EXAMINING MATHEMATICAL REASONING THROUGH ENACTED VISUALISATION

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ABSTRACT

This study sets out to analyse the co-emergence of visualisation and reasoning processes when selected learners engaged in solving word problems. The study argues that visualisation processes and mathematical reasoning processes are closely interlinked in the process of engaging in any mathematical activity.

This qualitative research project adopted a case study methodology embedded within a broader interpretative orientation. The research participants were a cohort of 17 mixed-gender and mixed-ability Grade 11 learners from a private school in southern Namibia. Data was collected in three phases and comprised of one-on-one task-based interviews in the first phase, focus group task-based interviews in the second, and semi-structured reflective interviews in the third. The analytical framework was informed by elements of enactivism and consisted of a hybrid of observable visualisation and mathematical reasoning indicators.

The study was framed by an enactivist perspective that served as a linking mediator to bring visualisation and reasoning processes together, and as a lens through which the co-emergence of these processes was observed and analysed. The key enactivist concepts of structural coupling and co-emergence were the two mediating ideas that enabled me to discuss the links between visualisation and reasoning that emerged whilst my participants solved the set word problems. The study argues that the visualisation processes enacted by the participants when solving these problems are inseparable from the reasoning processes that the participants brought to bear; that is, they co-emerged.

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DEDICATION

This thesis is dedicated to my two sets of parents.

My mother, Lavinia Ndaladipalekwa Dongwi and my father Jeremia Kondjeni Dongwi

and

My late Godmother, Maria Ndatoolewe Nashongo and my Godfather, Sakaria Ntinda Nashongo, for their guidance and good upbringing in accordance with Proverbs 22:6.

DECLARATION OF ORIGINALITY

I, Beata Lididimikeni Dongwi (Student number 609D6388), declare that this doctoral thesis entitled: “Examining mathematical reasoning through enacted visualisation”, is my own work written in my own words. Where I have drawn on the words or ideas of others, these have been fully acknowledged and referenced in the manner required by the Rhodes University Department of Education Guide to referencing.

	February 2018
Beata Dongwi (Signature)	(Date)

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LIST OF ABBREVIATIONS

4RPs	Four Reasoning Processes
5VIs	Five categories of Visual Imagery
CPI	Concrete Pictorial Imagery
DI	Dynamic Imagery
EVGRT	Enacted Visualisation Geometric Reasoning Tasks
GWP	Geometry Word Problems
KI	Kinaesthetic Imagery
MI	Memory Imagery
MoE	Ministry of Education
PI	Pattern Imagery
RPA	Reasoning Process Argumentation
RPE	Reasoning Process Explanation
RPG	Reasoning Process Generalisation
RPJ	Reasoning Process Justification

CHAPTER 1

INTRODUCTION

The purpose of this first chapter is to introduce the reader to the background, context and goals of the research, which focuses on the relationship between visualisation and reasoning processes when learners solve geometry word problems (GWP). I also address the theoretical underpinnings, methodology and significance of the research. The chapter ends with a chapter-by-chapter outline of the study.

1.1 RESEARCH CONTEXT AND BACKGROUND

Upon completion of the Senior Secondary Phase (SSP) in Namibia, learners are expected to be able to "use mathematical language and representation as a means of solving problems relevant to everyday life and to their further education and further careers" (Namibia. Ministry of Education [MoE], 2010b, p. 23). Word problem solving is one of the significant aspects of mathematical problem solving, incorporating actual problems and mathematical applications. However, learners "express great difficulties in handling a word or story problem" (Ahmad, Tarmizi, & Nawawi, 2010, p. 356). Since "mathematical concepts and relations are often based on visual mental representations attached to verbal information, the ability to generate, retain and manipulate abstract images is obviously important in mathematical problem solving" (Csíkos, Szitányi, & Kelemen, 2012, pp. 49–50). Hence successful problem solving requires the understanding of relevant textual information and the capacity to visualise the data (*ibid*., p. 49).

In my experience as a mathematics teacher for over 10 years, I have observed that children are curious beings who naturally learn to make sense of their world through exploration, questioning and reasoning. As they grow older, their questioning strategies and reasoning skills shift to suit the purpose of the phenomenon concerned. Like many mathematics teachers and researchers in the field of mathematics education, I am concerned by the difficulties that learners experience in expressing solutions to GWP. There often seems to be a gap between the learners' solution to such problems and their ability to explain how they obtained that solution. Learners appear to find it very difficult to make their reasoning explicit and talk about their problem-solving strategies, irrespective of the accuracy of their methods and/or solutions.

