GEOMETRIC STRUCTURES ON THE COMPLEMENT OF A PROJECTIVE ARRANGEMENT

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In memory of Peter Slodowy (1948–2002)

ABSTRACT

Consider a complex projective space with its Fubini-Study metric. We study certain one parameter deformations of this metric on the complement of an arrangement (= finite union of hyperplanes) whose Levi-Civita connection is of Dunkl type. Interesting examples are obtained from the arrangements defined by finite complex reflection groups. We determine a parameter interval for which the metric is locally of Fubini-Study type, flat, or complex-hyperbolic. We find a finite subset of this interval for which we get a complete orbifold or at least a Zariski open subset thereof, and we analyze these cases in some detail (e.g., we determine their orbifold fundamental group).

In this set-up, the principal results of Deligne-Mostow on the Lauricella hypergeometric differential equation and work of Barthel-Hirzebruch-Höfer on arrangements in a projective plane appear as special cases. Along the way we produce in a geometric manner all the pairs of complex reflection groups with isomorphic discriminants, thus providing a uniform approach to work of Orlik-Solomon.

Introduction

This article wants to be the child of two publications which saw the light of day in almost the same year. One of them is the book by Barthel-Hirzebruch-Höfer (1987) [1], which, among other things, investigates Galois coverings of $\mathbf{P}^2$ that ramify in a specified manner over a given configuration of lines and characterizes the ones for which a universal such cover is a complex ball (and thus make $\mathbf{P}^2$ appear as a—perhaps compactified—ball quotient). The other is a long paper by Deligne and Mostow (1986) [14], which completes the work of Picard and Terada on the (Appell-)Lauricella functions and which leads to a ball quotient structure on $\mathbf{P}^n$ relative to a hyperplane configuration of type A_{n+1}. Our reason for claiming such a descendence is that we develop a higher dimensional generalization of the work by Hirzebruch et al. in such a manner that it contains the cited work of Deligne-Mostow as a special case. In other words, this paper's subject matter is projective arrangements which can be understood as discriminants of geometric orbifold structures. Our approach yields new, and we believe, interesting, examples of ball quotients (which was the original goal) and offers at the same time a novel perspective on the material of the two parent papers.

It starts out quite simply with the data of a finite dimensional complex inner product space V in which is given a *hyperplane arrangement*, that is, a finite collection of (linear) hyperplanes. We write V° for the complement of the union of these hyperplanes and $\mathbf{P}(V^\circ) \subset \mathbf{P}(V)$ for its projectivization. The inner product determines a (Fubini-Study) metric on $\mathbf{P}(V)$ and the idea is to deform continuously (in a rather specific manner) the restriction of this metric to $\mathbf{P}(V^\circ)$ as to obtain a complex hyperbolic metric, i.e., a metric that makes $\mathbf{P}(V^\circ)$ locally isometric to a complex ball. We do this in two stages.

We first attempt to produce a one-parameter deformation ∇^t, $t \geq 0$, of the standard translation invariant connection ∇^0 on (the tangent bundle) of V restricted to V° *as a flat torsion free connection* on V°. For the reflection hyperplane arrangement of a finite Coxeter group such a deformation is given by Dunkl's construction and we try to imitate this. Although this is not always possible—the existence of such a deformation imposes strong conditions on the arrangement—plenty of examples do exist. For instance, this is always possible for the reflecting hyperplane arrangement of a complex reflection group. Besides, it is a property that is inherited by the arrangement that is naturally defined on a given intersection of members of the arrangement.

The inner product defines a translation invariant metric on V. Its restriction h^0 to V° is obviously flat for ∇^0 and the next step is to show that we can deform h^0 as a nonzero flat Hermitian form h^t *which is flat for* ∇^t (so that ∇^t becomes a Riemannian connection as long as h^t is nondegenerate). This is done in such a manner that scalar multiplication in V acts locally like homothety and as a consequence, $\mathbf{P}(V^\circ)$ inherits from V° a Hermitian form g^t. For $t = 0$ this gives us the Fubini-Study metric. We only allow t to move in an interval for which g^t stays positive definite. This still makes it possible for h^t to become degenerate or of hyperbolic signature as long as for every $p \in V^\circ$, the restriction of h^t to a hyperplane supplementary and perpendicular to $T_p(\mathbf{C}p)$ is positive definite. If $T_p(\mathbf{C}p)$ is the kernel of h^t (we refer to this situation as the *parabolic* case), then g^t is a flat metric, whereas when h^t is negative on $T_p(\mathbf{C}p)$ (the *hyperbolic* case), g^t is locally the metric of a complex ball. It is necessary to impose additional conditions of a simple geometric nature in order to have a neat global picture, that is, to have $\mathbf{P}(V^\circ)$ of finite volume and realizable as a quotient of a dense open subset of a flat space resp. a ball by a discrete group of isometries. We call these the *Schwarz conditions*, because they are reminiscent of the ones found by H.A. Schwarz which ensure that the Gauss hypergeometric function is algebraic.

Deligne and Mostow gave a modular interpretation of their ball quotients. Some of them (namely those with an arithmetic group of ball automorphisms) are in fact Shimura varieties and indeed, particular cases were already studied by Shimura and Casselman (who was then a student of Shimura) in the sixties [31],[5]. A natural question is whether such an interpretation also exists for the ball quotients introduced here. We know this to be the case for some of them, but we do not address this issue in the present paper.

We mention some related work, without however any pretension of attempted completeness. A higher dimensional generalization of Hirzebruch's original approach with Fermat covers and fixed weights along all hyperplanes and emphasizing the three dimensional case was developed by Hunt [18]. His paper with Weintraub [19] fits naturally in our framework; their Janus-like algebraic varieties are exactly related to the various ramification orders q allowed in the tables of our final Section 8. The articles by Holzapfel [20], [21] and Cohen-Wüstholz [8] contain applications to transcendence theory.

We now briefly review the contents of the separate sections of this paper. In the first section we develop a bit of the general theory of affine structures on complex manifolds, where we pay special attention to a simple kind of degeneration of such a structure along a normal crossing divisor. It is also a place where we introduce some terminology and notation.

Section two focuses on a notion which is central to this paper, that of a Dunkl system. We prove various hereditary properties and we give a number of examples. We show in particular that the Lauricella functions fit in this setting. In fact, in the last subsection we classify all the Dunkl systems whose underlying arrangement is a Coxeter arrangement and show that the Lauricella examples exhaust the cases of type A. For the other Coxeter arrangements of rank at least three the Dunkl system has automatically the symmetry of the corresponding Coxeter group, except for those of type B, for which we essentially reproduce the Lauricella series.

The next section is perhaps best characterized as to establish the applicability range of the principal results of later sections (for these do not formally depend on what we prove here). We discuss the existence of a nontrivial Hermitian form which is flat relative to the Dunkl connection and we prove among other things that such a form always exists in the case of a complex reflection arrangement and in the Lauricella case. We determine when this form is positive definite, parabolic or hyperbolic.

Section four is devoted to the Schwarz conditions. We show that when these conditions are satisfied, the holonomy cover extends as a ramified cover over the complement in V of a closed subset of V of codimension at least two, that the developing map extends to this ramified cover, and that the latter extension becomes a local isomorphism if we pass to the quotient by a finite group G (which acts as a complex reflection group on V, but lifts to the ramified cover). This might explain why we find it reasonable to impose such a condition. From this point onward we assume such conditions satisfied and concentrate on the situations that really matter to us.

Section five deals with the elliptic and the parabolic cases. The elliptic case can be characterized as having finite holonomy. It is in fact treated in two somewhat different situations: at first we deal with a situation where we find that $\mathbf{P}(G\backslash V)$ is the metric completion of $\mathbf{P}(G\backslash V^\circ)$ and acquires the structure of an elliptic orbifold. What makes this interesting is that this is not the natural G-orbifold structure that $\mathbf{P}(G\backslash V)$ has a priori: it is the structure of the quotient of a projective space by the holonomy group. This is also a complex reflection group, but usually differs from G. Still the two reflection groups are related by the fact that their discriminants satisfy a simple inclusion relation. We prove that all pairs of complex reflection groups with isomorphic discriminants are produced in this fashion and thus provide a uniform approach to the work of Orlik and Solomon on such pairs. The other elliptic case we discuss is when the metric completion of $\mathbf{P}(G\backslash V^\circ)$ differs from $\mathbf{P}(G\backslash V)$ but is gotten from the latter by means of an explicit blowup followed by an explicit blowdown. We have to deal with such a situation, because it is one which we encounter when we treat the hyperbolic case. The parabolic case presents little trouble and is dealt with in a straightforward manner.

Our main interest however concerns the hyperbolic situation and that is saved for last. We first treat the case when we get a compact hyperbolic orbifold, because it is relatively easy and takes less than half a page. The general case is rather delicate, because the metric completion of $\mathbf{P}(G\backslash V^\circ)$ (which should be a ball quotient of finite volume) may differ from $\mathbf{P}(G\backslash V)$. Deligne and Mostow used at this point geometric invariant theory for effective divisors on $\mathbf{P}^1$, but in the present situation this tool is not available to us and we use an argument based on Stein factorization instead. As it is rather difficult to briefly summarize the contents of our main theorem, we merely refer to 6.2 for its statement. It suffices to say here that it produces new examples of discrete complex hyperbolic groups of cofinite volume.

The short section seven is a miscellany of results: we give a uniform presentation of the holonomy groups in the cases we considered and we discuss the implications of our results for the allied algebra of automorphic forms.

The final section tabulates the elliptic, parabolic and hyperbolic examples of finite volume with the property that the associated arrangement is that of a finite reflection group of rank at least three (without requiring it to have the symmetry of that group). In the hyperbolic case we mention whether the holonomy group is cocompact.

This work has its origin in the thesis by the first author [9] at the University of Nijmegen (1994) written under the supervision of the second author. Although that project went quite far in carrying out the program described above, the results were never formally published, in part, because both felt that it should be completed first. This remained the state of affairs until around 2001, when the idea emerged that work of the third author [23] might be relevant here. After we had joined forces in 2002, the program was not only completed as originally envisaged, but we were even able to go well beyond that, including the adoption of a more general point of view and a change in perspective.

We dedicate this paper to the memory of our good friend and colleague Peter Slodowy.

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We also wish to acknowledge the valuable comments and suggestions of the referee (who clearly went with great scrutiny through this article), which led us to make a number of improvements.

0.1. *Terminological index.* — In alphabetical order.

Euler field	Proposition 2.2
exponent	∼ of a complex reflection group: Subsection 3.4
fractional divisor	Remark 6.6
germ	Notational conventions 0.3
Hecke algebra	Subsection 3.5
holonomy group	Definition 1.2
hyperbolic exponent	Theorem 3.1
hyperbolic structure	Definition 1.6
index	∼ of a Hermitian form: Lemma 3.22
infinitesimally	∼ simple degeneration of ...: Definition 1.9
irreducible	∼ arrangement, member of an ∼: Subsection 2.1
Lauricella	∼ connection, ∼ function: Proposition-definition 2.6
longitudinal	∼ Dunkl connection: Definition 2.20
logsingular	∼ function, ∼ differential: discussion preceding Lemma 3.9
monodromy group	Definition 1.2
nullity	Lemma 3.22
parabolic structure	Definition 1.6
reflection representation	Subsection 3.5
residue	∼ of a connection: Subsection 1.3
semisimple holonomy	∼ around a stratum: paragraph preceding Corollary 2.22
simple degeneration	∼ of an affine structure along a divisor: Definition 1.9
Schwarz	∼ condition, ∼ rotation group, ∼ symmetry group: Definition 4.2
special	∼ subball, ∼ subspace: Subsection 6.2
splitting	∼ of an arrangement: Subsection 2.1
stratum	∼ of an arrangement: Subsection 2.1
Stein	topological ∼ factorization: paragraph preceding Lemma 5.13
transversal	∼ Dunkl connection: Definition 2.20
weight property	Remarks 2.19

0.2. *List of notation.* — In order of appearance and restricted to items that occur in nonneighboring loci.

$\mathbf{P}$	Notational conventions 0.3: passage to a $\mathbf{C}^\times$-orbit space.
Aff_M	Subsection 1.1: the local system of locally affine-linear functions.
$\mathrm{Aff}(M)$	Subsection 1.1: the space of global sections of Aff_M.
Γ	Subsection 1.1: the holonomy group.
A	Subsection 1.1: the affine space which receives the developing map.
$\mathrm{Res}_D(\nabla)$	Subsection 1.3: Residue of a connection along D.
$\nu_{D/W}$	Lemma 1.7: normal bundle of D in W.
$D_{p,0}$	Corollary 1.14.
$D_{p,\lambda}$	Corollary 1.14.
$W_{p,\lambda}$	Corollary 1.14.
V°	Subsection 2.1: the complement of an arrangement in V.
$\mathcal{L}(\mathcal{H})$	Subsection 2.1: the intersection lattice of the arrangement $\mathcal{H}$.
$\mathcal{H}_L$	Subsection 2.1: the members of $\mathcal{H}$ containing L.
$\mathcal{H}^L$	Subsection 2.1: the intersections of the members of $\mathcal{H}-\mathcal{H}_L$ with L.
$\mathcal{L}_{\mathrm{irr}}(\mathcal{H})$	Subsection 2.1: the irreducible members of $\mathcal{L}(\mathcal{H})$.
$M(L)$	Lemma 2.1.
ϕ_H	Subsection 2.2: a linear from which defines the hyperplane H.
ω_H	Subsection 2.2: the logarithmic form defined by the hyperplane H.
∇^0	Subsection 2.2: the translation invariant connection on an affine space.
E_V	Proposition 2.2: the Euler vector field on a vector space V.
π_L	Subsection 2.4: the orthogonal projection in an inner product space with kernel L.
κ_L	Lemma 2.13.
∇^κ	paragraph preceding Corollary 2.17.
Ω^κ	paragraph preceding Corollary 2.17.
$\mathbf{C}^{\mathcal{H},\mathrm{flat}}$	Notation 2.16: the set of exponents κ for which ∇^κ is flat.
$\mathcal{H}_{L^\perp}$	Discussion preceding Lemma 2.18.

α^L	Lemma 2.18.
π_I^L	Lemma 2.18.
ω_I^L	Lemma 2.18.
$\mathrm{Bl}_L V$	Subsection 2.5: blowup of V in L.
$\tilde{\pi}_L^*$	Subsection 2.5.
h^0	Subsection 3.1: the Hermitian form defined by the inner product.
m_{hyp}	Theorem-definition 3.1: the hyperbolic exponent.
$\mathcal{H}(\mathcal{F})$	Lemma 3.2.
d_i	Subsection 3.4: the ith degree of a reflection group.
m_i	Subsection 3.4: the ith exponent of a reflection group.
d_i^*	Subsection 3.4: the ith codegree of a reflection group.
m_i^*	Subsection 3.4: the ith co-exponent of a reflection group.
$\mathrm{Ar}(M)$	Subsection 3.5: the Artin group attached to the Coxeter matrix M.
$\mathcal{H}(M,t)$	Subsection 3.5: the universal Hecke algebra.
R	Subsection 3.5: a domain associated to a Hecke algebra.
$\mathcal{H}(M)$	Subsection 3.5: the Hecke algebra over R.
ρ^{mon}	Subsection 3.5: the monodromy representation of $\mathcal{H}(M)$.
ρ^{refl}	Subsection 3.5: the reflection representation of $\mathcal{H}(M)$.
N	Subsection 3.6.
w_k	Subsection 3.6: a complex number of norm one attached to μ.
A_w	paragraph preceding Lemma 3.22: a hyperplane of $\mathbf{R}^{n+1}$.
Q_w	paragraph preceding Lemma 3.22: a quadratic form on A_w.
p_L, q_L	Definition 4.2: numerator resp. denominator of $1-\kappa_L$.
G_L	Definition 4.2: Schwarz rotation group.
G	Definition 4.2: Schwarz symmetry group.
V^f	Subsection 4.2: locus of finite holonomy in V.
ev_G	Theorem 4.5: a factor of an extension of the developing map.
$\mathcal{L}_{\kappa<1}$	Subsection 5.4.
$\mathcal{L}_{\kappa=1}$	Subsection 5.4.
$\mathcal{L}_{\kappa>1}$	Subsection 5.4 and Discussion 6.8.
$V^\sharp$	Discussion 5.9.
$V_{\kappa<1}$	Discussion 5.9.
$E(L)$	Discussion 5.9.
$D(L)$	Discussion 5.9.
$\mathbf{S}$	Discussion 5.9: the sphere of rays.
$E(L_\bullet)$	Discussion 5.9.
$S(L_\bullet)$	Discussion 5.9 and Discussion 6.8.
$\mathfrak{St}$	(as a subscript) paragraph preceding Lemma 5.13: formation of a Stein quotient.
$\mathbf{B}, A_{\mathbf{B}}$	Subsection 6.2.
$\mathbf{B}^\diamond, A_{\mathbf{B}}^\diamond$	Theorem 6.2.
$\mathbf{B}^\sharp$	Subsection 6.4: the Borel-Serre extension of $\mathbf{B}$.
$B^\sharp$	Discussion 6.8.

0.3. *Some notational conventions.* — If $\mathbf{C}^\times$ acts on a variety X, then we often write $\mathbf{P}(X)$ for the orbit space of the subspace of X where $\mathbf{C}^\times$ acts with finite isotropy groups. This notation is of course suggested by the case when $\mathbf{C}^\times$ acts by scalar multiplication on a complex vector space V, for $\mathbf{P}(V)$ is then the associated projective space. This example also shows that a $\mathbf{C}^\times$-equivariant map $f : X \to Y$ may or may not induce a morphism $\mathbf{P}(f) : \mathbf{P}(X) \to \mathbf{P}(Y)$.

If X is a space with subspaces A and Y, then the *germ* of Y at A is the filter of neighborhoods of A in X restricted to Y; we denote it by Y_A. Informally, Y_A may be thought of as an unspecified neighborhood of A intersected with Y. For instance, a *map germ* $Y_A \to Z$ is given by a pair $(U, f : U \cap Y \to Z)$, where U is some neighborhood of A, and another such pair $(U', f' : U' \cap Y \to Z)$ defines the same map-germ if f and f' coincide on $U'' \cap Y$ for some neighborhood U'' of A in $U \cap U'$.

CONTENTS

1. Affine structures with logarithmic singularities

We first recall a few basic properties regarding the notion of an affine structure.

1.1. *Affine structures.* — Let be given a connected complex manifold M of complex dimension n. An *affine structure* on M is an atlas (of complex-analytic charts) for which the transitions maps are complex affine-linear and which is maximal for that property. Given such an atlas, then the complex valued functions that are locally complex-affine linear make up a local system Aff_M of $\mathbf{C}$-vector spaces in the structure sheaf $\mathcal{O}_M$. This local system is of rank $n+1$ and contains the constants $\mathbf{C}_M$. The quotient $\mathrm{Aff}_M/\mathbf{C}_M$ is a local system whose underlying vector bundle is the complex cotangent bundle of M, hence is given by a flat connection $\nabla : \Omega_M \to \Omega_M \otimes \Omega_M$. This connection is torsion free, for it sends closed forms to symmetric tensors. (This is indeed equivalent to the more conventional definition which says that the associated connection on the tangent bundle is symmetric: for any pair of local vector fields X, Y on M, we have $\nabla_X Y - \nabla_Y X = [X, Y]$.)

Conversely, any flat, torsion free connection ∇ on the complex cotangent bundle of M defines an affine structure: the subsheaf $\mathrm{Aff}_M \subset \mathcal{O}_M$ of holomorphic functions whose total differential is flat for ∇ is then a local system of rank $n+1$ containing the constants and the atlas in question consists of the charts whose components lie in Aff_M. We sum up:

Lemma **1.1.** — *Given a complex manifold, then it is equivalent to give a affine structure on that manifold (compatible with the complex structure) and to give a (holomorphic) flat, torsion free connection on its cotangent bundle.*

We shall not make any notational distinction between a connection on the cotangent bundle and the one on any bundle associated to the cotangent bundle (such as the tangent bundle).

Observe that every connection on the cotangent bundle of a Riemann surface is flat and torsion free and hence defines an affine structure on that surface.

Definition **1.2.** — *Let M be a complex connected manifold endowed with an affine structure. The* monodromy group *of M is the monodromy of the local system whose bundle is the underlying tangent bundle of M, whereas the* holonomy group *of M is the monodromy group of the local system of local affine-linear functions on M. A* holonomy covering *of M is an (unramified) Galois covering of M with covering group the holonomy group.*

So the holonomy group is an extension of the monodromy group by a group of translations. Both groups are defined up to inner automorphism, of course.

Fix a complex connected manifold M endowed with an affine structure. Let $\widetilde{M} \to M$ be a holonomy covering and denote by Γ its Galois group. It is unique up to a covering transformation and has the property that the pull-back of Aff_M to this covering is generated by its sections. So the space of affine-linear functions on $\widetilde{M}$, $\mathrm{Aff}(\widetilde{M}) := H^0(\widetilde{M}, \mathrm{Aff}_{\widetilde{M}})$, is a Γ-invariant vector space of holomorphic functions on $\widetilde{M}$. This vector space contains the constant functions and the quotient $\mathrm{Aff}(\widetilde{M})/\mathbf{C}$ can be identified with the space of flat holomorphic differentials on $\widetilde{M}$; it has the same dimension as M. The set A of linear forms $\mathrm{Aff}(\widetilde{M}) \to \mathbf{C}$ which are the identity on $\mathbf{C}$ is an affine Γ-invariant hyperplane in $\mathrm{Aff}(\widetilde{M})^*$.

Lemma-definition **1.3.** — *Given a holonomy cover as above, then the holonomy group Γ acts faithfully on the affine space A as a group of affine-linear transformations. The image of Γ in the general linear group of the translation space of A is the monodromy group of the affine structure. The evaluation mapping* $\mathrm{ev} : \widetilde{M} \to A$ *which assigns to $\tilde{z}$ the linear form* $\mathrm{ev}_{\tilde{z}} : \tilde{f} \in \mathrm{Aff}(\widetilde{M}) \mapsto \tilde{f}(\tilde{z}) \in \mathbf{C}$ *is called the* developing map *of the affine structure; it is Γ-equivariant and a local affine isomorphism.* □

So a developing map determines a natural affine atlas on M whose charts take values in A and whose transition maps lie in Γ.

Definition **1.4.** — *Let M be a complex manifold endowed with an affine structure given by the torsion free, flat connection ∇. We call a nowhere zero holomorphic vector field E on M a* dilatation field with factor $\lambda \in \mathbf{C}$ *when for every local vector field X on M, $\nabla_X(E) = \lambda X$.*

Let us have a closer look at this property. If X is flat, then the torsion freeness yields: $[E, X] = \nabla_E(X) - \nabla_X(E) = -\lambda X$. In other words, Lie derivation with respect to E acts on flat vector fields simply as multiplication by $-\lambda$. Hence it acts on flat differentials as multiplication by λ. So E acts on Aff_M with eigenvalues 0 (on $\mathbf{C}$) and λ (on $\mathrm{Aff}_M / \mathbf{C}_M$).

Suppose first that $\lambda \neq 0$. Then the $f \in \mathrm{Aff}_M$ for which $E(f) = \lambda f$ make up a flat supplement of $\mathbf{C}_M$ in Aff_M. This singles out a fixed point $O \in A$ of Γ so that the affine-linear structure is in fact a *linear structure* and the developing map takes the lift of E on $\widetilde{M}$ to λ times the Euler vector field on A relative to O. This implies that locally the leaf space of the foliation defined by E is identified with an open set of the projective space of (A, O) (which is naturally identified with the projective space of the space of flat vector fields on $\widetilde{M}$). Hence this leaf space acquires a complex projective structure.

Suppose now that $\lambda = 0$. Then $\mathbf{C}$ need not be a direct summand of Aff_M. All we can say is that E is a flat vector field so that its lift to $\widetilde{M}$

maps a constant nonzero vector field on A. So locally the leaf space of the foliation defined by E has an affine-linear structure defined by an atlas which takes values in the quotient of A by the translation group generated by a constant vector field.

Example **1.5.** — The following example, although very simple, is perhaps helpful. Let $\kappa \in \mathbf{C}$ and define a connection ∇ on the cotangent bundle of $\mathbf{C}^\times$ by $\nabla(dz) = \kappa \frac{dz}{z} \otimes dz$. The new affine structure on $\mathbf{C}^\times$ has developing map (affine equivalent to) $w = z^{1-\kappa}$ $(\kappa \neq 1)$ or $w = \log z$ $(\kappa = 1)$. The Euler field $z\frac{\partial}{\partial z}$ is a dilatation field with factor $1-\kappa$ (and so flat when $\kappa = 1$). The monodromy of the connection sends dw to $\exp(2\pi\sqrt{-1}\kappa)dw$; the holonomy sends w to $\exp(2\pi\sqrt{-1}\kappa)w$ when $\kappa \neq 1$ and to $w - 2\pi\sqrt{-1}$ when $\kappa = 1$. In case $1-\kappa$ is a nonzero rational number: $1-\kappa = p/q$ with p, q relatively prime integers and $q > 0$, then the holonomy covering $\mathbf{C}^\times \to \mathbf{C}^\times$ is given by $z = \tilde{z}^q$ and the developing map by $w = \tilde{z}^p$. In the other cases (κ is irrational or equal to 1), the developing map defines an isomorphism of the universal cover of $\mathbf{C}^\times$ onto an affine line.

In anticipation of what will follow, it is also worthwhile to point out here a differential-geometric aspect in case κ is real and contained in the interval $[0, 1]$: then the standard Euclidean metric on the range of the developing map (i.e., $|dw|^2$) is invariant under the holonomy and hence determines a metric on $\mathbf{C}$, namely (up to constant) $|z|^{-2\kappa}|dz|^2$. This makes $\mathbf{C}$ a metric cone (having ∇ as its Levi-Civita connection) with total angle $2\pi - 2\pi\kappa$. So $2\pi\kappa$ has now an interpretation as the amount of curvature concentrated in $0 \in \mathbf{C}$.

1.2. *Admissible metrics.* — Let M be a connected complex manifold with an affine structure and let h be a flat Hermitian form on the tangent bundle of M. Such a form restricts to a Hermitian form h_p on a given tangent space T_pM which is invariant under the monodromy (and conversely, a monodromy invariant Hermitian form on T_pM extends to flat Hermitian form on M). The kernel of h is integrable to a foliation in M whose local leaf space comes with an affine structure endowed with a flat *nondegenerate* Hermitian form.

Suppose that we are also given a dilatation field E on M with factor λ such that $h(E, E)$ is nowhere zero. Then the leaf space M/E of the dimension one foliation defined by E inherits a Hermitian form $h_{M/E}$ in much the same way as the projective space of a finite dimensional Hilbert space acquires its Fubini-Study metric: If a point of the leaf space is represented by the orbit O, then a tangent vector v at that point is uniquely represented by a vector field X along O which is h-perpendicular to O. Notice that the functions $h(X, X)$ and $h(E, E)|O$ on O are both homogeneous of the same degree $-2\operatorname{Re}(\lambda)$ (relative to Lie derivation by E) and so $|h(E, E)|^{-1}h(X, X)$ is constant. We let this constant be the value of $h_{M/E}(v, v)$. It is easily seen that this defines a Hermitian form on M/E, which is nondegenerate when h is. We are especially interested in the case when $h_{M/E}$ is positive definite:

Definition **1.6.** — *Let be given an affine complex manifold M and a dilatation field E on M with factor λ. We say that a flat Hermitian form h on the tangent bundle of M is* admissible relative to E *if we are in one of the following three cases:*

$$\begin{aligned} \textit{elliptic: } & \lambda \neq 0 \ \textit{ and } h > 0, \\ \textit{parabolic: } & \lambda = 0 \ \textit{ and } h \geq 0 \ \textit{ with kernel spanned by } E, \\ \textit{hyperbolic: } & \lambda \neq 0,\ h(E, E) < 0 \ \textit{ and } h > 0 \ \textit{ on } E^{\perp}. \end{aligned}$$

The above argument shows that then the leaf space M/E acquires a metric $h_{M/E}$ of constant holomorphic sectional curvature, for it is locally isometric to a complex projective space with Fubini-Study metric, to complex-Euclidean space or to complex-hyperbolic space respectively.

1.3. *Logarithmic degeneration along a smooth hypersurface.* — In this subsection W is a complex manifold with a given affine structure ∇ on the complement $W - D$ of a smooth hypersurface D.

The cotangent bundle of $W - D$ has two extensions as a vector bundle over W that are of interest to us: the cotangent bundle of W and the logarithmic extension whose sheaf of local sections is $\Omega_W(\log D)$. The former is perhaps the more obvious choice, but the latter has better stability properties with respect to blowing up. We consider the degeneration of the affine structure in either extension.

We recall that if we are given a holomorphic vector bundle $\mathcal{V}$ on W, then a flat connection ∇ on $\mathcal{V}$ with a logarithmic pole along D is a map $\mathcal{V} \to \Omega_W(\log D) \otimes \mathcal{V}$ satisfying the usual properties of a flat connection. Then the residue map $\Omega_W(\log D) \to \mathcal{O}_D$ induces an $\mathcal{O}_D$-endomorphism $\operatorname{Res}_D(\nabla)$ of $\mathcal{V} \otimes \mathcal{O}_D$, called the *residue of the connection*. It is well-known that the conjugacy class of this endomorphism is locally constant constant along D. In particular, $\mathcal{V} \otimes \mathcal{O}_D$ decomposes according to the generalized eigenspaces of $\operatorname{Res}_D(\nabla)$. This becomes clear if we choose at $p \in D$ a chart $(t, u_1, \dots, u_n)$ such that D_p is given by $t = 0$: then $R := t\frac{\partial}{\partial t}, U_1 := \frac{\partial}{\partial u_1}, \dots, U_n := \frac{\partial}{\partial u_n}$ is a set of commuting vector fields, covariant derivation with respect to these fields preserves $\mathcal{V}_p$ (and since ∇ is flat, the resulting endomorphisms of $\mathcal{V}_p$ pairwise commute) and R induces in $\mathcal{V}_p \otimes \mathcal{O}_{D,p}$ the residue endomorphism. In particular, the kernel of R is preserved by U_i. The action of U_i on this kernel restricted to D_p only depends on the restriction of U_i to D_p. This shows that ∇ induces on the kernel of the residue endomorphism a flat connection. (A similar argument shows that the projectivization of the subbundle of $\mathcal{V} \otimes \mathcal{O}_D$ associated to an eigenvalue of $\operatorname{Res}_D(\nabla)$ comes with a projectively flat connection.)

Lemma **1.7.** — *Suppose that the affine structure ∇ on $W - D$ extends to Ω_W with a logarithmic pole. Letting $\nu_{D/W}$ stand for the normal bundle of D in W, then the residue of ∇ on Ω_W respects the natural exact sequence*

$$0 \to \nu^*_{D/W} \to \Omega_W \otimes \mathcal{O}_D \to \Omega_D \to 0$$

and induces the zero map in Ω_D.

If ∇ also extends with a logarithmic pole to $\Omega_W(\log D)$, then ∇ induces a connection on the cotangent bundle of D; this connection is torsion free and flat, so that D inherits an affine structure.

Proof. — By assumption, ∇ defines a map $\Omega_W \to \Omega_W(\log D) \otimes \Omega_W$. Since ∇ is torsion free, this extension takes values in

$$(\Omega_W(\log D) \otimes \Omega_W) \cap (\Omega_W \otimes \Omega_W(\log D)) \subset \Omega_W(\log D) \otimes \Omega_W(\log D).$$

If t be a local equation of D, then this intersection is spanned by $t^{-1}dt \otimes dt$ and $\Omega_W \otimes \Omega_W$. Hence the residue of ∇ on Ω_W maps $\Omega_W \otimes \mathcal{O}_D$ to the span of dt, that is, to $\nu^*_{D/W}$. This proves the first part. It also follows that the composite of $\nabla : \Omega_W \to \Omega_W(\log D) \otimes \Omega_W$ with the natural map $\Omega_W(\log D) \otimes \Omega_W \to \Omega_W(\log D) \otimes \Omega_D$ factors through a map $\Omega_W \to \Omega_W \otimes \mathcal{O}_D \to \Omega_D \otimes \Omega_D$. If ∇ also extends with a logarithmic pole to $\Omega_W(\log D)$, then it maps $t^{-1}dt$ to $\Omega_W(\log D) \otimes \Omega_W(\log D)$ and hence dt to $\Omega_W(\log D) \otimes t\Omega_W(\log D)$. Since this is just the kernel of $\Omega_W(\log D) \otimes \Omega_W \to \Omega_W(\log D) \otimes \Omega_D$, it follows that ∇ induces a map $\Omega_D \to \Omega_D \otimes \Omega_D$. This is a connection on the cotangent bundle of D which is torsion free and flat. □

Example **1.8.** — An affine structure may satisfy all the hypotheses of Lemma 1.7 and yet have nontrivial nilpotent monodromy. The following example is instructive since it is in essence a situation that we will later encounter when we deal with hyperbolic structures. Let $\alpha(u) = au + b$ be an affine-linear form on $\mathbf{C}$ with $b \neq 0$ and consider the singular affine structure on $\mathbf{C}^2$ at $(0,0)$ for which $(u,t) \mapsto (u, \alpha(u)\log t)$ is a developing map. It is clear that the holonomy near $(0,0)$ around the u-axis is given by $(u,\tau) \mapsto (u, \tau + 2\pi\sqrt{-1}\alpha(u))$ (where $\tau = \alpha \log t$). So the holonomy is nontrivial and the monodromy is nilpotent. The associated connection ∇ on the cotangent bundle is easily calculated for it is characterized by the property that du and $d(\alpha \log t) = \log t\, d\alpha + \alpha t^{-1} dt$ are flat; we find:

$$\nabla(du) = 0 \text{ and } \nabla\left(\frac{dt}{t}\right) = -\frac{dt}{t} \otimes \frac{d\alpha}{\alpha} - \frac{d\alpha}{\alpha} \otimes \frac{dt}{t}.$$

The last equality implies that

$$\nabla(dt) = \frac{dt}{t} \otimes dt - dt \otimes \frac{d\alpha}{\alpha} - \frac{d\alpha}{\alpha} \otimes dt.$$

So ∇ has along the u-axis a logarithmic singularity on both the sheaf of regular differentials and the sheaf of logarithmic differentials. The residue at $(0,0)$ on the former is the projection operator $du \mapsto 0$, $dt \mapsto dt$. But on the latter it is the nilpotent transformation $du \mapsto 0$, $t^{-1}dt \mapsto -ab^{-1}du$, which reflects the situation more faithfully, as it tells us whether or not $a = 0$. So this example speaks in favor of the logarithmic extension.

This is one of our reasons for only dealing with the logarithmic extension. In order to understand the behavior of an affine structure near a given smooth subvariety of its singular locus, it is natural to blow up that subvariety so that we are in the codimension one case. The simplest degenerating affine structures that we thus encounter lead to the following definition.

Definition **1.9.** — *Let D be a smooth connected hypersurface in an complex manifold W and let be given an affine structure on $W - D$. We say that the affine structure on $W - D$ has an* infinitesimally simple degeneration along D of logarithmic exponent $\lambda \in \mathbf{C}$ *if*

(i) ∇ extends to $\Omega_W(\log D)$ with a logarithmic pole along D,

(ii) the residue of this extension along D preserves the subsheaf $\Omega_D \subset \Omega_W(\log D) \otimes \mathcal{O}_D$ and its eigenvalue on the quotient sheaf $\mathcal{O}_D$ is λ and

(iii) the residue endomorphism restricted to Ω_D is semisimple and all of its eigenvalues are λ or 0.

When in addition

(iv) the connection has semisimple monodromy on the tangent bundle,

then we drop the adjective infinitesimally *and say that the affine structure on $W - D$ has a* simple degeneration along *D.*

For such a degenerating affine structure we have the following local model for the behavior of the developing map.

Proposition **1.10.** — *Let be given a be a smooth hypersurface D in an complex manifold W, an affine structure on $W-D$ and $p\in D$. Then the affine structure has an infinitesimally simple degeneration along D at p of logarithmic exponent $\lambda\in\mathbf{C}$ if and only if there exists a local equation t for D and a local isomorphism of the form*

$$(F_0,t,F_\lambda):W_p\to(T_0\times\mathbf{C}\times T_\lambda)_{(0,0,0)}$$

(ignore the third factor when $\lambda=0$), where T_0 and T_λ are vector spaces, such that the developing map near p is affine equivalent to the following multivalued map with range $T_0\times\mathbf{C}\times T_\lambda$ ($T_0\times\mathbf{C}$ when $\lambda=0$):

$$\begin{aligned}
\lambda\notin\mathbf{Z}&:(F_0,t^{-\lambda},t^{-\lambda}F_\lambda),\\
\lambda\in\mathbf{Z}_{>0}&:(F_0,t^{-\lambda},t^{-\lambda}F_\lambda)+\log t.(0,A\circ F_0),\\
&\quad\text{where } A:T_0\to\mathbf{C}\times T_\lambda \text{ is an affine-linear map,}\\
\lambda\in\mathbf{Z}_{<0}&:(F_0,t^{-\lambda},t^{-\lambda}F_\lambda)+\log t.t^{-\lambda}(B\circ F_\lambda,0,0),\\
&\quad\text{where } B:T_\lambda\to T_0 \text{ is an affine-linear map,}\\
\lambda=0&:(F_0,\log t.\alpha\circ F_0)\\
&\quad\text{for some affine-linear function } \alpha:T_0\to\mathbf{C} \text{ with } \alpha(0)\neq 0.
\end{aligned}$$

When $\lambda\notin\mathbf{Z}$, the holonomy around D_p (and hence the monodromy around D_p) is semisimple. When $\lambda\in\mathbf{Z}-\{0\}$, the monodromy is semisimple if and only if the associated affine-linear map A ($\lambda>0$) or B ($\lambda<0$) is zero (and in that case the holonomy is equal to the identity). When $\lambda=0$, the monodromy is semisimple if and only if α is constant and in that case the holonomy is a translation.

It is easy to verify that, conversely, these formulae define an affine structure on $T_0\times\mathbf{C}^\times\times T_\lambda$ with infinitesimally simple degeneration along $T_0\times\{0\}\times T_\lambda$. Notice that $\lambda=-1$ with $T_\lambda=\{0\}$ and $B=0$ describes the apparently dull case when the affine structure extends across D at p without singularities. But even here the proposition has something to offer, for it tells us that D must then be an affine hypersurface. We also observe that Example 1.8 corresponds to the case $\lambda=0$ and $\dim W=2$.

For the proof of Proposition 1.10 we need a few well-known facts about flat connections on vector bundles. In the following two lemma's $\mathcal{V}$ is a holomorphic vector bundle over the smooth germ W_p and ∇ is flat connection on $\mathcal{V}$ with a logarithmic pole along a smooth hypersurface germ D_p. So the connection has a residue endomorphism $\operatorname{Res}_p(\nabla)$ of the fiber $\mathcal{V}(p)$.

If A is an endomorphism of a finite dimensional complex vector space V, then we let t^A stand for the map $V\to V\{t\}$ defined by $\exp(A\log t)$.

Lemma **1.11.** — *The vector bundle $\mathcal{V}$ (with its flat connection) decomposes naturally according to the images in $\mathbf{C}/\mathbf{Z}$ of the eigenvalues of the residue endomorphism: $\mathcal{V} = \oplus_{\zeta\in\mathbf{C}^\times}\mathcal{V}^\zeta$, where $\mathcal{V}^\zeta$ has a residue endomorphism whose eigenvalues λ are such that $\exp(2\pi\sqrt{-1}\lambda) = \zeta$. If no two eigenvalues of the residue endomorphism $\mathrm{Res}_p(\nabla)$ differ by a nonzero integer, and a local equation t for D_p is given, then we have a $\mathbf{C}$-linear section $s : \mathcal{V}(p) \to \mathcal{V}$ of the reduction map such that $t^{-\mathrm{Res}_p(\nabla)} : \mathcal{V}(p) \to \mathcal{V}(p)\{t\}$ followed by $s\otimes 1$ takes values in the space of flat multivalued sections of $\mathcal{V}$.*

Proof. — The first statement is just the decomposition of $\mathcal{V}$ according to its generalized eigenvalues of the monodromy: $\mathcal{V}^\zeta$ is the submodule of $\mathcal{V}$ on which the monodromy has ζ as its unique eigenvalue. The second is well-known (see for instance [24]). □

There is also such a result for the general situation (in which eigenvalues can differ by a nonzero integer), but we do not state it here, as we shall only need the following special case.

