{(k,\ell)\in T_m \setminus T_{m-1}} \big(\widetilde{A}_{m,2,k-1,\ell} - \widetilde{A}_{m,2,k,\ell-1}\big)^2 \right) = 0.$$

Now, consider

$$\begin{aligned}
&\operatorname{Var}\left(\sum_{m=1}^{n} \sum_{(k,\ell)\in T_m \setminus T_{m-1}} \big(\widetilde{A}_{m,2,k-1,\ell} - \widetilde{A}_{m,2,k,\ell-1}\big)^2 \right) \qquad (6.8)\\
&= \sum_{m_1=1}^{n} \sum_{(k_1,\ell_1)\in T_{m_1} \setminus T_{m_1-1}} \sum_{m_2=1}^{n} \sum_{(k_2,\ell_2)\in T_{m_2} \setminus T_{m_2-1}} G_{m_1,m_2,k_1,\ell_1,k_2,\ell_2} \\
&= \sum_{m_1=1}^{n} \sum_{m_2=1}^{n} \Bigg(\sum_{k_1=-m_1+1}^{m_1} \sum_{k_2=-m_2+1}^{m_2} G_{m_1,m_2,k_1,m_1,k_2,m_2} \\
&\quad + \sum_{k_1=-m_1+1}^{m_1} \sum_{\ell_2=-m_2+1}^{m_2-1} G_{m_1,m_2,k_1,m_1,m_2,\ell_2} \\
&\quad + \sum_{\ell_1=-m_1+1}^{m_1-1} \sum_{k_2=-m_2+1}^{m_2} G_{m_1,m_2,m_1,\ell_1,k_2,m_2} + \sum_{\ell_1=-m_1+1}^{m_1-1} \sum_{\ell_2=-m_2+1}^{m_2-1} G_{m_1,m_2,m_1,\ell_1,m_2,\ell_2} \Bigg),
\end{aligned}$$

where

$$G_{m_1,m_2,k_1,\ell_1,k_2,\ell_2} := \operatorname{Cov}\Big(\big(\widetilde{A}_{m_1,2,k_1-1,\ell_1} - \widetilde{A}_{m_1,2,k_1,\ell_1-1}\big)^2, \big(\widetilde{A}_{m_2,2,k_2-1,\ell_2} - \widetilde{A}_{m_2,2,k_2,\ell_2-1}\big)^2\Big).$$

By representation (1.3) of $X_{k,\ell}$ and definition (4.4) of $\widetilde{A}_{m,2,k,\ell}$ we have that

$$\begin{aligned}
\widetilde{A}_{m,2,k-1,m} &= X_{k-1,m} - \sum_{i=-m+2}^{k-1} \alpha^{k-1-i} \varepsilon_{i,m}, && -m+2 \leqslant k \leqslant m;\\
\widetilde{A}_{m,2,k,m-1} &= X_{k,m-1}, && -m+1 \leqslant k \leqslant m;\\
\widetilde{A}_{m,2,m,\ell-1} &= X_{m,\ell-1} \sum_{j=-m+2}^{\ell-1} (1-\alpha)^{\ell-1-j} \varepsilon_{m,j}, && -m+2 \leqslant \ell \leqslant m-1;\\
\widetilde{A}_{m,2,m-1,\ell} &= X_{m-1,\ell}, && -m+1 \leqslant \ell \leqslant m-1.
\end{aligned}$$

Hence, e.g.

$$\begin{aligned}
&\sum_{k_1=-m_1+1}^{m_1} \sum_{k_2=-m_2+1}^{m_2} G_{m_1,m_2,k_1,m_1,k_2,m_2}\\
&= \sum_{k_1=-m_1+1}^{m_1} \sum_{k_2=-m_2+1}^{m_2} \operatorname{Cov}\Bigg(\Big(X_{k_1-1,m_1} - X_{k_1,m_1-1} - \sum_{i_1=-m_1+2}^{k_1-1} \alpha^{k_1-1-i_1} \varepsilon_{i_1,m_1}\Big)^2,\\
&\qquad\qquad \Big(X_{k_2-1,m_2} - X_{k_2,m_2-1} - \sum_{i_2=-m_2+2}^{k_2-1} \alpha^{k_2-1-i_2} \varepsilon_{i_2,m_2}\Big)^2\Bigg)\\
&\leqslant 4 \sum_{k_1=-m_1+1}^{m_1} \sum_{k_2=-m_2+1}^{m_2} G^{(1)}_{k_1,m_1,k_2,m_2} + G^{(2)}_{k_1,m_1,k_2,m_2} + G^{(2)}_{k_2,m_2,k_1,m_1} + G^{(3)}_{k_1,m_1,k_2,n_2},
\end{aligned}$$