The need to study mathematical reasoning as an essential tool in word problem solving in secondary school mathematics is widely recognised, as recent research studies in mathematics education attest (Boesen, Lithner, & Palm, 2010; Dejarnette & González, 2013; Mueller, Yankelewitz, & Maher, 2014). Malloy (1999) proposes the use of "mathematical questioning (or inquiry) by students and teachers as a strategy to help students use their innate reasoning abilities to sharpen and clarify their understanding of mathematical concepts" (p. 13). For English (1999), reasoning by analogy is also an important tool in problem solving and problem posing (p. 35) as it "entails understanding something new by comparison with something that is known" (p. 22). English (1999) observes that learners do not reason by analogy if they do not see the "connections and relationships among mathematical ideas and using these understandings to master new situations" (pp. 22–23).

When solving mathematical problems, learners go through different steps, methods and procedures to explain, justify, argue and generalise their solutions and problem-solving strategies. Visualisation is one of the methods that learners can employ during word problem solving. Zimmermann and Cunningham (1991) define mathematical visualisation as "the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated" (p. 1). Drawing a diagram is a commonly suggested strategy for solving word (story) problems in mathematics (Stylianou, 2002; Ahmad et al., 2010). In their study, David and Tomaz (2012, p. 413) presented an illustrative episode that shows how drawing geometrical figures can play a major part in structuring and modifying mathematical activity in the classroom. Chen and Herbst (2013) agree that diagrams can play an important role in students' geometrical reasoning and help them to make reasoned conjectures (p. 304). Drawings are important elements in the range of representations that can support abstraction (Jao, 2013).

Sternberg (1999) points out that "one cannot solve a problem until one identifies the nature of the problem to be solved" (p. 41). Having identified the nature of the problem, the learner then needs to "figure out a strategy that will effectively solve it" (*ibid.*). Thereafter, he/she needs to represent the problem using one or more of a multitude of possible visual representations and evaluate his/her work during and after problem solving to ensure that the solutions make sense, are error free and can be generalised. Sternberg (1999) emphasises that "a good assessment of mathematical reasoning will evaluate all aspects of mathematical reasoning" (*ibid.*, p. 43).

According to Burns (1985), learners' "classroom experiences need to lead them to make predictions, formulate generalisations, justify their thinking, consider how ideas can be expanded or shifted, look for alternate approaches" (p. 16). Huscroft-D'angelo, Higgins and Crawford (2014) views reasoning as a fundamental skill in mathematics and suggests that interventions focused on advancing student reasoning will be increasingly essential to mathematics education (p. 68). Mathematical reasoning is a broad topic that may be viewed from many perspectives. One perspective is construes learners' mathematical reasoning in terms of the reasoning processes of explanation, argumentation, justification and generalisation, as discussed in Section 2.2.1 of this study.

1.2 RESEARCH GOALS AND QUESTIONS

The aim of this study is two-fold:

1. To examine the mathematical reasoning of the selected Grade 11 learners when they are solving geometry word problems.
2. To analyse how enacted visualisation processes co-emerge with mathematical reasoning processes during collaborative groups.

Hence the study aims to answer the following research questions:

Main research question:

How do visualisation processes relate to mathematical reasoning processes when selected Grade 11 learners solve geometry word problems?

Sub-questions:

1. What visualisation processes are evident in all the selected Grade 11 participants when they solve geometry word problems?
2. How do visualisation and reasoning processes co-emerge when learners solve geometry word problems in small collaborative groups?

It was thus the purpose of this study, first, to determine the type of visualisation processes that the learners employed when solving geometry word problems. This occurred in the first phase of data collection and analysis. Secondly, once I had identified the participants whose preferred method of word problem solving was visual as opposed to algebraic, I analysed their visualisation processes vis-à-vis their reasoning processes and discussed how these co-emerged and related to each other. This occurred in the second phase of data collection and analysis.

1.3 THEORETICAL FRAMEWORK

The purpose of this section is to provide the reader with a brief overview of enactivism as a theory in mathematics education research, and indicate its significance to the study.

Enactivism has its roots in many parts of the world, but particularly in South America and Canada. It has gained traction as a theoretical framework in recent mathematics education research in Southern Africa, especially in South Africa and Namibia. Enactivism posits that "the individual knower is not simply an observer of the world but is bodily embedded in the world and is shaped both cognitively and as a whole physical organism by her interaction with the world" (Ernest, 2010, p. 42).