Lemma **1.12.** — *Assume that the residue map is semisimple with two eigenvalues λ and μ such that $\mu-\lambda$ a positive integer. If a local equation t for D_p is given, and $\mathcal{V}(p) = V_\lambda \oplus V_\mu$ is the eigenspace decomposition, then there exists a $\mathbf{C}$-linear section $(u, v) \in V_\lambda \oplus V_\mu \mapsto s_{\lambda,u} + s_{\mu,v} \in \mathcal{V}$ of the reduction map and a $C \in \mathrm{Hom}(V_\mu, V_\lambda)$ such that the image of*

$$\begin{aligned} &u \in V_\lambda \mapsto t^{-\lambda}s_{\lambda,u};\\ &v \in V_\mu \mapsto t^{-\mu}s_{\mu,v} - \log t . t^{-\lambda}s_{\lambda,C(v)}. \end{aligned}$$

spans the space of flat multivalued sections.

Proof. — This result is well-known, bus since we haven't found an exact reference, let us outline the proof (see however [25] and [24]). One begins with observing that the connection can be integrated in directions parallel to D; this allows us to restrict to the case $\dim W = 1$ with t as local coordinate. The idea is to find in that case a trivialisation of $\mathcal{V}$ given by a section $s : \mathcal{V}(p) \to \mathcal{V}$ of the reduction map on which the operator $t\nabla_{\partial/\partial t}$ is constant: this means that via the isomorphism of $\mathcal{O}_{W,p}$-modules $\mathcal{O}_{W,p} \otimes \mathcal{V}(p) \to \mathcal{V}$ defined by s, $t\nabla_{\partial/\partial t}$ is of the form $1 \otimes U$, with $U \in \mathrm{End}(\mathcal{V}(p))$. It is easily verified that this can be done formally with U differing from the residue operator by a linear map $V_\mu \to V_\lambda$. The formal solution can be made to converge and the lemma then follows. □

We also need a Poincaré lemma, the proof of which is left as an exercise.

Lemma **1.13.** — *Let $\lambda \in \mathbf{C}$ and $\omega \in \Omega_{W,p}(\log D)$ be such that $t^{-\lambda}\omega$ is closed. Then $t^{-\lambda}\omega = d(t^{-\lambda}f)$ for some $f \in \mathcal{O}_{W,p}$ unless λ is a nonnegative integer in which case $t^{-\lambda}\omega = d(t^{-\lambda}f + c\log t)$ for some $f \in \mathcal{O}_{W,p}$ and some $c \in \mathbf{C}$.*

Proof of Proposition 1.10. — Assume first that $\lambda \neq 0$. Choose a local equation t for D_p. Denote by V be the fiber of $\Omega_{W,p}(\log D)$ over p and let $V = V_0 \oplus V_\lambda$ be the eigenspace decomposition of the residue endomorphism.

If $\lambda \notin \mathbf{Z}$, then according to Lemma 1.11 there is a section $s = s_0 + s_\lambda : V_0 \oplus V_\lambda \to \Omega_{W,p}(\log D)$ of the reduction map such that s_0 resp. $t^{-\lambda}s_\lambda$ map to flat sections. Any flat section is closed, because the connection is symmetric. Since the residue has eigenvalue λ on the logarithmic differentials modulo the regular differentials, s_0 will take its values in the regular differentials. So by our Poincaré lemma 1.13 both s_0 and $t^{-\lambda}s_\lambda$ take values in the exact forms: there exists a linear $\tilde{s} = \tilde{s}_0 + \tilde{s}_\lambda : V_0 \oplus V_\lambda \to \mathcal{O}_{W,p}$ such that $d\tilde{s}_0 = s_0$ and $d(t^{-\lambda}\tilde{s}_\lambda) = t^{-\lambda}s_\lambda$. We put $T_0 := V_0^*$ and take for $F_0 : W_p \to T_0$ the morphism defined by s_0. Choose $v \in V_\lambda$ not in the cotangent space T_p^*D so that V_λ splits as the direct sum of $\mathbf{C}v \oplus (T_p^*D)_\lambda$. Then $\tilde{s}_\lambda(v)$ is a unit and so $t^{-\lambda}\tilde{s}_\lambda(v)$ is of the form $\tilde{t}^{-\lambda}$ for another defining equation $\tilde{t}$ of D_p. So upon replacing t by $\tilde{t}$ we can assume that $\tilde{s}_\lambda(v) = 1$. Then we take $T_1 = (T_pD)_\lambda$, and let $F_1 : W_p \to T_1$ be defined by the set of elements in the image of s_λ which vanish in p. The proposition then follows in this case.

Suppose now that λ is a positive integer n. Then Lemma 1.12 gives us a section $s_0 + s_n : V_0 \oplus V_n \to \Omega_{W,p}(\log D)$ and a linear map $C : V_n \to V_0$ such that the images of s_0 and $t^{-n}s_n - \log t.s_0C$ are flat. The image of s_0 consists of exact forms for the same reason as before so that we can still define $\tilde{s}_0 : V_0 \to \mathcal{O}_{W,p}$ and a flat morphism $F_0 : W_p \to T_0 = V_0^*$. If $u \in V_\lambda$, then $t^{-n}s_{n,u} - \log t.s_{0,C(u)}$ is flat and hence closed. Since $s_{0,C(u)} = d\tilde{s}_{0,C(u)}$ we have that $t^{-n}s_{n,u} + \tilde{s}_{0,C(u)}t^{-1}dt$ is also closed. Invoking our Poincaré lemma yields that this must have the form $d(\tilde{s}_{n,u} + c_u \log t)$ for some $\tilde{s}_{n,u} \in \mathcal{O}_{W,p}$ and $c_u \in \mathbf{C}$. So $\tilde{s}_{n,u} + \log t.(c_u - \tilde{s}_{0,C(u)})$ is a multivalued affine function. The argument is then concluded as in the previous case.

The case when λ is a negative integer is done similarly.

The case $\lambda = 0$ has nilpotent residue with kernel containing (and image contained in) the cotangent space of D at p. We can apply Lemma 1.11 here and the proceed as above; we leave the details to the reader.

It is clear that if the affine structure on $W - D$ has near p a developing map as given, the holonomy and the monodromy have the stated properties. It is also verified in a straightforward manner that then ∇ has a logarithmic pole on $\Omega_{W,p}(\log D)$ with the stated residue properties. $\square$

Let us rephrase (part of) Proposition 1.10 in more intrinsic, geometric terms.

Corollary **1.14.** — *Let be given an affine structure on the complement of a smooth hypersurface D in an complex manifold W which degenerates infinitesimally simply along D with logarithmic exponent λ and let $p \in D$. Then the decomposition of the tangent bundle of D into eigenspaces of the residue endomorphism integrates locally to a decomposition of germs $D_p = D_{0,p} \times D_{\lambda,p}$; the first factor has a natural affine structure and the second factor a natural projective structure.*

Suppose λ is not an integer ≤ 0. Then we have a natural retraction $W_p \to D_{0,p}$ in the affine category whose fibers are preserved by the holonomy; if λ is real and positive, then the projectivization of the developing map tends, as we approach D, locally to a univalued map which factors through a projective-linear embedding of $D_{\lambda,p}$; if λ is a positive integer, then the holonomy induces in every fiber a translation.

Suppose λ is not an integer ≥ 0. Then we have a natural affine foliation of W_p (the leaf space of which we denote by $W_{\lambda,p}$) whose fibers are preserved by the holonomy and which extends the foliation of D_p defined by the projection $D_p \to D_{\lambda,p}$ (so that we can regard $D_{\lambda,p}$ as a subgerm of $W_{\lambda,p}$); if λ is real and negative, then the developing map tends, as we approach D, locally to a univalued map which factors through an affine-linear embedding of $D_{0,p}$; if λ is a negative integer, then the holonomy induces in every leaf a translation.

Suppose $\lambda = 0$. Then we have a natural affine retraction $W_p \to D_p$ and W_p comes with a natural faithhul $\mathbf{C}^{\times}$-action which preserves the fibers of $W_p \to D_p$ and acts in each fiber by translations. If the degeneration is not *simple, then D_p has a natural flat codimension one foliation: a leaf parametrizes the fibers of the affine retraction in which the holonomy defines the same translation.*

Suppose $\lambda \notin \mathbf{Z}$, so that the above submersions define a decomposition of pairs $(W_p, D_p) = D_{0,p} \times (W_{\lambda,p}, D_{\lambda,p})$ in the affine category. Then $W_{\lambda,p}$ comes with a natural faithful $\mathbf{C}^{\times}$-action which acts by dilatations, has $D_{\lambda,p}$ as fixed point set and induces on $D_{\lambda,p}$ the projective structure mentioned above.

Proof. — The chart of W_p obtained at p restricts to a chart of D_p with values in $T_0 \times T_\lambda$. The corresponding decomposition of D_p, $D_p = D_{0,p} \times D_{\lambda,p}$ is canonical, because it integrates the eigenspace decomposition of the residue endomorphism restricted to the tangent bundle.

We now treat the case when λ is not an integer ≤ 0. Then the normal for given in Proposition 1.10 shows that the affine-linear elements of $\mathcal{O}_{W,p}$

define a foliation of W_p which has $D_{0,p}$ as a section. This amounts to a retraction $W_p \to D_{0,p}$ and an affine structure on $D_{0,p}$ which makes this map affine-linear. The normal form also shows that if $\lambda > 0$, then the projectivized developing map is given by

$$[F_0 : t^{-\lambda} + \log t.A' \circ F_0 : t^{-\lambda} F_\lambda + \log t.A'' \circ F_0] = \\ = [t^\lambda F_0 : 1 + t^\lambda \log t.A' \circ F_0 : F_\lambda + t^\lambda \log t.A'' \circ F_0]$$

(the log terms are absent if $\lambda \notin \mathbf{Z}$). So the limit for $t \to 0$ is given by the univalued map $[0 : 1 : F_1]$. This restricts to a local isomorphism $D_{\lambda,p} \cong \mathbf{P}(\mathbf{C} \times T_1)_{[1:0]}$ and thus endows $D_{\lambda,p}$ with a natural projective structure. It is also clear that if λ is a positive integer, then the holonomy is as stated.

The other cases are treated in a similar fashion. For instance, when λ is not an integer ≥ 0, then the affine-linear elements in $\mathcal{O}_{W,p}(\lambda D)$ (with its obvious interpretation) define the affine submersion $W_p \to W_{\lambda,p}$. Similarly, when $\lambda = 0$, the local, affine (multivalued) functions that lie in $\mathcal{O}_{W,p}$ define the canonical retraction $W_p \to D_{0,p}$ and the holonomy defines a translation in every fiber; if the the degeneration is not simple, then in our local model 1.10 we have $\alpha \neq 0$ and the foliation defined by $\alpha \circ F_0$ is natural.

If $\lambda \in \mathbf{C} - \mathbf{Z}$, then the affine structure on $W_{\lambda,p}$ is in fact a linear structure and has an associated $\mathbf{C}^\times$-action, which in terms of our chart acts on $\mathbf{C} \times T_\lambda$ by scalar multiplication. If $\lambda = 0$, then scalar multiplication in the second factor of $T_0 \times \mathbf{C}$, defines a $\mathbf{C}^\times$ acts on W_p by translations. □

Notice that for $\lambda \neq 0$, this corollary makes D locally appear as if it were the exceptional divisor of a blowup whose restriction to D_p is the projection $D_p \to D_{0,p}$.

1.4. *Logarithmic degeneration along a normal crossing divisor.* — We also need to understand what happens in case D is a normal crossing divisor is the compelx manifold W and the affine structure on $W - D$ degenerates infinitesimally simply along each irreducible component of D. Eventually we will only be interested in the case when each exponent is real and > -1, but in what follows we only assume that no exponent is a negative integer. We will also assume that the the holonomy around an irreducible component is semisimple unless its exponent is zero.

First we consider the case when D has only two smooth irreducible components D' and D''. Let $p \in S := D' \cap D''$. The vector bundle $\Omega_W(\log D) \otimes \mathcal{O}_{D'_p}$ over D' comes with a residue endomorphism R' and likewise over D''. These endomorphisms commute in $\Omega_W(\log D) \otimes \mathcal{O}_{S,p}$ and respect the exact residue sequence

$$0 \to \Omega_{S,p} \to \Omega_W(\log D) \otimes \mathcal{O}_{S,p} \to \mathcal{O}_{S_p} \oplus \mathcal{O}_{S_p} \to 0.$$

From this it is readily seen that the local eigenvalue foliations of $D'-S$ defined by R' extends across S and do so in such a manner that it is compatible with those coming from $D''-S$: we can find local coordinates at p such that each of the four subgerms $D'_{0,p},\dots,D''_{\lambda'',p}$ is defined by putting some of these coordinates equal to zero. The affine structure on $D'_{0,p}$ has an affine structure away from D'' and either we have infinitesimally simple degeneration along $D'_{0,p}\cap D''$ with exponent λ'' or the affine structure extends across this locus. The elements of $\mathcal{O}_W|D'-S$ that are affine-linear are invariant under the holonomy around D'' and so define an affine retraction $r':W_p\to D^1_{0,p}$. Likewise we have $r'':W_p\to D''_{0,p}$ and the two commute.

There eigenvalue pairs for (R',R'') in $\Omega_W(\log D)\otimes\mathbf{C}_p$ lie in $\{0,\lambda'\}\times\{0,\lambda''\}$. We are however only interested in the special case when the eigenvalue pair $(0,\lambda'')$ does not occur (so that in particular, $\lambda'\neq 0$). The reason is that in our applications D appears as the exceptional divisor of an iterated blowup, which makes that its irreducible components come in generations: in the case at hand, D'' comes after D'. The nonoccurrence of $(0,\lambda'')$ can also be stated in more geometric terms: it means that the formation of the affine local factor of D' as a quotient of its ambient germ persists as such (i.e., remains affine) at p: we have $r'r''=r'$.

Under these assumptions we find that when $\lambda''\neq 0$, there exist local equations t' resp. t'' for D' resp. D'' and a morphism $(F,F',F''):W_p\to(T\times T'\times T'')$ to a product of vector spaces wit $p\mapsto(0,0,0)$, such that (F,t',F',t'',F'') is a chart and the developing map is affine-equivalent to

$$\Big(F,(t')^{-\lambda'}(1,F'),(t')^{-\lambda'}(t'')^{-\lambda''}(1,F'')\Big):W_p\to T\times(\mathbf{C}\times T')\times(\mathbf{C}\times T'')$$

Notice that (F,F',F'') defines a chart for S_p; the resulting decomposition of S_p is the one alluded to above. When $\lambda''=0$, we find that exist local equations t' resp. t'' for D' resp. D'', a morphism $(F,F'):W_p\to T\times T'$ to a product of vector spaces with $p\mapsto(0,0)$ such that (F,t',F',t'') is a chart and the developing map is affine-equivalent to

$$\Big(F,(t')^{-\lambda'}\,(1,F',\log t'')\Big):W_p\to T\times(\mathbf{C}\times T'\times\mathbf{C}).$$

In this case (F,F') defines a chart for S_p; the corresponding decomposition of S_p is natural and inherited from D''. Notice that in this case the monodromy around D'' is not the identity; the associated foliation of D'' is defined by $t'|D''$.

This generalizes in a straightforward manner to the following proposition (To prevent notational overload, subscripts no longer refer to eigenvalues.)

Proposition **1.15.** — *Let W be an complex manifold, D a simple normal crossing divisor on W with smooth irreducible components $D_1, \dots, D_k$ ($k \geq 2$) and ∇ an affine structure on $W - D$ with infinitesimally simply degeneration along D_i of logarithmic exponent λ_i, $i = 1, \dots, k$. Suppose that no λ_i is a negative integer, that the holonomy around D_i is semisimple unless $\lambda_i = 0$ and that for any pair $1 \leq i < j \leq k$, the formation of the local affine quotient of the generic point of D_i extends across the generic point of $D_i \cap D_j$. Let $p \in \cap D_i$. Then $\lambda_i \neq 0$ for $i < k$ and the local affine retraction r_i at the generic point of D_i extends to $r_i : W_p \to D_{i,0}$ in such a manner that $r_i r_j = r_i$ for $i < j$. Furthermore, there exist for $i = 1, \dots k$ a local equation t_i for D_i and for $i = 1, \dots k{+}1$ (resp. $i = 1, \dots k$) a morphism to a vector space $F_i : W_p \to T_i$ with $F_i(p) = 0$ such that these are the components of a chart for W_p and are such that the developing map is affine equivalent to the multivalued map*

$$\Big(F_0, \big(t_1^{-\lambda_1} \cdots t_i^{-\lambda_i}(1, F_i)\big)_{i=1}^{k}\Big) : W_p \to T_0 \times \prod_{i=1}^{k} (\mathbf{C} \times T_i) \ \textit{resp.}$$

$$\Big(F_0, \big(t_1^{-\lambda_1} \cdots t_i^{-\lambda_i}(1, F_i)\big)_{i=1}^{k-1}, t_1^{-\lambda_1} \cdots t_{k-1}^{-\lambda_{k-1}} \log t_k\Big) : W_p \to T_0 \times \prod_{i=1}^{k-1} (\mathbf{C} \times T_i) \times \mathbf{C}.$$

2. Linear arrangements with a Dunkl connection

2.1. *Review of the terminology concerning linear arrangements.* — We adhere mostly to the notation used in the book by Orlik and Terao [28].

Let $(V, \mathcal{H})$ be a *linear hyperplane arrangement*, that is, a finite dimensional complex vector space V and a finite collection $\mathcal{H}$ of (linear) hyperplanes of V. We shall suppose that $\mathcal{H}$ is nonempty so that $\dim(V) \geq 1$. The *arrangement complement*, by which we shall mean the complement in V of the union of the members of $\mathcal{H}$, will be denoted by V°. We will also use the superscript $^\circ$ to denote such a complement in analogous situations (as for instance, in a projective setting), assuming that the arrangement is understood.

The collection of hyperplane intersections in V taken from subsets of $\mathcal{H}$ is denoted $\mathcal{L}(\mathcal{H})$ (this includes V itself as the intersection over the empty subset of $\mathcal{H}$). We consider it as a poset for the reverse inclusion relation: $L \leq M$ means $L \supseteq M$. (This is in fact a lattice with *join* $L \vee M = L \cap M$ and with *meet* $L \wedge M$ the intersection of the $H \in \mathcal{H}$ containing $L \cup M$.) The members of $\mathcal{H}$ are the minimal elements (the *atoms*) of $\mathcal{L}(\mathcal{H}) - \{V\}$ and $\cap_{H \in \mathcal{H}} H$ is the unique maximal element. For $L \in \mathcal{L}(\mathcal{H})$ we denote by $\mathcal{H}_L$ the collection of $H \in \mathcal{H}$ which contain L. We often think of $\mathcal{H}_L$ as defining a linear arrangement on V/L. This identifies $\mathcal{L}(\mathcal{H}_L)$ as a poset with the set of $M \in \mathcal{L}(\mathcal{H})$ with $M \leq L$. The assignment $L \mapsto \mathcal{H}_L$ identifies $\mathcal{L}(\mathcal{H})$ with a subposet of the lattice of subsets of $\mathcal{H}$ and we will often tacitly use that identification in our notation.

Given an $L \in \mathcal{L}(\mathcal{H})$, then each $H \in \mathcal{H} - \mathcal{H}_L$ meets L in a hyperplane of L. The collection of these hyperplanes of L is denoted $\mathcal{H}^L$. We call the arrangement complement $L^\circ \subset L$ defined by $\mathcal{H}^L$ an *$\mathcal{H}$-stratum*; these define a partition V.

A *splitting* of $\mathcal{H}$ is a nontrivial decomposition of $\mathcal{H}$ of the form $\mathcal{H} = \mathcal{H}_L \sqcup \mathcal{H}_{L'}$ with $L, L' \in \mathcal{L}(\mathcal{H})$ and $L + L' = V$. If no splitting exists, then we say that $\mathcal{H}$ is *irreducible*. A member $L \in \mathcal{L}(\mathcal{H})$ is called *irreducible* if $\mathcal{H}_L$ is. This amounts to the property that either L is a member of $\mathcal{H}$ (i.e., a hyperplane) or that there exist $(\operatorname{codim}(L)+1)$ hyperplanes from $\mathcal{H}_L$ such that L is the intersection of any $\operatorname{codim}(L)$-tuple out of them. Or equivalently, that the identity component of $\operatorname{Aut}(V/L, \mathcal{H}_L)$ is the group of scalars $\mathbf{C}^\times$. We denote by $\mathcal{L}_{\mathrm{irr}}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ the subposet of irreducible members.

Given $L \in \mathcal{L}(\mathcal{H})$, then an *irreducible component* of L is a maximal irreducible member of $\mathcal{L}(\mathcal{H}_L)$. If $\{L_i\}_i$ are the distinct irreducible components of L, then L is the transversal intersection of these in the sense that the map $V \to \oplus_i V/L_i$ is onto and has kernel L.

Lemma **2.1.** — *Given $L, M \in \mathcal{L}(\mathcal{H})$ with $M \subset L$, denote by $M(L) \in \mathcal{L}(\mathcal{H})$ the common intersection of the members of $\mathcal{H}_M - \mathcal{H}_L$. If $M \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^L)$, then $M(L)$ is the unique irreducible component of M in $\mathcal{L}(\mathcal{H})$ which is not an irreducible component of L. In particular, if $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ and $M \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^L)$, then either $M = M(L) \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ or $\{L, M(L)\}$ are the distinct irreducible components of M in $\mathcal{L}(\mathcal{H})$.*

Proof. — Left as an exercise. □

2.2. *Affine structures on arrangement complements.* — Let $\mathcal{H}$ be a linear arrangement in the complex vector space V. For $H \in \mathcal{H}$, we denote by ω_H (or ω_H^V, if a reference to the ambient space is appropriate) the unique meromorphic differential on V with divisor $-H$ and residue 1 along H. So $\omega_H = \phi_H^{-1} d\phi_H$, where ϕ_H is a linear equation for H.

Suppose ∇ is a torsion free flat connection on the tangent bundle of V°. If ∇^0 denotes the standard (translation invariant) flat connection on the tangent bundle of V restricted to V°, then $\nabla^0 - \nabla$ is a $\operatorname{End}(V)$-valued holomorphic differential on V°, which we denote by $\Omega \in H^0(V^\circ, \Omega_V) \otimes_{\mathbf{C}} \operatorname{End}(V)$. This differential is called the *connection form* of ∇. The associated (dual) connection on the cotangent bundle of V° (also denoted by ∇) is characterized by the property that the pairing between vector fields and differentials is flat. So its connection form is $-\Omega^*$.

Proposition **2.2.** — *Suppose that the torsion free flat connection ∇ on the tangent bundle of V° is invariant under scalar multiplication and has a logarithmic singularity along the generic point of every member of $\mathcal{H}$, when regarded as a meromorphic connection on the tangent bundle of V. Then for every $H \in \mathcal{H}$, $\mathrm{Res}_H(\nabla)$ is a constant endomorphism $\rho_H \in \mathrm{End}(V)$ whose kernel contains H and Ω has the form*

$$\Omega = \sum_{H\in\mathcal{H}} \omega_H \otimes \rho_H.$$

If E_V denotes the Euler vector field on V, then the covariant derivative of E_V with respect to the constant vector field parallel to a vector $v \in V$ is the constant vector field parallel to $v - \sum_{H\in\mathcal{H}} \rho_H(v)$.

If $\rho_H \neq 0$, then ∇ induces on $H \in \mathcal{H}$ a connection of the same type.

Proof. — The assumption that ∇ is invariant under scalar multiplication means that the coefficient forms of Ω in $H^0(V^\circ, \Omega_V)$ are $\mathbf{C}^\times$-invariant. This implies that these forms are $\mathbf{C}$-linear combinations of the logarithmic differentials ω_H and so Ω has indeed the form $\sum_{H\in\mathcal{H}} \omega_H \otimes \rho_H$ with $\rho_H \in \mathrm{End}(V)$. Following Lemma 1.7, ρ_H is zero or has kernel H. This lemma also yields the last assertion.

Finally, let ϕ_H be a defining linear form for H so that we can write $\omega_H = \phi_H^{-1} d\phi_H$ and $\rho_H(u) = \phi_H(u)v_H$ for some $v_H \in V$. Then

$$\omega_H(\partial_v)\rho_H(E_V) = \frac{\phi_H(v)}{\phi_H(z)}\phi_H(z)\partial_{v_H} = \partial_{\rho_H(v)}.$$

Since $\nabla^0_{\partial_v}(E_V) = \partial_v$, it follows that $\nabla_{\partial_v}(E_V) = \partial_v - \sum_{H\in\mathcal{H}} \partial_{\rho_H(v)}$. □

Proposition **2.3.** — *Suppose that for every $H \in \mathcal{H}$ we are given an endomorphism ρ_H of V with kernel H and put $\Omega := \sum_{H\in\mathcal{H}} \omega_H \otimes \rho_H$. Then the connection on the tangent bundle of V° defined by $\nabla := \nabla^\circ - \Omega$ is $\mathbf{C}^\times$-invariant and torsion free, and if we regard ∇ as a meromorphic connection on the tangent bundle of the projective completion of V, then it has logarithmic singularities (so that ∇ is regular-singular). Moreover, the following properties are equivalent:*

(i) ∇ *is flat,*
(ii) $\Omega \wedge \Omega = 0$,

(iii) for every pair $L, M \in \mathcal{L}(\mathcal{H})$ with $L \subset M$, the endomorphisms $\sum_{H\in\mathcal{H}_L} \rho_H$ and $\sum_{H\in\mathcal{H}_M} \rho_H$ commute,

(iv) for every $L \in \mathcal{L}(\mathcal{H})$ of codimension 2, the sum $\sum_{H\in\mathcal{H}_L} \rho_H$ commutes with each of its terms.

Proof. — The $\mathbf{C}^\times$-invariance of ∇ is clear. Let $\phi_H \in V^*$ have zero set H. Then there exist $e_H \in V$ such that

$$\Omega = \sum_{H\in\mathcal{H}} \phi_H^{-1} d\phi_H \otimes d\phi_H \otimes \partial_{e_H}$$

which plainly shows that Ω is symmetric in the first two factors. So ∇ is symmetric. The connection ∇ has on $\Omega_{\overline{V}}(\log(\mathbf{P}(V))$ visibly a logarithmic singularity along each member of $\mathcal{H}$ and so it remains to verify that this is also the case along $\mathbf{P}(V)$. It is clear that $\mathbf{P}(V)$ is pointwise fixed under the $\mathbf{C}^\times$-action. The generic point w of $\mathbf{P}(V)$ has a local defining equation u in $\overline{V}$ that is homogeneous of degree -1. The $\mathbf{C}^\times$-invariance of ∇ implies that its matrix has the form

$$\frac{du}{u} \otimes A(w) + \Omega'(w),$$

where A is a matrix and Ω' a matrix valued differential in the generic point of $\mathbf{P}(V)$.

The proof that the four properties are indeed equivalent can be found in [22]. □

Example **2.4** *(The case of dimension two).* — Examples abound in dimension two: suppose $\dim V = 2$ and let $\{\rho_i \in \mathrm{End}(V)\}_{i\in I}$ be a finite collection of rank one endomorphisms with distinct kernels. So if ω_i is the logarithmic differential defined by $\ker(\rho_i)$, then the connection defined by $\Omega = \sum_i \omega_i \otimes \rho_i$ is flat if and only if $\sum_i \rho_i$ commutes with each of its terms. When I has at least three elements, this is equivalent to $\sum_i \rho_i$ being a scalar operator (for it must preserve each line $\ker(\rho_i)$). In case I has just two elements, then this means that these two elements commute and hence are semisimple (because their kernels decompose V).

Example **2.5** *(Complex reflection groups).* — Irreducible examples in dimension ≥ 2 can be obtained from finite complex reflection groups. Let $G \subset \mathrm{GL}(V)$ be a finite irreducible subgroup generated by complex reflections and let $\mathcal{H}$ be the collection of fixed point hyperplanes of the complex reflections in G. Choose a G-invariant positive definite inner product on V and let for $H \in \mathcal{H}$, π_H be the orthogonal projection along H onto $H^\perp$. If $\kappa \in \mathbf{C}^{\mathcal{H}}$ is G-invariant, then the connection defined by the form $\sum_{H\in\mathcal{H}} \omega_H \otimes \kappa_H \pi_H$ is flat [22].

The next subsection describes a classical example.

2.3. *The Lauricella local system.* — Let V be the quotient of $\mathbf{C}^{n+1}$ by its main diagonal. Label the standard basis of $\mathbf{C}^{n+1}$ as $e_0, \dots, e_{n+1}$ and let for $0 \le i < j \le n$, H_{ij} be the hyperplane $z_i = z_j$ (either in $\mathbf{C}^{n+1}$ or in V) and $\omega_{ij} := (z_i - z_j)^{-1} d(z_i - z_j)$ the associated logarithmic form. We let $\mathcal{H}$ be the collection of these hyperplanes so that we can think of V° as the configuration space of $n+1$ ordered distinct points in $\mathbf{C}$, given up to translation.

Let be given positive real numbers $\mu_0, \dots, \mu_n$ and define an inner product $\langle\, ,\, \rangle$ on $\mathbf{C}^{n+1}$ by $\langle e_i, e_j \rangle = \mu_i \delta_{i,j}$. We may identify V with the orthogonal complement of the main diagonal, that is, with the hyperplane defined by $\sum_i \mu_i z_i = 0$. The line orthogonal to the hyperplane $z_i - z_j = 0$ is spanned by the vector $\mu_j e_i - \mu_i e_j$. (For this reason it is often convenient to use the basis $(e_i' := \mu_i^{-1} e_i)_i$ instead, for then the hyperplane in question is the orthogonal complement of $e_i' - e_j'$; notice that $\langle e_i', e_j' \rangle = \mu_i^{-1} \delta_{i,j}$.) So the endomorphism $\tilde{\rho}_{ij}$ of $\mathbf{C}^{n+1}$ which sends z to $(z_i - z_j)(\mu_j e_i - \mu_i e_j)$ is selfadjoint, has kernel H_{ij} and $\mu_j e_i - \mu_i e_j$ as eigenvector with eigenvalue $\mu_i + \mu_j$. In particular, $\tilde{\rho}_{ij}$ induces an endomorphism ρ_{ij} in V.

Proposition-definition **2.6.** — *The connection*

$$\nabla := \nabla^0 - \sum_{i<j} \omega_{ij} \otimes \rho_{ij}$$

is flat (we call it the Lauricella connection*) and has the Euler vector field on V as a dilatation field with factor $1 - \sum_i \mu_i$.*

Let γ be a path in $\mathbf{C}$ which connects z_i with z_j but otherwise avoids $\{z_0, \dots, z_n\}$ in $\mathbf{C}$. If both $\mu_i < 1$ and $\mu_j < 1$ and a determination of $(z_0 - \zeta)^{-\mu_0} \cdots (z_n - \zeta)^{-\mu_n}$ is chosen, then the integral

$$F_\gamma(z_0, \dots, z_n) := \int_\gamma (z_0 - \zeta)^{-\mu_0} \cdots (z_n - \zeta)^{-\mu_n} d\zeta$$

As a (multivalued) function of $(z_0, \dots, z_n)$, F_γ is translation invariant and thus defines a multivalued holomorphic (so-called Lauricella*) function on V°. This function is homogeneous of degree $1 - \sum_i \mu_i$ and its differential is flat for the Lauricella connection.*

Proof. — The first assertion follows from a straightforward computation based on Proposition 2.3: one verifies that for $0 \le i < j < k \le n$ the transformation $\tilde{\rho}_{ij} + \tilde{\rho}_{ik} + \tilde{\rho}_{jk}$ acts on the orthogonal complement of $e_i + e_j + e_k$

in the span of e_i, e_j, e_k as multiplication by $\mu_i + \mu_j + \mu_k$ so that this sum commutes with each of its terms.

The convergence and the translation invariance and the homogeneity property of the integral are clear. If F denotes the associated multivalued function, then the flatness of dF comes down to

$$\sum_{i,j} \frac{\partial^2 F}{\partial z_i \partial z_j} dz_i \otimes dz_j = -\sum_{i<j} \frac{1}{z_i - z_j}\Big(\mu_j \frac{\partial F}{\partial z_i} - \mu_i \frac{\partial F}{\partial z_j}\Big)(dz_i - dz_j) \otimes (dz_i - dz_j).$$

For $i<j$, we have

$$\begin{aligned}\frac{1}{z_i - z_j}\Big(\mu_j \frac{\partial F}{\partial z_i} - \mu_i \frac{\partial F}{\partial z_j}\Big) &= \frac{-\mu_i\mu_j}{z_i - z_j}\int_\gamma \Big(\frac{1}{z_i-\zeta} - \frac{1}{z_j - \zeta}\Big)\prod_{\nu=0}^{n}(z_\nu - \zeta)^{-\mu_\nu} d\zeta \\ &= \mu_i\mu_j \int_\gamma (z_i-\zeta)^{-1}(z_j-\zeta)^{-1}\prod_{\nu=0}^{n}(z_\nu-\zeta)^{-\mu_\nu}d\zeta = \frac{\partial^2 F}{\partial z_i \partial z_j}.\end{aligned}$$

If we combine this with the observation that $\sum_i \frac{\partial F}{\partial z_i} = 0$, we find the desired identity.

The proof that E_V is a dilatation field with factor $1 - \sum_i \mu_i$ is left to the reader. $\square$

Notice that if we take the μ_i's all equal to the same value $\frac{1}{2}\kappa$, then we get the A_n-case of Example 2.5. We shall later prove that this $(n+1)$-dimensional family exhausts the Dunkl systems having an underlying arrangement of type A_n.

Proposition 2.6 implies that locally, the Lauricella functions span a vector space of dimension $\leq n+1$ ($\leq n$ in case $\sum_i \mu_i \neq 1$). We can be more precise:

Proposition **2.7.** — *If $\mu_i < 1$ for all i, then the Lauricella functions span a vector space of dimension $\geq n$. So if $\sum_i \mu_i \neq 1$, then their differentials span the local system of Lauricella-flat 1-forms.*

Proof. — For $i = 1, \dots, n$, we choose a path γ_i from z_0 to z_i such that these paths have disjoint interior. We prove that the corresponding Lauricella functions $F_1, \dots, F_n$ are linearly independent. For this it is enough to show that F_n is not a linear combination of $F_1, \dots, F_{n-1}$. Let $T \subset \mathbf{C}$ be the union of the images of $\gamma_1, \dots, \gamma_n$ minus z_n. We fix $z_1, \dots, z_{n-1}$, but let z_n move along a path $z_n(s)$ in $\mathbf{C} - T$ that eventually follows a ray to infinity. Then $F_i(z_0, \dots, z_{n-1}, z_n(s))$ is for $s \to \infty$ approximately a constant times $z_n(s)^{-\mu_n}$ in case $i \neq n$, and a nonzero constant times $z_n(s)^{1-\mu_n}$ when $i = n$. The assertion follows. $\square$

2.4. *Connections of Dunkl type.* — The examples coming from complex reflection groups and the Lauricella examples suggest:

Definition **2.8.** — *Let $\mathcal{H}$ a finite hyperplane arrangement in the complex vector space V and let for every $H \in \mathcal{H}$ be given a $\rho_H \in \mathrm{End}(V)$ with kernel H such that there exists a positive definite inner product on V for which each ρ_H is selfadjoint (so if π_H denotes the orthogonal projection onto $H^\perp$, then $\rho_H = \kappa_H \pi_H$, where $\kappa_H \in \mathbf{C}$ is the trace of ρ_H). Put $\Omega := \sum_{H\in\mathcal{H}} \omega_H \otimes \rho_H$. If the connection ∇ on the tangent bundle of V° which has $-\Omega$ connection form is flat, then we say that Ω is a* Dunkl form, *∇ is of* Dunkl type *and that the pair (V, ∇) is a* Dunkl system.

So in the complex reflection example we have a connection of Dunkl type and the same is true for the Lauricella example. This last class shows that it is possible for not just the exponent function κ, but also for the Hermitian inner product (and hence the orthogonal projections π_H) to deform continuously in an essential manner while retaining the Dunkl property. We shall see in Subsection 2.6 that for the arrangement of type A_n, any connection of Dunkl type is essentially a Lauricella connection: its connection form is proportional to a Lauricella form.

Example **2.9.** — There are still many examples in dimension two. In order to understand the situation here, let be given a complex vector space V of dimension two and a finite set $\mathcal{H}$ of lines in V which comprises at least three elements.

Suppose that is given an inner product $\langle\, ,\, \rangle$ on V. Choose a defining linear form $\phi_H \in V^*$ for H of unit length relative the dual inner product and let $e_H \in V$ be the unique vector perpendicular to H on which ϕ_H takes the value 1. So e_H is also of unit length. By Proposition 2.3-iv, $\kappa \in (\mathbf{C}^\times)^{\mathcal{H}}$ defines a Dunkl form relative to this inner product if and only if the linear map

$$v \in V \mapsto \sum_{H\in\mathcal{H}} \kappa_H \phi_H(v) e_H \in V$$

commutes with each orthogonal projection π_H. This means that the map is multiplication by a scalar κ_0. Since $\langle v, e_H\rangle = \phi_H(v)$, we can also write this as

$$\sum_{H\in\mathcal{H}} \kappa_H \phi_H(v)\overline{\phi_H(v')} = \kappa_0 \langle v, v'\rangle.$$

This equality remains valid if we replace each coefficient by its real resp. imaginary part. Notice, that if every κ_H is real and positive, then $\kappa_H \phi_H \otimes \overline{\phi_H}$ can be thought of as an inner product on the line V/H.

Conversely, if we are given for every $H\in\mathcal{H}$ an inner product $\langle\, ,\,\rangle_H$ on V/H, and $a_H\in\mathbf{R}$ is such that $\langle\, ,\,\rangle:=\sum_{H\in\mathcal{H}}a_H\langle\, ,\,\rangle_H$ is an inner product on V, then we get a Dunkl system relative the latter with $\kappa_H=a_H\langle v,v\rangle_H/\langle v,v\rangle$ for a generator v of $H^\perp$.

Assumptions **2.10.** — Throughout the rest of this paper we assume that $\mathcal{H}$ is irreducible, that the common intersection of the members of $\mathcal{H}$ is reduced to $\{0\}$ (these are rather innocent) and that the residues ρ_H are nonzero and selfadjoint with respect to some inner product $\langle\, ,\,\rangle$ on V (this is more substantial).

Here are some first observations.

Lemma **2.11.** — *The irreducible components of any member of $\mathcal{L}(\mathcal{H})$ are pairwise perpendicular. In particular, if $L\in\mathcal{L}_{\rm irr}(\mathcal{H})$ and $M\in\mathcal{L}_{\rm irr}(\mathcal{H}^L)-\mathcal{L}_{\rm irr}(\mathcal{H})$, then $M(L)\supset L^\perp$.*

Every $L\in\mathcal{L}_{\rm irr}(\mathcal{H})$ of positive dimension contains a member of $\mathcal{L}_{\rm irr}(\mathcal{H})$ of codimension one in L. In particular there exists a complete flag $V>L_1>L_2>\cdots>L_n=\{0\}$ of irreducible intersections from $\mathcal{H}$.

Proof. — The first assertion amounts to: if $H',H''\in\mathcal{H}$ are distinct and such that their orthogonal complements are not perpendicular, then their intersection L is irreducible. The nonperpendicularity means that $\pi_{H'}$ and $\pi_{H''}$ do not commute and so by property (iv) of Proposition 2.2, $\mathcal{H}_L\neq\{H',H''\}$. Hence L is irreducible.

If all members of $\mathcal{H}-\mathcal{H}_L$ would contain $L^\perp$, then $\mathcal{H}$ would be reducible. So there exists a $H\in\mathcal{H}-\mathcal{H}_L$ which does contain $L^\perp$. It is clear that $L\cap H$ is then irreducible. □

For each linear subspace $L\subset V$ we denote by π_L the orthogonal projection with kernel L and image $L^\perp$. So each residue ρ_H is written as $\kappa_H\pi_H$ for some $\kappa_H\in\mathbf{C}$. The following lemma shows that π_L is independent of the inner product.