where

$$\begin{aligned}
G^{(1)}_{k_1,m_1,k_2,m_2} &:= \operatorname{Cov}\Big(\big(X_{k_1-1,m_1} - X_{k_1,m_1-1}\big)^2, \big(X_{k_2-1,m_2} - X_{k_2,m_2-1}\big)^2\Big);\\
G^{(2)}_{k_1,m_1,k_2,m_2} &:= \operatorname{Cov}\Bigg(\big(X_{k_1-1,m_1} - X_{k_1,m_1-1}\big)^2, \Big(\sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i} \varepsilon_{i,m_2}\Big)^2\Bigg)\\
&\leqslant 2\operatorname{Cov}\Bigg(\Big(X^2_{k_1-1,m_1} + X^2_{k_1,m_1-1}\Big), \Big(\sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i} \varepsilon_{i,m_2}\Big)^2\Bigg);\\
G^{(3)}_{k_1,m_1,k_2,m_2} &:= \operatorname{Cov}\Bigg(\Big(\sum_{i_1=-m_1+2}^{k_1-1} \alpha^{k_1-1-i_1} \varepsilon_{i_1,m_1}\Big)^2, \Big(\sum_{i_2=-m_2+2}^{k_2-1} \alpha^{k_2-1-i_2} \varepsilon_{i_2,m_2}\Big)^2\Bigg).
\end{aligned}$$

Thus, by Lemma 2.9 and representation (1.3) we have

$$G^{(2)}_{k_1,m_1,k_2,m_2} \leqslant 4M_4\mathrm{Cov}\Big(X_{k_1-1,m_1}, \sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i}\varepsilon_{i,m_2}\Big)^2 \qquad (6.9)$$

$$+4M_4\mathrm{Cov}\Big(X_{k_1,m_1-1}, \sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i}\varepsilon_{i,m_2}\Big)^2;$$

$$G^{(3)}_{k_1,m_1,k_2,m_2} \leqslant 2M_4\mathrm{Cov}\Big(\sum_{i_1=-m_1+2}^{k_1-1} \alpha^{k_1-1-i_1}\varepsilon_{i_1,m_1}, \sum_{i_2=-m_2+2}^{k_2-1} \alpha^{k_2-1-i_2}\varepsilon_{i_2,m_2}\Big)^2. \qquad (6.10)$$

From representation (1.3) follows that if $m_1 \geqslant m_2$ then

$$\mathrm{Cov}\Big(X_{k_1-1,m_1}, \sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i}\varepsilon_{i,m_2}\Big)$$

$$= \sum_{i=-(m_1-1)\wedge(m_2-2)}^{k_1\wedge k_2-1} \mathsf{P}\big(S^{(\alpha)}_{k_1+m_1-m_2-1-i} = k_1-1-i\big)\alpha^{k_2-1-i}$$

$$\leqslant \alpha^{k_2+m_1\wedge m_2} \sum_{i=-(m_1-1)\wedge(m_2-2)}^{k_1\wedge k_2-1} \frac{D_\alpha}{(k_1+m_1-m_2-1-i)^{1/2}} \leqslant D_\alpha \alpha^{k_2+m_1\wedge m_2}(k_1+m_1)^{1/2},$$

where the second inequality is a consequence of Theorem 2.5. Naturally, a similar upper bound can be derived for the second term in (6.9) and using this bound after a short calculation we obtain

$$\frac{1}{n^4}\sum_{m_1=1}^{n}\sum_{m_2=1}^{n}\sum_{k_1=-m_1+1}^{m_1}\sum_{k_2=-m_2+1}^{m_2} G^{(2)}_{k_1,m_1,k_2,m_2}$$

$$\leqslant \frac{16M_4D_\alpha^2}{n^4}\sum_{m_1=1}^{n}\sum_{m_2=1}^{m_1}\sum_{k_1=-m_1+1}^{m_1}\sum_{k_2=-m_2+1}^{m_2} \alpha^{2(k_2+m_1\wedge m_2)}(k_1+m_1)$$