Enactivist researchers in mathematics education bring a variety of perspectives to bear on the subject, some of which resonate with this study. First, Begg (2013) views enactivism as "a way of understanding how all organisms, including human beings, organise themselves and interact with their environments" (p. 81). Discussion about the interaction of social beings and their environments is essential to this study as the research participants solved GWP through social interaction in small collaborative groups. Secondly, Damiano (2012) referred to the dynamic interaction between the autopoietic system and its environment as structural coupling, which is another aspect of enactivism that resonates with this study. Structural coupling occurs as a result of the interaction between the organism, with his/her living and active body, and the environment. This interaction creates and is created by certain co-emergences (Rossi, Prenna, Giannandrea, & Magnoler, 2013, p. 38). Co-emergence is the third enactivist concept that resonates with the purpose of this study. In fact, it speaks to the second research sub-question, which seeks to unpack how visualisation processes and reasoning processes co-emerge in a social interaction. Reid (1995) provides the basis on which to build a conceptual understanding. He views the enactivist perspective on a problem-solving situation as one in which "the person and the situation co-emerge through their interaction and so the reasoning employed is both determined by the structure of the person, and occasioned by the sphere of possibilities implicit in the situation" (p. 10). This means that the actions that the problem-solver performs when solving a problem are intertwined with what the problem itself allows or involves. The one cannot exist without the other.

1.4 METHODOLOGY

Since this study aims to examine, analyse and interpret how visualisation processes are integral to the reasoning processes in the solving of geometry word problems, it is oriented within the interpretive paradigm. My aim is to understand how learners make sense of GWP in terms the visualisation processes they employ, and the type of reasoning processes that emerge as a result of these visualisation processes. To attain this understanding efforts were made to enter the worlds of selected learners by sharing their experiences, reflections and sense-making. From an enactivist perspective, meaning-making is not to be found in "elements belonging to the environment or in the internal dynamics of the agent, but . . . [in] the relational domain established between the two" (Di Paolo, De Jaegher, & Rohde, 2010, p. 40). The interpretive paradigm was therefore appropriate to this enactivist study.

The research takes the form of a qualitative case study. Using convenience sampling, I selected an initial cohort of 17 mixed-gender and mixed-ability Grade 11 learners to constitute the case and participate in the study. I made sure the sample was mixed gender and mixed ability because the use of visualisation processes is generic and not specific to a particular gender or ability group. Grade 11 learners were selected for this study because they were familiar with a wider variety of geometry concepts than Grade 10 learners, but still had time to participate in the study, as opposed to Grade 12 learners, whose school year is dominated by the end-of-year national examinations. Data was collected and analysed in three phases.

The first phase sought to identify the type of visualisation processes employed by the research participants when they solved GWP individually. Hence, data for this phase was collected from all 17 participants in the form of one-on-one interviews based on the 10 tasks of GWP. These word problems were developed alongside the Grades 11-12 Namibian mathematics curriculum and adapted from various sources. During these tasks, each learner was expected to use visualisation processes as part of their problem-solving strategy. The responses of each learner to each task – in the form of interview transcripts – were analysed using a coding system developed as an analytical framework. NVivo software was used to assist with coding and visualising the data. Data from the interviews and the problem-solving processes was both audio- and video-recorded, and the participants' solutions to the word problems were collected and scanned for electronic safekeeping.

In the second phase, data was collected from small collaborative discussion groups using a focus group, task-based interviews approach. Eight of the initial 17 research participants were purposively selected to participate in this phase. For the purpose of observing and analysing mathematical reasoning in this phase, the participants, whose preferred method of word problem solving involved visual methods as opposed to algebraic methods, were placed in small collaborative or focus groups. Although the participants were put into these small groups, the analysis of their reasoning processes was done individually. The group situation was meant to allow them greater freedom to converse than they might have enjoyed in the one-on-one interviews, where they might have been intimidated and not fully expressed themselves. Data for this phase was collected through focus group task-based interviews, recorded similarly to the Phase 1 data collection. The analysis was also initially done using NVivo software, which made it easier to run a matrix coding query for the relationship between the participants' visualisation processes and reasoning processes.

The third phase of the study was essentially for purposes of reflection. Data for this phase was collected by making use of a semi-structured reflective interview with the second-phase participants. The participants were asked to reflect on their experiences of having participated in the study with specific reference to how they solved the word problems in the first two phases of the study. Data analysis for this phase also made use of NVivo software to autocode the participants' responses and visualise these responses in a word cloud.

1.5 SIGNIFICANCE OF THE STUDY

The research contributes to the on-going debate about visualisation as an epistemological learning tool in mathematics education. In 2014 Presmeg (2014, pp. 151–152) repeated 13 possible research questions that she had generated over a decade earlier, pointing out that many of them still required proper investigation. Although a combination of some of Presmeg's 13 questions have been addressed in this study, further investigation into visualisation as a learning, problem-solving and reasoning tool is of utmost significance. It is an approach in line with the learner-centred approach advocated by the Namibian school curriculum, and it is to be hoped that mathematics teachers, policy makers and curriculum developers as well textbook authors take due note of the issues raised here.