Lemma **2.12.** — *Any inner product on V for which each of the ρ_H is selfadjoint is a positive multiple of $\langle\, ,\,\rangle$. (So the Dunkl form $\Omega:=\sum_H\omega_H\otimes\kappa_H\pi_H$ then determines both $\mathcal{H}$ and the inner product up to scalar.)*

Proof. — Suppose $\langle\ ,\ \rangle'$ is another Hermitian form on V for which the residues ρ_H are selfadjoint. Then $\langle\ ,\ \rangle'-c\langle\ ,\ \rangle$ will be degenerate for some $c\in\mathbf{R}$. We prove that this form is identically zero, in other words that its kernel K is all of V. Since ρ_H is selfadjoint for this form, we have

$\rho_H(K) \subset K$. This means that in terms of the old $\langle\ ,\ \rangle$, we either have $K \subset H$ or $K^\perp \subset H$. This distinction splits the arrangement: if $\mathcal{H}' \subset \mathcal{H}$ resp. $\mathcal{H}'' \subset \mathcal{H}$ denote the corresponding subsets, then for every pair $(H', H'') \in \mathcal{H}' \times \mathcal{H}''$, $H'^\perp \perp H''^\perp$. Since $\mathcal{H}$ is irreducible, this implies that either $\mathcal{H}' = \emptyset$ or $\mathcal{H}' = \mathcal{H}$. In the first case K lies in the common intersection of the $H \in \mathcal{H}$ and hence is reduced to $\{0\}$, contrary to our assumption. So we are in the second case: $K^\perp = \{0\}$, that is, $K = V$. □

Lemma **2.13.** — *Let ∇ be a Dunkl connection with residues $\kappa_H \pi_H$ and let $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$. Then the transformation $\sum_{H\in\mathcal{H}_L} \kappa_H \pi_H$ is of the form $\kappa_L \pi_L$, where*

$$\kappa_L = \frac{1}{\operatorname{codim}(L)} \sum_{H\in\mathcal{H}_L} \kappa_H.$$

(In the extremal case $L = V$, which corresponds to an intersection of an empty set of hyperplanes, the righthand side is zero and hence we must have $\kappa_V = 0$.)

Moreover, the Euler vector field E_V is a dilatation field for ∇ with factor $1 - \kappa_0$, so that when $\kappa_0 \neq 1$, the affine structure on V° is in fact a linear structure.

Proof. — It is clear that $\sum_{H\in\mathcal{H}_L} \kappa_H \pi_H$ is zero on L and preserves $L^\perp$. Since this sum commutes with each of its terms, it will preserve H and $H^\perp$, for each $H \in \mathcal{H}_L$. Since $\mathcal{H}_L$ contains $\operatorname{codim}(L) + 1$ members of which each $\operatorname{codim}(L)$-element subset is in general position, the induced transformation in $L^\perp$ will be scalar. This scalar operator must have the same trace as $\sum_{H\in\mathcal{H}_L} \kappa_H \pi_H$, and so the scalar equals the number κ_L above. Since $L^\perp$ is the span of the lines $H^\perp$, $H \in \mathcal{H}_L$, the first part of the lemma follows.

Suppose $H, H' \in \mathcal{H}$ are not perpendicular

The last assertion follows from Proposition 2.2. □

Example **2.14.** — In the Lauricella case a member L of $\mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ is simply given by a subset $I \subset \{0, \dots, n\}$: it is then the set of $z \in V$ for which $z_i - z_j = 0$ when $i, j \in I$. It is straightforward to verify that $\kappa_L = \sum_{i\in I} \mu_i$.

Remark **2.15.** — Later we will see that when $\kappa_H \in (0, 1]$ for all $H \in \mathcal{H}$, $1 - \kappa_0$ can often be understood as the combinatorial curvature of the projectivization $\mathbf{P}(V^\circ)$ of V°. By way of preview, we illustrate this here for the case when $\dim V = 2$. Suppose that $\kappa_H \in (0, 1]$ for all $H \in \mathcal{H}$. Assume that V° comes with an admissible Hermitian form h relative to the Euler field so that h induces on $\mathbf{P}(V^\circ)$ a constant curvature metric. The punctures are indexed by $\mathcal{H}$ and at a puncture p_H, $H \in \mathcal{H}$, the metric has a simple type of singularity described in Example 1.5: it is locally obtained by identifying the sides of a geodesic sector of total angle $2\pi(1-\kappa_H)$. The Gauss-Bonnet theorem (applied for instance to a geodesic triangulation of $\mathbf{P}(V)$ whose vertices include the punctures) says that the curvature integral equals $4\pi - 2\pi\sum_H \kappa_H = 4\pi(1-\kappa_0)$. This also shows that h must be positive definite when $\kappa_0 < 1$, positive with kernel spanned by the Euler field when $\kappa_0 = 1$ and hyperbolic when $\kappa_0 > 1$.

Notation **2.16.** — For $\kappa \in \mathbf{C}^{\mathcal{H}}$, put

$$\nabla^\kappa := \nabla^0 - \Omega^\kappa, \quad \Omega^\kappa := \sum_{H\in\mathcal{H}} \omega_H \otimes \kappa_H \pi_H.$$

Notice that the set of $\kappa \in (\mathbf{C}^\times)^{\mathcal{H}}$ for which ∇^κ is flat is the intersection of a linear subspace of $\mathbf{C}^{\mathcal{H}}$ with $(\mathbf{C}^\times)^{\mathcal{H}}$. We shall denote that subspace by $\mathbf{C}^{\mathcal{H},\mathrm{flat}}$ and for any subset $P \subset \mathbf{C}^{\mathcal{H}}$ we will write P^{flat} for $P \cap \mathbf{C}^{\mathcal{H},\mathrm{flat}}$.

Corollary **2.17.** — *Choose for every $H \in \mathcal{H}$ a unit vector $e_H \in V$ spanning $H^\perp$. Then the connection ∇^κ is flat if and only if for every $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ of codimension two we have*

$$\sum_{H\in\mathcal{H}_L} \kappa_H \langle v, e_H\rangle\langle e_H, v'\rangle = \kappa_L\langle \pi_L(v), \pi_L(v')\rangle$$

for some constant $\kappa_L \in \mathbf{C}$ (which is then necessarily given by the formula of Lemma 2.13). In particular, $\mathbf{C}^{\mathcal{H},\mathrm{flat}}$ is defined over $\mathbf{R}$. Moreover, any $\kappa \in (0,\infty)^{\mathcal{H},\mathrm{flat}}$ is monotonous in the sense that if $L, M \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ and M strictly contains L, then $\kappa_M < \kappa_L$.

Proof. — Lemma 2.13 and condition (iv) of Proposition 2.3 show that the flatness of ∇^κ is equivalent to the condition that for every $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ of codimension two, $\sum_{H\in\mathcal{H}_L} \kappa_H\pi_H$ be proportional to π_L, in other words that $\sum_{H\in\mathcal{H}_L} \kappa_H\langle v, e_H\rangle e_H$ be $\kappa_L\pi_L(v)$ for some $\kappa_L \in \mathbf{C}$. If we take the inner product with $v' \in V$, we see that this comes down to the stated equality. Since the terms $\langle v, e_H\rangle\langle e_H, v'\rangle$ and $\langle \pi_H(v), \pi_H(v')\rangle$ are Hermitian, this equality still holds if we replace the coefficients by their complex conjugates.

Finally, if $\kappa \in (0,\infty)^{\mathcal{H},\mathrm{flat}}$ and $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ then

$$\kappa_L\langle v, v\rangle = \sum_{H\in\mathcal{H}_L} \langle \kappa_H\pi_H(v), v\rangle = \sum_{H\in\mathcal{H}_L} \kappa_H|\langle v, e_H\rangle|^2.$$

If $M \in \mathcal{L}(\mathcal{H})$ strictly contains L, then $\mathcal{H}_L$ strictly contains $\mathcal{H}_M$, and from

$$(\kappa_L - \kappa_M)\langle v, v\rangle = \sum_{H\in\mathcal{H}_L-\mathcal{H}_M} \kappa_H|\langle v, e_H\rangle|^2$$

it follows (upon taking $v \in L^\perp$) that $\kappa_M < \kappa_L$. □

We shall discuss some heriditary properties of Dunkl connections. Therefore, we assume that in the remainder of this subsection Ω is of Dunkl type.

Proposition 2.3 shows that for every $L \in \mathcal{L}(\mathcal{H})$,

$$\Omega_L := \sum_{H\in\mathcal{H}_L} \omega_H \otimes \kappa_H \pi_H$$

defines a Dunkl-connection ∇_L in $(V/L)^\circ$. We shall see that L° also inherits such a connection.

Denote by $i_L : L \subset V$ the inclusion. Notice that if $H \in \mathcal{H} - \mathcal{H}_L$, then $i_L^*(\omega_H)$ is the logarithmic differential on L defined by $L \cap H$.

The set $\mathcal{H}^L$ of hyperplanes in L injects into $\mathcal{L}_{\rm irr}(\mathcal{H})$ by sending I to $I(L)$, the unique irreducible intersection such that $L \cap I(L) = I$ as in Lemma 2.1. Notice that if I is reducible (equivalently, $I(L) \neq I$), then $I(L)$ must be equal to the hyperplane $I + L^\perp$. The set of $I \in \mathcal{H}^L$ for which $I(L) \notin \mathcal{H}$ will be denoted $\mathcal{H}^L_{\rm irr}$ so that $\mathcal{H}^L - \mathcal{H}^L_{\rm irr}$ injects into $\mathcal{H} - \mathcal{H}_L$. It The image of the latter consists of the members of $\mathcal{H}$ which contain $L^\perp$ and we therefore denote that subset of $\mathcal{H}$ by $\mathcal{H}_{L^\perp}$.

Lemma **2.18.** — *Given $L \in \mathcal{L}(\mathcal{H})$, then the connection on the tangent bundle of V restricted to L° defined by*

$$i_L^*(\Omega - \Omega_L) = \sum_{H\in\mathcal{H}-\mathcal{H}_L} i_L^*\omega_H \otimes \kappa_H \pi_H.$$

is flat. Moreover, the decomposition $V = L^\perp \oplus L$ defines a flat splitting of this bundle; on the normal bundle (corresponding to the first summand) the connection is given by the differential $\alpha^L := \sum_{I\in\mathcal{H}^L_{\rm irr}} (\kappa_I - \kappa_L)\omega_I^L$, whereas on the tangent bundle of L (corresponding to the second summand) it is given by the End(L)*-valued* 1*-form*

$$\Omega^L := \sum_{I\in\mathcal{H}^L} \omega_I^L \otimes \kappa_{I(L)} \pi_I^L,$$

where π_I^L denotes the restriction of π_I to L. We thus have a natural affine structure on L° defined by a Dunkl connection ∇^L whose form is defined by restriction of the inner product to L and the function $\kappa^L : I \in \mathcal{H}^L \mapsto \kappa_{I(L)}$. The extension of that function to $\mathcal{L}_{\rm irr}(\mathcal{H}^L)$ (as defined by Lemma 2.13) is given by $M \in \mathcal{L}_{\rm irr}(\mathcal{H}^L) \mapsto \kappa_{M(L)}$.

If $M \in \mathcal{L}_{\rm irr}(\mathcal{H})$ and $L < M$, then $\kappa_M - \kappa_L = \sum_{I\in\mathcal{H}^L_{\rm irr}} (\kappa_I - \kappa_L)$.

Proof. — Let $M \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^L)$. We verify that $\sum_{H\in\mathcal{H}_M-\mathcal{H}_L} \kappa_H\pi_H$ commutes with π_L and that its restriction to L equals $\kappa_{M(L)}\pi_{M(L)}$. We first notice that

$$\sum_{H\in\mathcal{H}_M-\mathcal{H}_L} \kappa_H\pi_H = \begin{cases} \kappa_M\pi_M - \kappa_L\pi_L & \text{if } M \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}) \text{ irreducible,} \\ \kappa_{M(L)}\pi_{M(L)} & \text{otherwise.} \end{cases}$$

It is clear that the right-hand side restricted to L is $\kappa_{M(L)}\pi_M^L$ (for $M(L) = L$ in case M is irreducible) and restricted to $L^\perp$ the scalar $\kappa_L - \kappa_M$ if M is irreducible and zero otherwise (for then $M(L) \supset L^\perp$ by Lemma 2.11). On the other hand, by grouping the members of $\mathcal{H}_M - \mathcal{H}_L$ according to their intersection with L, we see that the same reasoning yields

$$\sum_{H\in\mathcal{H}_M-\mathcal{H}_L} \kappa_H\pi_H = \sum_{I\in\mathcal{H}_M^L} \sum_{H\in\mathcal{H}_I-\mathcal{H}_L} \kappa_H\pi_H = \sum_{I\in\mathcal{H}^L_{\mathrm{irr},M}} (\kappa_I\pi_I - \pi_L\kappa_L) + \sum_{H\in\mathcal{H}_{L^\perp}} \kappa_H\pi_H.$$

The restriction of this identity to L yields $\kappa_{M(L)}\pi_M^L = \sum_{I\in\mathcal{H}_M^L} \kappa_{I(L)}\pi_I^L$ and its restriction to $L^\perp$ yields for irreducible M the identity of scalars

$$\kappa_M - \kappa_L = \sum_{I\in\mathcal{H}^L_{\mathrm{irr},M}} (\kappa_I - \kappa_L)$$

(this restriction does not yield anything of interest in case M is reducible, for then the lefthand side becomes zero and $\mathcal{H}^L_{\mathrm{irr},M} = \emptyset$). This proves the last assertions of the lemma.

For the flatness of ∇^L we invoke criterion (iii) of Proposition 2.3: if $M, N \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^L)$ satisfy an inclusion relation, then it follows from the above, that the sums $\sum_{H\in\mathcal{H}_M-\mathcal{H}_L} \kappa_H\pi_H$ and $\sum_{H\in\mathcal{H}_N-\mathcal{H}_L} \kappa_H\pi_H$ commute and the flatness follows from this.

If we let M run over the members of $\mathcal{H}^L$ we get

$$\Omega - \Omega_L = \sum_{I\in\mathcal{H}^L_{\mathrm{irr}}} \omega_I^L \otimes \kappa_I\pi_I + \sum_{H\in\mathcal{H}_{L^\perp}} \omega_H \otimes \kappa_H\pi_H.$$

Since all the terms commute with π_L it follows that π_L is flat, when viewed as an endomorphism of the tangent bundle of V restricted to L. It also follows that the components of the connection are as asserted. □

Remarks **2.19.** — The last property mentioned in the lemma above turns out to impose a very strong condition on κ when viewed as a function on the poset $\mathcal{L}_{\mathrm{irr}}(\mathcal{H})$. Let us say that a function $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}) \mapsto \lambda_L \in \mathbf{C}$ has the *weight property* if for every $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$, $\mathrm{codim}(L)\lambda_L = \sum_{H\in\mathcal{H}_L} \lambda_H$. If we define $\lambda_L : \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^L) \to \mathbf{C}$ by $M \mapsto \lambda_{M(L)}$, then λ_L has the weight property precisely if $\lambda_M - \lambda_L = \sum_{I\in\mathcal{H}^L_{\mathrm{irr}}}(\lambda_I - \lambda_L)$. It turns out that this condition yields all the possible weights for Coxeter arrangements of rank at least three. We we will not pursue this here, since we will obtain this classification by a different method in Subsection 2.6.

The form restriction of $\Omega_{L^\perp}$ to L°, $\sum_{I\in\mathcal{H}_{L^\perp}} \omega^L_I \otimes \pi^L_I$, defines flat connection on L°. So does Ω^L and hence so does also every linear combination of these forms. In particular, their difference $\Omega^L_{\mathrm{irr}} := \sum_{I\in\mathcal{H}^L_{\mathrm{irr}}} \omega^L_I \otimes \pi^L_I$ defines a flat connection on L°.

Definition **2.20.** — *The Dunkl connection on $(V/L)^\circ$ resp. L° defined by Ω_L resp. Ω^L is called the* L-transversal *resp.* L-longitudinal *Dunkl connection.*

2.5. *Local triviality.* — One would perhaps hope the affine structure on V° to be locally trivial along the strata in the sense that for $L \in \mathcal{L}(\mathcal{H})$, there exists a decomposition $V_{L^\circ} \cong L^\circ \times (V/L)_0$ in the affine category. This is not always true (a case in point is when $\kappa_L = 1 \neq \kappa_0$), but in this subsection we give some sufficient conditions in order that this property (locally) holds. We analyze the affine structure near a point of L° by first blowing up L in V and then look at the behavior of the connection near the exceptional divisor.

Lemma **2.21.** — *Let $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ and denote by D the exceptional divisor of the blow-up of L in V. Then the affine structure on V° is of infinitesimally simple type along D° (in the sense of Definition 1.9) with logarithmic exponent $\kappa_L - 1$.*

When $\kappa_L \neq 0$, the natural local decomposition of D° of Corollary 1.14 is in fact globally defined: for $\kappa_L = 1$ it is the trivial decomposition and it is the obvious product decomposition $D^\circ = L^\circ \times \mathbf{P}(V/L)^\circ$ otherwise.

If $\kappa_L = 1 \neq \kappa_0$, then the degeneration is not simple and the associated codimension one foliation of D° (observed in Corollary 1.14) is the pull-back of one of L°.

Proof. — Let $\tilde p \in D^\circ$ lie over $p \in L^\circ$. We evaluate ∇ on two types of differentials: those of the form dy with $y \in V^*$ such that $y|L^\perp = 0$ and those of the form $x^{-1}dx$ with $x \in (V/L)^*$ and $\tilde p$ is not contained in the strict transform of the zero set of x (hence x is a local equation of $D_{\tilde p}$). A differential dy of the first type is regular in $\tilde p$ its form restriction to $D = L \times \mathbf{P}(V/L)$ is the pull-back of the form restriction of dy to L. A differential of the second type yields on $\mathrm{Bl}_L V = L \times \mathrm{Bl}_0(V/L)$ the pull-back of a logarithmic form on $\mathrm{Bl}_0(V/L)$ with residue constant 1 on the exceptional divisor $\mathbf{P}(V/L)$. Together these differentials generate $\Omega_{\mathrm{Bl}_L V,\tilde p}$.

The first type is easily dealt with: for $y \in (V/L^\perp)^*$ as above, we have $\pi_H^* dy = 0$ when $H \in \mathcal{H}_L$ and hence

$$\nabla(dy) = \sum_{H \in \mathcal{H} - \mathcal{H}_L} \omega_H \otimes \kappa_H \pi_H^*(dy).$$

Since ω_H is regular at $\tilde{p}$ when $H \in \mathcal{H} - \mathcal{H}_L$, this expression lies in $\Omega_{\mathrm{Bl}_L V, \tilde{p}} \otimes \Omega_{\mathrm{Bl}_L V, \tilde{p}}$ and in particular, $\mathrm{Res}_D(\nabla)(dy) = 0$. (The form restriction of $\nabla(dy)$ to D is in fact $\Omega^L(dy)$, the pull-back of longitudinal connection applied to dy.) A form of the second type requires more work.

Assertion: If we identify α^L (as defined in Lemma 2.18) with its pull-back along the orthogonal projection $1 - \pi_L : V \to L$, then

$$\nabla\left(\frac{dx}{x}\right) \in (\kappa_L - 1)\frac{dx}{x} \otimes \frac{dx}{x} + \frac{dx}{x} \otimes \alpha^L + \mathcal{I}_{D,\tilde{p}} \Omega_{\mathrm{Bl}_L V, \tilde{p}}(\log D) \otimes \Omega_{\mathrm{Bl}_L V, \tilde{p}}(\log D).$$

This assertion will complete the proof of the lemma: it follows that ∇ has a logarithmic singularity on $\Omega_{\mathrm{Bl}_L V, \tilde{p}}(\log D)$ and that its residue operator has the stated properties; for the case $\kappa_L = 1 \neq \kappa_0$ we invoke the last clause of Lemma 2.18 which implies that then $\kappa_L \neq \kappa_I$ for some $I \in \mathcal{H}^L_{\mathrm{irr}}$ so that $\alpha^L \neq 0$.

Let us prove the assertion. We compute modulo the $\mathcal{O}_{\mathrm{Bl}_L V, \tilde{p}}$-module

$$\mathcal{M} := \mathcal{I}_{D,\tilde{p}} \Omega_{\mathrm{Bl}_L V, \tilde{p}}(\log D) \otimes \Omega_{\mathrm{Bl}_L V, \tilde{p}}(\log D).$$

We have

$$\nabla\left(\frac{dx}{x}\right) = -\frac{dx}{x} \otimes \frac{dx}{x} + \sum_{H \in \mathcal{H}} \omega_H \otimes \frac{\kappa_H \pi_H^*(dx)}{x}.$$

We first consider the subsum over $\mathcal{H}_L$. If $H \in \mathcal{H}_L$, then $\omega_H - x^{-1}dx$ is regular at $\tilde{p}$ and so

$$\sum_{H \in \mathcal{H}_L} \omega_H \otimes \frac{\kappa_H \pi_H^*(dx)}{x} \equiv \sum_{H \in \mathcal{H}_L} \frac{dx}{x} \otimes \frac{\kappa_H \pi_H^*(dx)}{x} \equiv \kappa_L \frac{dx}{x} \otimes \frac{dx}{x} \pmod{\mathcal{M}}.$$

We group the members of $\mathcal{H} - \mathcal{H}_L$ according to their intersection with L and so we fix $I \in \mathcal{H}^L$. For $H \in \mathcal{H}_I - \mathcal{H}_L$, ω_H is regular at p, and so it is the factor x^{-1} that we have to worry about. Choose $\phi_I \in V^*$ such that $I + L^\perp$ is its zero hyperplane (so that ω_I^L, when identified with its pull-back along the orthogonal projection $V \to L$ equals $\phi_I^{-1} d\phi_I$). Any $H \in \mathcal{H}_I - \mathcal{H}_L$ has a unique defining linear form ϕ_H which has the same restriction to L as ϕ_I.

This implies that $(\phi_H/\phi_I) - 1 \in x\mathcal{O}_{\mathrm{Bl}_L V, \tilde p}$ and $d\phi_H - d\phi_I \in x\Omega_{\mathrm{Bl}_L V,\tilde p}(\log D)$. Let $e_H \in H^\perp$ be such that $\phi_H(e_H) = 1$, so that $\pi_H(z) = \phi_H(z)e_H$. Then

$$\omega_H \otimes \frac{\pi_H^* dx}{x} = \frac{d\phi_H}{\phi_H} \otimes \frac{x(e_H)d\phi_H}{x} \equiv$$
$$\equiv \frac{d\phi_H}{\phi_I} \otimes \frac{x(e_H)d\phi_I}{x} = \frac{\pi_H^*(dx)}{x} \otimes \frac{d\phi_I}{\phi_I} \pmod{\mathcal{M}}.$$

So if we multiply this congruence by κ_H, sum over $H \in \mathcal{H}_I - \mathcal{H}_L$ and keep in mind that $\sum_{H\in\mathcal{H}_I-\mathcal{H}_L} \kappa_H\pi_H$ equals $\kappa_I\pi_I - \kappa_L\pi_L$ or $\kappa_{I(L)}\pi_{I(L)}$, depending on whether or not I is irreducible (in the second case $I(L)$ is the hyperplane $I + L^\perp$), we find that

$$\sum_{H\in\mathcal{H}_I-\mathcal{H}_L} \omega_H \otimes \kappa_H \frac{\pi_H^*(dx)}{x} \equiv \begin{cases} (\kappa_L - \kappa_I)\frac{dx}{x} \otimes \frac{d\phi_I}{\phi_I} \pmod{\mathcal{M}} & \text{if } I \in \mathcal{H}^L_{\mathrm{irr}}, \\ 0 \pmod{\mathcal{M}} & \text{otherwise.} \end{cases}$$

If we now sum over $I \in \mathcal{H}^L$, the assertion follows and hence so does the lemma. □

The following corollary restates part of this lemma in more intrinsic terms.

Corollary **2.22.** — *Let $L \in \mathcal{L}_{\mathrm{irr}}$. If κ_L is not an integer ≤ 0, then there is a natural affine submersion $\mathrm{Bl}_L V_{L^\circ\times\mathbf{P}(V/L)^\circ} \to L^\circ$ which on the exceptional divisor extends the projection; if κ_L is not an integer ≥ 1, then there is a natural affine submersion $\mathrm{Bl}_L V_{L^\circ\times\mathbf{P}(V/L)^\circ} \to \mathrm{Bl}_0(V/L)_{\mathbf{P}(V/L)^\circ}$. (So if $\kappa_L \notin \mathbf{Z}$, then the two submersions define an affine decomposition of $\mathrm{Bl}_L V_{L^\circ\times\mathbf{P}(V/L)^\circ}$.)*

Proof. — We only prove the first half; the proof of the second half is similar. According to Lemma 2.21, the conditions (and hence the conclusions) of Proposition 1.10 and Corollary 1.14 are satisfied in the generic point of $\mathrm{Bl}_L V$ with $\lambda = \kappa_L - 1$. So there is a natural affine morphism

$$r : (\mathrm{Bl}_L V)_{L^\circ\times\mathbf{P}(V/L)^\circ} \to L^\circ$$

which extends the projection $L^\circ \times \mathbf{P}(V/L)^\circ \to L^\circ$. □

If we apply the preceding to $L = H \in \mathcal{H}$, then we find that the affine structure on V° degenerates infinitesimally simply along every $H \in \mathcal{H}$ (with logarithmic exponent $\kappa_H - 1$) and that we have a natural affine retraction $V_{H^\circ} \to H^\circ$ when κ_H is not at an integer ≤ 0 (which extends to a natural affine decomposition $V_{H^\circ} \cong H^\circ \times (V/H)_0$ when $\kappa_H \notin \mathbf{Z}$). Lemma 1.7 also implies that H° acquires an affine structure. This is of course the same affine structure that we found in Lemma 2.18. Here is a simple application.

Corollary **2.23.** — *If $\kappa_H \notin \mathbf{Z}$ for all $H \in \mathbf{Z}$ and κ_0 is not an integer ≤ 0, then every flat 1-form on V° is zero.*

Proof. — Let α be a flat 1-form on V°. Since the Dunkl connection is torsion free, α is closed. Let us verify that under the assumptions of the statement, α is regular in the generic point of $H \in \mathcal{H}$. Near the generic point of H, α is a linear combination of the pull-back of a differential on the generic point of H under the canonical retraction and a differential which is like $\phi^{-\kappa_H} d\phi$, where ϕ is a local defining equation for H. So if the latter appears in α with nonzero coefficient, then κ_H must be an integer and this we excluded. So α is regular in the generic point of H.

Hence α is regular on all of V. On the other hand, α will be homogeneous of degree $1-\kappa_0$. So if α is nonzero, then $1-\kappa_0$ is a positive integer. But this we excluded also. □

Remark **2.24.** — If V° has a nonzero Hermitian form h which is flat relative to ∇^κ, and $L \in \mathcal{L}(\mathcal{H})$, then such a form is often inherited by the transversal and longitudinal system associated to L. For instance, if L is irreducible and such that κ_L is not an integer, then the monodromy around L has the two distinct eigenvalues 1 and $\exp(2\pi\sqrt{-1}\kappa_L)$. These decompose the tangent space of a point near L° into two eigenspaces. This decomposition is orthogonal relative to h, since the latter is preserved by the monodromy. Both decompositions are flat and hence are integrable to foliations. It follows that the transversal system on V/L and the longitudinal system on L inherit from h a flat form. (But we cannot exclude the possibility that one of these is identically zero.)

The following will only be needed as of Subsection 5.4. Suppose we are given a flag $L_0 > \cdots > L_k > L_{k+1} = V$ in $\mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ and let $W \to V$ be the iterated blowup of these subspaces in the correct order: starting with L_0 and ending with L_k. Denote the exceptional divisor over L_i by E_i, so that the E_i's make up a normal crossing divisor. The common intersection S of the E_i's has a product decomposition

$$S \cong L_0 \times \mathbf{P}(L_1/L_0) \times \cdots \times \mathbf{P}(V/L_k).$$

We will abbreviate κ_{L_i} by κ_i.

Proposition **2.25.** — *Assume that $k \geq 1$, that no κ_i is an integer ≤ 0, that for $i = 1, \ldots k$ we have semisimple holonomy around L_i (so that $\kappa_i \neq 1$ for $i = 1, \ldots k$) and that either we have semisimple holonomy around L_{k+1} or that $\kappa_{k+1} = 1$. Let $z = (z_0, \ldots, z_{k+1})$ be a general point of S. Then there* *exists a local equation* t_i *for* E_i *(*$i = 1, \dots k$*) and a morphism* $F_i : W_z \to (T_i)_0$ *to a linear space germ (*$i = 1, \dots k+1$*) such that* $F_i|S_z$ *factors through an isomorphism* $\mathbf{P}(L_i/L_{i-1})_{z_i} \cong (T_i)_0$ *(and so if* $F_0 : W \to L_0$ *denotes the projection, then* $(F_0, t_0, F_1, \dots, t_k, F_{k+1})$ *is a chart for* W_z*), and the developing map at* z *is affine equivalent to the multivalued map*

$$\left(F_0, \left(t_0^{1-\kappa_0} t_1^{1-\kappa_1} \cdots t_{i-1}^{1-\kappa_{i-1}}(1, F_i)\right)_{i=1}^{k+1}\right),$$

resp.

$$\left(F_0, \left(t_0^{1-\kappa_0} t_1^{1-\kappa_1} \cdots t_{i-1}^{1-\kappa_{i-1}}(1, F_i)\right)_{i=1}^{k}, t_0^{1-\kappa_0} t_1^{1-\kappa_1} \cdots t_{k-1}^{1-\kappa_{k-1}}\left(\log t_k, F_{k+1}\right)\right).$$

Proof. — This follows from Lemma 2.21 and Proposition 1.15. To see this, we notice that the formation of the affine quotient of E_0 is its projection to L_0, hence defined everywhere on E_0. Likewise, the formation of the affine quotient of E_i is defined away from the union $\cup_{j<i} E_j$ of exceptional divisors of previous blowups and given by the projection $E_i - \cup_{j<i} E_j \to L_i - L_{i-1}$. (But notice that in Proposition 1.15 the notation is slightly different: the factor T_k in that proposition is here the product $T_k \times T_{k+1}$.) ☐

Remark **2.26.** — This proposition can also be used to strengthen part of Corollary 2.22: if for every $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}) - \{V\}$, $\kappa_L \notin \mathbf{Z}_{\leq 0}$ and the holonomy around L is semisimple when $\kappa_L \neq 0$, then for every $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ there is a natural affine retraction $r_L : V_{L^\circ} \to L^\circ$ such that these retractions satisfy the expected transitivity property: for every pair $L < M$ in $\mathcal{L}_{\mathrm{irr}}(\mathcal{H})$, $r_L r_M$ equals r_M wherever that makes sense. (The semisimplicity assumption is in fact superfluous.)

2.6. *Classification of Dunkl forms for reflection arrangements.* — Let be given be a complex vector space V in which acts a finite complex irreducible reflection group $G \subset \mathrm{GL}(V)$. We suppose that the action is essential so that $V^G = \{0\}$. Let $\mathcal{H}$ be the collection of reflecting hyperplanes of G in V. We want to describe the space of Dunkl connections on V°, where we regard the inner product as unknown. So we wish to classify the pairs $(\langle\, , \,\rangle, \kappa)$, where $\langle\, , \,\rangle$ is an inner product on V and $\kappa \in \mathbf{C}^{\mathcal{H}}$ is such that $\sum_{H\in\mathcal{H}} \omega_H \otimes \kappa_H \pi_H$ is a Dunkl form (with π_H being the projection with kernel H that is orthogonal relative to $\langle\, , \,\rangle$). We shall see that in case G is irreducible of rank ≥ 3, any such Dunkl system is G-invariant unless G is of type A or B; for these two series we also get (a variant of) the Lauricella forms.

We begin with a lemma.

Lemma **2.27.** — *Let V be a complex inner product space of dimension two and let $\mathcal{H}$ be a collection of lines in V.*

($A_1{}^2$) If $\mathcal{H}$ consists of two distinct elements, then a compatible Dunkl system exists if and only if the lines are perpendicular.

(A_2) If $\mathcal{H}$ consists of three distinct elements, then a compatible Dunkl form exists if and only if the corresponding three points in $\mathbf{P}(V)$ lie on a geodesic (with respect to the Fubini-Study metric). Such a form is unique up to scalar.

(B_2) Let (ϕ_1, ϕ_2) be a basis of V^* such that $\mathcal{H}$ consists of the lines H_1, H_2, H', H'' defined by the linear forms $\phi_1, \phi_2, \phi' := \phi_1+\phi_2, \phi'' := \phi_1-\phi_2$. Suppose that $\langle\, ,\,\rangle$ is an inner product on V for which H_1 and H_2 are perpendicular. Let μ_i be the square norm of ϕ_i relative to the inverse inner product on V^*. Then for every system $(\kappa_1, \kappa_2, \kappa', \kappa'')$ of exponents of a compatible Dunkl system there exist $a, b \in \mathbf{C}$ such that $\kappa' = \kappa'' = b(\mu_1 + \mu_2)$ and $\kappa_i = a + 2b\mu_i$ for $i = 1, 2$.

Proof. — The proofs are simple calculations. The first statement is easy and left to the reader. To prove the second: let H_1, H_2, H_3 be the three members of $\mathcal{H}$. Choose a defining linear form $\phi_i \in V^*$ for H_i in such a way that $\phi_1+\phi_2+\phi_3 = 0$. The triple (ϕ_1, ϕ_2, ϕ_3) is then defined up to a common scalar factor. Let $V(\mathbf{R})$ be the set of v on which each ϕ_i is $\mathbf{R}$-valued. This is a real form of V and the image P of $V(\mathbf{R})-\{0\}$ in $\mathbf{P}(V)$ is the unique real projective line which contains the three points defined by H_i's. The functions $\phi_1^2, \phi_2^2, \phi_3^2$ form a basis of the space of quadratic forms on V and so if $\langle\, ,\,\rangle$ is an inner product on V, then its real part restricted to $V(\mathbf{R})$ is the restriction of $\sum_i a_i\phi_i^2$ for unique $a_i \in \mathbf{R}$. Then P is a geodesic for the associated Fubini-Study metric on $\mathbf{P}(V)$ if and only if complex conjugation with respect to $V(\mathbf{R})$ interchanges the arguments of the inner product. The latter just means that $\langle\, ,\,\rangle = \sum_i a_i\phi_i \otimes \overline{\phi_i}$. According to Example 2.9 this is equivalent to: $\langle\, ,\,\rangle$ is part of a Dunkl system with $\kappa_i = a_i|\phi_i(v)|^2/\langle v, v\rangle$, where v is a generator of $H_i^\perp$ (and any other triple $(\kappa_1, \kappa_2, \kappa_3)$ is necessarily proportional to this one).

To prove the last statement, let (e_1, e_2) be the basis of V dual to (ϕ_1, ϕ_2). Since $e_1 \pm e_2$ has square length $\mu_1^{-1} + \mu_2^{-1}$, a quadruple $(\kappa_1, \kappa_2, \kappa', \kappa'')$ is a system of exponents if and only if there exist a $\lambda \in \mathbf{C}$ such for all $v \in V$:

$$\lambda v = \mu_1\kappa_1\langle v, e_1\rangle e_1 + \mu_2\kappa_2\langle v, e_2\rangle e_2 + \kappa'\frac{\mu_1\mu_2}{\mu_1+\mu_2}\langle v, e_1+e_2\rangle(e_1+e_2) + \kappa''\frac{\mu_1\mu_2}{\mu_1+\mu_2}\langle v, e_1-e_2\rangle(e_1-e_2).$$

Subsituting e_1 and e_2 for v shows that this amounts to:

$$\kappa' = \kappa'', \quad \lambda = \kappa_1 + \frac{\mu_2(\kappa' + \kappa'')}{\mu_1 + \mu_2} = \kappa_2 + \frac{\mu_1(\kappa' + \kappa'')}{\mu_1 + \mu_2}.$$

Now put $b := \kappa'(\mu_1+\mu_2)^{-1} = \kappa''(\mu_1+\mu_2)^{-1}$ so that $\kappa_1 + 2b\mu_2 = \kappa_2 + 2b\mu_1$. The assertion follows with $a := \kappa_1 - 2b\mu_1 = \kappa_2 - 2b\mu_2$. □

Remark **2.28.** — This lemma derives part of its power from the following observation. If $\mathcal{H}$ is the collection of reflection hyperplanes of a finite irreducible Coxeter group of rank ≥ 3, then the equivalence relation on $\mathcal{H}$ generated by '$H_1 \cap H_2$ defines a A_2-subsystem' (in the sense that there are precisely three members of $\mathcal{H}$ containing $H_1 \cap H_2$) has at most two equivalence classes, with two only occurring for the types B_n and F_4. This follows from the classification.

Recall that on A_n, we have the Lauricella systems: for positive real $\mu_0, \dots, \mu_n$ we define an inner product $\langle\, , \rangle$ on $\mathbf{C}^{n+1}$ by $\langle e_i, e_j \rangle = \mu_i \delta_{i,j}$ and the hyperplanes $H_{i,j} = (z_i = z_j)$, $0 \leq i < j \leq n$, restricted to the orthogonal complement $V = (\sum_i \mu_i z_i = 0)$ of the main diagonal, then make up a Dunkl system with $\kappa_{i,j} = \mu_i + \mu_j$. Is is convenient to switch to $\phi_i := \mu_i z_i$ so that $\sum_i \phi_i$ vanishes on V and each n-element subset of $\{\phi_0, \dots, \phi_n\}$ is a coordinate system. The group G permutes the ϕ_i's (it is the full permutation group on them) and the inner product is now $\sum_i \mu_i^{-1} \phi_i \otimes \overline{\phi_i}$. There are choices for the μ_i's that are not all positive for which $\sum_i \mu_i^{-1} \phi_i \otimes \overline{\phi_i}$ is nevertheless positive definite on V. We then still have a Dunkl system and in what follows we shall include such cases when we refer to the term *Lauricella system.*

Proposition **2.29.** — *If G is of type A_n, $n \geq 2$, then any Dunkl form is proportional to a Lauricella form.*

Proof. — For the case $n = 2$, it easily follows from Lemma 2.27 that the Lauricella systems exhaust all examples. So assume $n \geq 3$ and consider the space $\mathbf{H}(V)$ of Hermitian forms on V and regard it as a real representation of $G = \mathcal{S}_{n+1}$. Its decomposition into its irreducible subrepresentations has three summands: one trivial representation, one isomorphic to the natural real form of V, and another indexed by the numerical partition $(n-1, 2)$ of $n+1$. By Lemma 2.27, an inner product underlying a Dunkl system must have the property that the summands of every $A_1 \times A_1$-subsystem are orthogonal. The Hermitian forms on V with this property make up a subrepresentation of $\mathbf{H}(V)$; it is in fact the sum of the trivial representation and the one isomorphic to V and consists of the forms $\sum_{i=0}^n c_i |\phi_i|^2$ with $c_i \in \mathbf{R}$

restricted to the hyperplane $\sum_{i=0}^n \phi_i = 0$. The inner products in this subset are those of Lauricella type (with $\mu_i = c_i^{-1}$). According to Lemma 2.27 such an inner product determines κ on every A_2-subsystem up to scalar. Hence it determines κ globally up to scalar. This implies that the Dunkl form is proportional to one of Lauricella type. □

Let now G be of type B_n with $n \geq 3$. We use the standard set of positive roots: in terms of the basis $e_1, \dots, e_n$ of $\mathbf{C}^n$ these are the basis elements themselves $e_1, \dots, e_n$ and the $e_i \pm e_j$, $1 \leq i < j \leq n$.

Proposition **2.30.** — *Let $\mu_1, \dots, \mu_n$ be positive real numbers and let $a \in \mathbf{C}$. Then relative to this hyperplane system of type B_n and the inner product defined by $\langle e_i, e_j\rangle = \mu_i^{-1}\delta_{i,j}$, the exponents $\kappa_{i,\pm j} := \mu_i + \mu_j$, $\kappa_i := a + 2\mu_i$ define a Dunkl form. In this case, $\kappa_0 = a + 2\sum_i \kappa_i$. Any Dunkl form is proportional to one of this kind for certain $\mu_1, \dots, \mu_n; a$. In particular, it is always invariant under reflection in the mirrors of the short roots.*

Proof. — The Dunkl property is verified for the given data by means of Proposition 2.3-iv and the computation of κ_0 is straightforward.