$$\leqslant \frac{16M_4D_\alpha^2}{(1-\alpha)(1-\alpha^2)n} \to 0$$

as $n \to \infty$. Now, (6.10) implies

$$\frac{1}{n^4}\sum_{m_1=1}^{n}\sum_{m_2=1}^{n}\sum_{k_1=-m_1+1}^{m_1}\sum_{k_2=-m_2+1}^{m_2} G^{(3)}_{k_1,m_1,k_2,m_2}$$

$$\leqslant \frac{8M_4}{n^4}\sum_{m=1}^{n}\sum_{k_1=-m+1}^{m}\sum_{k_2=-m+1}^{m}\sum_{i=-m+2}^{k_1\wedge k_2-1} \alpha^{2k_1+2k_2-4-4i} \leqslant \frac{16M_4}{(1-\alpha^2)^2n^2} \to 0,$$

as $n \to \infty$ and the same can be proved for the remaining three terms of (6.8). Thus,

$$\begin{aligned}\mathrm{Var}\left(\sum_{m=1}^{n} V_m\right) &\leqslant \sum_{m_1=1}^{n} \sum_{(k_1,\ell_1)\in T_{m_1}\setminus T_{m_1-1}} \sum_{m_2=1}^{n} \sum_{(k_2,\ell_2)\in T_{m_2}\setminus T_{m_2-1}} G^{(1)}_{k_1,\ell_1,k_2,\ell_2} + H_n \\ &\leqslant \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} G^{(1)}_{k_1,\ell_1,k_2,\ell_2} + H_n,\end{aligned}$$

where $n^{-4}H_n \to 0$ as $n \to \infty$. As $X_{k-1,\ell} - X_{k,\ell-1}$ is also a linear combination of the variables $\{\varepsilon_{i,j} : (i,j) \in T_{k,\ell}\}$, by Lemma 2.9 we have

$$\begin{aligned}\sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} G^{(1)}_{k_1,\ell_1,k_2,\ell_2} \leqslant \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} 2M_4 L^{(1)}_{k_1,\ell_1,k_2,\ell_2} + (M_4-3)^+ L^{(2)}_{k_1,\ell_1,k_2,\ell_2} \\ + (M_4-3)^+ \sum_{\ell=-n+1}^{n} \sum_{k_1=-\ell+2}^{n} \sum_{k_2=-\ell+2}^{n} L^{(3)}_{k_1,k_2,\ell}, \quad (6.11)\end{aligned}$$

where

$$\begin{aligned}L^{(1)}_{k_1,\ell_1,k_2,\ell_2} &:= \mathrm{Cov}\big(X_{k_1-1,\ell_1} - X_{k_1,\ell_1-1}, X_{k_2-1,\ell_2} - X_{k_2,\ell_2-1}\big)^2 \\ L^{(2)}_{k_1,\ell_1,k_2,\ell_2} &:= \sum_{(i,j)\in T_{k_1\wedge k_2-1,\ell_1\wedge\ell_2-1}} \Big(\mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j} = k_1-i\big) - \mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j} = k_1-1-i\big)\Big)^2 \\ &\qquad \times \Big(\mathsf{P}\big(S^{(\alpha)}_{k_2+\ell_2-1-i-j} = k_2-i\big) - \mathsf{P}\big(S^{(\alpha)}_{k_2+\ell_2-1-i-j} = k_2-1-i\big)\Big)^2 \\ L^{(3)}_{k_1,k_2,\ell} &:= \sum_{i=-\ell+1}^{k_1\wedge k_2-1} \big(\alpha^{2k_1+2k_2-4-4i} + (1-\alpha)^{2k_1+2k_2-4-4i}\big) \leqslant \frac{1}{\alpha(1-\alpha)}.\end{aligned}$$

Obviously,

$$\begin{aligned}&\frac{1}{n^4} \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} L^{(1)}_{k_1,\ell_1,k_2,\ell_2} \\ &\quad = \iint_T \iint_T \Big(\sqrt{n}\,\mathrm{Cov}\big(Z^{(n)}_{0,1}(s_1,t_1) - Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{0,1}(s_2,t_2) - Z^{(n)}_{1,0}(s_2,t_2)\big)\Big)^2,\end{aligned}$$

where due to Propositions 2.3, 2.6 and the dominated convergence the right hand side converges to 0 as $n \to \infty$. Furthermore,

$$\frac{1}{n^4} \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} L^{(2)}_{k_1,\ell_1,k_2,\ell_2} = \iint_T \iint_T L^{(2)}_{[ns_1],[nt_1],[ns_2],[nt_2]}$$