Of further significance is the contribution that the study makes to the growing enactivist discourse. The use of enactivism in empirical studies is relatively novel and it is hoped that this study enriches this discourse, especially in Southern African mathematics education communities. Furthermore, the links between visualisation, reasoning and enactivism are interesting but have not attracted much attention in the context of mathematics education, either empirically or theoretically. Therefore, it is important that the findings of this study contribute to these links and their further development. Lastly, the linking of visualisation and enactivism constructs is unique in the Namibian context, and it is hoped that further studies from Namibia and beyond contribute to this discourse.

1.6 THE STRUCTURE OF THE THESIS

Chapter 2 – Literature review

This chapter discusses and reviews the literature pertinent to the study. First, the construct of mathematical reasoning is unpacked in relation to word problem solving in mathematics. Four reasoning processes are identified and discussed. Secondly, the literature on visualisation in canvassed, insofar as it relates to both mathematical reasoning and geometry word problems. A working definition of visualisation is adopted and, together with five categories of visual imagery. Lastly in this chapter, the theoretical approach which frames the study is reviewed in terms of its relation to embodied cognition, social interaction and how it relates to constructs of visualisation and mathematical reasoning. The chapter concludes with a critique of enactivism as the theoretical underpinning of the study.

Chapter 3 – Methodology

This chapter provides a detailed overview of the research methodology which was used to collect and analyse data for this study. The research paradigm and choice of methodological approaches are discussed and justified within the context of the theoretical framework of the study. The chapter concludes with a discussion of ethical considerations, validity, reliability, and positionality.

Chapter 4 – Data analysis

This chapter presents and discusses the findings of the study in relation to the study's research goals, questions and methodological design. First, the results of Phase 1 are presented and commented on. These are followed by a summary of the phase. Secondly, a vertical analysis of Phase 2 results is presented in terms of the reasoning processes employed by the research participants during the focus group task-based interviews and in terms of the relationship between these reasoning processes and the learners' visualisation processes. A horizontal, fine-grained analysis of this relation is then offered in order to help answer the main research question of the study. Finally, a reflective analysis of Phase 3 data is presented and discussed in relation to the research participants' experiences with the whole research project.

Chapter 5 – Conclusion and recommendations

The purpose of this final chapter is to consolidate the findings of the study with reference to the original research question, sub-research questions and within the context of the theoretical and methodological frameworks. The limitations, significance and contribution of the study are also discussed. Some recommendations for various stakeholders are made and briefly commented on. The chapter concludes with some personal reflections.

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

As a mathematics teacher for over a decade, it has always been my concern that despite the accuracy of my learners' responses to mathematical problems, many of them struggle to provide reasons for their solutions. This goes for both mathematically strong and mathematically weak learners. The most common responses I get from these learners include: 'I know what the answer is, but I don't know how I got it', 'I used a calculator, can I show you?', 'I don't know how to say it, but I did it', 'I don't know how I got the answer, I just know that it is right'. Such responses have worried me over the years and inspired me to delve deeper into the matter. I decided to study mathematical reasoning in the context of enacted visualisation to determine whether the use of visualisation processes to solve geometry word problems (GWP) could enhance learners' mathematical reasoning. By presenting a review of relevant literature, this chapter provides a contextual background to the study. The review begins with a discussion of mathematical reasoning and visualisation. The assumption is that visualisation is beneficial, perhaps even essential, to learners' mathematical reasoning during word problem solving if they can explain, argue, justify and generalise their solutions and solution strategies during collaborative group work. I then unpack the theoretical framework of enactivism that underpins the study. The chapter concludes with a discussion of the significance of enactivism for the study as well as a brief critique of enactivism as a theoretical framework in mathematics education.

2.2 MATHEMATICAL REASONING

Reasoning and sense-making in mathematics education provide a way for learners to participate in the activities of the discipline of mathematics (Dejarnette & González, 2013, p. 5). One of the aims of the mathematics curriculum in Namibia is to enable learners to use mathematics as a means of communication, with an emphasis on the use of clear expressions and developing "the abilities to reason logically, to classify, to generalise and to prove" when presented with real-life situations (Namibia. MoE, 2010a, p. 2). Stein, Grover and Henningsen (1996) promote the use of task features which support higher cognitive demands on learners, including reasoning and sense-making. These features include "the existence of multiple-solution strategies, the extent to which the task lends itself to multiple representations and the extent to which the task demands explanations and/or justifications from the students" (p. 461). Brodie (2010) argues that "tasks that support multiple voices, disagreements, and challenges also support mathematical reasoning, when used appropriately" (p. 7).