Suppose now that we are given a Dunkl form defined by the inner product $\langle\, ,\,\rangle$ and the system $(\kappa_i, \kappa_{i,\pm j})$. For $1 \leq i < j < n$ and $\varepsilon \in \{1, -1\}$ the hyperplanes $z_i + \varepsilon z_j = 0$ and $z_n = 0$ make up a $A_1 \times A_1$ system that is saturated (i.e., not contained in a larger system of rank two). So these hyperplanes are orthogonal. By letting i and j vary, we find that $\langle e_i, e_n\rangle = 0$ for all $i < n$. This generalizes to: $\langle e_i, e_j\rangle = 0$ when $i \neq j$. Hence the inner product has the stated form. For every pair of indices $1 \leq i < j \leq n$ we have a subsystem of type B_2 with positive roots $e_i, e_j, e_i \pm e_j$. We can apply 2.27-iii to that subsystem and find that there exist $a_{ij}, b_{ij} \in \mathbf{C}$ such that $\kappa_{i,j} = \kappa_{i,-j} = b_{ij}(\mu_i + \mu_j)$ and $\kappa_i = a_{ij} + 2b_{ij}\mu_i$ and $\kappa_j = a_{ij} + 2b_{ij}\mu_j$. It remains to show that both a_{ij} and b_{ij} do not depend on their indices. For the b_{ij}'s this follows by considering a subsystem of type A_2 defined by $z_1 = z_2 = z_3$: our treatment of that case implies that we must have $b_{12} = b_{13} = b_{23}$ and this generalizes to arbitrary index pairs. If we denote the common value of the b_{ij} by b, then we find that $a_{ij} = \kappa_i - 2b\mu_i = \kappa_j - 2b\mu_j$. This implies that a_{ij} is also independent of its indices. □

Corollary **2.31.** — *A Dunkl system of type B_n in $\mathbf{C}^n$, $n \geq 3$, has A_1^n-symmetry and the quotient by this group is a Dunkl system of type A_n. If the parameters of B_n-system (as in Proposition 2.30) are given by $(\mu_0, \dots, \mu_n; a)$, then those of the quotient A_n-system are $(\mu_0, \mu_1 \dots, \mu_n)$ with $\mu_0 = \frac{1}{2}(a + 1)$.*

Proof. — The quotient of the Dunkl connection by the symmetry group in question will be a flat connection on $\mathbf{C}^n$ with logarithmic poles and is $\mathbf{C}^\times$-invariant. So by Proposition 2.2, its the connection form has the shape $\sum_{H\in\mathcal{H}}\omega_H\otimes\rho_H$, with ρ_H a linear map. A little computation shows that the nonzero eigenspace of $\rho_{(z_i-z_j=0)}$ is spanned by e_i-e_j with eigenvalue μ_i+u_j. □

Remark **2.32.** — A B_n-arrangement appears in a A_{2n}-arrangement as the restriction to a linear subspace not contained in a A_{2n}-hyperplane as follows. Index the standard basis of $\mathbf{C}^{2n+1}$ by the integers from $-n$ through n: $e_{-n},\dots,e_n$, and let V be the hyperplane in $\mathbf{C}^{2n}$ defined by $\sum_{i=-n}^n z_i=0$. An arrangement $\mathcal{H}$ of type A_{2n} in V is given by the hyperplanes in V defined by $z_i=z_j$, $-n\le i<j\le n$. The involution ι of $\mathbf{C}^{2n+1}$ which interchanges e_{-i} and $-e_i$ (and so sends e_0 to $-e_0$) leaves V and the arrangement invariant; its fixed point subspace in V is parametrized by $\mathbf{C}^n$ by: $(w_1,\dots,w_n)\mapsto(-w_n,\dots,-w_1,0,w_1,\dots,w_n)$. The members of $\mathcal{H}$ meet V^ι as follows: for $1\le i<j\le n$, $w_i=w_j$ is the trace of the $A_1\times A_1$-subsystem $\{z_i=z_j,z_{-i}=z_{-j}\}$ on V^ι, likewise $w_i=-w_j$ is the trace for $\{z_i=z_{-j},z_{-i}=z_j\}$, and $w_i=0$ is the trace of the A_2-system $z_{-i}=z_i=z_0$. This shows that $\mathcal{H}|V^\iota$ is of type B_n. Suppose that we are given a Dunkl form on V which is invariant under ι. This implies that V° contains $V^\circ\cap V^\iota$ as a flat subspace, so that the Dunkl connection on V induces one on V^ι. The values of κ on the hyperplanes of V^ι are easily determined: since the inner product on V comes from an inner product on $\mathbf{C}^{2n}$ in diagonal form: $\langle e_i,e_j\rangle=\mu_i^{-1}\delta_{i,j}$ for certain positive numbers $\mu_{\pm i}$, $i=1,\dots,n$, we must have $\mu_{-i}=\mu_i$. Up to scalar factor we have $\kappa_{(z_i=z_j)}=\mu_i+\mu_j$ for $-n\le i<j\le n$. So with that proviso, $\kappa_{(w_i\pm w_j=0)}=\mu_i+\mu_j$, $1\le i<j\le n$ and $\kappa_{(w_i=0)}=2\mu_i+\mu_0$, which shows that we get the Dunkl form described in Proposition 2.30 with $a=\mu_0$.

We complete our discussion of the Coxeter case with

Proposition **2.33.** — *Suppose that G is a finite, irreducible Coxeter group of rank ≥ 3 which is not of type A or B. Then every Dunkl system with the reflecting hyperplanes of G as its polar arrangement is G-invariant.*

We shall see in Subsection 3.5 that the local system associated to such a Dunkl system can be explicitly described in terms of the Hecke algebra of G.

We first prove:

Lemma **2.34.** — *If the finite Coxeter group G contains a reflection subgroup of type D_4, but not one of type B_4, then any Dunkl form relative to $\mathcal{H}$ is necessarily G-invariant.*

Proof. — We prove this with induction on the dimension of V. To start this off, let us first assume that G is of type D_4. We use the standard root basis $(e_1-e_2, e_2-e_3, e_3-e_4, e_3+e_4)$ from [3]. The four roots $\{e_1 \pm e_2, e_3 \pm e_4\}$ define a subsystem of type $(A_1)^4$. So by the first clause of Lemma 2.27, these roots are mutually perpendicular: the inner product on V has the shape

$$\langle v, v\rangle = a|v_1 - v_2|^2 + b|v_1 + v_2|^2 + c|v_3 - v_4|^2 + d|v_3 + v_4|^2$$

for certain positive a, b, c, d. Any $g \in G$ sends a $(A_1)^4$-subsystem to another such, and so must transform $\langle\, ,\, \rangle$ into a form of the same type (with possibly different constants $a, \dots, d$). From this we easily see that $a = b = c = d$, so that $\langle v, v\rangle = a\sum_i |v_i|^2$. This form is G-invariant. If we apply 2.27 to any subsystem of type A_2, we find that κ is constant on such subsystem. Since $\mathcal{H}$ is connected by its A_2-subsystems, it follows that κ is constant.

In the general case, let $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ be such that its normal system contains a system of type D_4. By our induction hypothesis, the Dunkl system transversal to L is invariant under the subgroup of $g \in G$ which stabilizes L pointwise. An inner product is already determined by its restriction to three distinct hyperplanes; since we have at least three such L, it follows that the inner product is G-invariant. The A_2-connectivity (Remark 2.28) of $\mathcal{H}$ implies that κ is constant. □

Proof of Proposition 2.33. — By Lemma 2.34 this is so when G contains a subsystem of type D_4. The remaining cases are those of type F_4, H_3 and H_4. In each case the essential part of the proof is to show that the inner product $\langle\, ,\, \rangle$ is G-invariant. Let us first do the case F_4. If we have two perpendicular roots of different length, then they generate a saturated $A_1 \times A_1$ subsystem. So the corresponding coroots must be perpendicular for the inverse inner product. It is easily checked that any such an inner product must be G-invariant. Lemma 2.27 then shows see that the exponents are constant on any subsystem of type A_2. Since a G-orbit of reflecting hyperplanes is connected by its A_2 subsystems (Remark 2.28), it follows that the Dunkl form is G-invariant.

The cases H_3 and H_4 are dealt with in a similar fashion: any inner product with the property that the summands of a $A_1 \times A_1$ subsystem (all are automatically saturated) are orthogonal must be G-invariant. The A_2-connectivity of the set of reflecting hyperplanes implies that every such hyperplane has the same exponent. □

3. From Dunkl to Levi-Civita

3.1. *The hyperbolic exponent.* — According to Lemma 2.12, the inner product $\langle\, ,\, \rangle$ is unique up to a scalar factor. An inner product on V determines a (Fubini-Study) metric on $\mathbf{P}(V)$ and two inner products determine the same metric if and only if they are proportional. So we are then basically prescribing a Fubini-Study metric on $\mathbf{P}(V)$.

The inner product $\langle\, ,\, \rangle$ defines a translation invariant (Kähler) metric on the tangent bundle of V; its restriction to V° (which we shall denote by h^0) has ∇^0 as Levi-Civita connection. We shall see that we can often deform h^0 with the connection.

The theorem below is the main result of this subsection. It generalizes to some extent Remark 2.15. We shall later see that in all cases of interest we can satisfy its hypotheses.

Theorem **3.1.** — *Let* $\dim V \geq 2$ *and* $\kappa \in (0,\infty)^{\mathcal{H},\mathrm{flat}}$ *be such that* $\kappa_0 = 1$. *Assume we are given for every* $s \geq 0$ *a nonzero Hermitian form* h_s *on* V° *which is flat for* $\nabla^{s\kappa}$, *is equal to the given positive definite form for* $s = 0$ *and is real-analytic in* s. *Then:*

(i) for $s < 1$, $h_s > 0$,

(ii) we have $h_1 \geq 0$ *and its kernel is spanned by the Euler field,*

(iii) there exists a $m_{\mathrm{hyp}} \in (1,\infty]$ *characterized by the property that for* $s \in (1, m_{\mathrm{hyp}})$, h_s *is of hyperbolic type and* $h_{m_{\mathrm{hyp}}}$ *is degenerate in case* m_{hyp} *is finite.*

Moreover, h_s *is admissible for* $s \in (1, m_{\mathrm{hyp}})$ *provided that* $s\kappa_H \leq 1$ *for all* $H \in \mathcal{H}$: h_s *is then negative on the Euler field. We call* m_{hyp} *the* hyperbolic exponent *of the family.*

It is likely that h_s is admissible for all $s \in (1, m_{\mathrm{hyp}})$.

The proof requires some preparation. We begin with a lemma.

Lemma **3.2.** — *Let* $\kappa \in (0,\infty)^{\mathcal{H},\mathrm{flat}}$ *and let* $\mathcal{F}$ *be a vector subbundle of rank* r *of the holomorphic tangent bundle of* V° *which is flat for* ∇^κ. *Let* $\mathcal{H}(\mathcal{F})$ *denote the set of* $H \in \mathcal{H}$ *for which the connection on* $\mathcal{F}$ *becomes singular (relative to its natural extension across the generic point of* H *as a vector subbundle of the tangent bundle). Then there exists an* r*-vector field* X *on* V *with the following properties:*

(i) $X|V^\circ$ *defines* $\mathcal{F}$ *and the zero set of* X *is contained in the union of the codimension two intersections from* $\mathcal{H}$,

(ii) X *is homogeneous of degree* $r(\kappa_0 - 1) - \sum_{H\in\mathcal{H}(\mathcal{F})}\kappa_H$ *and multiplication of* X *by* $\prod_{H\in\mathcal{H}(\mathcal{F})}\phi_H^{\kappa_H}$ *yields a flat multivalued form.*

In particular, $\sum_{H\in\mathcal{H}(\mathcal{F})}\kappa_H \le r\kappa_0$ *(so that* $\mathcal{H}(\mathcal{F}) \ne \mathcal{H}$ *when* $r < \dim V$*). Moreover, in the case of a line bundle (*$r=1$*), the degree of* X *is nonnegative and is zero only when* $\mathcal{F}$ *is spanned by the Euler field of* V*.*

Likewise there exists a regular $(\dim V - r)$*-form* η *on* V *satisfying similar properties relative to the annihilator of* $\mathcal{F}$*:*

(iii) $\eta|V^\circ$ *defines the annihilator of* $\mathcal{F}$ *and the zero set of* η *is contained in the union of the codimension two intersections from* $\mathcal{H}$*,*

(iv) η *is homogeneous of degree* $(\dim V - r)(1-\kappa_0) + \sum_{H\in\mathcal{H}-\mathcal{H}(\mathcal{F})}\kappa_H$ *and multiplication of* η *by* $\prod_{H\in\mathcal{H}-\mathcal{H}(\mathcal{F})}\phi_H^{-\kappa_H}$ *yields a flat multivalued form.*

Remark **3.3.** — We will use this lemma in the first instance only in the case of a line bundle. When $r = \dim V$, then clearly $\mathcal{H}(\mathcal{F}) = \mathcal{H}$ and so the lemma then tells us that for any $X \in \wedge^{\dim V} V$, $(\prod_{H\in\mathcal{H}}\phi_H^{\kappa_H})X$ is flat section of the determinant bundle of TV° for ∇^κ.

Proof of Lemma 3.2. — Let us first observe that $\mathcal{F}$ will be invariant under scalar multiplication. It extends as an holomorphic vector subbundle of the tangent bundle over the complement of the union of the codimension two intersections from $\mathcal{H}$ and it is there given by a section X of the rth exterior power of the tangent bundle of V. Since $\mathcal{F}$ is invariant under scalar multiplication, we can assume X to be homogeneous. The local form 1.10 of ∇^κ along the generic point of $H \in \mathcal{H}$ implies that $\mathcal{F}$ is in this point either tangent or perpendicular to H. In the first case the connection ∇^κ restricted to $\mathcal{F}$ is regular there, whereas in the second case it has there a logarithmic singularity with residue $-\kappa_H$. So if $D\pi_H$ denotes the action of π_H on polyvectors as a derivation (i.e., it sends an r-polyvector $X_1 \wedge \cdots \wedge X_r$ to $\sum_i X_1 \wedge \cdots \wedge \pi_{H*}X_i \wedge \cdots \wedge X_r$), then ϕ_H divides $D\pi_H(X)$ or $D\pi_H(X) - X$ according to whether $H \in \mathcal{H} - \mathcal{H}(\mathcal{F})$ or $H \in \mathcal{H}(\mathcal{F})$. Consider the multivalued function $\Phi := \prod_{H\in\mathcal{H}(\mathcal{F})}\phi_H^{\kappa_H}$ on V°. Locally we can find a holomorphic function f on V° such that $f\Phi X$ is flat for ∇^κ; we then have

$$-\frac{df}{f}\otimes X = \nabla^0(X) - \sum_{H\in\mathcal{H}-\mathcal{H}(\mathcal{F})}\kappa_H d\phi_H \otimes \phi_H^{-1} D\pi_H(X) - \sum_{H\in\mathcal{H}(\mathcal{F})}\kappa_H d\phi_H \otimes \phi_H^{-1}(D\pi_H(X) - X).$$

We have arranged things in such a manner that the right-hand side of this identity is regular. Hence so is the left-hand side. Since X is nonzero in codimension one, it follows that df/f is the restriction of a regular, globally defined (closed) differential on V. This can only happen if f is a nonzero constant. Hence $\exp(-a)\Phi X$ is a flat multivalued r-vector field on V°. Such a field must be homogeneous of degree $r(\kappa_0 - 1)$. Since ΦX is homogeneous, so is $\exp(a)$. It follows that a is a scalar and that the degree of X is $r(\kappa_0 - 1) - \sum_{H\in\mathcal{H}(\mathcal{F})}\kappa_H$. The fact that X must have a degree of homogeneity at least $\dim V - r$ implies that $\sum_{H\in\mathcal{H}(\mathcal{F})}\kappa_H \le r\kappa_0$.

The assertions regarding the annihilator of $\mathcal{F}$ are proved in a similar fashion.

Now assume $r = 1$ so that X is a vector field. Its degree cannot be -1, for then X would be a constant vector field, that is, given by some nonzero $v \in V$. But then $v \in \cap_{H\in\mathcal{H}-\mathcal{H}(\mathcal{F})} H$, whereas $\mathcal{H}(\mathcal{F})$ is empty or consists of $v^\perp$, and this contradicts the irreducibility of $\mathcal{H}$.

If X is homogeneous of degree zero, then clearly $\mathcal{H}(\mathcal{F}) = \emptyset$ (in other words, X is tangent to each member of $\mathcal{H}$) and $\kappa_0 = 1$. If we think of X as a linear endomorphism Ξ of V, then the tangency property amounts to $\Xi^* \in \mathrm{End}(V^*)$ leaving each line in V^* invariant which is the annihilator of some $H \in \mathcal{H}$. Since $\mathcal{H}$ is irreducible, there are $1+\dim V$ such lines in general position and so Ξ^* is must be a scalar. This means that X is proportional to the Euler vector field of V. ☐

Proposition **3.4.** — *Let* $\dim V \ge 2$, $\kappa \in (0,\infty)^{\mathcal{H},\mathrm{flat}}$, $\kappa_0 \le 1$ *and let* h *be a nonzero Hermitian form on* V° *which is flat for* ∇^κ *and* ≥ 0. *Then for* $\kappa_0 < 1$ *we have* $h > 0$ *and in case* $\kappa_0 = 1$, *the kernel of* h *is spanned by the Euler vector field.*

Proof. — Suppose first $\dim V = 2$. If h is degenerate, then its kernel defines a flat line field on V°. According to Lemma 3.2 this implies that $\kappa_0 = 1$ and the line field is spanned by the Euler field.

We now verify the theorem by induction on $\dim V$ and hence suppose that $\dim V > 2$.

Pick a $H \in \mathcal{H}$ and choose $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ such that $L \subset H$ and $\dim L = 1$ (such an L exists by Lemma 2.11). Then we have $\kappa_L \in (0,1)$ by the monotonicity property of κ. So by Corollary 2.22 and Remark 2.24 we have a natural affine decomposition $V^\circ_{L^\circ} = L^\circ \times (V/L)^\circ_0$ of Hermitian germs. Our induction hypothesis implies that the form on $(V/L)^\circ$ is > 0 or is identically zero.

Claim: *The form on* $(V/L)^\circ$ *is* > 0 *and the one on* L° *is* ≥ 0 *with equality if and only if* $\kappa_0 = 1$.

To prove this, we first note that if $\{0\} \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^H)$, then the value of $\kappa^H : \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^H) \to \mathbf{C}$ on this singleton is, by Lemma 2.18, equal to κ_0. We have a decomposition of Hermitian germs $V_{H^\circ} = H^\circ \times (H/L)_0$ as for L. By induction hypothesis, H° carries a form which is proportional to one which is >0 or, in case $\{0\} \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^H)$ and $\kappa_0 = 1$ to one which is ≥ 0 with kernel spanned by the Euler field. The form on H° cannot be identically zero: for then the one on $(H/L)^\circ$ is zero also, and hence the one on $(V/L)^\circ$ will be degenerate. But this implies that the form on $(V/L)^\circ$ is identically zero and hence that $h \equiv 0$, contrary to our assumption. So the form on H° is either >0 or has constant sign with kernel spanned by the Euler field (in which case $\{0\} \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^H)$ and $\kappa_0 = 1$). In the latter situation the form on L° is zero, but in either case the form on $(H/L)^\circ$ will be nonzero. Hence the form on $(V/L)^\circ$ will be nonzero also, and therefore it is >0. The claim follows.

So if $\kappa_0 < 1$, the forms on $(V/L)^\circ$ and H° must be >0 and hence $h>0$. If $\kappa_0 = 1$, it follows that $h \geq 0$ with kernel of dimension one. This kernel defines a flat line field on V° and according to Lemma 3.2 this can only happen if the line field is spanned by the Euler vector field. □

For the proof of Theorem 3.1 we need:

Lemma **3.5.** — *Let T be a finite dimensional complex vector space, $L \subset T$ a line and $(H_s)_s$ a real-analytic family of Hermitian forms on T parametrized by a neighborhood of 0 in $\mathbf{R}$. Suppose that $H_s > 0$ if and only if $s < 0$ and that $H_0 \geq 0$ with kernel L. Then there exists a $\varepsilon > 0$ such that when $s \in (0, \varepsilon)$, H_s is of hyperbolic type and negative on L.*

Proof. — Let $T' \subset T$ be a supplement of L in T. Then H_0 is positive definite on T' and we may therefore just as well assume that every H_s restricted to T' is positive. A Gram-Schmidt process then produces an orthonormal basis $(e_1(s), \dots, e_m(s))$ for H_s restricted to T' which depends real-analytically on s. Let $e \in T$ generate L, so that $(e, e_1(s), \dots, e_m(s))$ is a basis for T. The determinant of H_s with respect this basis is easily calculated to be $H_s(e,e) - \sum_{i=1}^m |H_s(e, e_i(s))|^2$. We know that this determinant changes sign at $s=0$. This can only happen if $H_s(e,e)$ is the dominating term and (hence) changes sign at $s=0$. □

Proof of 3.1. — According to Proposition 3.4 h_s is positive for $s<1$. We first show that h_s changes sign at $s=1$. If this is not the case, then there exists $s>1$ such that $s\kappa_L < 1$ for all $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}) - \{0\}$ and $h_s > 0$. Choose $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ of dimension two. Then h_s induces on L° a Hermitian form >0 which is flat for $s\kappa^L$. We have $s\kappa_0^L = s > 1$, whereas for every line $M \in \mathcal{H}^L$, we have $s\kappa_M^L = s\kappa_{I(M)} < 1$. But this contradicts Remark 2.15, which says that L° cannot carry a $\nabla^{s\kappa^L}$-flat Hermitian form which is >0.

Since h_s changes sign at $s = 1$, Lemma 3.5 (applied to the tangent space of V° at some point) implies that there exists an $m' > 1$ such that for all $s \in (1, m')$, h_s is hyperbolic. We take for m_{hyp} the supremum of the values m' for which this is true. This means that if m_{hyp} is finite, then $h_{m_{\mathrm{hyp}}}$ must be degenerate, thus proving the first part of the theorem.

Let m'_{hyp} be the minimum of m_{hyp} and the $1/\kappa_H$, $H \in \mathcal{H}$. We show that the function

$$\ell : V^\circ \times \mathbf{R}_+ \to \mathbf{R}, \quad \ell(p, s) = h_s(E_V, E_V)(p),$$

is negative on $V^\circ \times (1, m'_{\mathrm{hyp}})$. This almost suffices for it leaves us only to consider the case $s = m'_{\mathrm{hyp}} < m_{\mathrm{hyp}}$.

Notice that ℓ is a real-analytic function which satisfies the homogeneity property $\ell(\lambda p, s) = \lambda^{2-2s}\ell(p, s)$. For a fixed $p \in V^\circ$, Proposition 3.4 and Lemma 3.5 (applied to the restriction of h_s to T_pV and the line in T_pV spanned by $E_V(p)$), imply that there exists an $m' \in (1, m'_{\mathrm{hyp}}]$ such that for all $s \in (1, m')$, $\ell(p, s) < 0$. Let $m(p) \in (1, m'_{\mathrm{hyp}}]$ be the supremum of the values m' for which this is true. We first show that $m(p)$ is constant in $p \in V^\circ$.

Because of the homogeneity property of ℓ, m factors through $\mathbf{P}(V^\circ)$. We consider the situation near H°, $H \in \mathcal{H}$. Let $s \in [0, m'_{\mathrm{hyp}})$. Since $s\kappa_H \in (0, 1)$, we have according to Lemma 2.22 and Remark 2.24 an orthogonal decomposition $V_{H^\circ} \cong H^\circ \times (V/H)_0$ in the Hermitian category. This decomposition is natural and so it maps the Euler field to the sum of the Euler fields of the factors. For the same reason it depends continuously (even holomorphically) on s. This implies that the one-dimensional factor $(V/H)_0$ stays positive definite for all $s \in [0, m'_{\mathrm{hyp}})$. The length of the Euler field on $(V/H)_0$ is homogeneous of degree $1 - s\kappa_H$ and so when $s\kappa_H < 1$, $h_s(E_V, E_V)(p)$ decreases as p tends to a point of H°. We use this to prove that m is constant on any linear subspace of dimension two $P \subset V$ which meets V° and is in general position with respect to $\mathcal{H}$ (in the sense that no point of $P - \{0\}$ is contained in two distinct members of $\mathcal{H}$). This will certainly imply that m is constant.

Consider the function

$$\ell_P : (P - \{0\}) \times \mathbf{R}_+ \to \mathbf{R}, \quad (p, s) \mapsto h_s(E_V, E_V)(p).$$

Since every element of $P - \{0\}$ is either in V° or in some H°, it follows from the preceding discussion (and the compactness of $\mathbf{P}(P)$) that there exists a $m' \in (1, m'_{\mathrm{hyp}}]$ such that ℓ_P is negative on $(P-\{0\}) \times (1, m')$. Let $m(P)$ be the supremum of the m' for which this is true. Suppose that $m(P) < m(p_0)$ for some $p_0 \in P \cap V^\circ$. Then $h_{m(P)}$ is of hyperbolic type, whereas $p \in P - \{0\} \mapsto$

$\ell_P(p, m(P))$ has 0 as its supremum. Our local considerations near a member of $\mathcal{H}$ show that this supremum is in fact a maximum: it is taken at some point of $P \cap V^\circ$. This means that the developing map for $\nabla^{m(P)\kappa}$ is affine-equivalent to a morphism from a cover of $P \cap V^\circ$ to the subset of $\mathbf{C}^n$ defined by $|z_1|^2 + \cdots + |z_{n-1}|^2 \leq |z_n|^2$, and such that the inequality is an equality at some point. This, however, contradicts a convexity property of this subset as is shown by Lemma 3.6 below.

We conclude that m is constant on V°. Suppose this constant value (also denoted by m) is smaller than m'_{hyp}. Then h_m is hyperbolic and so the developing map is equivalent to a morphism from a cover of V° to the subset of $\mathbf{C}^n$ defined by $|z_1|^2 + \cdots |z_{n-1}|^2 = |z_n|^2$. This is impossible, since latter subset is not open in $\mathbf{C}^n$.

It remains to treat the case $s = m'_{\text{hyp}} < m_{\text{hyp}}$. By continuity, $\ell(p, m'_{\text{hyp}}) \leq 0$ for all $p \in V^\circ$ and so the developing map is for $s = m'_{\text{hyp}}$ affine equivalent to a morphism from a cover of V° to the subset of $\mathbf{C}^n$ defined by $|z_1|^2 + \cdots + |z_{n-1}|^2 \leq |z_n|^2$. We conclude as before that it will then map to $|z_1|^2 + \cdots |z_{n-1}|^2 < |z_n|^2$ so that h_s is admissible. □

Lemma **3.6.** — *Let $f = (f_1, \dots, f_n) : U \to \mathbf{C}^n$ be a holomorphic map from a connected complex manifold U such that $|f_1|^2 + \cdots + |f_{n-1}|^2 \leq |f_n|^2$. Then the latter inequality is strict unless f maps to a line.*

Proof. — We may assume that f_n is not constant equal to zero so that each $g_i := f_i/f_n$ is a meromorphic function. Since $g := (g_1, \dots, g_{n-1})$ takes values in the closed unit ball, the g_i's are bounded and hence, by the Riemann extension theorem, holomorphic. The maximum principle implies that g maps to the open unit ball unless it is constant. □

3.2. *The Lauricella integrand as a rank two example.* — We do not know whether a Dunkl system with real exponents always admits a nontrivial flat Hermitian form, not even in the case $\dim V = 2$ (the answer is probably no). However, if $\dim V = 2$ and $\kappa_0 = 1$, then there is natural choice. In order to avoid conflicting notation, let us write P instead of V, let $H_0, \dots, H_{n+1}$ be the distinct elements of $\mathcal{H}$ (so that $|\mathcal{H}| = n+2$) and write μ_i for κ_{H_i} (so that $\sum_i \mu_i = 2$). Recall from Lemma 3.2 that if α is a translation invariant 2-form, then $(\prod_{H \in \mathcal{H}} \phi_H^{-\kappa_H})\alpha$ is a flat multivalued 2-form. Since $\kappa_0 = 1$, the Euler field E_P is flat, and so if ω denotes the 1-form obtained by taking the inner product of E_P with α, then $(\prod_{i=0}^{n+1} \phi_i^{-\mu_i})\omega$ is a flat multivalued 1-form. Hence its absolute value,

$$h := |\phi_0|^{-2\mu_0} \cdots |\phi_{n+1}|^{-2\mu_{n+1}} |\omega|^2,$$

is then a nontrivial flat Hermitian form. It is positive semidefinite with kernel spanned by the Euler field.

This is intimately connected with an observation due to Thurston [33], about which we will have more to say later on. Since $\kappa_0 = 1$, the punctured Riemann sphere $\mathbf{P}(P^\circ)$ acquires an affine structure. The form h is a pull-back from $\mathbf{P}(P^\circ)$ so that $\mathbf{P}(P^\circ)$ has in fact a Euclidean (parabolic) structure. If we assume that $\mu_i \in (0,1)$ for all i, then $\mathbf{P}(P)$ is a Euclidean *cone manifold* in Thurston's sense: at the point $p_i \in \mathbf{P}(P)$ defined by H_i, the metric is conical with total angle $2\pi(1-\mu_i)$. In such a point is concentrated a certain amount of curvature, its *apex curvature* $2\pi\mu_i$, which is its contribution to the Gauss-Bonnet formula (the sum of these is indeed 4π, the area of the unit sphere). On the other hand, the multivalued form $(\prod_{H\in\mathcal{H}} \phi_H^{-\kappa_H})\omega$ is directly related to the Lauricella integrand. To see this, choose an affine coordinate z on $\mathbf{P}(V)$ such that if $z_i := z(p_{H_i})$, then $z_{n+1} = \infty$. Then $(\prod_{i=0}^{n+1} \phi_i^{-\mu_i})\omega$ is up to a constant factor the pull-back of a constant times $\prod_{i=0}^{n}(z_i - \zeta)^{-\mu_i} dz$, which we recognize as the Lauricella integrand.

Of course, the $(n+1)$-tuple $(z_0, \dots, z_n) \in \mathbf{C}^{n+1}$ is defined only up to an affine-linear transformation of $\mathbf{C}$. This means that if V is the quotient of $\mathbf{C}^{n+1}$ by its main diagonal (as in Subsection 2.3), then only the image of $(z_0, \dots, z_n)$ in $\mathbf{P}(V^\circ)$ matters. Thus $\mathbf{P}(V^\circ)$ can be understood as the moduli space of Euclidean metrics on the sphere with $n+2$ conical singularities which are indexed by $0, \dots, n+1$ with prescribed apex curvature $2\pi\mu_i$ at the ith point.

3.3. *Flat Hermitian forms for reflection arrangements.* — The following theorem produces plenty of interesting situations to which the results of Subsection 3.1 apply. It may very well hold in a much greater generality.

Theorem **3.7.** — *Suppose that $\mathcal{H}$ is the reflection arrangement of a finite complex reflection group G. Then there exists a map from $(\mathbf{R}^{\mathcal{H}})^G$ to the space of nonzero Hermitian forms on the tangent bundle of V° (denoted $\kappa \mapsto h^\kappa$) with the following properties: for every $\kappa \in (\mathbf{R}^{\mathcal{H}})^G$,*

(i) h^κ is flat for ∇^κ and invariant under G.

(ii) $t \in \mathbf{R} \mapsto h^{t\kappa}$ is smooth (notice that h^0 was already defined) and the associated curve of projectivized forms, $t \mapsto [h^{t\kappa}]$ is real-analytic.

Moreover, h^κ is unique up to scalar multiple (relative to (i) and (ii)).

Likewise there is a map from $(\mathbf{R}^{\mathcal{H}})^G$ to the space of nonzero Hermitian forms on the cotangent bundle of V° with analogous properties.

Example **3.8.** — For $V = \mathbf{C}$ and $\Omega = \kappa z^{-1}dz$, we can take $h^\kappa(z) := |z|^{-2\kappa}|dz|^2$. Notice that we can expand this in powers of κ as

$$h^\kappa(z) = \sum_{k=0}^{\infty} \kappa^k \frac{(-\log|z|^2)^k}{k!}|dz|^2.$$

We shall first prove that in the situation of Theorem 3.7 we can find such an h^κ formally at $\kappa = 0$. For this we need the following notion, suggested by Example 3.8. Let be given a complex manifold M and a smooth hypersurface $D \subset M$. We say that a function f on a neighborhood of p in $M - D$ is *logsingular of order* $\leq k$ *along* D if there is a defining equation ϕ of D at p such that f can be written as a polynomial of degree $\leq k$ in $\log|\phi|$ with coefficients that are real-analytic on M_p. Since ϕ is unique up to a unit as a factor, $\log|\phi|$ is unique up to the addition of a real-analytic function on M_p and so any generator of $\mathcal{O}_{M,p}(-D)$ will do for this purpose.

We say that a differential η on a neighborhood of p in $M - D$ is *logsingular of order* $\leq k$ *along* D at p if it is a linear combination of real-analytic forms on M_p and $|\phi|^{-1}d|\phi| = \mathrm{Re}(d\phi/\phi)$ and with logsingular functions of order $\leq k$ as coefficients, but with the coefficient of $|\phi|^{-1}d|\phi|$ being of order $\leq k-1$ (read equal to zero when $k = 0$).

Lemma **3.9.** — *In this situation we have:*

(i) $\log|w|$ *is algebraically independent over the ring of real-analytic functions on* M_p.

(ii) Any logsingular differential of order $\leq k$ *at* p *that is closed is the differential of a logsingular function of order* $\leq k$ *at* p.

Proof. — The proof of (i) is left to the reader. For the proof of (ii) we use a retraction $M_p \to D_p$ in order to identify M_p with $D_p \times \mathbf{C}_0$ and employ polar coordinates (r, θ) on the second coordinate. If η is a logsingular differential of order $\leq k$ at p, then we can write

$$\eta = \sum_{i=0}^{k}(\log r)^i(\alpha_i + g_i r d\theta) + \sum_{i=0}^{k-1}(\log r)^i f_i r^{-1}dr,$$

with α_i a differential on D_p whose coefficients are real-analytic functions on M_p and f_i and g_i real-analytic on $D_p \times \mathbf{R}_0 \times S^1$. Since $\lim_{r\downarrow 0} r(\log r)^i = 0$ for $i \geq 0$, the integral of such a form over the circle $\{(p,r)\} \times S^1$ tends to zero as r tends to zero. Hence, if η is closed, then the integral over any such circle is closed and the form can be integrated to a function on $(M - D)_p$. A straightforward verification shows that this function is logsingular of order $\leq k$. $\square$

Lemma **3.10.** — *In the situation of Theorem 3.7, let $\kappa \in (\mathbf{R}^{\mathcal{H}})^G$. Then there exists a formal expansion $h^{s\kappa} = \sum_{k=0}^{\infty} s^k h_k$ in G-invariant Hermitian forms with initial coefficient $h_0 = h^0$ and such that h_k is logsingular of order $\leq k$ along the smooth part of the arrangement and with the property that $h^{s\kappa}$ is flat for $\nabla^{s\kappa}$.*

Proof. — The flatness of $h^{s\kappa}$ means that for every pair $v, v' \in V$ (when regarded as translation invariant vector fields on V) we have

$$d(h^{s\kappa}(v, v')) = -sh^{s\kappa}(\Omega^{\kappa}(v), v') - sh^{s\kappa}(v, \Omega^{\kappa}(v')),$$

where $\Omega^{\kappa}(v) = \sum_H \kappa_H \pi_H(v) \otimes \omega_H$, which boils down to

$$(*) \quad d(h_{k+1}(v, v')) = -h_k(\Omega^{\kappa}(v), v') - h_k(v, \Omega^{\kappa}(v')), \quad k = 0, 1, 2, \ldots$$

In other words, we must show that we can solve (*) inductively by G-invariant forms. In case we can solve (*), then it is clear that a solution will be unique up to a constant.

The first step is easy: if we choose our defining equation $\phi_H \in V^*$ for H to be such that $\langle \phi_H, \phi_H \rangle = 1$, then

$$h_1(v, v') := -\kappa_H \sum_H \langle \pi_H(v), \pi_H(v') \rangle \log |\phi_H|^2.$$

will do. Suppose that for some $k \geq 1$ the forms $h_0, \ldots, h_k$ have been constructed. In order that $(*)$ has a solution for h_{k+1} we want the right-hand side (which we shall denote by $\eta_k(v, v')$) to be exact. It is certainly closed: if we agree that $h(\omega \otimes v, \omega' \otimes v')$ stands for $h(v, v')\omega \wedge \overline{\omega'}$, then

$$\begin{aligned} d\eta_k(v, v') = h_{k-1}(\Omega^{\kappa} \wedge \Omega^{\kappa}(v), v') - h_{k-1}(\Omega^{\kappa}(v), \Omega^{\kappa}(v'))+ \\ + h_{k-1}(\Omega^{\kappa}(v), \Omega^{\kappa}(v')) + h_{k-1}(v, \Omega^{\kappa} \wedge \Omega^{\kappa}(v')) = \\ = h_{k-1}(\Omega^{\kappa} \wedge \Omega^{\kappa}(v), v') + h_{k-1}(v, \Omega^{\kappa} \wedge \Omega^{\kappa}(v')) = 0 \end{aligned}$$

(since $\Omega^{\kappa} \wedge \Omega^{\kappa} = 0$). So in order to complete the induction step, it suffices by Lemma 3.9 that to prove that η_k is logsingular of order $\leq k+1$ along the arrangement: since the complement in V of the singular part of the arrangement is simply connected, we then write η_k as the differential of a Hermitian form h_{k+1} on V that is logsingular of order $\leq k+1$ along the arrangement and averaging such h_{k+1} over its G-transforms makes it G-invariant as well.

Our induction assumption says that near H° we can expand h_k in $\log |\phi_H|$ uniquely as:

$$h_k = \sum_{i=0}^{k} (\log |\phi_H|)^i h_{k,i}$$

where $h_{k,i}$ is a real-analytic Hermitian form on TV_{H°. Now

$$\eta_k = -\sum_{H'\in\mathcal{H}}\sum_{i=0}^{k}\kappa_{H'}(\log|\phi_{H'}|)^i\Big(h_{k,i}(\pi_{H'}(v),v')\omega_{H'}+h_{k,i}(v,\pi_{H'}(v')\overline{\omega_{H'}}\Big).$$

The terms indexed by $H'\neq H$ are linear combinations differentials that are regular at H° with logsingular coefficients of order $\le k$ and hence are logsingular of order $\le k$ along H°. So it remains to show that $h_{k,i}(\pi_H(v),v')\omega_H + h_{k,i}(v,\pi_H(v')\overline{\omega_H}$ is logsingular of order $\le k+1$ along H°. To see this, let G_H be the group of complex reflections in G with mirror H. Each term $h_{k,i}$ is invariant under G_H (acting on the germ V_{H°), because h_k and $|\phi_H|$ are (and the uniqueness of the expansion in powers of $\log|w|$). Since π_H is the projection onto an eigenspace of G_H, we find that if $a\perp H$ and $b\in H$, then the real-analytic function $h_{k,i}(a,b)$ on M_p transforms under G_H with the same character as ϕ_H. Hence this function is divisible by ϕ_H with real-analytic quotient. So

$$\alpha := h_{k,i}(\pi_H(v),v')\omega_H - h_{k,i}(\pi_H(v),\pi_H(v'))\omega_H = h_{k,i}(\pi_H(v),v'-\pi_H(v'))\frac{d\phi_H}{\phi_H}$$

is real-analytic. Hence

$$\begin{aligned}h_{k,i}(\pi_H(v),v')\omega_H + h_{k,i}(v,\pi_H(v')\overline{\omega_H} &=\\ &= h_{k,i}(\pi_H(v),\pi_H(v'))(\omega_H+\overline{\omega_H})+\alpha+\overline{\alpha}\\ &= 2h_{k,i}(\pi_H(v),\pi_H(v'))\frac{d|\phi_H|}{|\phi_H|}+\alpha+\overline{\alpha}.\end{aligned}$$

This shows that η_k is logsingular of order $\le k+1$. □

In order to prove Theorem 3.7, we begin with a few generalities regarding conjugate complex structures. Denote by $V^\dagger$ the complex vector space V with its conjugate complex structure: scalar multiplication by $\lambda\in\mathbf{C}$ acts on $V^\dagger$ as scalar multiplication by $\overline{\lambda}\in\mathbf{C}$ in V. Then $V\oplus V^\dagger$ has a natural real structure for which complex conjugation is simply interchanging arguments. The ensuing conjugation on $\mathrm{GL}(V\oplus V^\dagger)$ is, when restricted to $\mathrm{GL}(V)\times\mathrm{GL}(V^\dagger)$, also interchanging arguments, whereas on the space of bilinear forms on $V\times V^\dagger$, it is given by $h^\dagger(v,v') := \overline{h(v',v)}$. So a real point of $(V\otimes V^\dagger)^*$ is just a Hermitian form on V.