and

$$L^{(2)}_{[ns_1],[nt_1],[ns_2],[nt_2]} \leqslant \sum_{m=1}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]-2} \sum_{i=m-[nt_1]\wedge[nt_2]+1}^{[ns_1]\wedge[ns_2]-1} \Big(\mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]-1-m} = [ns_1]-i\big)^2 + \mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]-1-m} = [ns_1]-1-i\big)^2\Big) \quad (6.12)$$
$$\times \Big(\mathsf{P}\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]-1-m} = [nt_2]-m+i\big)^2 + \mathsf{P}\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]-1-m} = [nt_2]-1-m+i\big)^2\Big).$$

Consider e.g. the first term of the right hand side of (6.12). Theorem 2.5 implies that there exists a constant D_α such that

$$\sum_{m=1}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]-2} \sum_{i=m-[nt_1]\wedge[nt_2]+1}^{[ns_1]\wedge[ns_2]-1} \mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]-1-m} = [ns_1]-i\big)^2 \times \mathsf{P}\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]-1-m} = [ns_2]-m+i\big)^2$$
$$\leqslant \sum_{m=1}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]-2} \frac{D_\alpha^2}{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]-1-m}$$
$$\times \sum_{i=m-[nt_1]\wedge[nt_2]+1}^{[ns_1]\wedge[ns_2]-1} \mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]-1-m} = [ns_1]-i\big)\mathsf{P}\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]-1-m} = [ns_2]-m+i\big)$$
$$\leqslant D_\alpha^2 \sum_{m=1}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]-2} \frac{\mathsf{P}\big(S_{[ns_1]+[nt_1]-1-m,[ns_2]+[nt_2]-1-m} = [ns_1]+[nt_2]-m\big)}{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]-1-m}$$
$$\leqslant D_\alpha^3 \sum_{m=0}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]-3} \frac{1}{m+1}\frac{1}{([ns_1]\vee[ns_2]+[nt_1]\vee[nt_2]+m+1)^{1/2}}$$
$$\leqslant D_\alpha^3 \sum_{m=0}^{\infty} \frac{1}{(m+1)^{3/2}} \leqslant \infty.$$

From the other hand,

$$\sum_{m=1}^{[ns_1]\wedge[ns_2]+[nt_1]\wedge[nt_2]-2} \sum_{i=m-[nt_1]\wedge[nt_2]+1}^{[ns_1]\wedge[ns_2]-1} \mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]-1-m} = [ns_1]-i\big)^2 \times \mathsf{P}\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]-1-m} = [ns_2]-m+i\big)^2$$
$$\leqslant \sqrt{n}\operatorname{Cov}\big(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{1,0}(s_2,t_2)\big)$$

which by Proposition 2.3 converges to 0 as $n\to\infty$ if $(1-\alpha)(s_1-s_2)\neq\alpha(t_1-t_2)$. Similar results can be derived for the remaining three terms of the right hand side of (6.12), so by the dominated convergence theorem

$$\lim_{n\to\infty}\frac{1}{n^4}\sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n} L^{(2)}_{k_1,\ell_1,k_2,\ell_2} = 0.$$

At the end we have

$$\frac{1}{n^4}\sum_{\ell=-n+1}^{n}\sum_{k_1=-\ell+2}^{n}\sum_{k_2=-\ell+2}^{n} L^{(3)}_{k_1,k_2,\ell} \leqslant \frac{8}{\alpha(1-\alpha)n} \to 0 \qquad \text{as} \quad n\to\infty,$$

which completes the proof. □

Proof of Proposition 6.2. Similarly to the proof of Proposition 4.2 it suffices to show that

$$\frac{1}{n^4}\sum_{m=1}^{n} \mathsf{E}\left((Q_m - Q_{m-1})^4 \,\middle|\, \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0 \qquad \text{as} \quad n\to\infty. \tag{6.13}$$

Using decomposition (6.5) we obtain

$$(Q_m - Q_{m-1})^4 \leqslant 2^3\big(A^{(1)}_{m,1} - A^{(2)}_{m,1}\big)^4 + 2^3\bigg(\varepsilon_{k,\ell}\big(\widetilde{A}_{m,2,k-1,\ell} - \widetilde{A}_{m,2,k,\ell-1}\big)\bigg)^4.$$