Brodie (2010) views mathematical reasoning as a means to sense-making of and in mathematical activity. She maintains that "only through making sense of the mathematics can we truly move to sense-making as a worthwhile everyday life activity" (p. 59). Bjuland (2007) regards sense-making, conjecturing, convincing, reflecting and generalising as interrelated processes of mathematical thinking and reasoning (p. 2). Brodie (2010) elaborates:

> Mathematical reasoning is what mathematicians do – it involves forming and communicating a path between one idea or concept and the next. When students form these paths they come to enjoy mathematics, understand the reasons why ideas work, and develop a connected and powerful form of knowledge. When students do not engage in reasoning, they often do not know that there are paths between different ideas in mathematics and they come to believe, dangerously, that mathematics is a set of isolated facts and methods that need to be remembered. (p. v)

Sternberg (1999) identifies a set of high-order processes underlying the analytical, creative and practical aspects of what is entailed in mathematical reasoning. These are:

(a) the identification of the problem
(b) formulating a strategy for solving the problem
(c) mentally representing information about a problem
(d) allocating resources
(e) monitoring and evaluating solutions. (pp. 41–43)

Sternberg (1999) argues that "one cannot solve a problem until one identifies the nature of the problem to be solved" (p. 41). Hence, having determined the nature of the problem, the learner needs to "figure out a strategy that will effectively solve it" (*ibid.*). Thereafter, she needs to represent the problem to herself, perhaps using one or more from among a multitude of potential visual representations. Finally, she needs to evaluate her work during and after solving the problem to ensure that the solutions make sense, are error free and can be generalised. Sternberg (1999) emphasises that "a good assessment of mathematical reasoning will incorporate all [the five mentioned] aspects of mathematical reasoning" (*ibid*., p. 43).

I concur with Sternberg (1999) that we have in many ways created a closed system in our mathematics classrooms that consistently rewards learners who are skilled in memory and analytical abilities, but often fails to reward learners who are skilled in creative and practical abilities (p. 38). It is no wonder that students who seem to have the "analytical mathematical-reasoning skills" still do not seem to know "how to apply these skills in a creative manner" (*ibid.*). Sternberg (1999) asserts that "mathematics will continue to matter in the lives of most of our students, not in the test scores or course grades, but in their ability to apply the mathematics they learn to practical everyday problems" (Sternberg, 1999, p. 38). It is therefore crucial that "teachers give the time needed for children not only to work through activities that promote thinking but also to reflect on that thinking whenever possible" (Burns, 1985, p. 16). Furthermore, "classroom experiences must extend beyond the goal of arriving at correct answers. Children must be asked to judge the reasonableness of their thinking, to defend their solutions" (*ibid*.) and be able to express the reasoning behind a mathematical solution.

Burns (1985) declares that learners' classroom experiences need to lead them to make predictions, formulate generalisations, justify their thinking and consider how ideas can be expanded, transformed or shifted. They should also be able to look for alternate approaches (p. 16). For Huscroft-D'angelo, Higgins and Crawford (2014), reasoning is a fundamental skill in mathematics and interventions focused on advancing student reasoning will be increasingly essential in mathematics education (p. 68). Reid (2002, p. 25) points out that reasons are expected in many domains of human experience, but in mathematics the reasons are of a particular kind. He urges us to pay attention to the different ways of reasoning learners use and the degree of formality in their reasoning (Reid, 2002, p. 27). In this study, patterns of reasoning in the learners' responses to geometry word problems are viewed in terms of the explanation, argumentation, justification and generalisation they involve, whether the learners are working alone or in a group.

2.2.1 Reasoning processes

The reasoning processes of explanation, justification, argumentation and generalisation are viewed as important aspects of this study as it was through them that mathematical reasoning was defined and analysed to answer the main research question. These processes are discussed below.

2.2.1.1 Explanation

Yackel (2001) observes that teachers and learners give mathematical explanations to clarify aspects of their mathematical thinking that they think might not be readily apparent to others (p. 13). Proofs and explanations fundamentally rest on their acceptance by individuals or groups. Thus in a successful explanation, the truth of a conclusion is accepted by the participants (Nussbaum, 2008, p. 349). Maturana and Poerksen (2004) claim that "if we accept something, we always, consciously or subconsciously, apply a criterion of validation in order to decide about the acceptability of what is to be proved and explained" (p. 54). Hence, the nature of acceptable mathematical explanation is not something that can be outlined in advance for learners simply to apply. Instead, it is formed in and through the interactions of participants in the classroom (Yackel, 2001, p. 14).

In her study, Webb (1991) notes that explanations in cooperative groups often lead to individual learning in mathematics, with the highest growth associated with those individuals who generate detailed explanations of a problem or exercise. Furthermore, content-related explanations consist of descriptions of how to solve a problem or part of a problem that include some elaboration of the solution process (p. 368). Burns (2005) urges mathematics teachers to ask learners to explain their answers, whether or not the answers are correct. Engaging learners in this way enables other learners in the same group to 'give' and 'receive' help (Webb, 1991, p. 368).