Fix a base point $*\in V^\circ$ and identify T_*V° with V. For $\kappa\in(\mathbf{C}^{\mathcal{H}})^G$, we denote the monodromy representation of ∇^κ by $\rho^\kappa\in\mathrm{Hom}(\pi_1(V^\circ,*),\mathrm{GL}(V))$. Notice that ρ^κ depends holomorphically on κ. Then the same property must hold for

$$\kappa\in(\mathbf{C}^{\mathcal{H}})^G\mapsto(\rho^{\overline{\kappa}})^\dagger\in\mathrm{Hom}(\pi_1(V^\circ,*),\mathrm{GL}(V^\dagger)).$$

Recall from 2.17 that $(\mathbf{C}^{\mathcal{H}})^G$ is invariant under complex conjugation.

Lemma **3.11.** — *Let* $\mathbf{H}$ *be the set of pairs* $(\kappa, [h]) \in (\mathbf{C}^{\mathcal{H}})^G \times \mathbf{P}((V \otimes V^\dagger)^*)$*, where* $h \in V \times V^\dagger \to \mathbf{C}$ *is invariant under* $\rho^\kappa \otimes (\rho^{\overline{\kappa}})^\dagger$ *and let* $p_1 : \mathbf{H} \to (\mathbf{C}^{\mathcal{H}})^G$ *be the projection. Then* $\mathbf{H}$ *resp.* $p_1(\mathbf{H})$ *is a complex-analytic set defined over* $\mathbf{R}$ *(in* $(\mathbf{C}^{\mathcal{H}})^G \times \mathbf{P}((V \otimes V^\dagger)^*)$ *resp.* $(\mathbf{C}^{\mathcal{H}})^G$*) and we have* $p_1(\mathbf{H}(\mathbf{R})) = (\mathbf{R}^{\mathcal{H}})^G$*.*

Proof. — That $\mathbf{H}$ is complex-analytic and defined over $\mathbf{R}$ is clear. Since p_1 is proper and defined over $\mathbf{R}$, $p_1(\mathbf{H})$ is also complex-analytic and defined over $\mathbf{R}$. If $\kappa \in (\mathbf{R}^{\mathcal{H}})^G$ is in the image of $\mathbf{H}$, then there exists a nonzero bilinear map $h : V \times V^\dagger \to \mathbf{C}$ invariant under $\rho^\kappa \otimes (\rho^{\overline{\kappa}})^\dagger$. But then both the 'real part' $\frac{1}{2}(h+h^\dagger)$ and the 'imaginary part' $\frac{1}{2\sqrt{-1}}(h-h^\dagger)$ of h are Hermitian forms invariant under ρ^κ and clearly one of them will be nonzero. The lemma follows. ◻

Proof of Theorem 3.7. — Now let $L \subset (\mathbf{C}^{\mathcal{H}})^G$ be a line defined over $\mathbf{R}$. By the preceding discussion, there is a unique irreducible component $\tilde{L}$ of the preimage of L in $\mathbf{H}$ which contains $(0, [h^0])$. The map $\tilde{L} \to L$ is proper and the preimage of 0 is a singleton. Hence $\tilde{L} \to L$ is an complex-analytic isomorphism. Since L is defined over $\mathbf{R}$, so are $\tilde{L}$ and the isomorphism $\tilde{L} \to L$. The forms parametrized by $\tilde{L}(\mathbf{R})$ define a real line bundle over $L(\mathbf{R})$. Such a line bundle is trivial in the smooth category and hence admits a smooth generating section with prescribed value in 0. We thus find a map $\kappa \mapsto h^\kappa$ with the stated properties. The proof for the map the Hermitian form on the cotangent bundle is similar. ◻

If h is a nondegenerate Hermitian form on the tangent bundle of V° which is flat for the Dunkl connection, then ∇ must be its Levi-Civita connection of h (for ∇ is torsion free); in particular, h determines ∇. Notice that to give a flat Hermitian form h amounts to giving a monodromy invariant Hermitian form on the translation space of A. So h will be homogeneous in the sense that the pull-back of h under scalar multiplication on V° by $\lambda \in \mathbf{C}^\times$ is $|\lambda|^{2-2\,\mathrm{Re}(\kappa_0)}h$.

3.4. *The hyperbolic exponent of a complex reflection group.* — In case $\mathcal{H}$ is a complex reflection arrangement of a finite reflection group G, we can estimate the hyperbolic exponent. According to the theorem of Shephard-Todd and Chevalley, the graded algebra of G-invariants $\mathbf{C}[V]^G$ is a polynomial algebra. Choose a set of homogeneous generators, $f_1, \dots, f_n$, ordered by their degrees: $\deg(f_1) \le \cdots \le \deg(f_n)$. Although the generators are not unique, their degrees are and we put $d_i := \deg(f_i)$. The number $m_i := d_i - 1$, which is the degree of the coefficients of df_i on a basis of constant differentials on V, is called the ith *exponent* of G. It is known that the subalgebra of G-invariants in the exterior algebra $\mathbf{C}[V] \otimes \wedge^\bullet V^*$ of regular forms on V is generated as such by $df_1, \dots, df_n$ [32]. In particular any invariant n-form is proportional to $df_1 \wedge \dots \wedge df_n$.

The geometric content of the Shephard-Todd-Chevalley theorem is the assertion that the orbit space $G\backslash V$ is an affine space, a fact which never stops to surprise us. The union of the members of $\mathcal{H}$ is also the union of the irregular orbits and hence is the singular locus of the orbit map $\pi : V \to G\backslash V$. The image of this orbit map is a hypersurface in $G\backslash V$, the *discriminant* of G. It is defined by a suitable power of the jacobian of $(f_1, \dots, f_n)$.

A vector field on $G\backslash V$ lifts to V precisely when it is tangent to the discriminant and in this manner we get all the G-invariant vector fields on V. The G-invariant regular vector fields make up a graded $\mathbf{C}[V]^G$ -module and it is known (Lemma 6.48 of [28]) that this module is free. As with the Chevalley generators, we choose a system of homogeneous generators $X_1, \dots, X_n$ ordered by their degree: $\deg(X_1) \leq \dots \leq \deg(X_n)$. We put $d_i^* := \deg(X_i)$ and $m_i^* := 1 + \deg(X_i)$ (so that m_i^* is the degree of the coefficients of X_i on a basis of constant vector fields on V). The generator of smallest degree is proportional to the Euler field. Hence $d_1^* = 0$ and $m_1^* = 1$. The number m_i^* is called the ith *co-exponent* of G. It usually differs from m_i, but when G is a Coxeter group the two are equal, because the defining representation of G is self-dual.

A polyvector field on $G\backslash V$ lifts to V if and only if it does so in codimension one (that is, in the generic points of the discriminant) and we thus obtain all the G-invariant polyvector fields on V. As in the case of forms, the subalgebra of G-invariants in the exterior algebra $\mathbf{C}[V] \otimes \wedge^\bullet V$ of regular polyvector fields on V is generated as such by $X_1, \dots, X_n$ (Proposition 6.47 of [28]).

Theorem **3.12.** — *Suppose that $\mathcal{H}$ is the reflection arrangement of a finite complex reflection group G which is transitive on $\mathcal{H}$. Then the hyperbolic exponent for the ray $((0,\infty)^{\mathcal{H}})^G$ (which is defined in view of Theorem 3.7) is $\geq m_2^*$.*

Remark **3.13.** — We shall later show (Corollary 3.19) that for constant $\kappa : \mathcal{H} \to (0, 1)$, the flat Hermitian form is degenerate if and only if κ_0 is a (co-)exponent. (Without proof we also mention that this remains true for primitive complex reflection groups of rank ≥ 3, provided that we use the co-exponent.)

Proof of Theorem 3.12. — Let $\kappa \in ((0,1)^{\mathcal{H}})^G$ be such that $\kappa_0 = 1$ and let h_s be the family of Hermitian forms on the tangent bundle of V° whose existence is asserted by Theorem 3.7. Let $m \in (1, \infty]$ be its hyperbolic exponent. If $m = \infty$ there is nothing to show, so let us assume that $m < \infty$. This means that h_m is degenerate. So its kernel defines a nontrivial subbundle $\mathcal{F}$ of the tangent bundle of V° (of rank r, say) which is flat for $\nabla^{m\kappa}$. This bundle is G-invariant. So the developing map maps to a vector space A endowed with a monodromy invariant Hermitian form H_m with a kernel of dimension r. Since H_m is nontrivial, so is $H_m(E_A, E_A)$ and hence so is $h_m(E_V, E_V)$. In other words, $\mathcal{F}$ does not contain the Euler field.

Let X be the associated r-vector field on V as in Lemma 3.2. That lemma asserts that $\mathcal{H}(\mathcal{F}) \neq \mathcal{H}$. Since $\mathcal{H}(\mathcal{F})$ is G-invariant, this implies that $\mathcal{H}(\mathcal{F}) = \emptyset$ so that X has degree $r(m-1)$. We prove that X is G-invariant. Since X is unique up to a constant factor it will transform under G by means of a character. For this it is enough to show that X is left invariant under any complex reflection. Let $H \in \mathcal{H}$. The splitting $V = H \oplus H^\perp$ defines one of $\wedge^r V$: $\wedge^r V = \wedge^r H \oplus (H^\perp \otimes \wedge^{r-1} H)$. This splitting is the eigenspace decomposition for the action of the cyclic group G_H of $g \in G$ which leave H pointwise fixed. So if $X|H^\circ$ is viewed as a map to $\wedge^r V$, then its image lies in one of these summands. If it is the second summand, then X is transversal to H°, which contradicts the fact that $H \notin \mathcal{H}(\mathcal{F})$. So $X|H^\circ$ maps to the first summand. It follows that X is invariant under G_H indeed. Now write X out in terms of our generators:

$$X = \sum_{1 \leq i_1 < \cdots < i_r \leq n} a_{i_1,\dots,i_r} X_{i_1} \wedge \cdots \wedge X_{i_r}, \quad a_{i_1,\dots,i_r} \in \mathbf{C}[V]^G.$$

Since $\mathcal{F}$ does not contain the Euler field, X is not divisible by X_1 and so a term with $i_1 \geq 2$ appears with nonzero coefficient. This means that the degree of X will be at least $d_2^* + \cdots + d_{r+1}^* \geq r(d_2^*) = r(m_2^* - 1)$. It follows that $m \geq m_2^*$, as asserted. □

Remarks **3.14.** — Only two primitive complex reflection groups of rank > 2 are excluded by the hypothesis of Theorem 3.12: type F_4 and the extended Hesse group (no. 26 in the Shephard-Todd list). The former is a Coxeter group and the latter is an arrangement known to have the same discriminant as the Coxeter group of type B_3 (in the sense of Corollary 5.4). Since we deal with Coxeter groups in a more concrete manner in the next Subsection 3.5, we shall have covered these cases as well.

We shall see that for the case of a Coxeter group and constant $\kappa : \mathcal{H} \to (0,1)$, the Hermitian form h^κ is degenerate precisely when κ_0 is an exponent (Corollary 3.19).

3.5. *A Hecke algebra approach to the Coxeter case.* — The monodromy representation of ∇^κ and its invariant form h^κ can be determined up to equivalence in case the Dunkl connection is associated to a finite Coxeter group.

Let W be an irreducible finite reflection group in a real vector space $V(\mathbf{R})$ without a nonzero fixed vector. We take for $\mathcal{H}$ the collection of reflecting hyperplanes of W in V. It is clear that the orthogonal projection $\pi_H \in \mathrm{End}(V)$ with kernel H is simply $\frac{1}{2}(\mathbf{1} - s_H)$, where s_H is the reflection in H. Choose $\kappa \in \mathbf{R}^{\mathcal{H}}$ to be W-invariant. We know that then $\nabla^\kappa = \nabla^\circ - \kappa \sum_{H\in\mathcal{H}} \omega_H \otimes \pi_H$ is a flat W-invariant connection. We account for the W-invariance by regarding ∇^κ as a connection on the tangent bundle of $W\backslash V^\circ$ (the group W acts freely on V°). So if we fix a base point $* \in W\backslash V^\circ$, then we have a monodromy representation $\rho^{\mathrm{mon}} \in \mathrm{Hom}(\pi_1(W\backslash V^\circ, *), \mathrm{GL}(V))$. It is convenient to let the base point be the image of a real point $x \in V(\mathbf{R})^\circ$. So x lies in a chamber C of W. Let I be a set that labels the (distinct) supporting hyperplanes of C: $\{H_i\}_{i\in I}$ and let us write s_i for s_{H_i}. Then I has $\dim V$ elements. Let $m_{i,j}$ denote the order of $(s_i s_j)$, so that $M := (m_{i,j})_{i,j}$ is the *Coxeter matrix* of W. Then the *Artin group* $\mathrm{Ar}(M)$ associated to M has a generating set $(\sigma_i)_{i\in I}$ with defining relations (the *Artin relations*)

$$\underbrace{\sigma_i\sigma_j\sigma_i\cdots}_{m_{i,j}} = \underbrace{\sigma_j\sigma_i\sigma_j\cdots}_{m_{i,j}},$$

where both members are words comprising $m_{i,j}$ letters. The Coxeter group W arises as a quotient of $\mathrm{Ar}(M)$ by introducing the additional relations $\sigma_i^2 = 1$; σ_i then maps to s_i. According to Brieskorn [4] this lifts to an isomorphism of groups $\mathrm{Ar}(M) \to \pi_1(W\backslash V^\circ, *)$ which sends σ_i to the loop is represented by the path in V° from x to $s_i(x)$ which stays in the contractible set $V^\circ \cap (V(\mathbf{R}) + \sqrt{-1}\bar{C})$.

As long as $|\kappa_i| < 1$, $\rho^{\mathrm{mon}}(\sigma_i)$ is semisimple and acts as a complex reflection over an angle $\pi(1+\kappa_i)$. So if we put $t_i := \exp(\frac{1}{2}\pi\kappa_i\sqrt{-1})$, then σ_i satisfies the identity $(\sigma - 1)(\sigma + t_i^2) = 0$. Although the monodromy need not be semisimple for $\kappa_i = 1$, this equation then still holds (for $t_i^2 = -1$) . In other words, when $-1 < \kappa_i \leq 1$, ρ^{mon} factors through the quotient of of the group algebra $\mathbf{C}[\mathrm{Ar}(M)]$ by the two-sided ideal generated by the elements $(\sigma_i - 1)(\sigma_i + t_i^2)$, $i \in I$. These relations are called the *Hecke relations* and the algebra thus defined is known as the *Hecke algebra* attached to the matrix M with parameters $t = (t_i)_i$. (It is more traditional to use the elements

$-\sigma_i$ as generators; for these the Artin relations remain valid, but the Hecke relations take the form $(\sigma_i+1)(\sigma_i-t_i^2)=0$.)

From now on we regard the t_i's merely as unknowns indexed by the conjugacy classes of reflections in W: so $t_i=t_j$ if s_i and s_j are conjugate in W and t_i are t_j are algebraically independent otherwise. This makes the Hecke algebra one over the polynomial ring in these unknowns. There are at most two conjugacy classes of reflections in W. This results in a partition of I into at most two subsets; we denote by $J\subset I$ a nonempty part. We have two conjugacy classes (i.e., $J\neq I$) only for a Coxeter group of type $I_2(\text{even})$, F_4 and $B_{l\geq 3}$. We denote the associated variables t and t' (when the latter is defined).

If we put all $t_i=1$, then the Hecke algebra reduces to the group algebra $\mathbf{C}[W]$, which is why the Hecke algebra for arbitrary parameters can be regarded as a deformation of this group algebra.

For us is relevant the *reflection representation* of the Hecke algebra introduced in [11]. Since we want the reflections to be unitary relative to some nontrivial Hermitian form we need to adapt this discussion for our purposes. We will work over the domain R obtained from $\mathbf{C}[t_i|\, i\in I]$ by adjoining the square root of $(t_it_j)^{-1}$ for each pair $i,j\in I$. So either $R=\mathbf{C}[t,t^{-1}]$ or $R:=\mathbf{C}[t,t',(tt')^{-1/2}]$, depending on whether W has one or two conjugacy classes of reflections and R contains $t_i^kt_j^l$ if k and l are half integers which differ by an integer. So $T:=\operatorname{Spec}(R)$ is a torus of dimension one or two. Complex conjugation in $\mathbf{C}$ extends to an anti-involution $r\in R\mapsto \overline{r}\in R$ which sends t_i to t_i^{-1} and $(t_it_j)^{1/2}$ to $(t_it_j)^{-1/2}$. This gives T a real structure for which T is anisotropic (i.e., $T(\mathbf{R})$ is compact). We denote by $\Re: R\to R$ 'taking the real part': $\Re(r):=\frac{1}{2}(r+\overline{r})$.

Let $\mathcal{H}(M)$ stand for the Hecke algebra as defined above with coefficients taken in R (so this is a quotient of $R[\operatorname{Ar}(M)]$). For $i,j\in I$ distinct, we define a real element of R:

$$\lambda_{i,j}:=\Re\big(\exp(\pi\sqrt{-1}/m_{i,j})t_i^{1/2}t_j^{-1/2}\big).$$

Notice that $\lambda_{i,j}=\cos(\pi/m_{i,j})$ if $t_i=t_j$. If W has two orbits in $\mathcal{H}$, then there is a unique pair $(j_0,j_1)\in J\times(I-J)$ with $m_{j_0,j_1}\neq 2$. Then m_{j_0,j_1} must be even and at least 4 and we write m for m_{j_0,j_1}, and λ resp. λ' for $\lambda_{j,j'}$ resp. $\lambda_{j',j}$. So $\lambda=\Re(\exp(\pi\sqrt{-1}/m)t^{1/2}t'^{-1/2})$ and $\lambda'=\Re(\exp(\pi\sqrt{-1}/m)t^{-1/2}t'^{1/2})$.

Define for every $i\in I$ a linear form $l_i:R^I\to R$ by

$$l_i(e_j)=\begin{cases}1+t_i^2 & \text{if } i=j,\\ -2\lambda_{i,j}t_i & \text{if } i\neq j.\end{cases}$$

Let $\rho^{\text{refl}}(\sigma_i)$ be the pseudoreflection in R^I defined by

$$\rho^{\text{refl}}(\sigma_i)(z) = z - l_i(z)e_i.$$

We claim that this defines a representation of $\mathcal{H}(M)$. First observe that the minimal polynomial of $\rho^{\text{refl}}(\sigma_i)$ is $(X-1)(X+t_i^2)$. For $i \neq j$, we readily verify that

$$l_i(e_j)l_j(e_i) = t_i^2 + t_j^2 + 2t_it_j\cos(2\pi/m_{i,j}),$$

This implies that the trace of $\rho^{\text{refl}}(\sigma_i)\rho^{\text{refl}}(\sigma_j)$ on the plane spanned by e_i and e_j is equal to $2t_it_j\cos(2\pi/m_{i,j})$. Since its determinant is $t_i^2t_j^2$, it follows that the eigenvalues of $\rho^{\text{refl}}(\sigma_i)\rho^{\text{refl}}(\sigma_j)$ in this plane are $t_it_j\exp(2\pi\sqrt{-1}/m_{i,j})$ and $t_it_j\exp(-2\pi\sqrt{-1}/m_{i,j})$. In particular $\rho^{\text{refl}}(\sigma_i)$ and $\rho^{\text{refl}}(\sigma_j)$ satisfy the Artin relation. So ρ^{refl} defines a representation of $\mathcal{H}(M)$.

Lemma **3.15.** — *Fix $p \in T$ and consider the reflection representation of the corresponding specialization $\mathcal{H}(M)(p)$ on $\mathbf{C}^I$. Then $(\mathbf{C}^I)^{\mathcal{H}(M)(p)}$ is the kernel of the associated linear map $(l_i)_i : \mathbf{C}^I \to \mathbf{C}^I$. Moreover, if K is a proper invariant subspace of $\mathbf{C}^I$ which is not contained in $(\mathbf{C}^I)^{\mathcal{H}(M)(p)}$, then $J \neq I$ and $\lambda\lambda' = 0$ and K equals $\mathbf{C}^J$ resp. $\mathbf{C}^{I-J}$ modulo $(\mathbf{C}^I)^{\mathcal{H}(M)(p)}$ when $\lambda' = 0$ resp. $\lambda = 0$.*

Proof. — The first statement is clear.

Since $K \not\subset (\mathbf{C}^I)^{\mathcal{H}(M)(p)}$, some l_i with will be nonzero on K; suppose this happens for $i \in J$. Let $z \in K$ be such $l_i(z) \neq 0$. From $z - \rho^{\text{refl}}(\sigma_i)(z) = l_i(z)e_i$ it follows that $e_i \in K$. Since $t \neq 0$, our formulas then imply that $K \supset \mathbf{C}^J$. Since K is a proper subspace of $\mathbf{C}^I$, $J \neq I$ and l_j vanishes on K for all $j \in I - J$ (otherwise the same argument shows that $K \supset \mathbf{C}^{I-J}$). This implies in particular that $\lambda' = 0$. □

By sending κ_H to $\exp(\frac{1}{2}\pi\sqrt{-1}\kappa_H)$ we obtain a universal covering

$$\tau : (\mathbf{C}^{\mathcal{H}})^W \to T.$$

Let $\Delta \subset (\mathbf{C}^{\mathcal{H}})^W$ denote the locally finite union of affine hyperplanes defined by: $\kappa_H \in \mathbf{Z}$ and $\kappa_0 \in \{0, -1, -2, \dots\}$.

Proposition **3.16.** — *The map τ lifts to a holomorphic intertwining morphism $\tilde{\tau}$ from the monodromy representation ρ^{mon} of $\mathrm{Ar}(M)$ to the reflection representation ρ^{refl} of $\mathcal{H}(M)$ in such a manner that it is an isomorphism away from Δ and nonzero away from a codimension two subvariety $(\mathbf{C}^{\mathcal{H}})^W$ contained in Δ.*

Proof. — Suppose first $\kappa \notin \mathit{\Delta}$.

Since each κ_H is nonintegral, $\rho^{\mathrm{mon}}(\sigma_i)$ is semisimple and acts in V as a complex reflection (over an angle $\pi(1+\kappa_i)$). Hence $\mathbf{1} - \rho^{\mathrm{mon}}(\sigma_i)$ is of the form $v_i \otimes f_i$ for some $v_i \in V$ and $f_i \in V^*$. The individual f_i and v_i are not unique, only their tensor product is. But we have $f_i(v_i) = 1 + t_i^2 = l_i(e_i)$ and the fact that σ_i and σ_j satisfy the Artin relation implies that $f_i(v_j)f_j(v_i) = t_i^2 + t_j^2 + 2t_i t_j \cos(2\pi/m_{i,j}) = l_i(e_j)l_j(e_i)$.

We claim that the v_i's are then independent and hence form a basis of V. For if that were not the case, then there would exist a nonzero $\phi \in V^*$ which vanishes on all the v_i's. This ϕ will be clearly invariant under the monodromy representation. But this is prohibited by Corollary 2.23 which says that then $\kappa_0 - 1$ must be a negative integer.

Since the Coxeter graph is a tree, we can put a total order on I such that that if $i \in I$ is not the smallest element, there is precisely one $j < i$ with $m_{i,j} \neq 2$. Our assumption implies that whenever $m_{i,j} \neq 2$, at least one of $\lambda_{i,j}$ and $\lambda_{j,i}$ is nonzero. This means that in such a case one of $l_i(e_j)$ and $l_j(e_i)$ is nonzero. On the other hand, it is clear that $l_i(e_j) = 0$ when $m_{i,j} = 2$. We can now choose f_i and e_i in such a manner that $f_i(v_j) = l_i(e_j)$ for all i, j: proceed by induction on i: The fact that for exactly one $j < i$ we have that one of $l_i(e_j)$ and $l_j(e_i)$ is nonzero can be used to fix v_i or f_i and since $v_i \otimes f_i$ is given, one determines the other. This prescription is unambiguous in case both $l_i(e_j)$ and $l_j(e_i)$ are nonzero, for as we have seen, $f_i(v_j)f_j(v_i) = l_i(e_j)l_j(e_i)$.

We thus obtain an intertwining isomorphism $\tilde{\tau}(\kappa) : V \to \mathbf{C}^I$, $e_i \mapsto v_i$, which depends holomorphically on κ and is meromorphic along $\mathit{\Delta}$. Since we are free to multiply $\tilde{\tau}$ by a meromorphic function on $(\mathbf{C}^{\mathcal{H}})^W$, we can arrange that $\tilde{\tau}$ extends holomorphically and nontrivially over the generic point of each irreducible component of $\mathit{\Delta}$. □

Remark **3.17.** — With a little more work, one can actually show that the preceding proposition remains valid if we alter the definition of $\mathit{\Delta}$ by letting κ_H only be an *odd* integer.

We define a Hermitian form H on R^I (relative to our anti-involution) preserved by ρ^{refl}. This last condition means that we want that for all $i \in I$,

$$l_i(z)H(e_i, e_i) = (1 + t_i^2)H(z, e_i).$$

In case all the reflections of W belong to a single conjugacy class so that all t_i take the same value t, then the form defined by

$$H(e_i, e_j) := \begin{cases} \Re(t) & \text{if } i = j, \\ -\cos(\pi/m_{i,j}) & \text{if } i \neq j \end{cases}$$

is as desired. In case we have two conjugacy classes of reflections, then

$$H(e_i,e_j)=\begin{cases}\lambda'\Re(t) & \text{if } i=j\in J,\\ \lambda\Re(t') & \text{if } i=j\in I-J,\\ -\lambda'\cos(\pi/m_{i,j}) & \text{if } i,j\in J \text{ are distinct},\\ -\lambda\cos(\pi/m_{i,j}) & \text{if } i,j\in I-J \text{ are distinct},\\ -\lambda\lambda'\cos(\pi/m_{i,j}) & \text{otherwise.}\end{cases}$$

will do. If we specialize in some $p\in T$, then the kernel of H is of course $\mathcal{H}(M)(p)$-invariant. If $\lambda'(p)=0$ resp. $\lambda(p)=0$, then the formulas show that this kernel contains $\mathbf{C}^J$ resp. $\mathbf{C}^{I-J}$. The zero loci of λ' and λ are disjoint and so no specialization of H is trivial, unless I is a singleton and $t^2=-1$.

We conclude from Proposition 3.16:

Corollary **3.18.** — *Suppose that κ takes values in $(0,1)$. Then the monodromy representation is isomorphic to the reflection representation and thus comes via such an isomorphism with a nonzero W-invariant Hermitian form.*

At points where all the t_i's take the same value (so this is all of T in case $J=I$ and the locus defined by $t=t'$ otherwise), there is a neat formula for the determinant of H, which goes back to Coxeter and appears as Exercise 4 of Ch. V, § 6 in Bourbaki [3]:

$$\det(H(e_i,e_j)_{i,j})=\prod_{j=1}^{|I|}(\Re(t)-\cos(\pi m_j/h)),$$

where h is the Coxeter number of W and the m_j's are the exponents of W. Since $\mathrm{Re}(t)=\cos(\frac{1}{2}\pi\kappa)$. So if $t=\exp(\frac{1}{2}\sqrt{-1}\pi\kappa)$, we see that H is degenerate precisely when $\kappa/4\equiv m_j/2h \pmod{\mathbf{Z}}$ for some m_j. Since the m_j's are distinct and in the interval $\{1,\dots,h-1\}$, the nullity of H is 1 in that case. The cardinality of $\mathcal{H}$ is $h|I|/2$ ([3], Ch. V,§ 6, no. 2, Th. 1), so that $\kappa_0=h\kappa/2$. Hence H is degenerate precisely when $\kappa_0\equiv m_j \pmod{2h\mathbf{Z}}$. If we combine this with the results of Subsection 3.1 and 3.16, we find:

Corollary **3.19.** — *In case $\kappa:\mathcal{H}\to(0,1)$ is constant, then the flat Hermitian form of the associated Dunkl connection is degenerate precisely when κ_0 equals some exponent m_j. In particular, m_2 is the hyperbolic exponent.*

This raises the following

Question **3.20.** — Assuming that I is not a singleton, can we find a system of generators $X_1, \dots, X_{|I|}$ of the $\mathbf{C}[V]^W$-module of W-invariant vector fields on V of the correct degrees $(m_1 - 1, \dots, m_{|I|} - 1)$ such that the ones in degree m_j generate the kernel of the flat Hermitian metric we found for the constant map $\kappa : \mathcal{H} \to (0, 1)$ characterized by $\kappa_0 = m_j$?

It makes sense to ask this question more generally for a complex reflection group (where we should then take the co-exponents as the appropriate generalization). (We checked by an entirely different technique that the Hermitian form attached to a constant map $\kappa : \mathcal{H} \to (0, 1)$ is degenerate precisely when κ_0 is a co-exponent, at least when the group is primitive of rank at least three.)

3.6. *A flat Hermitian form for the Lauricella system.* — Let H be a monodromy invariant Hermitian form on the translation space of A and denote by h the corresponding flat Hermitian form on V°. Suppose that $\kappa_0 \neq 1$, so that we can think of H as a Hermitian form on the vector space (A, O). Then the associated 'norm squared' function, $H(a, a)$, evidently determines H. If we regard H as a translation invariant form on A, then we can express this fact by saying that $\frac{1}{2}\sqrt{-1}\partial\bar{\partial}(H(E_A, E_A)) = \mathrm{Im}(H)$, where E_A is the Euler vector field on (A, O). Since the developing map sends E_V to $(1 - \kappa_0)E_A$, this property is transfered to V° as: if $N : V^\circ \to \mathbf{R}$ is defined by $N := h(E_V, E_V)$, then

$$\frac{\sqrt{-1}}{2}\partial\bar{\partial}N = |1 - \kappa_0|^2 \,\mathrm{Im}(h).$$

So if h is nondegenerate, then the Dunkl connection is also determined by N. It would be interesting to find N explicitly, or at least to characterize the functions N on V° that are thus obtained. We can do this for the Lauricella example:

We consider the Lauricella system 2.3. For the moment we choose all the parameters $\mu_i \in (0, 1)$ as usual, but we now also require that $\mu_0 + \cdots + \mu_n > 1$ (recall that here $\mu_0 + \cdots + \mu_n = \kappa_0$). We write z_{n+1} for ∞ and put $\mu_{n+1} := 2 - \sum_{i=0}^{n} \mu_i$, so that $\mu_{n+1} < 1$ also. Observe that $-\mu_{n+1}$ is 'order of vanishing' of the multivalued Lauricella differential

$$\eta := (z_0 - \zeta)^{-\mu_0} \cdots (z_n - \zeta)^{-\mu_n} d\zeta$$

at z_{n+1} in the sense that there is a local parameter u of $\mathbf{P}^1$ at ∞, such that η is given there by $u^{-\mu_{n+1}}du$. We also notice that $\eta \wedge \overline{\eta}$ is a univalued 2-form and that the conditions imposed on the μ_i's guarantee that it is integrable, provided that $(z_0,\dots,z_n)\in V^\circ$. Since $\frac{\sqrt{-1}}{2}d\zeta\wedge d\bar\zeta$ is the area element of $\mathbf{C}$,

$$N(z_0,\dots,z_n):=-\frac{\sqrt{-1}}{2}\int_{\mathbf{C}}\eta\wedge\overline{\eta}$$

is *negative*. We will show that N is a Hermitian form in Lauricella functions. This implies that the Levi form of N is flat and hence defines a flat Hermitian form on V°.

For this purpose we choose a smoothly embedded oriented interval γ on the Riemann sphere connecting z_0 with $z_{n+1}=\infty$ and passing through $z_1,\dots,z_n$ (in this order). On the complement of γ, η is representable by a holomorphic univalued differential which we extend to $\mathbf{P}^1-\{z_0,\dots,z_{n+1}\}$ by taking on γ the limit 'from the left' relative to the orientation of γ. We continue to denote this differential by η, but this now makes η discontinuous along γ: its limit from the right on the relative interior of the stretch γ_k from z_{k-1} to z_k is easily seen to be $\exp(-2\pi\sqrt{-1}(\mu_0+\cdots+\mu_{k-1}))\eta$. We find it convenient to put $w_0=1$ and $w_k:=\exp(\pi\sqrt{-1}(\mu_0+\cdots+\mu_{k-1}))$ for $k=1,\dots,n$ so that the limit in question can be written $\bar w_k^2\eta$. We put

$$\varPhi_\gamma(\zeta):=\int_{z_0}^{\zeta}\eta,\quad \zeta\in\mathbf{P}^1-\gamma,$$

where the path of integration begins at z_0, but is otherwise not allowed to cross γ. The integral is unambiguously defined, because η is single-valued on the complement of γ (the convergence is not an issue, because η has order $-\mu_{n+1}\geq -1$ at z_{n+1}). In case $z_0,\dots,z_n$ are all real and ordered by size, then a natural choice for γ is the straight line on the real axis which goes from z_0 in the positive direction to ∞. On the interval (z_{k-1},z_k) a natural choice of determination of the integrand is then the one which is real and positive:

$$\eta_k:=(\zeta-z_0)^{-\mu_0}\cdots(\zeta-z_{k-1})^{-\mu_{k-1}}(z_k-\zeta)^{-\mu_k}\cdots(z_n-\zeta)^{-\mu_n}d\zeta.$$

As $\eta_k=\bar w_k\eta$, this suggests to introduce

$$F_{\gamma_k}(z):=\bar w_k\int_{\gamma_k}\eta,\quad k=1,\dots,n+1,$$

in general. Here γ_k is the part of γ which connects z_{k-1} with z_k. So apart from scalar factor $\bar w_k$, this is a Lauricella function. For $\zeta\in\gamma_k$, $k=1,\dots,n+1$, we have

$$\varPhi_\gamma(\zeta)=\sum_{j=1}^{k-1}\int_{\gamma_j}\eta+\int_{z_{k-1}}^{\zeta}\eta=\sum_{j=1}^{k-1}w_jF_{\gamma_k}(z)+w_k\int_{\gamma_k(\zeta)}\eta_k,$$

where $\gamma_k(\zeta)$ is the piece of γ_k that ends in ζ. (For $k=1$ the sum is zero, because its index set is empty.)

Lemma **3.21.** — *Under the above assumptions (so $\mu_k \in (0,1)$ for all k and $\sum_{k=0}^n \mu_k > 1$) the Lauricella functions F_{γ_k} satisfy the linear relation $\sum_{k=1}^{n+1} \operatorname{Im}(w_k) F_{\gamma_k} = 0$ and we have $N(z) = \sum_{1\le j<k\le n+1} \operatorname{Im}(w_j\bar{w}_k)\bar{F}_{\gamma_j}F_{\gamma_k}$.*

Proof. — The limiting value of Φ_γ on γ_k from the right is equal to

$$\sum_{j=0}^{k-1} \bar{w}_j F_{\gamma_j} + \bar{w}_k \int_{\gamma_k(\zeta)} \eta.$$

For $k = n+1$ and $\zeta = z_{n+1}$, this gives $\Phi_\gamma(z_{n+1}) = \sum_{k=1}^{n+1} \bar{w}_k F_{\gamma_k}$. As this is also equal to $\sum_{k=1}^{n+1} w_k F_{\gamma_k}$, we find that $\sum_{k=1}^{n+1} \operatorname{Im}(w_k) F_{\gamma_k} = 0$.

In order to derive the formula for $N(z_0,\dots,z_n)$, we first observe that it is the integral of the exterior derivative of the differential $\frac{\sqrt{-1}}{2}\bar{\Phi}_\gamma \eta$. So by the theorem of Stokes (applied to $\mathbf{C}-\gamma$), $N(z_0,\dots,z_n)$ is equal to the sum $\frac{\sqrt{-1}}{2}\sum_{k=1}^{n+1}\int_{\gamma_k}\delta_k$, where δ_k is the 1-form on γ_k which is the difference between $\bar{\Phi}_\gamma\eta$ and its limiting value from the right. On γ_k,

$$\bar{\Phi}_\gamma \eta = \left(\sum_{j=1}^{k-1} \bar{w}_j \bar{F}_{\gamma_j} + \bar{w}_k \int_{\gamma_k(\zeta)} \bar{\eta}_k\right) w_k \eta_k = \sum_{j=1}^{k-1} \bar{w}_j w_k \bar{F}_{\gamma_j} \eta_k + \left(\int_{\gamma_k(\zeta)} \bar{\eta}_k\right) \eta_k$$

and its limit from the right is there obtained from the above expression by replacing $\bar{w}_j w_k$ by $w_j \bar{w}_k$. Hence

$$N(z_0,\dots,z_n) = \frac{\sqrt{-1}}{2}\int_\gamma \delta = \sum_{1\le j<k\le n+1} \operatorname{Im}(w_j\bar{w}_k)\bar{F}_{\gamma_j}F_{\gamma_k}.$$

□

We regard the F_{γ_i} as (affine-)linear functions on the receiving affine space A of the developing map which satisfy $\sum_{i=1}^{n+1} \operatorname{Im}(w_k) F_{\gamma_k} = 0$. The preceding lemma tells us that N defines a Hermitian form on A that is invariant under the holonomy group. This suggests to consider for any $(n+1)$-tuple $w = (w_1,\dots,w_{n+1})$ of complex numbers of norm one which are not all real, the hyperplane A_w of $\mathbf{R}^{n+1}$ with equation $\sum_{k=1}^{n+1}\operatorname{Im}(w_k)a_k = 0$ and the quadratic form on $\mathbf{R}^{n+1}$ defined by

$$Q_w(a) := \sum_{1\le j<k\le n+1} \operatorname{Im}(w_j\bar{w}_k)(a_j a_k).$$

We determine the signature of Q_w.

Lemma **3.22.** — *Represent* $w_1, \dots, w_{n+1}$ *by real numbers* $\mu_0, \dots, \mu_n$ *as before in the sense that* $w_k = \exp(\pi\sqrt{-1}(\mu_0 + \dots + \mu_{k-1}))$, $k = 1, \dots, n+1$. *Then the* nullity *(that is, the number of zero eigenvalues) of* Q_w *on* A_w *is equal to the number of integers in the sequence* $\mu_0, \dots, \mu_n, \sum_{i=0}^n \mu_i$ *and its* index *(that is, the number of negative eigenvalues) is equal to* $-1 + [\sum_{i=0}^n \mu_i] - \sum_{i=0}^n [\mu_i]$.

Proof. — It is clear that (A_w, Q_w) only depends on the reduction of $\mu_0, \dots, \mu_n$ modulo 2, but the isomorphism type of (A_w, Q_w) only depends on their reduction modulo 1: if we replace μ_k by $\mu_k + 1$, then the new values w'_j of w_j are: $w'_j = w_j$ for $j \le k$ and $w'_j = -w_j$ for $j > k$ and we note that $(a_1, \dots, a_{n+1}) \mapsto (a_1, \dots, a_k, -a_{k+1}, \dots, -a_{n+1})$ turns (A_w, Q_w) into $(A_{w'}, Q_{w'})$. So without loss of generality we may assume that $0 \le \mu_k < 1$ for all k.

For $n = 1$, we get

$$Q_w(a) = \mathrm{Im}(w_1 \bar{w}_2) a_1 a_2 = -\mathrm{Im}(w_1 \bar{w}_2) \frac{\mathrm{Im}(w_1)}{\mathrm{Im}(w_2)} a_1^2 = \frac{\sin(\pi\mu_0)\sin(\pi\mu_1)}{\sin(\pi(\mu_0 + \mu_1))} a_1^2$$

(read zero when $\mu_0 + \mu_1 = 1$) and so the sign of Q_w is the sign of $1 - \mu_0 - \mu_1$.