By the independence of $A_{m,1}$ and $\mathcal{F}_{m-1}$ we have

$$\mathsf{E}\big(\big(A^{(1)}_{m,1} - A^{(2)}_{m,1}\big)^4 \,\big|\, \mathcal{F}_{m-1}\big) = \mathsf{E}\big(A^{(1)}_{m,1} - A^{(2)}_{m,1}\big)^4 \leqslant 4\mathsf{E}\big\|A_{m,1}\big\|^4,$$

while the measurability of $A_{m,2,k,\ell}$ with respect to $\mathcal{F}_{m-1}$, implies

$$\begin{aligned}\mathsf{E}\Bigg(\bigg(\sum_{(k,\ell)\in T_m\setminus T_{m-1}} &\varepsilon_{k,\ell}\big(\widetilde{A}_{m,2,k-1,\ell} - \widetilde{A}_{m,2,k,\ell-1}\big)\bigg)^4 \,\bigg|\, \mathcal{F}_{m-1}\Bigg)\\ &\leqslant \big((M_4-3)^+ + 3\big)\bigg(\sum_{(k,\ell)\in T_m\setminus T_{m-1}} \big(\widetilde{A}_{m,2,k-1,\ell} - \widetilde{A}_{m,2,k,\ell-1}\big)^2\bigg)^2.\end{aligned}$$

From (4.13) follows that in order to prove (6.13), it suffices to show

$$\lim_{n\to\infty}\frac{1}{n^4}\sum_{m=1}^{n}\mathsf{E}\bigg(\sum_{(k,\ell)\in T_m\setminus T_{m-1}} \big(\widetilde{A}_{m,2,k-1,\ell} - \widetilde{A}_{m,2,k,\ell-1}\big)^2\bigg)^2 = 0. \tag{6.14}$$

Using the same arguments as in the proof of (6.7) we obtain

$$\begin{aligned}\mathsf{E}\bigg(\sum_{(k,\ell)\in T_m\setminus T_{m-1}} &\big(\widetilde{A}_{m,2,k-1,\ell} - \widetilde{A}_{m,2,k,\ell-1}\big)^2\bigg)^2\\ &= \sum_{(k_1,\ell_1)\in T_m\setminus T_{m-1}}\ \sum_{(k_2,\ell_2)\in T_m\setminus T_{m-1}} \mathsf{E}\Big(\big(\widetilde{A}_{m,2,k_1-1,\ell_1} - \widetilde{A}_{m,2,k_1,\ell_1-1}\big)^2\\ &\qquad\qquad \times \big(\widetilde{A}_{m,2,k_2-1,\ell_2} - \widetilde{A}_{m,2,k_2,\ell_2-1}\big)^2\Big)\\ &\leqslant \sum_{(k_1,\ell_1)\in T_m\setminus T_{m-1}}\ \sum_{(k_2,\ell_2)\in T_m\setminus T_{m-1}} \widetilde{G}_{k_1,\ell_1,k_2,\ell_2} + \widetilde{H}_m,\end{aligned}$$

where

$$\widetilde{G}_{k_1,\ell_1,k_2,\ell_2} := \mathsf{E}\Big(\big(X_{k_1-1,\ell_1} - X_{k_1,\ell_1-1}\big)^2 \big(X_{k_2-1,\ell_2} - X_{k_2,\ell_2-1}\big)^2\Big)$$

and

$$\frac{1}{n^4}\sum_{m=1}^{n} \widetilde{H}_m \to 0 \qquad \text{as} \quad n\to\infty. \tag{6.15}$$

Furthermore, by applying Lemma 2.9 and a decomposition similar to (6.11) we have

$$\widetilde{G}_{k_1,\ell_1,k_2,\ell_2} \leqslant 3M_4 \mathsf{E}\big(X_{k_1-1,\ell_1} - X_{k_1,\ell_1-1}\big)^2 \mathsf{E}\Big(X_{k_2-1,\ell_2} - X_{k_2,\ell_2-1}\big)^2\Big) + C_\alpha,$$

with some constant $C_\alpha > 0$. Besides this, from Proposition 2.6 immediately follows that there exists a constant K_α such that for $k,\ell \in \mathbb{Z}$, $k+\ell \geqslant 1$,

$$\mathsf{E}\big(X_{k-1,\ell} - X_{k,\ell-1}\big)^2 = \operatorname{Var}(X_{k-1,\ell}) + \operatorname{Var}(X_{k,\ell-1}) - 2\operatorname{Cov}(X_{k-1,\ell}, X_{k,\ell-1}) \leqslant 2K_\alpha.$$