Webb (1991) maintains that learners have the potential to give understandable and timely explanations. As they share a similar language, they are able to translate difficult vocabulary and expressions and use language that fellow learners can understand. The person receiving an explanation also has an opportunity to use the explanation to correct his or her misunderstanding or lack of understanding about the work (368). However, care should be taken to ensure that teammates do not unnecessarily interrupt the member they are trying to help with continual suggestions and corrections. They should allow him/her time to complete the work (*ibid.*).

2.2.1.2 Justification

Justification is a practice at the heart of mathematics, particularly mathematical reasoning. According to Staples, Bartlo and Thanheiser (2012), justification is used to validate claims, provide insight into a result or phenomenon, and systematise knowledge, among other purposes (p. 447). Staples et al. (2012) define justification as “an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning” (p. 448).

Kilpatrick, Swafford and Findell (2001) place justification and generalisation at the heart of adaptive reasoning. They stressed that the key element in adaptive reasoning is “to provide sufficient reason for” (p. 130). Students need to be able to justify and explain ideas in order to make their reasoning clear, hone their reasoning skills and improve their conceptual understanding (*ibid.*, p. 130). Brodie (2010) confirms that “communication is fundamental to mathematical reasoning, both for an individual working with previously produced texts to produce a new one, and for groups working together to produce an argument” (p. 7). It therefore makes sense for learners to discuss their reasoning with others to enable them to justify their solutions (Brodie, 2010, p. 20).

As a learning practice, “justification is a means by which students enhance their understanding of mathematics and their proficiency at doing mathematics; it is a means to learn and do mathematics . . . both the process of justifying and also the end point of having constructed a justification are relevant for thinking about the purposes and value of justification in the classroom” (Staples et al., 2012, pp. 447, 448). In addition to demonstrating the truth of a mathematical claim, justification as a learning practice also “promotes understanding among those engaged in justification – both the individual offering a justification and the audience of that justification” (*ibid.*, p. 449). The role of justification in the context of GWP is to provide a convincing argument, such as why making a series of visual representations is a valid method for determining the answer to a given word problem.

2.2.1.3 Argumentation

Although it is merely one of the reasoning processes in this study, argumentation is central to reasoning in mathematical problem solving. Lithner (2000) defines argumentation as the “substantiation, the part of reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate” (p. 166). Along similar lines, Dove (2009, p. 139) suggests that argumentation is a verbal, social and rational activity aimed at convincing a reasonable critic of the acceptability of a conclusion by foregrounding a pattern of propositions justifying or refuting the proposition expressed in the conclusion. “An argument is a sequence of statements/sentences/propositions/formulas such that each is either a premise or the consequence of (some set of) previous lines, and the last of which is the conclusion" (Dove, 2009, p. 138).

Burns (1985) emphasises the need to encourage learners to make conjectures and examine the validity of their thoughts. She states that children need to decide on the reasonableness of their solutions, justify their procedures, verbalise their processes, reflect on their thinking and search for convincing arguments that support their conjectures (p. 14). This contributes to the development of mathematical thinking and reasoning, as the learners' structures and experience determine their next step in solving word problems. Nussbaum (2008) adds that "argumentation theory provides an analytic framework for assessing the quality of discussions and student artefacts in terms of depth of reasoning, amount of backing for claims, and consideration of counterarguments" (p. 348). It should also be noted that the type of reasoning used in the argument must be a mathematical form of reasoning (Staples et al., 2012, p. 448).

Nussbaum (2008) distinguishes between two senses of the word argument: argument as a product, consisting of a series of propositions in which a conclusion is inferred from premises; and argument as a process, referring to the social processes in which two or more individuals engage in a dialogue where arguments are constructed and critiqued (p. 348). A classroom discussion, in which learners are making and evaluating one another's arguments, would be a form of argument as process (*ibid.*). Arguments as products, on the other hand, are tangible artefacts that are the products of learners' individual or collaborative reasoning. Such reasoning can be observed in learner discussions or, for individual reasoning, through "think aloud-protocols" (Nussbaum, 2008, p. 348).