We proceed by induction on n. So we suppose $n \ge 2$ and the lemma proved for smaller values of n. This allows us to restrict ourselves to the case when $0 < \mu_k < 1$ for all k: if $\mu_k = 0$, then $w_k = w_{k+1}$ and so if $w' := (w_1, \dots, w_k, w_{k+2}, \dots, w_n)$, then (A_w, Q_w) is the pull-back of $(A_{w'}, Q_{w'})$ under $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_{k-1}, a_k + a_{k+1}, a_{k+2}, \dots, a_n)$.

We now let $w' := (w_1, \dots, w_n)$. First assume that $w_n \notin \mathbf{R}$ so that $\sum_{k=0}^{n-1} \mu_k \notin \mathbf{Z}$. According to our induction hypothesis this means that the quadratic space $(A_{w'}, Q_{w'})$ is nondegenerate of index $-1 + [\sum_{i=0}^{n-1} \mu_i]$. There exist unique $s, t \in \mathbf{R}$ such that $w_{n+1} = s w_n + t$. The fact that $0 < \mu_n < 1$ implies that $t \neq 0$. We set $a' := (a_1, \dots, a_{n-1}, a_n + s a_{n+1})$. Then we have

$$\sum_{k=1}^{n+1} \mathrm{Im}(w_k) a_k - \sum_{k=1}^{n} \mathrm{Im}(w'_k) a'_k = \mathrm{Im}(w_{n+1} a_{n+1} - w_n a_{n+1} s) = \mathrm{Im}(t a_{n+1}) = 0$$

so that $a \in A_w$ if and only if $a' \in A_{w'}$. A similar calculation shows that

$$Q_w(a) = Q_{w'}(a') - t\,\mathrm{Im}(w_{n+1}) a_{n+1}^2, \quad a \in A_w.$$

If $w_{n+1} \notin \mathbf{R}$, then from the equality $t = -s w_n + w_{n+1}$ and the fact that $-w_n$ makes a positive angle (less than π) with w_{n+1}, we see that $t\,\mathrm{Im}(w_{n+1}) > 0$ if and only if the signs $\mathrm{Im}(w_n)$ and $\mathrm{Im}(w_{n+1})$ are opposite. The latter amounts to $[\mu_0 + \dots + \mu_n] = [\mu_0 + \dots + \mu_{n-1}] + 1$, and so here the induction hypothesis yields the lemma for (A_w, Q_w). This is also the case when $w_{n+1} \in \mathbf{R}$, for then $\sum_{i=0}^n \mu_i \in \mathbf{Z}$.

Suppose $w_n \in \mathbf{R}$, in other words, that $\sum_{i=0}^{n-1} \mu_i \in \mathbf{Z}$. If we let $w'' = (w_1, \dots, w_{n-1})$, then $Q_{w'}(a_1, \dots, a_n) = Q_{w''}(a_1, \dots, a_{n-1})$. We may assume that $n \geq 2$, so that $A_{w''}$ is defined. By induction, $(A_{w''}, Q_{w''})$ is nondegenerate of index $-1 + [\sum_{i=0}^{n-2} \mu_i]$. It is now easy to check that (A_w, Q_w) is isomorphic to the direct sum of $(A_{w''}(\mathbf{R}), Q_{w''})$ and a hyperbolic plane. Hence (A_w, Q_w) is nondegenerate of index $[\sum_{i=0}^{n-2} \mu_i]$. This last integer is equal to $-1 + \sum_{i=0}^{n-1} \mu_i$ and hence also equal to $-1 + [\sum_{i=0}^{n} \mu_i]$. □

Corollary **3.23.** — *The function N defines an invariant Hermitian form on the Lauricella system whose isomorphism type is given by Lemma 3.22. If $0 < \mu_k < 1$ for all k, then the form is admissible of elliptic, parabolic, hyperbolic type for $\kappa_0 < 1$, $\kappa_0 = 1$, $1 < \kappa_0 < 2$ respectively. The hyperbolic exponent of this family is equal to* 2.

Proof. — Lemma 3.22 tells us that the Hermitian form is positive for $0 < \kappa_0 < 1$, positive semidefinite with nullity one for $\kappa = 1$, nondegenerate hyperbolic for $1 < \kappa_0 < 2$, and degenerate for $\kappa_0 = 2$. The admissibility follows from Theorem 3.1 for the case $\kappa_0 = 1$. For hyperbolic range $1 < \kappa_0 < 2$ it is a consequence of the fact that N is negative. □

Remark **3.24.** — In the hyperbolic case: $\mu_i \in (0, 1)$ for all i and $\sum_i \mu_i \in (1, 2)$, we observed with Thurston in Subsection 3.2 that $\mathbf{P}(V^\circ)$ can be understood as the moduli space of Euclidean metrics on the sphere with $n+2$ conical singularities with a prescribed total angle. The hyperbolic form induces a natural complex hyperbolic metric on $\mathbf{P}(V^\circ)$. The modular interpretation persists on the metric completion of $\mathbf{P}(V^\circ)$: in this case we allow some of the singular points to collide, that is, we may include some the diagonal strata. This metric completion is quite special and is of the same nature as the objects it parametrizes: it is what Thurston calls a cone manifold.

Remark **3.25.** — If each μ_i is positive and rational, then the associated Lauricella system with its Hermitian form can also be obtained as follows. Let q be a common denominator, so that $\mu_i := p_i/q$ for some positive integer p_i, and put $p := \sum_i p_i$. Consider the Dunkl system on the Coxeter arrangement of type A_{p-1} defined by the diagonal hyperplanes in the hyperplane V_p in $\mathbf{C}^p$ defined by $\sum_{i=1}^{q} z_i = 0$ and with κ constant equal to $1/q$. Let $V_P \subset V_p$ be the intersection of hyperplanes defined by the partition $P := (p_0, p_1, \dots, p_n)$ of p. Then the Lauricella system can be identified with longitudinal system on V_P. The Hermitian form that we have on the ambient system via the Hecke algebra approach 3.5 is inherited by V_P (as a flat Hermitian form). This approach is taken (and consistently followed) by B. Doran in his thesis [16].

3.7. *The degenerate hyperbolic case.* — By a *degenerate hyperbolic form* on a vector space we simply mean a degenerate Hermitian form which is a hyperbolic form on the quotient of this vector space by kernel of the form. If H is such a form on the vector space A with kernel K, then the subset $\mathbf{B} \subset \mathbf{P}(A)$ defined by $H(a,a) < 0$ is fibered by affine spaces over a complex ball: since H induces a nondegenerate form H' on $A' := A/K$, there is a ball $\mathbf{B}'$ defined in $\mathbf{P}(A')$ by $H'(a',a') < 0$ and the projection $A \to A'$ induces a fibration $\pi : \mathbf{B} \to \mathbf{B}'$ whose fibers are affine spaces of the same dimension as K. The vector group $\mathrm{Hom}(A', K)$ acts as a group of bundle automorphisms of π (over the identity on $\mathbf{B}'$), but this action is not proper. So if the holonomy preserves a form of this type it might not act properly on $\mathbf{B}$. Yet, this sometimes happens; in such cases the foliation of V° defined by the kernel of H are often orbits of a 'hidden symmetry' of the system and the holonomy acts on $\mathbf{B}'$ via a discrete group of automorphisms. A case in point is the Lauricella system with $n = 11$ and all μ_i's are equal to $\frac{1}{6}$: then Lemma 3.22 tells us that the Hermitian form in question is degenerate hyperbolic with kernel of dimension one.

In order to understand the extra symmetry, let us consider more generally case when $\mu_i \in (0,1)$ for $i = 0, \dots, n$ and $\sum_i \mu_i = 2$ (so that $\mu_{n+1} = 0$). As before, we put $w = (w_k := e^{\pi\sqrt{-1}(\mu_0 + \cdots + \mu_{k-1})})_{k=1}^{n+1}$ (so that $w_{n+1} = 1$). On the hyperplane $A_w \subset \mathbf{R}^{n+1}$ defined by $\sum_{i=1}^{n+1} \mathrm{Im}(w_i) a_i = 0$ we have defined the quadratic form $Q_w : A_w \to \mathbf{C}$, $Q_w(a) = \sum_{1 \le j < k \le n+1} \mathrm{Im}(w_j \bar{w}_k) a_j a_k$. According to Lemma 3.22, Q_w is degenerate hyperbolic with one dimensional kernel. In fact, if $w' := (w_1, \dots, w_n)$, then omission of the last coordinate, $a = (a_1, \dots, a_{n+1}) \mapsto a' := (a_1, \dots, a_n)$, defines a projection $A_w \to A_{w'}$, we have $Q_w(a) = Q_{w'}(a')$ and $Q_{w'}$ is nondegenerate of hyperbolic signature (see the proof of Lemma 3.22). This describes the situation at the receiving end of the developing map. Now let us interpret this in the domain. The projection $A_w \otimes \mathbf{C} \to A_{w'} \otimes \mathbf{C}$ amounts to ignoring the Lauricella function $F_{\gamma_{n+1}}$; this is the only one among the $F_{\gamma_1}, \dots, F_{\gamma_{n+1}}$ which involves an integral with $z_{n+1} = \infty$ as end point. Recall that the condition $\sum_i \mu_i = 2$ implies that ∞ is not a singular point of the Lauricella form $\eta = (z_0 - \zeta)^{-\mu_0} \cdots (z_n - \zeta)^{-\mu_n} d\zeta$. This suggests an invariance property with respect to Möbius transformations. This is indeed the case: a little exercise shows that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{C})$ transforms η into $(cz_0 + d)^{\mu_0} \cdots (cz_n + d)^{\mu_n} \eta$. Hence the first n coordinates of the developing map $(F_{\gamma_1}, \dots, F_{\gamma_{n+1}})$ (with values in $A_w \otimes \mathbf{C}$) all get multiplied by the same factor: for $k = 1, \dots, n$ we have

$$F_{\gamma_k}\Big(\frac{az_0 + b}{cz_0 + d}, \dots, \frac{az_n + b}{cz_n + d}\Big) = (cz_0 + d)^{\mu_0} \cdots (cz_n + d)^{\mu_n} F_{\gamma_k}(z_0, \dots, z_n).$$

In geometric terms this comes down to the following. Embed $\mathbf{C}^{n+1}$ in $(\mathbf{P}^1)^{n+1}$ in the obvious manner and let the Möbius group $\mathrm{PSL}(2,\mathbf{C})$ act on $(\mathbf{P}^1)^{n+1}$ diagonally. This defines a birational action of $\mathrm{PSL}(2,\mathbf{C})$ on $(\mathbf{C}^{n+1})^\circ$. Recall that V° stands for the quotient of $(\mathbf{C}^{n+1})^\circ$ by the main diagonal. The obvious map $(\mathbf{C}^{n+1})^\circ \to \mathbf{P}(V^\circ)$ is the formation of the orbit space with respect to the group of affine-linear transformations of $\mathbf{C}$. Hence a $\mathrm{PSL}(2,\mathbf{C})$-orbit in $(\mathbf{P}^1)^{n+1}$ which meets $(\mathbf{C}^{n+1})^\circ$ maps to a rational curve in $\mathbf{P}(V^\circ)$. Thus the fibration of $\mathbf{P}(V^\circ)$ can (and should) be thought of as the forgetful morphism $\mathcal{M}_{0,n+2} \to \mathcal{M}_{0,n+1}$ which ignores the last point: it is represented by $(\mathbf{P}^1; z_0,\dots,z_n,\infty) \mapsto (\mathbf{P}^1; z_0,\dots,z_n)$. In particular, the fiber is an $(n+1)$-pointed rational curve; it can be understood as the curve on which is naturally defined the Lauricella form η (up to a scalar multiple). Thus we have before us the universal family for the Lauricella integral. We conclude:

Proposition **3.26.** — *The fibration $\mathcal{M}_{0,n+2} \to \mathcal{M}_{0,n+1}$ integrates the distribution defined by the kernel of the flat Hermitian form so that we have a commutative diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{0,n+2} & \longrightarrow & \mathbf{B}_w \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}_{0,n+1} & \longrightarrow & \mathbf{B}_{w'} \end{array}$$

where on the left we have the holonomy cover of $\mathcal{M}_{0,n+2} \to \mathcal{M}_{0,n+1}$ and on the right $\mathbf{B}_w$ and $\mathbf{B}_{w'}$ are the open subsets of $\mathbf{P}(A_w \otimes \mathbf{C})$ resp. $\mathbf{P}(A_{w'} \otimes \mathbf{C})$ defined by the Hermitian forms.

The holonomy along a fiber of $\mathcal{M}_{0,n+2} \to \mathcal{M}_{0,n+1}$ is understood as follows. Let $C := \mathbf{P}^1 - \{z_0,\dots,z_n\}$ represent a point of $\mathcal{M}_{0,n+1}$. The map $H_1(C;\mathbf{Z}) \to \mathbf{R}$ which assigns to a small circle centered at z_i the value μ_i defines an abelian covering of C; it is a covering on which the Lauricella integrand becomes single valued. Yet another abelian cover may be needed to make this single valued form exact. The resulting nilpotent cover $\widetilde{C} \to C$ appears as a fiber of $\widetilde{\mathcal{M}}_{0,n+2} \to \widetilde{\mathcal{M}}_{0,n+1}$ and the developing map restricted to this fiber is essentially the function $\widetilde{C} \to \mathbf{C}$ which integrates the Lauricella integrand.

4. The Schwarz conditions

4.1. *The Schwarz symmetry groups.* — We begin with recalling the basic Example 1.5.

Example **4.1** *(Example 1.5 revisited).* — Here V equals $\mathbf{C}$, $\mathcal{H}$ consists of the origin and $\Omega = \kappa z^{-1}dz$. We assume that we have finite holonomy, so that we can write $1-\kappa = p/q$ with p, q relatively prime integers and $q > 0$. The holonomy cover extends with ramification over the origin as the q-fold cover $\tilde{V} \to V$ defined by $\tilde{z}^q = z$. The developing map $\tilde{V} - \{0\} \to \mathbf{C}$ is given by $w = \tilde{z}^p$ and hence extends across the origin only if $p > 0$, that is, if $\kappa < 1$. Assuming this to be the case, it is then injective only if $p = 1$. But rather than imposing this condition, we could note that the connection is invariant under the pth roots of unity μ_p; the μ_p-orbit space of V is covered by the μ_p-orbit space of $\widetilde{V}$ and the developing map factors through the latter as an isomorphism onto $\mathbf{C}$. This observation motivates the definition below.

Definition **4.2.** — *Given a Dunkl system, then we say that $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ satisfies the* Schwarz condition *if κ_L is a rational number, which is either* 1 *or has the following property: if we write $1 - \kappa_L = p_L/q_L$ with p_L, q_L relatively prime and $q_L > 0$, then the Dunkl system is invariant under the group G_L of unitary transformations of V which fix L pointwise and act as scalar multiplication in $L^\perp$ by a $|p_L|$th root of unity. We call G_L the* Schwarz rotation group *of L. The* Schwarz symmetry group *is the subgroup of the unitary group of V generated by the Schwarz rotation groups G_L of the $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ which satisfy the Schwarz condition; we will usually denote it by G. We say that the Dunkl system* satisfies the Schwarz condition *if every $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ satisfies the Schwarz condition and that it* satisfies the Schwarz condition in codimension one *if this is only known to be true for every member of $\mathcal{H}$.*

Notice that the Schwarz symmetry group is finite: this follows from the fact that the group of projective-linear transformations of $\mathbf{P}(V)$ which leave $\mathcal{H}$ invariant is finite (since $\mathcal{H}$ is irreducible) and the fact that the determinants of the generators of G are roots of unity. This group may be trivial or be reducible nontrivial (despite the irreducibility of $\mathcal{H}$). If the Schwarz symmetry group is generated in codimension one, then according to Shephard-Todd-Chevalley theorem, the orbit space $G\backslash V$ is isomorphic to affine space.

It it clear that $\{0\}$ always satisfies the Schwarz condition.

Example **4.3.** — For the Lauricella system discussed in Subsection 2.3, the Schwarz condition in codimension one amounts to: for $0 \leq i < j \leq n$, $1 - \mu_i - \mu_j$ is a positive rational number with numerator 1 or 2 with 2 only allowed if $\mu_i = \mu_j$. This last possibility is precisely Mostow's ΣINT-condition [26].

Let $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$. If a Dunkl system satisfies the Schwarz condition, then this property is clearly inherited by the L-transversal Dunkl system. This is also true for the L-longitudinal Dunkl system:

Lemma **4.4.** — *Suppose that the Dunkl system satisfies the Schwarz condition. Then for every $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$, the longitudinal Dunkl system on L° also satisfies the Schwarz condition.*

Proof. — Let $M \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^L)$. Either M is irreducible in $\mathcal{H}$ or M is reducible with two components L and M'. The exponent of M relative to $\mathcal{H}^L$ is then κ_M and $\kappa_{M'}$ respectively. It is clear that the Schwarz symmetry group of M resp. M' preserves L. ☐

4.2. *An extension of the developing map.* — We can think of the holonomy of the Dunkl system as a homomorphism from the fundamental group of V° to a group of affine-linear transformations, which is given up to an inner automorphism of the fundamental group. The holonomy group Γ is the image of that homorphism. Every point of V determines a conjugacy class of subgroups in the fundamental group of V° (namely the image of the map on fundamental groups of the inclusion in V° of the trace on V° of a small convex neighborhood of that point), hence also determines a conjugacy class in the holonomy group Γ of the Dunkl connection. If the latter is a conjugacy class of finite subgroups we say that we have *finite holonomy at this point*. The set $V^f \subset V$ of the points at which we have finite holonomy is a union of $\mathcal{H}$-strata which contains V° and is open in V (the subscript f stands for finite). We denote the corresponding subset of $\mathcal{L}(\mathcal{H})$ by $\mathcal{L}^f(\mathcal{H})$. The holonomy covering extends uniquely to a ramified Γ-covering $\widetilde{V^f} \to V^f$ and V^f is the maximal subset of V for which this is the case.

If each κ_H is rational $\neq 1$, then $\mathcal{L}^f(\mathcal{H})$ contains $\mathcal{H}$ and so $V - V^f$ is everywhere of codimension ≥ 2.

Theorem **4.5.** — *Assume that κ takes values in the rational numbers. Then the action of the Schwarz symmetry group G on V° is free and lifts naturally to one on $\widetilde{V^f}$. The latter action commutes with the Γ-action and the developing map $\mathrm{ev} : \widetilde{V^\circ} \to A$ is constant on G-orbits: it factors through a morphism $\mathrm{ev}_G : G\backslash\widetilde{V^\circ} \to A$.*

If $\kappa_0 \neq 1$ and $1 - \kappa_0$ is written as a fraction p_0/q_0 with p_0, q_0 relatively prime and $q_0 > 0$ as usual, then $G \cap \mathbf{C}^\times$ consists of the $|p_0|$-th roots of unity and both $\widetilde{V^f}$ and $G\backslash\widetilde{V^f}$ come with natural effective $\mathbf{C}^\times$-actions such that $\widetilde{V^f} \to V^f$ is homogeneous of degree q_0, $\widetilde{V^\circ} \to G\backslash\widetilde{V^\circ}$ is homogeneous of degree p_0 and $\mathrm{ev}_G : G\backslash\widetilde{V^\circ} \to A$ is homogeneous of degree one.

In case $\kappa_0 = 1$, then the lift of the Euler vector field generates a free action of $\mathbf{C}^+$ on $G\backslash\widetilde{V^\circ}$ such that ev_G is equivariant with respect to a one-dimensional translation subgroup of A.

Proof. — Since G preserves the Dunkl connection, it preserves the local system Aff_{V°. So G determines an automorphism group Γ_G of $\widetilde{V^\circ}$ (with its affine structure) which contains the holonomy group Γ and has G as quotient acting in the given manner on V°. This group acts on A as a group of affine-linear transformations. Denote by $\tilde{G}$ the kernel of this representation. Since Γ acts faithfully on A, $\tilde{G}\cap\Gamma = \{1\}$ and so the map $\tilde{G}\to G$ is injective. On the other hand, if $L\in\mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ satisfies the Schwarz condition, then the local model near the blowup of L in V shows that $\tilde{G}$ contains a cyclic subgroup of order $|p_L|$ which maps onto G_L. Since G is generated by the subgroups G_L, this proves that $\tilde{G}\to G$ is an isomorphism, in other words, that the action of G on V° lifts naturally to $\widetilde{V^\circ}$. It also follows that Γ_G is the direct product of Γ and $\tilde{G}$ and that the developing map factors as asserted. Since the developing map is a local isomorphism on $\widetilde{V^\circ}$, the action of G on $\widetilde{V^\circ}$ must be free.

Suppose now $\kappa_0\neq 1$. The holonomy of Aff_{V° along a $\mathbf{C}^\times$-orbit in V° is of order q_0 and so $\widetilde{V^\circ}$ comes with an effective $\mathbf{C}^\times$-action for which its projection to V° is homogeneous of degree q_0. The developing map $\mathrm{ev}:\widetilde{V^\circ}\to A$ is constant on the orbits of the order $|p_0|$ subgroup of $\mathbf{C}^\times$, but not for any larger subgroup. The infinitesimal generators of the $\mathbf{C}^\times$-actions on $\widetilde{V^\circ}$ and A are compatible and so ev is homogeneous of degree p_0 and there is a (unique) effective $\mathbf{C}^\times$-action on $G\backslash\widetilde{V^\circ}$ which makes $\widetilde{V^\circ}\to G\backslash\widetilde{V^\circ}$ homogeneous of degree p_0. Then $\mathrm{ev}_G : G\backslash\widetilde{V^\circ}\to A$ will be homogeneous of degree one. These actions extend to $\widetilde{V^f}$ and $G\backslash\widetilde{V^f}$ respectively.

The last assertion follows from the fact that the holonomy along a $\mathbf{C}^\times$-orbit in V is a nontrivial translation. $\square$

Theorem **4.6.** — *Suppose that every κ_H is a rational number smaller than 1 and that the Dunkl system satisfies the Schwarz condition in codimension one. Then the developing map $\widetilde{V^\circ}\to A$ extends to $\widetilde{V^f}$ and this extension drops to a local isomorphism $\mathrm{ev}_G : G\backslash\widetilde{V^f}\to A$. In particular, $G\backslash\widetilde{V^f}$ is smooth and the G-stabilizer of a point of $\widetilde{V^f}$ acts near that point as a complex reflection group. Moreover, every $L\in\mathcal{L}_{\mathrm{irr}}(\mathcal{H})\cap\mathcal{L}^f(\mathcal{H})$ satisfies the Schwarz condition and has $\kappa_L < 1$.*

Proof. — The local model of the connection near the generic point of $H\in\mathcal{H}$ shows that $H^\circ\subset V^f$ and that the developing map extends over

H° and becomes a local isomorphism if we pass to the G_H-orbit space. So the developing map extends to $\widetilde{V^f}$ in codimension one. Hence it extends to all of $\widetilde{V^f}$ and the resulting extension of ev_G to $G\backslash\widetilde{V^f}$ will even be a local isomorphism.

Now let $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}) \cap \mathcal{L}^f(\mathcal{H})$. Then the composite of ev with a generic morphism $(\mathbf{C}, 0) \to (V, L^\circ)$ is of the form $z \mapsto z^{1-\kappa_L}$ plus higher order terms (for $\kappa_L \neq 1$) or $z \mapsto \log z$ plus higher order terms (for $\kappa_L = 1$). As the developing map extends over L°, we must have $\kappa_L < 1$. Since the developing map is in fact a local isomorphism at L°, L must satisfy the Schwarz condition. □

Remark **4.7.** — The orbit spaces $G\backslash V$ and $G\backslash\widetilde{V^f}$ are both smooth. Notice that $G\backslash V^f$ underlies two affine orbifold structures: one of these is obtained as the G-orbit space of V^f and has orbifold fundamental group G, whereas the other inherits this structure from the Dunkl connection and has $\mathrm{ev}_G : G\backslash\widetilde{V^f} \to A$ as developing map. In the next two sections we shall refine and extend this 'Januslike' feature of $G\backslash V^f$ within the realm of complex differential geometry.

5. Geometric structures of elliptic and parabolic type

5.1. *Dunkl connections with finite holonomy.* — In case Γ is finite, then the vector space (A, O) admits a Γ-invariant Hermitian positive definite inner product. In particular, the tangent bundle of V° admits a positive definite inner product invariant under the holonomy group of the Dunkl connection. Since the Dunkl connection is torsion free, the latter is then the Levi-Civita connection of this metric. Conversely:

Theorem **5.1.** — *Suppose that $\kappa \in (0,1)^{\mathcal{H}}$, that the Dunkl system satisfies the Schwarz condition in codimension one and that there is a flat positive definite Hermitian form on the tangent bundle of V°. Then the holonomy of the affine structure defined by the Dunkl connection is finite and so we are in the situation where* ev_G *is a Γ-equivariant isomorphism of $G\backslash\widetilde{V}$ onto A and $\kappa_0 < 1$. In particular, this map descends to an isomorphism of orbit spaces of reflection groups $G\backslash V \to \Gamma\backslash A$ via which $\mathbf{P}(G\backslash V)$ acquires another structure as a complete elliptic orbifold.*

The proof of Theorem 5.1 uses the following topological lemma. We state it in a form that makes it applicable to other cases of interest.

Lemma **5.2.** — *Let $f : X \to Y$ be an continuous map with discrete fibers between locally compact Hausdorff spaces and let $Y' \subset Y$ be an open subset of which the topology is given by a metric. Suppose that there is a symmetry group Γ of this situation (i.e., Γ acts on X and Y, f is Γ- equivariant and Γ preserves Y' and acts there as a group of isometries) for which the following properties hold:*

(i) The action of Γ on X is cocompact.

(ii) For every $y \in Y$ and neighborhood V of y in Y there exists an $\varepsilon > 0$ and a neighborhood V' of y such that the ε-neighborhood of $V' \cap Y'$ is contained in V.

Then there exists an $\varepsilon > 0$ such that every $x \in f^{-1}Y'$ has a neighborhood which is proper over the ε-ball in Y' centered at $f(x)$. In particular, if f is a local homeomorphism over Y' and Y' is connected and locally connected, then f is a covering projection over Y'.

Proof. — Let $x \in X$. Since the fiber through x is discrete, we can find a compact neighborhood K of x such that $f(x) \notin f(\partial K)$. Put $U_x := K \setminus f^{-1}f(\partial K)$ and $V_x := Y - f(\partial K)$ so that U_x is a neighborhood of x, V_x a neighborhood of $f(x)$ and f maps U_x properly to V_x. By (ii) there exist a neighborhood V'_x of $f(x)$ and a $\varepsilon_x > 0$ such that such that for every $y \in V'_x \cap Y'$ the ε_x-neighborhood of y is contained in V_x. We let U'_x be the preimage of V'_x in U_x. It has the property that any ε_x-ball centered at a point of $f(U'_x) \cap Y'$ has a preimage in U_x that is proper over that ball.

Let $C \subset X$ be compact and such that $\Gamma.C = X$. Then C is covered by $U'_{x_1}, \dots, U'_{x_N}$, say. We claim that $\varepsilon := \min_{i=1}^N\{\varepsilon_{x_i}\}$ has the required property. Given any $x \in f^{-1}Y'$, then $\gamma x \in U'_{x_i}$ for some i and $\gamma \in \Gamma$. By construction, the ε-ball centered at $f(\gamma x)$ is contained in V_{x_i} and its preimage in U_{x_i} is proper over that ball. Now take the translate over γ^{-1} and we get the desired property at x. $\square$

Proof of Theorem 5.1. — We have already verified this when $\dim(V) = 1$. So we take $\dim(V) \geq 2$ and assume inductively the theorem proved for lower values of $\dim(V)$. The induction hypothesis implies that V^f contains $V' = V - \{0\}$. By Theorem 4.6 ev_G is then a local isomorphism on preimage $G\backslash\widetilde{V'}$. On $G\backslash\widetilde{V'}$ we have an effective $\mathbf{C}^\times$-action for which ev_G is homogeneous of nonzero degree. Since ev_G is a local isomorphism, it maps $G\backslash\widetilde{V'}$ to $A - \{O\}$ and is the $\mathbf{C}^\times$-action on $G\backslash\widetilde{V'}$ without fixed points. So ev_G induces a local isomorphism of $\mathbf{C}^\times$-orbit spaces $G\backslash\mathbf{P}(\widetilde{V'}) \to \mathbf{P}(A)$. The action of Γ on $G\backslash\mathbf{P}(\widetilde{V'})$ is discrete and the orbit space of this action is a finite quotient of $\mathbf{P}(V)$ and hence compact. So $G\backslash\mathbf{P}(\widetilde{V'}) \to \mathbf{P}(A)$ satisfies the hypotheses of Lemma 5.2 (with $Y' = Y = \mathbf{P}(A)$), hence is a covering map. Then $\mathrm{ev}_G : G\backslash\widetilde{V'} \to A - \{O\}$ is also a covering map. But $A - \{O\}$ is simply connected and so this must be an isomorphism. Such a map extends across the origin and so the degree of homogeneity is positive: $1 - \kappa_0 > 0$. It also follows that the subgroup Γ of $\mathrm{GL}(A)$ acts properly discretely on $A - \{O\}$ so that Γ is finite. □

5.2. *A remarkable duality.* — Suppose that the holonomy of the Dunkl connection is finite. Then according to Theorem 4.6, we have $\kappa_L < 1$ for all $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ and the developing map defines a isomorphism of $G\backslash V$ onto $\Gamma\backslash A$. So $G\backslash V$ has two orbifold structures, one with orbifold fundamental group G, another with Γ.

There is a simple relation between the invariant theory of the groups G and Γ, which was observed earlier by Orlik and Solomon [27] in a somewhat different and more special setting.

The $\mathbf{C}^\times$-action on (A, O) descends to a $\mathbf{C}^\times$-action on $\Gamma\backslash A$ with kernel $\Gamma \cap \mathbf{C}^\times$. Let $1 \le d_1(\Gamma) \le d_2(\Gamma) \le \cdots \le d_{\dim A}(\Gamma)$ be the set of weights of this action, ordered by size. The degrees > 1 are the degrees of the basic invariants of Γ. Their product $\prod_i d_i(\Gamma)$ is the degree of $A \to \Gamma\backslash A$, that is, the order of Γ. The situation for the G-action is likewise. The isomorphism between the two orbit spaces is $\mathbf{C}^\times$-equivariant once we pass to the corresponding effective actions. This implies that the weights of these groups are proportional:

$$d_i(\Gamma) = (1 - \kappa_0)^{-1} d_i(G), \quad i = 1, \ldots, \dim V.$$

So the degrees of Γ are readily computed from the pair (κ, G). In particular, we find that

$$|\Gamma| = (1 - \kappa_0)^{-\dim V} |G|.$$

The isomorphism $G\backslash V \to \Gamma\backslash A$ maps the G-orbit space of the union of the hyperplanes from $\mathcal{H}$ onto a hypersurface in A whose preimage in A is a Γ-invariant union of hyperplanes containing the reflecting hyperplanes of Γ. If we denote that linear arrangement in A by $\mathcal{H}'$, then we have bijection between the G-orbits in $\mathcal{H}$ and the Γ-orbits in the $\mathcal{H}'$.

We can also go in the opposite direction, that is, start with the finite reflection group Γ on A and define a compatible Γ-invariant Dunkl connection on A whose holonomy group is G and has a developing map equal to the inverse of the developing map of the Dunkl connection on V. The following theorem exhibits the symmetry of the situation. At the same time it shows that all pairs of reflection groups with isomorphic discriminants arise from Dunkl connections.

Theorem **5.3.** — *Let for $i = 1, 2$, $G_i \subset \mathrm{GL}(V_i)$ be a finite complex reflection group and $D_i \subset V_i$ its union of reflection hyperplanes. Then any isomorphism of orbit spaces $f : G_1\backslash V_1 \to G_2\backslash V_2$ which maps $G_1\backslash D_1$ onto $G_2\backslash D_2$ and is $\mathbf{C}^\times$-equivariant relative the natural* effective *$\mathbf{C}^\times$-actions on range and domain is obtained from the developing map of a G_1-invariant Dunkl connection on $V_1 - D_1$ (and then likewise for f^{-1}, of course).*

Proof. — The ordinary (translation invariant) flat connection on V_2 descends to a flat connection on $G_2\backslash(V_2 - D_2)$. Pull this back via f to a flat connection on $G_1\backslash(V_1 - D_1)$ and lift the latter to a G-invariant flat connection ∇ on $V_1 - D_1$. It is clear that ∇ is $\mathbf{C}^\times$-invariant. A straightforward local computation at the generic point of a member of the arrangement shows that ∇ extends to the tangent bundle of V_1 with a logarithmic poles and semisimple residues. So by Proposition 2.2 it is a Dunkl connection. It is clear that f realizes its developing map. □

Corollary **5.4.** — *Let for $i = 1, 2$, $G_i \subset \mathrm{GL}(V_i)$ be a finite irreducible complex reflection group and $D_i \subset V_i$ its union of reflecting hyperplanes. If the germs of $G_1\backslash D_1$ and $G_2\backslash D_2$ at their respective origins are isomorphic, then the two are related by the above construction: one is obtained from the other by means of the developing map of a Dunkl connection.*

Proof. — Any isomorphism of germs $f : G_1\backslash(V_1, D_1, 0) \to G_2\backslash(V_2, D_2, 0)$ takes the effective $\mathbf{C}^\times$-action on $G_1\backslash V_1$ to an effective $\mathbf{C}^\times$-action on the germ $G_2\backslash(V_2, D_2, 0)$. A finite cover of this action lifts to an effective action on the germ $(V_2, D_2, 0)$ which commutes with the action of G_2. Restrict this action to the tangent space of V_2 at the origin. This action preserves D_2. Since D_2 is the union of hyperplanes of an irreducible arrangement, the $\mathbf{C}^\times$-action in the tangent space T_0V_2 must be given by a single weight (i.e., by scalar multiplication). So if we identify this tangent space with V, then we get another isomorphism $f_0 : (G_1\backslash V_1, D_1, 0) \to (G_2\backslash V_2, D_2, 0)$ which is $\mathbf{C}^\times$-equivariant (and hence extends globally as such). Now apply Theorem 5.3 □

Remark **5.5.** — The group G_i acts on $\mathcal{L}(\mathcal{H}_i)$ as a group of poset automorphisms and we have a quotient poset $G_i\backslash\mathcal{L}(\mathcal{H}_i)$. The ramification function induces $\kappa_i : G_i\backslash\mathcal{L}_{\mathrm{irr}}(\mathcal{H}_i) \to \mathbf{Q}$. If z_i is the function on $G_i\backslash\mathcal{L}_{\mathrm{irr}}(\mathcal{H}_i)$ which assigns to $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}_i)$ the order of the group of scalars in the image of $Z_{G_i}(L)$ in V_i/L, then the isomorphism f of this theorem induces an isomorphism of posets $G_1\backslash\mathcal{L}(\mathcal{H}_1) \cong G_2\backslash\mathcal{L}(\mathcal{H}_2)$ which takes z_2 to $(1 - \kappa_1)z_1$ and z_1 to $(1 - \kappa_2)z_2$.

5.3. *Dunkl connections with almost-Heisenberg holonomy.* — This subsection treats the parabolic case. The principal result is as follows.

Theorem **5.6.** — *Let be given a Dunkl system with $\kappa \in (0,\infty)^{\mathcal{H}}$ and $\kappa_0 = 1$, which satisfies the Schwarz condition in codimension one and admits a nontrivial flat Hermitian form ≥ 0. Then:*

(i) the kernel of the Hermitian form generated by the Euler field,

(ii) $V^f = V - \{0\}$, the monodromy group Γ/Γ_0 of the connection on $G \cdot \mathbf{C}^\times\backslash V^\circ$ is finite and Γ_0 is an integral Heisenberg group,

(iii) the developing map identifies the Γ/Γ_0-cover of $G\backslash V - \{0\}$ in a $\mathbf{C}^\times$-equivariant fashion with an anti-ample $\mathbf{C}^\times$-bundle over an abelian variety,

(iv) $G\backslash\widetilde{V - \{0\}} \to A$ is a Γ-isomorphism and the Dunkl connection satisfies the Schwarz condition.

(v) The Hermitian form gives $\mathbf{P}(G\backslash V)$ the structure of a complete parabolic orbifold: if K is the kernel of the Hermitian form on the translation space of A, then Γ acts in $K\backslash A$ via a complex crystallographic space group and the developing map induces an isomorphism between $\mathbf{P}(G\backslash V)$ and the latter's orbit space.

Proof. — The first assertion follows from Proposition 3.4. Upon replacing the flat form by its negative, we assume that it is positive semidefinite; we denote this form by h. The monodromy around every member of $\mathcal{L}_{\mathrm{irr}}(\mathcal{H}) - \{0\}$ leaves invariant a positive definite form and hence is finite by Theorem 5.1. This implies that $V^f \supset V - \{0\}$; it also shows that the monodromy of the connection is finite. Since $\kappa_0 = 1$, the Euler field E_V is flat and determines a nonzero translation T A such that $2\pi\sqrt{-1}T$ is the monodromy around a $\mathbf{C}^\times$-orbit in V°. In particular, the monodromy around such an orbit is not of finite order, so that $V^f = V - \{0\}$.

The Euler field (resp. T) generates a faithful $\mathbf{C}^+$-action on $\widetilde{V - \{0\}}$ (resp. A) such that the developing map descends to a local isomorphism $\mathbf{C}^+ \cdot G\backslash\widetilde{V - \{0\}} \to \mathbf{C}^+\backslash A$. Observe that the translation space of $\mathbf{C}^+\backslash A$ has a Γ-invariant positive definite Hermitian form: if the kernel of h is spanned by E_V this is clear and if h is positive definite we simply identify the translation space in question with the orthogonal complement of T in the translation space of A. The group $\Gamma/(2\pi\sqrt{-1}T)$ acts on $\mathbf{C}^+.G\backslash\widetilde{V - \{0\}}$ through a group which acts properly discretely. The orbit space of this action can be identified with $G\backslash\mathbf{P}(V)$, hence is compact. So the assumptions of Lemma 5.2

are fulfilled (with $Y' = Y = \mathbf{C}^+\backslash A$) and we conclude that

$$\mathbf{C}^+.G\backslash\widetilde{V - \{0\}} \to \mathbf{C}^+\backslash A$$

is a covering. Since the range is an affine space (hence simply connected), this must be an isomorphism. It follows that the action of Γ on A is properly discrete and cocompact. It also follows that the developing map defines a Γ-equivariant isomorphism of $G\backslash\widetilde{V - \{0\}}$ onto A.

Let Γ_0 be the subgroup of $\gamma \in \Gamma$ that act as a translation in $\mathbf{C}^+\backslash A$. This subgroup is of finite index in Γ and our assumption implies that the projection $\Gamma_0\backslash A \to \Gamma_0 \cdot \mathbf{C}^+\backslash A$ has the structure of a flat $\mathbf{C}^\times$-bundle over a complex torus. The developing map induces an isomorphism $\Gamma_0\backslash A \cong \Gamma_0 \cdot G\backslash\widetilde{V - \{0\}}$; the latter is finite over $G\backslash V - \{0\}$ and extends therefore as a finite cover over $G\backslash V$. This means that the associated line bundle over the complex torus has contractible zero section. Hence this line bundle is anti-ample and Γ_0 is a Heisenberg group.

Property (iv) is almost immediate from Theorem 4.6. $\square$

5.4. *Dunkl connections with finite holonomy (continued).* — In this subsection we concentrate on a situation where we want to establish finite holonomy without the hypothesis that $\kappa_H < 1$ for all $H \in \mathcal{H}$. We denote the collection of $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ for which $1 - \kappa_L$ is positive, zero, negative by $\mathcal{L}_{\kappa<1}(\mathcal{H})$, $\mathcal{L}_{\kappa=1}(\mathcal{H})$, $\mathcal{L}_{\kappa>1}(\mathcal{H})$ respectively (the superscript refers here to the sign of the transversal curvature); we have a likewise interpretation for $\mathcal{H}_{\kappa<1}, \cdots$. According to Corollary 2.17, κ is monotonous, and so the union $V_{\kappa<1}$ of the strata L° with $L \in \mathcal{L}_{\kappa<1}(\mathcal{H})$ is an open subset of V (recall that $\kappa_V = 0$ and so $V^\circ \subset V_{\kappa<1}$).