Hence,

$$\frac{1}{n^4}\sum_{m=1}^{n} \sum_{(k_1,\ell_1)\in T_m\setminus T_{m-1}} \sum_{(k_2,\ell_2)\in T_m\setminus T_{m-1}} \widetilde{G}_{k_1,\ell_1,k_2,\ell_2} \leqslant (14M_4K_\alpha + C_\alpha)\frac{1}{n^4}\sum_{m=1}^{n}(2m-1)^2 \to 0$$

as $n\to\infty$, which together with (6.15) implies (6.14). □

References

[1] Anderson, T. W. (1959). On asymptotic distributions of estimates of parameters of stochastic difference equations. *Ann. Math. Statist.* **30**, 676–687.

[2] Baran, S., Pap, G. and Zuijlen, M. v. (2004). Asymptotic inference for an unstable spatial AR model. *Statistics* **38**, 465–482.

[3] Basu, S. and Reinsel, G. C. (1992). A note on properties of spatial Yule–Walker estimators. *J. Statist. Comput. Simulation* **41**, 243–255.

[4] Basu, S. and Reinsel, G. C. (1993). Properties of the spatial unilateral first-order ARMA model. *Adv. in Appl. Probab.* **25** 631–648.

[5] Basu, S. and Reinsel, G. C. (1994). Regression models with spatially correlated errors. *J. Amer. Statist. Assoc.* **89**, 88–99.

[6] Bateman, H. and Erdélyi, A. (1953). *Higher Transcendental Functions. Volume 1.* Mc Graw-Hill, New York, Toronto, London.

[7] Bhattacharyya, B. B., Khalil, T. M. and Richardson, G. D. (1996). Gauss–Newton estimation of parameters for a spatial autoregression model. *Statist. Probab. Lett.* **28**, 173–179.

[8] Bhattacharyya, B. B., Richardson, G. D. and Franklin, L. A. (1997). Asymptotic inference for near unit roots in spatial autoregression. *Ann. Statist.* **25**, 1709–1724.

[9] Besag, J. E. (1972). On the correlation structure of some two dimensional stationary processes. *Biometrika* **59** 43–48.

[10] Chan, N. H. and Wei, C. Z. (1987). Asymptotic inference for nearly nonstationary AR(1) processes. *Ann. Statist.* **15**, 1050–1063.

[11] Cullis, B. R. and Gleeson, A. C. (1991). Spatial analysis of field experiments — an extension to two dimensions. *Biometrics* **47**, 1449–1460.

[12] Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58** 13–30.

[13] Jacod, J. and Shiryaev, A. N. (1987). *Limit Theorems for Stochastic Processes.* Springer–Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.

[14] Jain, A. K. (1981). Advances in mathematical models for image processeng. *Proc. IEEE* **69**, 502–528.

[15] Mann, H. B. and Wald, A. (1943). On the statistical treatment of linear stochastic difference equations. *Econometrica* **11**, 173–220.

[16] Martin, R. J. (1979). A subclass of lattice processes applied to a problem in planar sampling. *Biometrika* **66**, 209–217.

[17] Martin, R. J. (1990). The use of time-series models and methods in the analysis of agricultural field trials. *Comm. Statist. Theory Methods* **19**, 55–81.

[18] Petrov, V. V. (1975). *Sums of independent random variables*, Springer–Verlag, Berlin, Heidelberg, New York.

[19] Phillips, P. C. B. (1987). Towards a unified asymptotic theory for autoregression. *Biometrika* **74**, 535–547.

[20] Tjøstheim, D. (1978). Statistical spatial series modelling. *Adv. in Appl. Probab.* **10** 130–154.

[21] Tjøstheim, D. (1981). Autoregressive modelling and spectral analysis of array data in the plane. *IEEE Trans. on Geosciences and Remote Sensing* **19**, 15–24.

[22] Tjøstheim, D. (1983). Statistical spatial series modelling II: some further results on unilateral processes. *Adv. in Appl. Probab.* **15**, 562–584.

[23] White, J. S. (1958). The limiting distribution of the serial correlation coefficient in the explosive case. *Ann. Math. Statist.* **29**, 1188–1197.

[24] Whittle, P. (1954). On stationary processes in the plane. *Biometrika* **41** 434–449.