2.2.1.4 Generalisation

To generalise means to introduce new ideal objects, to overcome objective constraints (Otte, Mendonça, & de Barros, 2015, p. 144), to identify the operators and the sequence of operations that are common among specific cases and extend them to the general case (Swafford & Langrall, 2000, p. 91). A generalisation of a problem situation may be presented verbally or symbolically. Narrative descriptions of the general case are verbal representations of the generalisation, whereas representations using variables are symbolic representations (Swafford & Langrall, 2000). In their study, Hodnik Čadež and Manfreds Kolar (2015) distinguish two aspects of generalisation: "seeing the general in the particular or seeing the particular in the general" (p. 286). These two aspects, according to Hodnik Čadež and Manfreds Kolar (2015), enable the creation of a generalisation schema: "students should first solve a problem, usually by observing a particular case, creating new cases, observing the pattern and generalising. When addressing the related problems students can solve them by applying their schemas, that is, by seeing the particular in the general" (p. 286). Hodnik Čadež and Manfreds Kolar (2015) further assert that from a mathematical point of view, the first aspect of generalisation, "'seeing the general in the particular' can be understood as inductive reasoning, which is a very prominent manner of scientific thinking, providing for mathematically valid truths on the basis of concrete cases" (p. 286). Inductive reasoning is a method of identifying the properties of phenomena and of finding regularities in a logical way (Hodnik Čadež & Manfreds Kolar, 2015, p. 286). The second aspect of generalisation, 'seeing the particular in the general', refers to deductive reasoning: "It is the process of inferring conclusions from the known information (premises) based on formal logic rules, whereby conclusions are necessarily drawn from the given information, and there is no need to validate them by experiments" (Hodnik Čadež & Manfreds Kolar, 2015, p. 287).

Generalisation, as reiterated by Fahlgren and Brunström (2014), is also part of the last phase in the problem solving process. They emphasise the importance of learners' sticking to a problem when they think that they have solved it. Learners should utilise the opportunity to elaborate the problem further, and try to learn more from the result and the method they used. "It is instructive for students to investigate if there are related problems and if it is possible to generalise the result" (p. 291). In summary, "the literature suggests that it could be instructive for students to explore a statement further by asking 'what if…' or 'what if not…' questions and systematically varying key aspects to make the statement more general" (Fahlgren & Brunström, 2014, p. 291).

Otte et al. (2015) conclude that "the processes of generalisation and application of mathematics are essential to understanding what mathematics is and how it works in the context of cultural history or individual cognitive development" (p. 162). It is also essential to note that the product of a reasoning process is a text that represents a conclusion that is acceptable within the community that is producing the argument (Brodie, 2010, p. 7): "As students explain, justify and convince others of their ideas, representations are often re-examined and certain features of the representations emerge" (Sweetman, Walter, & Ilaria, 2002, p. 2).

2.2.2 Collaborative/collective argumentation and discourse

Nussbaum (2008) defines collaborative argumentation as a social process in which individuals work together to construct and critique arguments (p. 348). According to Prusak, Hershkowitz and Schwarz (2012), a collaborative argument refers to a "situation in which two or more people learn or attempt to learn something together" (p. 23). Collective argumentation also refers to participation in discussions in a distinctively mathematical way that involves multiple people arriving at a conclusion, often by consensus (Conner, Singletary, Smith, Wagner, & Francisco, 2014, p. 401). Conner et al. (2014) argue that collective argumentation is defined very broadly to enable the inclusion of any instance where learners and teachers make a mathematical claim and provide evidence to support it (p. 404).

As stated earlier, communication plays an integral role in the mathematical reasoning of both individual learners and learners working in small groups (Brodie, 2010). In their study, Kieran and Dreyfus (1998) find that two to three learners may work together on a given task, make reasonable progress toward a solution while talking to each other, probably even listening to each other, but without any proper interaction to speak of; that is, their interaction does not reflect their thinking. Each one appears to be contemplating the problem for himself, in his own way, within his own universe; when trying to convince the other, they use not reasoning so much as simple statements, forcefully made (p. 115). Hence, Kieran and Dreyfus (1998) observe that "work[ing] in pairs (and groups) is not unproblematic, and that students should be given ample time to work and to think on their own" (p. 119).

Nussbaum (2008) notes a growing body of evidence that collaborative student discourse, which includes reflective discussions among learners about academic content, can sometimes promote deep and meaningful learning (p. 348). From an enactivist perspective, the world of meaning is not in us or in the physical world around us. It is in our interactions with each other, with the subject matter and the environment – in mutually affective relationships (Proulx, 2008a, p. 21). In their study, Dekker and Elshout-Mohr (1998) argue that letting students work together on mathematical problems can mean that the contradictions that arise are very close to their own work and methods (p. 311). Therefore, in order to incorporate collaborative argumentation in this study, it is imperative that proper interaction during small-group word problem solving is nurtured and encouraged, to ensure that processes of reasoning are apparent as learners work together to produce an argument.