The result that we are aiming at is the following. We will need it when we treat the hyperbolic case.

Theorem **5.7.** — *Let be given a Dunkl system for which κ takes positive rational values on $\mathcal{H}$, for which $\mathcal{L}_{\kappa=1}(\mathcal{H})$ is empty and which satisfies the following two conditions:*

(i) the members of $\mathcal{H}_{\kappa<1}$ and the lines in $\mathcal{L}_{\kappa>1}(\mathcal{H})$ satisfy the Schwarz condition and

(ii) $\mathcal{L}_{\kappa>1}(\mathcal{H})$ is closed under intersections: the intersection of any two members of $\mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ on which $\kappa > 1$ is irreducible.

If the system admits a flat Hermitian form which is positive definite, then it has a finite holonomy group, satisfies the Schwarz condition, and the developing map induces an isomorphism $G\backslash V_{\kappa<1} \cong \Gamma\backslash A^\circ$, where A° is a linear *arrangement complement in A. This gives $\mathbf{P}(G\backslash V_{\kappa<1})$ the structure of an elliptic orbifold whose completion can be identified with $\Gamma\backslash\mathbf{P}(A)$.*

Remark **5.8.** — Observe that we are not making the assertion here that the developing map extends across a cover of V. In fact, if we projectivize, so that we get a Fubini metric on $\mathbf{P}(V^\circ)$, then we will see that the metric completion of $\mathbf{P}(V^\circ)$ may involve some blowing up and blowing down on $\mathbf{P}(V)$. The modification of $\mathbf{P}(V)$ that is involved here is discussed below in a somewhat more general setting. After that we take up the proof of the theorem.

Discussion **5.9.** — We here describe a resolution of V (resp. $\mathbf{P}(V)$) that will help us to understand the (projectivized) developing map. We work here under hypotheses Theorem 5.7. It follows from Theorem 5.1 (applied to the system transversal to any $L \in \mathcal{L}_{\kappa<1}(\mathcal{H})$) that the holonomy cover extends to a Γ-cover $\widetilde{V_{\kappa<1}} \to V_{\kappa<1}$, that the developing map extends to that cover and factors through a local isomorphism $G\backslash\widetilde{V_{\kappa<1}} \to A$.

Let $f : V^\sharp \to V$ be obtained by the blowing up of the members of $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$ in their natural (partial) order (so starting with the origin first). We shall identify $V_{\kappa<1}$ with its preimage in $V^\sharp$. Notice that the group G naturally acts on $V^\sharp$.

Every $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$ defines an exceptional divisor $E(L)$ and these exceptional divisors intersect normally. If we write $1-\kappa_L = p_L/q_L$ as usual (so p_L and q_L are relatively prime integers with $q_L > 0$ and hence $p_L < 0$), then the holonomy around $E(L)$ is of finite order q_L. So the holonomy covering extends to a ramified covering $\widetilde{V^\sharp} \to V^\sharp$. Notice that the preimage of $\cup_L E(L)$ in $\widetilde{V^\sharp}$ is also a normal crossing divisor. According to Lemma 2.21 the affine structure on V° degenerates infinitesimally simply along $E(L)$ with logarithmic exponent $\kappa_L - 1 > 0$ and the associated affine foliation of $E(L)$ is given by its projection onto L. Since we have a flat positive hermitian form, we can omit the restrictive 'infinitesimally'.

The divisor $E(L)$ contains a (unique) open-dense stratum $E^\circ(L)$ which can be identified with an open subset of $L \times \mathbf{P}(V/L)$. The behavior of the developing map near $E^\circ(L)$ is well-understood: if $z = (z_0, z_1) \in E^\circ(L)$, then there exist a submersion $F_0 : V^\sharp_z \to L_{z_0}$ (extending the projection $E(L)_z \to L_{z_0}$), a submersion $F_1 : V^\sharp_z \to T_0$ (with T a vector space) and a defining equation t of $E(L)_z$ in $V^\sharp_z$ such that (F_0, t, F_1) is a chart for $V^\sharp_z$ and the developing map is given by $(F_0, t^{1-\kappa_L}, t^{1-\kappa_L}F_1)$. We write $1-\kappa_L = p_H/q_H$ as a fraction as usual (so with $q_H > 0$ and hence $p_H < 0$). If $\tilde{z} \in \widetilde{V^\sharp}$ lies over z, and $\widetilde{E(L)}$ denotes the connected component of $\tilde{z}$ in the pre-image of $E(L)$

in $\widetilde{V^\sharp}$, then $\widetilde{E(L)}_{\tilde{z}}$ has a defining equation τ for which $\tau^{q_H} = t$, (F_0, τ, F_1) is a chart for $\widetilde{V^\sharp}_{\tilde{z}}$ and the developing map is given there by $(F_0, \tau^{p_L}, \tau^{p_L} F_1)$. (Notice that if the Schwarz condition is fulfilled at L, then (F_0, τ^{-p_L}, F_1) is a chart for $G_z \backslash \widetilde{V^\sharp}_{\tilde{z}}$.) The developing map followed by projectivization of the range, $A - \{0\} \to \mathbf{P}(A)$, is $[F_0 : \tau^{p_L} : \tau^{p_L} F_1] = [\tau^{-p_L} F_0 : 1 : F_1]$ and so this map extends across $\widetilde{V^\sharp}_{\tilde{z}}$ with restriction to $\widetilde{E(L)}_{\tilde{z}}$ is essentially given by F_1, in other words, by a covering of the projection $E(L)_z \to \mathbf{P}(V/L)_{z_1}$.

The divisors $E(L)$ determine a simple type of stratification of $V^\sharp$. An arbitrary stratum is described inductively: the collection of divisors defined by a subset of $\mathcal{L}^-(\mathcal{H})$ has a nonempty intersection if and only if that subset makes up a flag: $L_\bullet : L_0 > L_1 > \cdots > L_k > V$. Their common intersection $E(L_\bullet)$ contains a stratum $E^\circ(L_\bullet)$ as an open-dense subset. The latter can be identified with the product

$$E^\circ(L_\bullet) = (L_0)_{\kappa<1} \times \mathbf{P}((L_1/L_0)_{\kappa<1}) \times \cdots \times \mathbf{P}((V/L_k)_{\kappa<1}).$$

We already observed that the developing map does not extend unless we projectivize. That is why we shall focus on the central exceptional divisor E_0, which we will also denote by $\mathbf{P}(V^\sharp)$. Notice that $\mathbf{P}(V^\sharp)$ is a projective manifold and that $\widetilde{V^\sharp} \to V^\sharp$ restricts to a Γ-covering $\mathbf{P}(\widetilde{V^\sharp}) \to \mathbf{P}(V^\sharp)$. Each $E(L)$ with $L \in \mathcal{L}_{\kappa>1}(\mathcal{H}) - \{0\}$ meets $\mathbf{P}(V^\sharp)$ in a smooth hypersurface of $\mathbf{P}(V^\sharp)$ and these hypersurfaces intersect normally in $\mathbf{P}(V^\sharp)$. The open dense stratum of $\mathbf{P}(V^\sharp)$ is $\mathbf{P}(V_{\kappa<1})$, of course.

A stratum of $\mathbf{P}(V^\sharp)$ is given by a flag $L_\bullet$ as above with $L_0 = \{0\}$. Let us now write E_i for E_{L_i}, κ_i for κ_{L_i} etc. According to Proposition 2.25, the developing map is then at a point $z = (z_1, \dots, z_{k+1}) \in E^\circ(L_\bullet)$ linearly equivalent to a map of the form:

$$V_z^\sharp \to \prod_{i=1}^{k+1} (\mathbf{C} \times T_i), \quad \Big(t_0^{1-\kappa_0} \cdots t_{i-1}^{1-\kappa_{i-1}} (1, F_i)\Big)_{i=1}^{k+1},$$

where $F_i : V_z^\sharp \to (T_i)_0$ is a submersion to a vector space T_i whose restriction to $E^\circ(L_\bullet)_z$ is the projection $E^\circ(L_\bullet)_z \to \mathbf{P}(L_i/L_{i-1})_{z_i}$ followed by an isomorphism $\mathbf{P}(L_i/L_{i-1})_{z_i} \cong (T_i)_0$, and t_i is a local equation for E_i. So $(t_{i-1}, F_i)_{i=1}^{k+1}$ is a chart for $V^\sharp$ at z. We restrict the projectivized developing map to $\mathbf{P}(V^\sharp)$ (which is defined by $t_0 = 0$). The preceding shows that this restriction is projectively equivalent to the map with coordinates

$$\left[t_i^{\kappa_i - 1} \cdots t_k^{\kappa_k - 1} (1, F_i)\right]_{i=1}^{k+1}.$$

(Notice that the term for $i = k+1$ is $(1, F_{k+1})$.) Let $\tilde{z} \in \mathbf{P}(\widetilde{V^\sharp})$ lie over z and denote by $\tilde{D}_i$ resp. $\widetilde{E}^\circ(L_\bullet)$ the connected component of $\tilde{z}$ in the preimage of $D_i = E_i \cap \mathbf{P}(\widetilde{V^\sharp})$ resp. $E^\circ(L_\bullet)$. If $i > 0$, then near $\tilde{z}$, $V^\sharp$ is simply given by extracting the q_ith root of t_i (recall that $1 - \kappa_i = p_i/q_i$): $\tau_i^{q_i} := t_i$. The projectivized developing map is at $\tilde{z}$ given in terms of this chart and an affine chart in $\mathbf{P}(A)$ by

$$\left[\tau_i^{-p_i} \cdots \tau_k^{-p_k}(1, F_i))\right]_{i=1}^{k+1}.$$

Since each p_i is negative, the projectivization defines a regular morphism $\mathbf{P}(\widetilde{V^\sharp}) \to \mathbf{P}(A)$ whose restriction to $\widetilde{E}^\circ(L_\bullet)$ factors through a covering of the last factor $\mathbf{P}((V/L_k)_{\kappa<1})$. The fiber through $\tilde{z}$ is here defined by putting $\tau_k = 0$ and F_{k+1} constant. It follows that the connected component of this fiber lies in $\tilde{D}_k$, more precisely, that it lies in a connected component of a fiber of the natural map $\widetilde{D}_k \to D_k = \mathbf{P}(L_k^\sharp) \times \mathbf{P}((V/L_k)^\sharp) \to \mathbf{P}((V/L_k)^\sharp)$. We also see that $\tilde{z}$ is isolated in its fiber if and only if the flag is reduced to $L_0 = \{0\} > L_1$ with $\dim L_1 = 1$; in that case, the map above is simply given by $(\tau_1^{-p_1}, F_1)$. By assumption, the Schwarz condition is then satisfied at L_1, and so the latter is also a chart for the orbit space $\mathbf{P}(G\backslash\widetilde{V^\sharp})$ so that the projectivized developing map modulo G is then a local isomorphism at the image of $\tilde{z}$.

The holonomy near $E^\circ(L_\bullet)$ decomposes as a product and so a connected component $\widetilde{E}^\circ(L_\bullet)$ of the preimage of $E^\circ(L_\bullet)$ in $\mathbf{P}(\widetilde{V^\sharp})$ decomposes as a product as well: $\widetilde{E}^\circ(L_\bullet) = \mathbf{P}(\widetilde{(L_1)_{\kappa<1}}) \times \mathbf{P}(\widetilde{(L_2/L_1)_{\kappa<1}}) \times \cdots \times \mathbf{P}(\widetilde{(V/L_k)_{\kappa<1}})$. Its closure is an irreducible component of the preimage of $E(L_\bullet)$; the normalisation of that closure decomposes accordingly:

$$\widetilde{E}(L_\bullet) = \mathbf{P}(\widetilde{L_1^\sharp}) \times \mathbf{P}(\widetilde{(L_2/L_1)^\sharp}) \cdots \times \mathbf{P}(\widetilde{(V/L_k)^\sharp}).$$

We shall want the latter to be compact. In other words, we want that each factor $\mathbf{P}(\widetilde{(L_i/L_{i-1})_{\kappa<1}})$ has finite holonomy.

The proof of 5.7 proceeds by induction on $\dim V$. The induction starts trivially.

Since the form is positive definite, we shall (by simple averaging) assume that it is invariant under the Schwarz symmetry group G.

Lemma **5.10.** — *For every $L \in \mathcal{L}_{\kappa>1}(\mathcal{H}) - \{0\}$, the longitudinal holonomy in L° is finite.*

Proof. — We show that the affine structure on L° satisfies the hypotheses of theorem that we want to prove, so that we can invoke the induction hypothesis. The flat metric on V° determines one on L°. It remains to verify Conditions (i) and (ii) on L. Let $M \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}^L)$. Recall that if $M(L)$ denotes the intersection of the members of $\mathcal{H}_M - \mathcal{H}_L$, then $M(L) \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ and $\kappa^L_M = \kappa_{M(L)}$.

If $M \in \mathcal{L}_{\kappa<1}(\mathcal{H}^L)$, then $M(L) \in \mathcal{L}_{\kappa<1}(\mathcal{H})$ and so $M(L)$ satisfies the Schwarz condition by Theorem 4.6. This implies that M satisfies the Schwarz condition when viewed as a member of $\mathcal{L}_{\kappa<1}(\mathcal{H}^L)$.

We claim that if $M \in \mathcal{L}_{\kappa>1}(\mathcal{H}^L)$, then $M = M(L)$. For if that were not the case, then L and $M(L)$ would be two members of $\mathcal{L}_{\kappa>1}(\mathcal{H})$ whose intersection M is reducible, contradicting assumption (ii).

This immediately implies that property (ii) is inherited by L. It also follows that if M is a line in $\mathcal{L}_{\kappa>1}(\mathcal{H}^L)$, then it is a line in $\mathcal{L}_{\kappa>1}(\mathcal{H})$ and hence satisfies the Schwarz condition. This proves (i) on L. □

Corollary **5.11.** — *The connected components of the fibers of the projectivized developing map $\mathbf{P}(\widetilde{V^\sharp}) \to \mathbf{P}(A)$ are compact (and hence the same is true for the induced map $\mathbf{P}(G\backslash\widetilde{V^\sharp}) \to \mathbf{P}(A)$).*

Proof. — Over $\mathbf{P}(V_{\kappa<1})$, the projectivized developing map is locally finite and so in these points the claim is clear. Let us therefore examine the situation over another stratum $S(L_\bullet)$ (as in the Discussion 5.9). Since the stratum is not open, we have $k \geq 1$. We observed that the connected component of a fiber through $\tilde{z}$ lies in the fiber over $z_k \in \mathbf{P}((V/L_k)_{\kappa<1})$ of the composite

$$\tilde{D}_k \to D_k = \mathbf{P}(L_k^\sharp) \times \mathbf{P}((V/L_k)^\sharp) \to \mathbf{P}((V/L_k)^\sharp)$$

The holonomy in the last factor $\mathbf{P}((L_k)^\sharp)$ is longitudinal and hence finite. This implies that every irreducible component in $\tilde{D}_k$ over $\mathbf{P}((L_k)_{\kappa<1}) \times \{z_k\}$ is compact. □

We need to understand better the topology of the map $\mathbf{P}(G\backslash\widetilde{V^\sharp}) \to \mathbf{P}(A)$. For this it is convenient to have at our disposal the following

Definition **5.12.** — *Let $f : X \to Y$ be a continuous map between topological spaces. The* topological Stein factorization *of f is the factorization through the quotient $X \to X_f$ of X defined by the partition of X into connected components of fibers of f. The second factor is denoted $f_{\mathfrak{St}} : X_f \to Y$ and if f is understood, we usally write $X_{\mathfrak{St}}$ for X_f.*

So the first factor $X \to X_f$ has connected fibers and the second factor $f_{\mathfrak{St}} : X_f \to Y$ has discrete fibers in case the fibers of f are locally connected. Here is a useful criterion for a complex-analytic counterpart.

Lemma **5.13.** — *Let $f : X \to Y$ be a morphism of connected normal analytic spaces. Suppose that the connected components of the fibers of f are compact. Then the Stein factorization of f,*

$$f : X \longrightarrow X_{\mathfrak{St}} \xrightarrow{f_{\mathfrak{St}}} Y,$$

is in the complex-analytic category. More precisely, $X \to X_{\mathfrak{St}}$ is a proper morphism with connected fibers to a normal analytic space $X_{\mathfrak{St}}$ and $f_{\mathfrak{St}}$ is a morphism with discrete fibers. If in addition, Y is smooth, f is a local isomorphism in every point that is isolated in its fiber and such points are dense in X, then $f_{\mathfrak{St}}$ is a local isomorphism.

Proof. — The first part is well-known and standard in case f is proper ([17], Ch. 10, §6). The second part perhaps less so, but we show that it is a consequence of the first part. Since $f : X \to Y$ is then a morphism from a normal analytic space to a smooth space of the same dimension which contracts its singular locus, $f_{\mathfrak{St}} : X_{\mathfrak{St}} \to Y$ will be a local isomorphism outside a subvariety of $X_{\mathfrak{St}}$ of codimension one. But then there is no ramification at all, since a ramified cover of a smooth variety has as its ramification locus a hypersurface.

So it remains to show that we can reduce to the proper case. We do this by showing that if $K \subset X$ is a connected component of the fiber $f^{-1}(y)$, then there exist open neighborhoods U of K in X and V of y in Y such that $f(U) \subset V$ and $f : U \to V$ is proper. This indeed suffices, for if $y' \in V$, then $f^{-1}(y') \cap U$ is open and closed in $f^{-1}(y')$, and hence a union of connected components of $f^{-1}(y')$.

Choose a compact neighborhood C of K which does not meet $f^{-1}(y) - K$. Clearly, for every neighborhood V of y in Y, $f : f^{-1}V \cap C \to V$ is proper. So it is enough to show that $f^{-1}V \cap C$ is open in X (equivalently, $f^{-1}V \cap \partial C = \emptyset$) for V small enough. If that were not the case, then we could find a sequence of points $(x_i \in \partial C)_{i=1}^{\infty}$ whose image sequence converges to y. Since ∂C is compact, a subsequence will converge, to $x \in \partial C$, say. But clearly $f(x) = y$ and so $x \in K$. This cannot be since $K \cap \partial C = \emptyset$. □

Corollary **5.14.** — *The Stein factorization of $\mathbf{P}(G\backslash\widetilde{V^\sharp}) \to \mathbf{P}(A)$,*

$$\mathbf{P}(G\backslash\widetilde{V^\sharp}) \longrightarrow \mathbf{P}(G\backslash\widetilde{V^\sharp})_{\mathfrak{St}} \longrightarrow \mathbf{P}(A),$$

is complex-analytic and the Stein factor $\mathbf{P}(G\backslash\widetilde{V^\sharp})_{\mathfrak{St}} \to \mathbf{P}(A)$ is a local isomorphism.

Proof. — In Corollary 5.11 and the Discussion 5.9 we established that the conditions in both clauses of the Lemma 5.13 are satisfied. □

Proof of Theorem 5.7. — We first prove that $\mathbf{P}(G\backslash\widetilde{V^\sharp})_{\mathfrak{St}} \to \mathbf{P}(A)$ is a Γ-isomorphism. For this we verify that the hypotheses of Lemma 5.2 are verified for that map with $Y' = Y = \mathbf{P}(A)$. By Corollary 5.14, $\mathbf{P}(G\backslash\widetilde{V^\sharp})_{\mathfrak{St}} \to \mathbf{P}(A)$ is a local isomorphism. We know that Γ acts properly discontinuously on $\mathbf{P}(\widetilde{V^\sharp})$ with compact fundamental domain. This is then also true for its action on $\mathbf{P}(G\backslash\widetilde{V^\sharp})_{\mathfrak{St}}$. Since Γ acts on $\mathbf{P}(A)$ as a group of isometries, Condition (ii) of 5.2 is fulfilled as well. So $\mathbf{P}(G\backslash\widetilde{V^\sharp})_{\mathfrak{St}} \to \mathbf{P}(A)$ is a covering projection. But $\mathbf{P}(A)$ is simply connected, and so this must be an isomorphism. It follows that $\mathbf{P}(\widetilde{V^\sharp})$ is compact, so that Γ must be finite.

An irreducible component $\tilde{D}(L)$ over $D(L)$ gets contracted if $\dim L > 1$, with image in $\mathbf{P}(A)$ a subspace of codimension equal to the dimension of L. In particular, we get a divisor in case $\dim L = 1$ and so the image of a covering of $\mathbf{P}(V_{\kappa<1})$ is mapped to an arrangement complement, $\mathbf{P}(A^\circ)$, say. So the developing map $\mathrm{ev}_G : G\backslash\widetilde{V_{\kappa<1}} \to A^\circ$ becomes an isomorphism if we pass to $\mathbf{C}^\times$-orbit spaces. According to Theorem 4.5 ev_G is homogeneous of degree one. It follows that this map as well as the induced map $G\backslash V_{\kappa<1} \to \Gamma\backslash A^\circ$ are isomorphisms.

Finally we verify the Schwarz condition for any $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$. We know already that this is the case when $L \in \mathcal{L}_{\kappa<1}(\mathcal{H})$. For $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$ this is seen from the simple form of the projectivized developing map at a general point of $D(L)$: in terms of a local chart (τ, F, F') of $\mathbf{P}(\widetilde{V^\sharp})$ at such a point it is given by $(\tau^{-p_L}, \tau^{-p_L}F, F')$. Since $(G\backslash\mathbf{P}(\widetilde{V^\sharp})_{\mathfrak{St}} \to \mathbf{P}(A)$ is an isomorphism, G must contain the group of $|p_L|$th roots of unity acting on the transversal coordinate τ. This just tells us that L satisfies the Schwarz condition. □

6. Geometric structures of hyperbolic type

In this section we consider Dunkl systems of admissible hyperbolic type. So the affine space A in which the evaluation map takes its values is in fact a vector space (it comes with an origin) equipped with a nondegenerate Hermitian form of hyperbolic signature. We denote by $A_{\mathbf{B}} \subset A$ the set of vectors of negative self-product and by $\mathbf{B} \subset \mathbf{P}(A)$ its projectivization. Notice that $\mathbf{B}$ is a complex ball and that $A_{\mathbf{B}}$ can be thought of as a $\mathbf{C}^\times$-bundle over $\mathbf{B}$. The admissibility assumption means that the evaluation map takes its values in $A_{\mathbf{B}}$ so that its projectivization takes its values in $\mathbf{B}$.

6.1. *The compact hyperbolic case.* — This is a relatively simple case and for that reason we state and prove it separately. The result in question is the following.

Theorem **6.1.** — *Suppose that the Dunkl system is of admissible hyperbolic type, satisfies the Schwarz condition in codimension one and is such that $0 < \kappa_L < 1$ for all $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H}) - \{0\}$. Then the Dunkl system satisfies the Schwarz condition, the discrete group Γ acts on $\mathbf{B}$ properly with compact fundamental domain and the developing map induces an isomorphism $G\backslash V \cong \Gamma\backslash A_{\mathbf{B}}$. Thus $\mathbf{P}(G\backslash V)$ acquires the structure of a complete hyperbolic orbifold isomorphic to $\Gamma\backslash\mathbf{B}$.*

Proof. — Arguing as in the proof of Theorem 5.6 we find that $V^f = V - \{0\}$. It follows from Theorem 4.6 that the Dunkl system satisfies the Schwarz condition. So the developing map descends to a local isomorphism $G\backslash\mathbf{P}(\widetilde{V^f}) \to \mathbf{P}(A)$. It takes values in the complex ball $\mathbf{B}$. The latter comes with a Γ-invariant Kähler metric. The Γ-orbit space of $G\backslash\mathbf{P}(\widetilde{V^f})$ can be identified with $G\backslash\mathbf{P}(V)$, and hence is compact. So the assumptions of Lemma 5.2 are fulfilled and we conclude that $G\backslash\mathbf{P}(\widetilde{V^f}) \to \mathbf{B}$ is a covering. Since the range is simply connected, this must be an isomorphism. In particular, Γ acts properly on $\mathbf{B}$ with compact fundamental domain.

It also follows that $G\backslash V \cong \Gamma\backslash A_{\mathbf{B}}$ becomes an isomorphism if we pass to $\mathbf{C}^\times$-orbit spaces. It then follows that the map itself is an isomorphism, because G contains by definition all the scalars which leave the developing map invariant. $\square$

6.2. *Statement of the main theorem.* — The general hyperbolic case concerns the situation where the holonomy group is of cofinite volume (rather than cocompact) in the automorphism group of a complex ball. This is substantially harder to deal with.

Let be given a Dunkl system for which the flat Hermitian form $h = h^\kappa$ is of hyperbolic type (i.e., nondegenerate of index one, so that h defines a complex ball $\mathbf{B}$ in the projective space at infinity $\mathbf{P}(A)$ of A). We retain some of the notation introduced in Subsection 5.4, such as $\mathcal{L}_{\kappa<1}(\mathcal{H})$, $\mathcal{L}_{\kappa=1}(\mathcal{H})$, $\mathcal{L}_{\kappa>1}(\mathcal{H})$, $\mathcal{H}_{\kappa<1}$, $V_{\kappa<1}, \cdots$.

If $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$, then if we approach L° from V° along a curve, the image of a lift in $\widetilde{V^\circ}$ of this curve under the developing map tends to infinity with limit a point of $\mathbf{P}(A)$. These limit points lie in a well-defined Γ-orbit of linear subspaces of $\mathbf{P}(A)$ of codimension $\dim(L)$. We call such space a *special subspace* in $\mathbf{P}(A)$ and its intersection with $\mathbf{B}$ a *special subball.* We use the same terminology for the corresponding linear subspace of A.

The main goal of this section is to prove:

Theorem **6.2.** — *Let be given a Dunkl system for which κ is positive on $\mathcal{H}$ and which comes with a flat admissible form h of hyperbolic type. Suppose that every member of $\mathcal{H}_{\kappa<1}$ and every line in $\mathcal{L}_{\kappa>1}(\mathcal{H})$ satisfies the Schwarz condition. Then:*

(i) The group Γ, considered as a subgroup of the unitary group $\mathrm{U}(h)$ of h, is discrete and has cofinite volume in $\mathrm{U}(h)$.

(ii) The system satisfies the Schwarz condition and the collection of special hyperplanes is locally finite in $A_{\mathbf{B}}$.

(iii) If $A_{\mathbf{B}}^{\diamond}$ denotes the complement in $A_{\mathbf{B}}$ of the union of the special hyperplanes, then the developing map defines a Γ-equivariant isomorphism $G\backslash\widetilde{V_{\kappa<1}} \to A_{\mathbf{B}}^{\diamond}$ which drops to an isomorphism $G\backslash V_{\kappa<1} \to \Gamma\backslash A_{\mathbf{B}}^{\diamond}$ of normal analytic spaces.

Thus, if $\mathbf{B}^{\diamond}$ denotes the complement in $\mathbf{B}$ of the union of the special hyperplanes, then $\mathbf{P}(G\backslash V_{\kappa<1})$ can be identified with $\Gamma\backslash\mathbf{B}^{\diamond}$ and acquires the structure of a hyperbolic orbifold whose metric completion is $\Gamma\backslash\mathbf{B}$.

Remarks **6.3.** — Our proof yields more precise information, for it tells us how $\mathbf{P}(G\backslash V)$ is obtained from the Baily-Borel compactification of $\Gamma\backslash\mathbf{B}$ by a blowup followed by a blowdown. This is in fact an instance of the construction described in [23]. The proof also shows that if $\kappa_H \le 1$ for every $H \in \mathcal{H}$, then $V_{\kappa<1} = V^f$.

Couwenberg gives in his thesis [9] a (presumably complete) list of the cases for which $\mathcal{H}$ is a Coxeter arrangement and G is the associated Coxeter group. The Schwarz condition for the lines then amounts to: if L is a line which is the fixed point subspace of an irreducible Coxeter subgroup of G and such that $\kappa_L > 1$, then $(\kappa_L - 1)^{-1}$ is an integer or, when $L^{\perp} \in \mathcal{H}$, half an integer. The fact that the list is substantial gives the theorem its merit. In particular, it produces new examples of discrete hyperbolic groups of cofinite volume.

6.3. *Connection with the work of Deligne-Mostow.* — Theorem 6.2 implies one of the main results of Deligne-Mostow [14] and Mostow [26].

Theorem **6.4** *(Deligne-Mostow).* — *Let be given a Lauricella system of dimension n whose parameters $\mu_0, \dots, \mu_n$ are positive reals with sum in $(1,2)$ so that $\mu_{n+1} := 2 - \sum_{k=0}^{n} \mu_k \in (0,1)$. Suppose that whenever $1 - \mu_i - \mu_j > 0$ for $0 \le i < j \le n+1$, then $1 - \mu_i - \mu_j$ is a rational number with numerator 1 or 2, allowing the latter only in case $j \le n$ and $\mu_i = \mu_j$. Then the system satisfies* *the Schwarz condition, the Schwarz symmetry group is the group G of permutations of $\{0,\dots,n\}$ which preserves the weight function $\mu:\{0,1,\dots,n\}\to\mathbf{R}$, the collection of special hyperplanes is locally finite on $\mathbf{B}$, Γ is a lattice in the unitary group of A and the developing map identifies $\mathbf{P}(G\backslash V_{\kappa<1})$ with $\Gamma\backslash\mathbf{B}^{\diamond}$.*

Proof. — We verify the hypotheses of Theorem 6.2. First of all, every $H_{i,j}$ on which κ is <1 satisfies the Schwarz condition: for recall from Example 4.3 that this means that for every pair $0\le i<j\le n$ with $1-\mu_i-\mu_j$ positive, this is a rational number with numerator 1 or 2, allowing the latter only in case $\mu_i=\mu_j$.

The Schwarz condition is also fulfilled by a line in $\mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ with $\kappa_L>1$: such a line is given by an n-element subset of $\{0,\dots,n\}$, say as the complement of the singleton $\{i\}$, such that $\sum_{0\le j\le n, j\ne i}\mu_j>1$. The Schwarz condition for this line amounts to $-1+\sum_{0\le j\le n, j\ne i}\mu_j$ being the reciprocal of an integer. This comes down to: if $1-\mu_i-\mu_{n+1}$ is positive, then it is the reciprocal of an integer. The rest follows from easily from Theorem 6.2. □

Remark **6.5.** — The conditions imposed here imply the ΣINT-condition of Mostow: this is the condition which says that for any pair $0\le i<j\le n+1$ with $\mu_i+\mu_j<1$, $1-\mu_i-\mu_j$ must be a rational number with numerator 1 or 2, allowing the latter only in case $\mu_i=\mu_j$. Clearly, this condition is more symmetric as it does not attribute a special role to μ_{n+1}. This symmetry is understood as follows. We can regard of $\mathbf{P}(V^\circ)$ as parametrizing the collection of *mutually distinct* $(n+1)$-tuples $(z_0,\dots,z_n)$ in the affine line $\mathbf{C}$ given up to an affine-linear transformation. But it is better to include ∞ and to think of $\mathbf{P}(V^\circ)$ as the moduli space of *mutually distinct* $(n+2)$-tuples $(z_0,\dots,z_{n+1})$ on the projective line $\mathbf{P}^1$ given up to a projective-linear transformation, that is, to identify $\mathbf{P}(V^\circ)$ with $\mathcal{M}_{0,n+2}$. This makes evident an action of the permutation group of $\{0,\dots,n+1\}$ on $\mathbf{P}(V^\circ)$. It is conceivable that there are cases for which the ΣINT-condition is satisfied and ours aren't, even after permutation. The table in [34], lists 94 systems $(\mu_0\ge\mu_1\ge\dots\ge\mu_{n+1}>0)$ satisfying the ΣINT-condition. Most likely, it is complete. In this list, there is precisely one case which escapes us and that is when $n+2=12$ and all μ_i's equal to $\frac{1}{6}$. With little extra effort, we can get around this (and at the same time avoid resorting to this list) if we let the group of permutations of $\{0,\dots,n+1\}$ which leave $\mu:\{0,\dots,n+1\}\to\mathbf{Q}$ invariant act from the outset. This group contains G and the elements not in G act nonlinearly on $\mathbf{P}(V_{\kappa<1})$. An alternative approach starts with analyzing the developing map of a Dunkl system with a degenerate hyperbolic form (see Subsection 3.7). Indeed, this is a class worth studying its own right.

Remark **6.6.** — Deligne and Mostow show that there is a modular interpretation of the Baily-Borel compactification of $\Gamma\backslash\mathbf{B}$. Given positive rational numbers $\mu_0,\dots,\mu_{n+1}$ with sum 2, then let us say that an *effective fractional anticanonical divisor* on $\mathbf{P}^1$ of type μ simply consists of $n+2$ points of $\mathbf{P}^1$ endowed with the weights $\mu_0,\dots,\mu_{n+1}$ and given up to order. We do not require the points to be distinct. So such a divisor determines a *support function* $\mathbf{P}^1\to\mathbf{Q}_+$ which is zero for all but finitely many points and whose sum (over $\mathbf{P}^1$) of its values is two. It is said to be *stable* (resp. *semistable*) if this function is everywhere less than (resp. at most) one. The projective linear group acts on the variety of the semistable fractional divisors and this action is proper on the (open) subvariety of the stable ones. So a stable orbit is always closed. Any other minimal semistable orbit is represented by a fractional divisor whose support consists of two distinct points, each with weight 1. The points of its Hilbert-Mumford quotient are in bijective correspondence with the minimal semistable orbits. We thus get a projective compactification $\mathcal{M}_{0,n+2}\subset\overline{\mathcal{M}}^{\mu}_{0,n+2}$. A period map enters the picture by imitating the familiar approach to the elliptic integral, that is, by passing to a cyclic cover of $\mathbf{P}^1$ on which the Lauricella integrand becomes a regular differential. Concretely, write $\mu_i=m_i/m$ with m_i,m positive integers such that the m_i's have no common divisor, and write ν_i for the denominator of μ_i. Consider the cyclic cover $C\to\mathbf{P}^1$ of order m which has ramification over z_i of order ν_i. In affine coordinates, C is given as the normalization of the curve defined by

$$w^m=\prod_{i=0}^{n}(z_i-\zeta)^{m_i}.$$

The Lauricella integrand pulls back to a regular differential $\tilde\eta$ on C, represented by $w^{-1}d\zeta$. Over $z_i\in\mathbf{P}^1$ we have m/ν_i distinct points in each of which $\tilde\eta$ has a zero of order $\nu_i(1-\mu_i)-1$. This form transforms under the Galois group by a certain character χ and up to a scalar factor, $\tilde\eta$ is the only regular form with that property: $H^{1,0}(C)^\chi$ is a line spanned by $\tilde\eta$. It turns out that such Hodge data are uniformized by a complex ball. Although the holonomy group need not map to an arithmetic group, much of Shimura's theory applies here. Indeed, Shimura (see for instance [31]) and Casselman [5] (who was Shimura's student at the time) had investigated in detail the case for which m is prime before Deligne and Mostow addressed the general situation. A (if not *the*) chief result of Deligne-Mostow [14] is a refined Torelli theorem: if their INT condition is satisfied, then

(i) the holonomy group maps to a subgroup of automorphisms of the Hodge period ball which is discrete and of cofinite volume,

(ii) the corresponding orbit space admits a compactification of Baily-Borel type (this adds a finite number of points, the cusps),

(iii) the map described above identifies $\overline{\mathcal{M}}^{\mu}_{0,n+2}$ with this Baily-Borel compactification, making the minimal semistable nonstable orbits correspond to the cusps.

This is essentially the content of their Theorem (10.18.2). They also determine when the holonomy group is arithmetic.

6.4. *The Borel-Serre extension.* — Before we begin the proof the main theorem, we first make a few observations regarding the unitary group $\mathrm{U}(h)$ of h (since A has an origin, we regard this as a group operating in A). Suppose we have a unipotent transformation $g \in \mathrm{U}(h)$ that is not the identity. Let $E \subset A$ be the fixed point space of g. Then $E^{\perp}$ is g-invariant and hence contains eigenvectors. So $E \cap E^{\perp}$ is nontrivial. In other words, E contains an isotropic line I. Now g induces in $I^{\perp}/I$ a transformation that will preserve the form induced by h. Since this form is positive definite and g is unipotent, g will act trivially on $I^{\perp}/I$. The unitary transformations which respect the flag $\{0\} \subset I \subset I^{\perp} \subset A$ and act trivially on the successive quotients form a Heisenberg group N_I whose center is parametrized as follows. Notice that the one-dimensional complex vector space $I \otimes \overline{I}$ has a natural real structure which is oriented: it is defined by the 'positive' ray of the elements $e \otimes e$, where e runs over the generators of I. This line parametrizes a one parameter subgroup of $\mathrm{GL}(A)$:

$$\exp : I \otimes \overline{I} \to \mathrm{GL}(A), \quad \exp(\lambda e \otimes e) : z \in A \mapsto z + \lambda h(z,e)e, \quad e \in I, \lambda \in \mathbf{C}.$$

Since

$$h(\exp(\lambda e \otimes e)(z), \exp(\lambda e \otimes e)(z)) = h(z,z) + 2|h(z,e)|^2 \operatorname{Re}(\lambda),$$

the transformation $\exp(\lambda e \otimes e)$ is unitary relative to h if λ is purely imaginary: exp maps $\sqrt{-1} I \otimes \overline{I}(\mathbf{R})$ to a one-parameter subgroup of $\mathrm{U}(h)$. This one-parameter subgroup is the center of the Heisenberg group N_I above. The group N_I is parametrized by pairs $(a,e) \in I^{\perp} \times I$: any element of this group is written

$$g_{a,e} : z \in A \mapsto z + h(z,a)e - h(z,e)a - \tfrac{1}{2} h(a,a) h(z,e) e.$$

This is not quite unique since $g_{a+\lambda e, e} = g_{a,e}$ when $\lambda \in \mathbf{R}$. But apart from that we have uniqueness: N_I modulo its center can be identified with vector group $I^{\perp}/I \otimes \overline{I}$ by assigning to (a,e) its image in $I^{\perp}/I \otimes \overline{I}$.

Let T be a subspace of A on which h is degenerate with kernel I: so $I \subset T \subset I^{\perp}$. We suppose that $T \neq I$. Clearly, N_I preserves T. Suppose that

g acts trivially on A/T and induces in the fibers of $A \to A/T$ a translation. So if we write g in the above form: $g = g_{a,e}$, then we see that a must be proportional to e: $a = \lambda e$ with λ purely imaginary, in other words g is in the center of N_I.

For $I \subset A$ as above and $e \in I$ a generator, the above formula shows that when λ is a negative real number, $\exp(\lambda e \otimes e)$ is not unitary, but will still map $\mathbf{B}$ into itself. In fact, the orbits of the ray of positive elements in $I \otimes \overline{I}$ are (oriented) geodesic rays in $\mathbf{B}$ which tend to $[I] \in \partial\mathbf{B}$. Perhaps a more concrete picture is gotten by fixing a generator $e \in I$ so that every point of $\mathbf{B}$ can be represented in the affine hyperplane in V defined by $h(z, e) = 1$: under the realization of $\mathbf{B}$ in this hyperplane, the geodesic ray action becomes simply the group of translations over negative multiples of e. We regard the space $\mathbf{B}(I)$ of these rays as a quotient space of $\mathbf{B}$ so that we have a fibration by rays $\pi(I) : \mathbf{B} \to \mathbf{B}(I)$. The Borel-Serre topology on the disjoint union $\mathbf{B} \sqcup \mathbf{B}(I)$ is generated by the open subsets of $\mathbf{B}$ and the subsets of the form $U \sqcup \pi(I)(U)$, where U runs over the open subsets of $\mathbf{B}$ invariant under N_I and the positive ray in $I \otimes \overline{I}$. This adds a partial boundary to $\mathbf{B}$ so that it becomes a manifold with boundary. Let $\mathbf{B}^\sharp \supset \mathbf{B}$ be the *Borel-Serre extension* associated to Γ: for every isotropic line $I \subset V$ for which $\Gamma \cap N_I$ is discrete and cocompact, we do the above construction. That makes $\mathbf{B}^\sharp$ a manifold with boundary, the boundary having an infinite number of connected components (or being empty). Notice that the action of Γ on this boundary is properly discrete and cocompact. Indeed, this is the main justification for its introduction.

6.5. *Proof of the main theorem.* — We now turn to the proof of Theorem 6.2. Throughout this section the assumptions of that theorem are in force.

We begin with a lemma in which we collect a number of useful properties.

Lemma **6.7.** — *We have:*

(i) For any $L \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$, the restriction of h to the fibers of the natural retraction $r : V_{L^\circ} \to L^\circ$ is positive, semipositive with one-dimensional kernel, hyperbolic according to whether $1 - \kappa_L$ is positive, zero, or negative.

(ii) The intersection of any two distinct members L_1, L_2 of $\mathcal{L}_{\kappa\geq 1}(\mathcal{H})$ is irreducible and (hence) belongs to $\mathcal{L}_{\kappa\geq 1}(\mathcal{H})$.

(iii) If $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$, then the longitudinal Dunkl connection on L° has finite holonomy and L satisfies the Schwarz condition (so that the system satisfies the Schwarz condition).

Proof. — When $0 < \kappa_L < 1$, then the developing map has a well-defined limit on L°, namely the developing map for the affine structure on L°; since the developing map must takes values in $A_{\mathbf{B}}$, so does its limit (we here use the convexity property (3.6), and so h induces on L° a hyperbolic form. This implies that h positive definite on the fibers of r. If on the other hand, $\kappa_l > 1$, then the developing map blows up along L; if we identify the exceptional divisor with $L \times \mathbf{P}(V/L)$, then the projectivized developing map has a limit on $L^\circ \times \mathbf{P}((V/L)^\circ)$ which is essentially the projectivized developing map of the second factor. Since that limit takes its values in $\mathbf{B}$, it follows that h is hyperbolic on the fibers of r. Finally, if $\kappa_L = 1$, the consider the situation near the exceptional divisor D of the blowup $\mathrm{Bl}_L V$. According to Lemma 2.21 the holonomy around D is nontrivial unipotent, preserves every fiber of r and acts in such a fiber as a translation (the translation is constant on the leaves of a codimension one foliation of L°). This implies that h must be degenerate on a fiber of r.

Now let L_1, L_2 be as in the lemma and suppose that $L_1 \cap L_2$ is reducible. So the plane V/L decomposes as $(V/L_1) \times (V/L_2)$ in the Hermitian category. Since V/L_i is negative semidefinite, the same is true for V/L. But this is impossible, because a hyperbolic form cannot be negative semidefinite on a plane.

Let now $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$. Then the longitudinal holonomy in L° has a flat positive Hermitian form. The desired properties follow from Theorem 5.7: in view of the way κ^L is defined, and part (ii) any one-dimensional member in $\mathcal{L}_{\kappa>1}(\mathcal{H}^L)$ is in fact a member of $\mathcal{L}_{\kappa>1}(\mathcal{H})$ and so satisfies the Schwarz condition and any codimension one member in $\mathcal{L}_{\kappa<1}(\mathcal{H}^L)$ comes from a member of $\mathcal{L}_{\kappa<1}(\mathcal{H})$ and hence satisfies the Schwarz condition. □

Discussion **6.8.** — In 5.9 we introduced a blowup $V^\sharp$ of V under the assumption that $\mathcal{L}_{\kappa=1}(\mathcal{H})$ is empty and that there exists a flat positive definite Hermitian form. We now do this in the present situation, where $\mathcal{L}_{\kappa=1}(\mathcal{H})$ might be nonempty and there is given a flat hyperbolic Hermitian form of admissible type.

Our $V^\sharp$ will be obtained by blowing up the members of $\mathcal{L}_{\kappa>1}(\mathcal{H})$ first (in the usual order), and then blowing up each $L \in \mathcal{L}_{\kappa=1}(\mathcal{H})$ in a *real-oriented manner*. This is unambiguously defined since by Lemma 6.7-(ii) the intersection of two such members lies in $\mathcal{L}_{\kappa>1}(\mathcal{H})$ and so their strict transforms will not meet. It is clear that $V^\sharp$ is a manifold with smooth boundary $\partial V^\sharp$ whose manifold interior $V^\sharp - \partial V^\sharp$ is a quasiprojective variety. The latter contains $V_{\kappa<1}$ as an open-dense subset and the complement of $V_{\kappa<1}$ in $V^\sharp - \partial V^\sharp$ is a normal crossing divisor whose closure in $V^\sharp$ meets the boundary transversally.

Any $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$ defines a divisor $E(L)$ in $V^\sharp$ and any $L \in \mathcal{L}_{\kappa=1}(\mathcal{H})$ defines a boundary component $\partial_L V^\sharp$. These cross normally in an obvious sense so that we get a natural stratification of $V^\sharp$. Let us describe the strata explicitly. For $L \in \mathcal{L}_{\kappa\geq 1}(\mathcal{H})$ we define $L_{\kappa<1}$ as in Discussion 5.9:

$$L_{\kappa<1} := L - \cup\{M \ : \ M \in \mathcal{L}_{\kappa\geq 1}(\mathcal{H}), L < M\}.$$

So every $M \in \mathcal{L}_{\mathrm{irr}}(\mathcal{H})$ which meets $L_{\kappa<1}$ but does not contain L belongs to $\mathcal{L}_{\kappa<1}(\mathcal{H})$. In particular, $L_{\kappa<1}$ is contained in the subset L^f of L defined by the longitudinal connection. The preimage of $L_{\kappa<1}$ in $V^\sharp$ is a union of strata and trivial as a stratified space over $L_{\kappa<1}$. It has a unique open-dense stratum which can be identified with the product $L_{\kappa<1} \times \mathbf{P}((V/L)_{\kappa<1})$ in case $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$. If $L \in \mathcal{L}_{\kappa=1}(\mathcal{H})$, then we must replace the factor $\mathbf{P}((V/L)_{\kappa<1})$ by $\mathbf{S}(V/L)$, where $\mathbf{S}$ assigns to a (real) vector space the sphere of its real half lines. (There is no need to write $(V/L)_{\kappa<1}$ here, since the latter equals $V/L - \{0\}$.)

An arbitrary stratum is described inductively: the collection of divisors and boundary walls defined by a subset of $\mathcal{L}_{\kappa\geq 1}(\mathcal{H})$ has a nonempty intersection if and only if that subset makes up a flag: $L_\bullet : L_0 > L_1 > \cdots > L_k > L_{k+1} = V$. Their common intersection contains a stratum $S(L_\bullet)$ which decomposes as

$$S(L_\bullet) = (L_0)_{\kappa<1} \times \prod_{i=1}^{k} \mathbf{P}((L_i/L_{i-1})_{\kappa<1}) \times \mathbf{P}((V/L_k)_{\kappa<1}),$$

at least, when $L_k \in \mathcal{L}_{\kappa>1}(\mathcal{H})$; if $L_k \in \mathcal{L}_{\kappa=1}(\mathcal{H})$, we must replace the last factor by $\mathbf{S}(V/L_k)$. It is clear that $G.\mathbf{C}^\times$ naturally acts on $V^\sharp$. The covering $\widetilde{V_{\kappa<1}} \to V_{\kappa<1}$ extends naturally to a ramified covering $\widetilde{V^\sharp} \to V^\sharp$ with $\Gamma \times G$-action. Since the holonomy along $S(L_\bullet)$ decomposes according to its factors, a connected component $\tilde{S}(L_\bullet)$ of the preimage of a stratum $S(L_\bullet)$ decomposes as a product of coverings of the factors of $S(L_\bullet)$. By Lemma 6.7, the covers of these factors are finite except for the last, which is the holonomy cover of $\mathbf{P}((V/L_n)_{\kappa<1})$ or $\mathbf{S}(V/L_n)$.

The preimage $\mathbf{P}(V^\sharp)$ of the origin of V in $V^\sharp$ is a compact manifold with boundary $\mathbf{P}(\partial V^\sharp)$. Let us write $B^\sharp$ for $\mathbf{P}(V^\sharp)$, $\partial B^\sharp$ for $\mathbf{P}(\partial V^\sharp)$ and denote the manifold interior $B^\sharp - \partial B^\sharp$ simply by B. The latter is a quasiprojective manifold which contains $\mathbf{P}(V_{\kappa<1})$ as the complement of a normal crossing divisor. The strata in $B^\sharp$ are given by the flags $L_\bullet$ which begin with $L_0 = \{0\}$. We denote by $D(L)$ the exceptional divisor in $B^\sharp$ defined by $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$. (It is easy to see that $D(L) = \mathbf{P}(L_{\kappa<1}) \times \mathbf{P}((V/L)_{\kappa<1})$.) The group Γ acts on $\widetilde{B^\sharp}$ properly discontinuously with compact orbit space $B^\sharp$.

Proposition **6.9.** — *The projectivized developing map extends to this covering as a continuous Γ-equivariant map $\widetilde{B^\sharp} \to \mathbf{B}^\sharp$ which is constant on the G-orbits. It has the following properties:*

(i) It maps every boundary component of $\widetilde{B^\sharp}$ to a Borel-Serre boundary component of $\mathbf{B}^\sharp$ and the restriction $\widetilde{B} \to \mathbf{B}$ is a holomorphic map.

(ii) Every irreducible component of the preimage in $\widetilde{B}$ of an exceptional divisor $D(L)$, $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$, is mapped to an open subset of special subball of $\mathbf{B}$ of codimension $\dim(L)$ and the resulting map from such irreducible components to special subballs reverses the inclusion relation.

(iii) Every connected component of a fiber of the map $\widetilde{B^\sharp} \to \mathbf{B}^\sharp$ is compact. If that connected component is a singleton, then at the image of this singleton in $G\backslash\widetilde{B^\sharp}$, the map $G\backslash\widetilde{B^\sharp} \to \mathbf{B}^\sharp$ is local isomorphism.

Proof. — The proof amounts to an analysis of the behavior of the projectivized developing map on $\widetilde{B^\sharp}$. Since we did this already in the case without boundary components in the proof of Theorem 5.7, we shall now concentrate on the case of a boundary stratum. Such a stratum is given by a flag $L_\bullet = (\{0\} = L_0 > L_1 > \cdots > L_k > L_{k+1} = V)$, for which $L_i \in \mathcal{L}_{\kappa>1}(\mathcal{H})$ for $i < k$ and $L_k \in \mathcal{L}_{\kappa=1}(\mathcal{H})$:

$$S(L_\bullet) \cong \mathbf{P}((L_1/L_0)_{\kappa<1}) \times \cdots \times \mathbf{P}((L_k/L_{k-1})_{\kappa<1}) \times \mathbf{S}(V/L_k)$$

Let us write $\partial B^\sharp$ for the boundary component of $B^\sharp$ defined by L_k. If we had not blown up the strict transform of L_k in a real-oriented fashion, but in the conventional manner, then the last factor would be $\mathbf{P}(V/L_k)$. On a point over that stratum, the developing map is according to Proposition 2.25 affine-linearly equivalent to a map taking values in $\mathbf{C} \times T_1 \times \mathbf{C} \cdots \times T_k \times \mathbf{C}$ with components

$$\left(\left(t_0^{1-\kappa_0} t_1^{1-\kappa_1} \cdots t_{i-1}^{1-\kappa_{i-1}}(1, F_i)\right)_{i=1}^{k}, t_0^{1-\kappa_0} t_1^{1-\kappa_1} \cdots t_{k-1}^{1-\kappa_{k-1}}(\log t_k, F_{k+1})\right).$$

Here F_i is a morphism at a point of this conventional blowup to a linear space germ $(T_i)_0$, t_i defines the ith exceptional divisor and the map $(t_0, F_1, \ldots, F_k, t_k, F_{k+1})$ constitutes a chart. On the real-oriented blowup, $\log t_k$ is a coordinate: its imaginary part $\arg t_k$ helps to parametrize the ray space $\mathbf{S}(V/L_k)$ and its real part $\log|t_k|$ must be allowed to take the value $-\infty$ (its value on the boundary). We denote this coordinate τ_k. On a connected component $\tilde{S}(L_\bullet)$ of the preimage of $S(L_\bullet)$ in $\widetilde{B^\sharp}$, we have defined roots of the normal coordinates: $t_i = \tau_i^{q_i}$, $i = 0, \ldots, k-1$, so that $(F_1, \tau_1 \ldots, F_k, \tau_k, F_{k+1})$

is a chart for $\widetilde{B^\sharp}$. In terms of this chart, the projectivized developing map becomes

$$\left(\left(\tau_i^{-p_i}\cdots\tau_{k-1}^{-p_{k-1}}(1,F_i)\right)_{i=1}^{k-1}, 1, F_k, \tau_k, F_{k+1}\right),$$

where we recall that $-p_i$ is a positive integer and the constant component 1 reminds us of the fact that we are mapping to an affine chart of a projective space). We use this to see that the projectivized developing map extends to $\widetilde{B^\sharp} \to \mathbf{B}^\sharp$. A chart of $\mathbf{B}^\sharp$ is (implicitly) given by the affine hyperplane $A_1 \subset A$ defined by $h(-,e)=1$, where e is minus the unit vector corresponding to the slot occupied by τ_k (the geodesic action is then given by translation over negative multiples of e). This normalization is here already in place, for the coordinate in question is in the slot with constant 1. So we then have in fact a chart of the Borel-Serre compactification, provided that we remember that τ_k takes its values in $[-\infty,\infty)+\sqrt{-1}\mathbf{R}$. In particular, we have the claimed extension $\widetilde{B^\sharp} \to \mathbf{B}^\sharp$. It sends the boundary stratum $\tilde{S}(L_\bullet)$ to the Borel-Serre boundary (for $\mathrm{Re}(\tau_k)$ takes there the value $-\infty$) with image herein the locus defined by putting all but the last three slots equal zero. The fiber passing through $\tilde z$ is locally given by putting $\tau_{k-1}=0$ and fixing the values of (F_k,F_{k+1}) and $\tau_k \in \infty+\sqrt{-1}\mathbf{R}$. In particular, this fiber is smooth at $\tilde z$. This is true everywhere, and hence a connected component of that fiber is also an irreducible component. Let us denote the irreducible component passing through $\tilde z$ by $\Phi_{\tilde z}$. So $\Phi_{\tilde z}$ lies over $\partial B^\sharp$.

If $k=1$, then $\Phi_{\tilde z}=\{\tilde z\}$ and the extension is at $\tilde z$ simply given by $(1,F_1,\tau_1,F_2)$ and hence is there a local isomorphism. If $k>1$, then since the pair $(F_k,\mathrm{Im}(\tau_k),F_{k+1})$ defines a chart for the product $\mathbf{P}((L_{k-1}/L_k)_{\kappa<1}) \times \mathbf{S}((V/L_k)_{\kappa<1})$, $\Phi_{\tilde z}$ is an irreducible component of a fiber of the natural map

$$\begin{aligned}
&\partial\widetilde{B^\sharp} \cap \widetilde{D(L_{k-1})} \to \partial B^\sharp \cap D(L_{k-1}) = \\
&\qquad = \mathbf{P}((L_{k-1})_{\kappa<1}) \times \mathbf{P}((L_{k-1}/L_k)_{\kappa<1}) \times \mathbf{S}((V/L_k)_{\kappa<1}) \to \\
&\qquad\qquad \to \mathbf{P}((L_{k-1}/L_k)_{\kappa<1}) \times \mathbf{S}((V/L_k)_{\kappa<1}).
\end{aligned}$$

Since L_{k-1}° has finite longitudinal holonomy by Lemma 6.7, the irreducible components of the fibers of this map are compact. If $\Phi_{\tilde z}=\{\tilde z\}$, then we must have either $k=1$ or $k=2$ and $\dim L_1=1$. The former case we already discussed: our extension of the developing map is then a local isomorphism. In the second case we have $T_1=\{0\}$ and so (τ_1,F_2,τ_2,F_3) is a chart of $\widetilde{B^\sharp}_{\tilde z}$. Since G_{L_1} acts on the first component as multiplication by $|p_1|$th roots of unity, $(\tau_1^{-p_1},F_2,\tau_2,F_3)$ is a chart of $G_{L_1}\backslash\widetilde{B^\sharp}_{\tilde z}$. The extension of the developing map at $\tilde z$ is given by $(\tau_1^{-p_1},F_2,\tau_2,F_3)$ and so factors through $G_{L_1}\backslash\widetilde{B^\sharp}_{\tilde z}$ as a local isomorphism. The proof of the proposition is now complete. $\square$

Proof of Theorem 6.2. — According to Proposition 6.9, the mapping $G\backslash\widetilde{B^\sharp}\to\mathbf{B}^\sharp$ has the property that the connected components of its fibers are compact, that the preimage of the Baily-Borel boundary is in the boundary of the domain and that where this map is locally finite it is in fact a local isomorphism. So Lemma 5.13 can be applied (in its entirity) to this situation and we find that for the topological Stein factorization of $G\backslash\widetilde{B^\sharp}\to\mathbf{B}^\sharp$,

$$G\backslash\widetilde{B^\sharp}\longrightarrow(G\backslash\widetilde{B^\sharp})_{\mathfrak{St}}\longrightarrow\mathbf{B}^\sharp,$$

the second map is a local isomorphism over $\mathbf{B}$. Since $(G\backslash\widetilde{B^\sharp})_{\mathfrak{St}}$ can be identified with $G\backslash(\widetilde{B^\sharp}_{\mathfrak{St}})$, we drop the parentheses in the notation. We first prove that $G\backslash\widetilde{B}_{\mathfrak{St}}\to\mathbf{B}$ is a Γ-isomorphism. For this we verify that the hypotheses of Lemma 5.2 are verified for the Stein factor $G\backslash\widetilde{B^\sharp}_{\mathfrak{St}}$ with $Y':=\mathbf{B}$.

We know that Γ acts properly discontinuously on $\widetilde{B^\sharp}$ with compact fundamental domain. The first Stein factor is proper and Γ-equivariant and so Γ acts also properly discontinuously on $G\backslash\widetilde{B^\sharp}_{\mathfrak{St}}$. Since Γ acts on $\mathbf{B}$ as a group of isometries, Condition (ii) of 5.2 is fulfilled as well. The lemma tells us that $G\backslash\widetilde{B}_{\mathfrak{St}}\to\mathbf{B}$ is then a covering projection. But $\mathbf{B}$ is simply connected, and so this must be an isomorphism. It is easy to see that $G\backslash\widetilde{B^\sharp}_{\mathfrak{St}}\to\mathbf{B}^\sharp$ is then a Γ-homeomorphism. Since Γ acts on the domain discretely and cocompactly, the same is true on its range. This implies that Γ is discrete and of cofinite volume in the unitary group of h.

The irreducible components of the preimages in $\tilde{B}$ of the exceptional divisors $D(L)$ are locally finite in $\widetilde{B}$; since $\widetilde{B}\to G\backslash\widetilde{B}_{\mathfrak{St}}$ is proper, the image of these in $\widetilde{B}_{\mathfrak{St}}$ are also locally finite. An irreducible component $\tilde{D}(L)$ over $D(L)$ gets contracted if $\dim L>1$, and its image in $\mathbf{B}$ is the intersection of $\mathbf{B}$ with a special subspace of codimension equal to the dimension of L. The irreducible components of the preimages of the divisors $D(L)$ in $\widetilde{B^\sharp}$ are locally finite. Hence their images in $\mathbf{B}$ are locally finite in $\mathbf{B}$. We get a divisor precisely when $\dim L=1$. It follows that the collection of special hyperplanes is locally finite on $\mathbf{B}$, and that $G\backslash\mathbf{P}(V_{\kappa<1})\subset G\backslash B_{\mathfrak{St}}$ maps isomorphically onto the complement of the special hyperball arrangement modulo Γ, $\Gamma\backslash\mathbf{B}^\circ$.

Since $G\backslash\mathbf{P}(V_{\kappa<1})\to\Gamma\backslash\mathbf{B}^\circ$ is an isomorphism, so is $G\backslash V_{\kappa<1}\to\Gamma\backslash A^\circ_{\mathbf{B}}$. $\square$

7. Supplementary results and remarks

In this section, $(V,\mathcal{H},\nabla)$ is a Dunkl system satisfying the Schwarz condition and endowed with a flat Hermitian form h of admissible type. We adhere to our earlier notation; for instance, $G \subset \mathrm{U}(V)$ denotes the Schwarz symmetry group of the system.

7.1. *A presentation for the holonomy group.* — The holonomy group Γ is the image of a representation of the fundamental group $\pi_1(G\backslash V^\circ, *)$. In case G is a Coxeter group and $\mathcal{H}$ is its set of reflecting hyperplanes, then $\pi_1(G\backslash V^\circ, *)$ is the Artin group of G that we encountered in Subsection 3.5. But as the Lauricella systems show, $\mathcal{H}$ may very well be bigger than the set of reflecting hyperplanes of G. We describe a set of generators of the kernel of the holonomy representation and thus obtain a presentation of the holonomy group Γ in case we have one of $\pi_1(G\backslash V^\circ, *)$.

Let us first note that any $L \in \mathcal{L}(\mathcal{H})$ unambiguously determines a conjugacy class in the fundamental group of V°: blow up L in V and take the conjugacy class of a simple loop around the generic point of the exceptional divisor in (the preimage of) V°. If we pass to the orbit space $G\backslash V^\circ$, then L° determines a stratum in $G\backslash V$. This stratum determines in the same way a conjugacy class in $\pi_1(G\backslash V^\circ, *)$. If L is irreducible and $\alpha_L \in \pi_1(G\backslash V^\circ, *)$ is a member of this conjugacy class, then $\alpha_L^{|G_L|}$ is in the conjugacy class of $\pi_1(V^\circ, *)$ defined above. If $\kappa_L \neq 1$, then the holonomy around this stratum in $G\backslash V^\circ$ has order q_L, where q_L is the denominator of $1 - \kappa_L$. So $\alpha_L^{q_L}$ is then the smallest power of α_L which lies in the kernel of the monodromy representation.

Theorem **7.1.** — *Suppose that we are in the elliptic, parabolic or hyperbolic case, that is, in one the cases covered by Theorems 5.1, 5.7, 5.6 and 6.2. Then Γ is obtained from $\pi_1(G\backslash V^\circ, *)$ by imposing the relations $\alpha_L^{q_L} = 1$ for $L \in \mathcal{H}_{\kappa<1}$ and for $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$ of dimension ≤ 1.*

Proof. — We limit ourselves to the hyperbolic case, since the others are easier. By adding $\mathbf{B}$ to $A_{\mathbf{B}}$ at infinity we obtain a line bundle $\mathbf{L}$ over $\mathbf{B}$ that has $\mathbf{B}$ as the zero section. Theorem 6.2 shows that $G\backslash V^\circ$ can be identified with an open subset of $\Gamma\backslash\mathbf{L}$. Since $\mathbf{L}$ is a contractible (hence simply connected) complex manifold, Γ is the orbifold fundamental group of $\Gamma\backslash\mathbf{L}$. Hence the quotient $\pi_1(G\backslash V^\circ, *) \to \Gamma$ can be understood as the map on (orbifold) fundamental groups of the map $G\backslash V^\circ \to \Gamma\backslash\mathbf{L}$. It is well-known (and easy to see) that the kernel of such a map is generated by the powers of the conjugacy classes in the fundamental group of $G\backslash V^\circ$ defined by irreducible components of codimension one of the complement of the image, $\Gamma\backslash\mathbf{L} - G\backslash V^\circ$, the power in question being the order of local fundamental group at a general point of such an irreducible component. These irreducible components are naturally indexed by the strata of $G\backslash V$ of the type described in the theorem: the strata of codimension one of $G\backslash V$ yield the irreducible components meeting $G\backslash V_{\kappa<1}$, the zero dimensional stratum corresponds the image of the zero section $\Gamma\backslash\mathbf{B}\subset\Gamma\backslash\mathbf{L}$ and the strata of dimension one on which $\kappa>1$ correspond to the remaining irreducible components. The powers are of course as stated in the theorem. □

Remarks **7.2.** — Notice that for the complete elliptic and parabolic cases 5.1 and 5.6 the relations of the second kind do not occur.

Once we seek to apply this theorem in a concrete case, we need of course to have at our disposal a presentation of the fundamental group of $G\backslash V^\circ$ in which the elements α_L can be identified. For G a Coxeter group, this is furnished by the Brieskorn-Tits presentation [4], [13]; this produces in the elliptic range the presentations of the associated complex reflection groups that are due to Coxeter [10], Sections 12.1 and 13.4. For the case of an arbitrary finite complex reflection group, one may use a presentation of the fundamental group due to Bessis [2].

7.2. *Automorphic forms and invariant theory.* — According to Theorem 4.5, the developing map $\widetilde{V_{\kappa<1}}\to A^\diamond_{\mathbf{B}}$ is homogeneous of negative degree p_0 (recall that p_0 is the numerator of the negative rational number $1-\kappa_0$). We can express this in terms of orbifold line bundles as follows: if $\mathcal{O}_{\Gamma\backslash\mathbf{B}^\diamond}(-1)$ denotes the Γ-quotient of the automorphic line bundle $\mathcal{O}_{\mathbf{B}^\diamond}(-1)$ over $\Gamma\backslash\mathbf{B}^\diamond$, then the pull-back of this bundle over $\mathbf{P}(V_{\kappa<1})$ is isomorphic to $\mathcal{O}_{\mathbf{P}(V_{\kappa<1})}(-p_0)$. Now $\mathbf{P}(V)-\mathbf{P}(V_{\kappa<1})$ is a closed subset of $\mathbf{P}(V)$ which is everywhere of codimension >1 and so for any $k\geq 0$, the space of sections of $\mathcal{O}_{\mathbf{P}(V_{\kappa<1})}(k)$ is the space $\mathbf{C}[V]_k$ of homogeneous polynomials on V of degree k. We conclude that we have an isomorphism of graded algebras

$$\oplus_{n\geq 0}H^0(\mathbf{B}^\diamond,\mathcal{O}(-n))^\Gamma\cong\oplus_{n\geq 0}\mathbf{C}[V]^G_{-np_0}.$$

In particular, the lefthand side is finitely generated and its Proj can be identified with $G\backslash\mathbf{P}(V)$. In [23] a systematic study was made of algebras of meromorphic automorphic forms of the type under consideration here. The upshot is that the Proj of the lefthand side is explicitly described as a modification of the Baily-Borel compactification of $\Gamma\backslash\mathbf{B}$ which leaves $\Gamma\backslash\mathbf{B}^\diamond$ untouched.

To be more explicit, let us start out with the data consisting of the ball $\mathbf{B}$, the group Γ , and the collection of special hyperplanes. Let us also make the rather modest assumptions that $\dim V\geq 3$, so that $\dim\mathbf{B}\geq 2$ and that $\kappa_H<1$ for all $H\in\mathcal{H}$. The following lemma verifies the central hypothesis of Corollary 5.8 of [23] (where the Hermitian form is given the opposite signature).

Lemma **7.3.** — *Every* 1*-dimensional intersection of special hyperplanes is positive definite.*

Proof. — Any 1-dimensional intersection K of special hyperplanes which is negative semidefinite defines a point on the closure of $\mathbf{B}$. If K is negative (which defines an interior point of $\mathbf{B}$), then K is a special subspace and hence corresponds to a member of $\mathcal{L}_{\kappa>1}(\mathcal{H})$ of codimension one, that is, a member $H \in \mathcal{H}$. Since $\kappa_H \le 1$, this is impossible. If K is isotropic, then choose a 2-dimensional intersection P of special hyperplanes which contains K. Since the projectivization of P meets $\mathbf{B}$, it is a special subspace and hence corresponds to a member $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$ of codimension 2. The transversal Dunkl system in V/L has a projectivized developing map taking values in $\mathbf{B} \cap \mathbf{P}(P)$. So $\mathcal{H}_L$ contains a member H with $\kappa_H = 1$. But this we excluded also. □

Although Corollary 5.8 of [23] does not apply as it stands—Γ need not be arithmetic—one can verify that the arguments to prove it only require Γ to be discrete and of cofinite volume in the relevant unitary group. It then tells us something we already know via our main theorem, namely that the algebra of automorphic forms on $\mathbf{B}$ with arbitrary poles along the special hyperplanes is finitely generated with positive degree generators and that the Proj of this graded algebra defines a certain projective completion of $\Gamma\backslash\mathbf{B}^\circ$: in the present situation the latter is just $\mathbf{P}(G\backslash V)$. But in [23] the completion is explicitly described as a blowup followed by a blowdown of the Baily-Borel compactification of $\Gamma\backslash\mathbf{B}$. If we go through the details of this, we find that this intermediate blowup is almost $G\backslash B^\sharp$: the difference is that we now must blow up the parabolic $L \in \mathcal{L}_{\kappa=1}(\mathcal{H})$ in the standard manner and not in the real-oriented sense.

Question **7.4.** — The algebra of Γ-automorphic forms (of fractional degree) must appear in $\mathbf{C}[V]^G$ as a subalgebra. It is in fact the subalgebra of G-invariant polynomials which in degree n vanish on each $L \in \mathcal{L}_{\kappa>1}(\mathcal{H})$ of order $\ge n(\kappa_L - 1)/(\kappa_0 - 1)$. It is only via our main theorem that we can give a geometric interpretation of the Proj of this subalgebra as a modification of $\mathbf{P}(G\backslash V)$. In the Lauricella case, this can be done directly by means of geometric invariant theory, but is this possible in general?

8. Classification of orbifolds for reflection arrangements

Our aim is to list the Dunkl systems whose underlying arrangement is that of a finite reflection group and for which the holonomy is as studied in the previous chapters: elliptic, parabolic or hyperbolic with a discrete holonomy group of cofinite volume. More precisely, we classify the cases for which the hypotheses of the Theorems 5.1, 5.6 and 6.2 are satisfied.

In order to display the information in an efficient way, we elaborate a little on Remark 2.32. Given a Dunkl system of type A_n with the parameters $\mu_0, \dots, \mu_n$ on $V = \mathbf{C}^{n+1}/(\text{main diagonal})$, then for $m = 0, \dots, n$ we have a map

$$s_m : \mathbf{C}^n \to V, \quad (u_1, \dots, u_n) \mapsto (u_1^2, \dots, u_{m-1}^2, 0, u_m^2, \dots, u_n^2).$$

Remark 2.32 tells us that pulling back the Dunkl system along this map yields a Dunkl system of type B_n; we refer to this way of producing a B_n-system as *reduction of the A_n-system at index m*. Notice that any type B_k subsystem of the B_n-system determines a $k+1$-element subset $I \subset \{0, \dots, n\}$ which contains m (and vice versa) with κ taking the value $-1 + 2\mu_I$ on its fixed point subspace (where $\mu_I := \sum_{i \in I} \mu_i$). On the other hand, any type A_k subsystem is contained in a unique subsystem of type B_{k+1} and so determines $(k+1)$-element subset of $J \subset \{0, \dots, n\} - \{m\}$; κ takes then value μ_J on its fixed point subspace.

If we only wish to consider non-negative weights on arrangements, then reduction at index m is allowed only if $\frac{1}{2} \le \mu_i + \mu_m < 1$ for all $i \ne m$. Since the Dunkl system is invariant under reflection in the short roots, we see that the Schwarz condition on the weight κ for a B-type intersection becomes: for all $I \ni m$, $1 - \mu_I$ is zero or the reciprocal of an integer. In particular the weights on B_n that satisfy the Schwarz conditions are all obtained by reduction at an index on A_n that satisfies the Schwarz conditions.

The tables below list all the weights (with values in $(0, 1)$) for arrangements of type A and B that satisfy the Schwarz conditions. The parameters μ_i are defined by n_i/d where n_i and d appear in the table. If a parameter n_m is typeset in bold then the weight obtained by reduction at position m satisfies the Schwarz conditions for type B. If additionally $n_i + n_m = d/2$ for all $i \ne m$ then the reduced weight can be considered as a weight on an arrangement of type D. Note that such a weight is then invariant under the Weyl-group of type D. In the "remark" column "ell" stands for elliptic, "par" for parabolic and "cc" for co-compact. If no remark indicates otherwise, the group will be hyperbolic and acts with cofinite volume.

We omit the case $\kappa = 0$ from our tables. There is one additional series, corresponding to the full monomial groups, that is obtained as follows. Take integers $n \ge 1$, $q \ge 2$ and define a weight on A_n by $\mu_0 = \ldots = \mu_{n-1} = 0$, $\mu_n = 1 - 1/q$. This weight can be reduced at index n and satisfies both the Schwarz conditions for type A and type B.

TABLE 8.1. — Types A_* and B_*

#	d	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	remark
1	3	**1**	**1**	**1**	**1**							
2	4	**1**	**1**	**1**	**1**							par
3	4	**1**	**1**	**1**	**2**							
4	5	**2**	**2**	**2**	**2**							cc
5	6	1	1	1	1							ell
6	6	1	1	1	**2**							ell
7	6	1	1	1	**3**							par
8	6	1	1	1	**4**							
9	6	1	1	**2**	**2**							par
10	6	1	1	**2**	**3**							
11	6	**1**	**2**	**2**	**2**							
12	6	**1**	**2**	**2**	**3**							
13	6	**2**	**2**	**2**	**3**							
14	8	**1**	**3**	**3**	**3**							cc
15	8	**2**	**2**	**2**	**5**							cc
16	8	**3**	**3**	**3**	**3**							cc
17	8	**3**	**3**	**3**	**4**							cc
18	9	**2**	**4**	**4**	**4**							cc
19	9	**4**	**4**	**4**	**4**							cc
20	10	**1**	**4**	**4**	**4**							cc
21	10	**2**	3	3	3							cc
22	10	**2**	3	3	**6**							cc
23	10	3	3	3	3							cc
24	10	3	3	3	**5**							cc
25	10	3	3	3	**6**							cc
26	12	**1**	**5**	**5**	**5**							cc
27	12	2	2	2	**7**							cc
28	12	2	2	2	**9**							cc
29	12	2	2	**4**	**7**							cc
30	12	**2**	**4**	**4**	**7**							cc
31	12	**3**	**3**	**3**	**5**							cc
32	12	**3**	**3**	**3**	**7**							cc
33	12	**3**	**3**	**3**	**8**							cc
34	12	**3**	**3**	**5**	**5**							cc

35	12	**3**	**3**	**5**	**6**	
36	12	**3**	**5**	**5**	**5**	cc
37	12	**3**	**5**	**5**	**6**	cc
38	12	**4**	**4**	**4**	**5**	
39	12	**4**	**4**	**4**	**7**	
40	12	**4**	**4**	**5**	**5**	cc
41	12	**4**	**4**	**5**	**6**	cc
42	12	**4**	**5**	**5**	**5**	cc
43	12	**4**	**5**	**5**	**6**	cc
44	12	**5**	**5**	**5**	**5**	cc
45	12	**5**	**5**	**5**	**6**	cc
46	14	**2**	5	5	5	cc
47	14	5	5	5	5	cc
48	15	**4**	**6**	**6**	**6**	cc
49	15	**4**	**6**	**6**	**8**	cc
50	15	**6**	**6**	**6**	**8**	cc
51	18	**1**	**8**	**8**	**8**	cc
52	18	**2**	7	7	7	cc
53	18	**2**	7	7	**10**	cc
54	18	3	3	3	**13**	cc
55	18	3	3	3	**14**	cc
56	18	**5**	7	7	7	cc
57	18	7	7	7	7	cc
58	18	7	7	7	**10**	cc
59	20	**5**	**5**	**5**	**11**	cc
60	20	**5**	**5**	**5**	**14**	cc
61	20	6	6	6	**9**	cc
62	20	6	6	6	**13**	cc
63	20	6	6	**9**	**9**	cc
64	20	6	6	**9**	**10**	cc
65	24	4	4	4	**17**	cc
66	24	4	4	4	**19**	cc
67	24	**7**	**9**	**9**	**9**	cc
68	24	**7**	**9**	**9**	**14**	cc
69	24	**9**	**9**	**9**	**14**	cc
70	30	5	5	5	**19**	cc

71	30	5	5	5	**22**			cc
72	30	5	5	5	**23**			cc
73	30	9	9	9	**11**			cc
74	42	7	7	7	**29**			cc
75	42	7	7	7	**34**			cc
76	42	**13**	15	15	15			cc
77	42	15	15	15	**26**			cc
78	3	**1**	**1**	**1**	**1**	**1**		
79	4	**1**	**1**	**1**	**1**	**1**		
80	4	**1**	**1**	**1**	**1**	**2**		
81	6	1	1	1	1	1		ell
82	6	1	1	1	1	**2**		par
83	6	1	1	1	1	**3**		
84	6	1	1	1	1	**4**		
85	6	1	1	1	**2**	**2**		
86	6	1	1	1	**2**	**3**		
87	6	1	1	**2**	**2**	**2**		
88	6	1	1	**2**	**2**	**3**		
89	6	**1**	**2**	**2**	**2**	**2**		
90	6	**1**	**2**	**2**	**2**	**3**		
91	6	**2**	**2**	**2**	**2**	**3**		
92	8	**1**	**3**	**3**	**3**	**3**		cc
93	8	**3**	**3**	**3**	**3**	**3**		cc
94	10	**2**	3	3	3	3		cc
95	10	3	3	3	3	3		cc
96	10	3	3	3	3	**6**		cc
97	12	2	2	2	2	**7**		cc
98	12	2	2	2	2	**9**		cc
99	12	2	2	2	**4**	**7**		cc
100	12	**3**	**3**	**3**	**3**	**5**		
101	12	**3**	**3**	**3**	**3**	**7**		
102	12	**3**	**3**	**3**	**5**	**5**		cc
103	12	**3**	**3**	**5**	**5**	**5**		cc
104	4	**1**	**1**	**1**	**1**	**1**	**1**	
105	4	**1**	**1**	**1**	**1**	**1**	**2**	
106	6	1	1	1	1	1	1	par

107	6	1	1	1	1	1	**2**					
108	6	1	1	1	1	1	**3**					
109	6	1	1	1	1	1	**4**					
110	6	1	1	1	1	**2**	**2**					
111	6	1	1	1	1	**2**	**3**					
112	6	1	1	1	**2**	**2**	**2**					
113	6	1	1	1	**2**	**2**	**3**					
114	6	1	1	**2**	**2**	**2**	**2**					
115	10	3	3	3	3	3	3					cc
116	12	2	2	2	2	2	**7**					cc
117	4	**1**	**1**	**1**	**1**	**1**	**1**	**1**				
118	6	1	1	1	1	1	1	1				
119	6	1	1	1	1	1	1	**2**				
120	6	1	1	1	1	1	1	**3**				
121	6	1	1	1	1	1	1	**4**				
122	6	1	1	1	1	1	**2**	**2**				
123	6	1	1	1	1	1	**2**	**3**				
124	6	1	1	1	1	**2**	**2**	**2**				
125	6	1	1	1	1	1	1	1	1			
126	6	1	1	1	1	1	1	1	**2**			
127	6	1	1	1	1	1	1	1	**3**			
128	6	1	1	1	1	1	1	**2**	**2**			
129	6	1	1	1	1	1	1	1	1	1		
130	6	1	1	1	1	1	1	1	1	**2**		
131	6	1	1	1	1	1	1	1	1	1	1	

Tables 2–5 list all remaining cases for the arrangements of the exceptional real and complex reflection groups. The Shephard groups G_{25}, G_{26} and G_{32} are omitted because these are already covered by the tables for types A_3, B_3 and A_4 respectively.

Only in the F_4 case the group has more than one orbit in its mirror arrangement. This number is then two, which means that its discriminant has two irreducible components; we write q_1 and q_2 for the ramification indices along these components, while we use a single q in all other cases. The weight κ on the arrangement is obtained by setting $\kappa_H = 1 - 2/q_H$ where q_H is the ramification index along the image of the mirror H in the orbit space.

All listed cases correspond to a hyperbolic reflection group except $q_1 = 2$, $q_2 = 3$ for type F_4 which is of parabolic type. If a number q or q_i is typeset in bold then the corresponding group acts co-compactly on a hyperbolic ball, otherwise it acts with co-finite volume. All the obtained hyperbolic groups for the *real* exceptional root systems are arithmetic.

TABLE 8.2. — Types E_n

n	6	7	8
q	3, 4	3	3

TABLE 8.3. — Type F_4

q_1	2	3	4	6
q_2	3, 4, **5**, 6, **8**, **12**	3, **4**, 6, **12**	4	6

The case $q_1 = 2$, $q_2 = 3$ is of parabolic type.

TABLE 8.4. — Types H_n

n	3	4
q	**3**, **4**, **5**, **10**	**3**, **5**

TABLE 8.5. — Shephard-Todd groups G_n

n	24	27	29	31	33	34
q	**3**, 4, **5**, 6, **8**, **12**	**3**, 4, **5**	3, 4	3, **5**	3	3

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