This study is particularly interested in participants' interactions within their small groups that provide evidence of the co-emergence of visualisation and mathematical reasoning during word problem solving. Many research outcomes support the idea that small group work promotes mathematical understanding. However, much of the research on learning in groups lacks a focus on the progress of the individual student (Dekker & Elshout-Mohr, 1998, p. 304). Dekker and Elshout-Mohr (1998) developed an interesting process model for the analysis of interaction between two students working on a mathematical task, with the focus on the individual learning process. I adopted components of this model and developed a hybrid of observable indicators that enabled me to analyse data for mathematical reasoning purposes (see Table 3.2 for the analytical tool that comprises these observable indicators).

Like the reasoning processes of explanation, argumentation, justification and generalisation, visualisation is an element of mathematical reasoning (Arcavi, 2003). This brings us to a discussion of visualisation as one of the pivotal components of mathematical reasoning in this study.

2.3 VISUALISATION

"We don't know what we see, we see what we know" – Goethe

"Everything said is said by an observer" – Maturana and Poerksen (2004)

These are two important and inspiring statements to consider when unpacking the concept of visualisation. In his seminal panoramic interview with his friend Bernhard Poerksen, Maturana insist that "the observer is the source of everything. Without the observer, there is nothing" (Maturana & Poerksen, 2004, p. 28). Adapting this to a visualisation perspective, I argue that there is no visualisation of an object unless somebody observes the object. However, Arcavi (2003) cautions that "as biological and as socio-cultural beings, we are encouraged and aspire to 'see' not only what comes 'within sight', but also what we are unable to see" (pp. 215–216), and "visualisation offers a method of seeing the unseen" (*ibid.*, p. 216).

Arcavi (2003) asserts that mathematics, as a human and cultural creation dealing with objects and entities quite different from physical phenomena, relies heavily on visualisation in its different forms and at different levels, far beyond the obviously visual field of geometry, (pp. 216–217). The importance of visualising geometrical objects and the necessity of spatial intuition for successful mathematical teaching and learning is emphasised in the Namibian mathematics curriculum (Namibia. Ministry of Education [MoE]., 2010b). One of the aims of the junior mathematics curriculum is to enable students to "develop an understanding of spatial concepts and relationships" (Namibia. Ministry of Education [MoE]., 2010a, p. 2). The use of multiple representations can be a powerful tool to facilitate learners' understanding of geometry word problems. The process of problem posing and solving that happens around the representations can foster mathematical learning (Tripathi, 2008, p. 444). Arcavi (2003) adds that "visualisation is no longer related to the illustrative purposes only, but is also being recognised as a key component of reasoning (deeply engaging with the conceptual and not merely the perceptual), problem solving, and even proving" (p. 235).

Guzmán (2002) suggests that mathematical concepts, ideas and methods have a great richness of visual relationships that are intuitively representable in a variety of ways, and can be clearly beneficial when solving problems (n.p.). According to Duval (2014), it is not so much knowing which kinds of representations learners use that matters, but rather "how to enable them to use different mathematical representations for the same object or the same relation" (p. 168). Therefore, "using these different representations is like examining the concept through a variety of lenses, with each lens providing a different perspective that makes the picture (concept) richer and deeper" (Tripathi, 2008, p. 439).

Duval (2014) regards visual representations as all kinds of representations that are used in mathematics teaching and learning, to fulfil quite different functions, such as unpacking mathematical nuances, for heuristic exploration in problem solving, and as an educational tool for helping in the acquisition of mathematical concepts (pp. 159–160). Tripathi (2008) encouraged us to think of a visual representation as "a form of an idea that allows us to interpret, communicate, and discuss the idea with others" (p. 438). Visual representations include "concrete, verbal, numerical, graphical, contextual, pictorial, or symbolic components that depict aspects of the concept" (*ibid.*). The Namibian mathematics curriculum supports these ideas as it aims to "use mathematics as a means of communication with emphasis on the use of clear expression" (Namibia. Ministry of Education [MoE]., 2010a, p. 2). Furthermore, using visualisation processes may offer new resources for the teacher that could help learners to become aware of their mental processes and of the importance of using appropriate visualisation methods in solving word problems, thereby "saying farewell to the drilling practice in the world of word problems" (Csíkos et al., 2012) and encouraging multiple representations.

Gal and Linchevski (2010) view visualisation as the ability to represent, transform, generalise, communicate, document, and reflect on visual information, which means that it clearly plays a major role in the understanding of geometry (p. 165). Hence, “when students are translating a mathematical text into a visual representation by drawing an auxiliary figure or making a modification of a figure, they employ the strategy of visualising” (Bjuland, 2007, p. 3). In this study, I have adopted Arcavi's (2003) definition of visualisation:

> Visualisation is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding. (p. 217)

In this light, according to Rosken and Rolka (2006), visualisation can be seen as an effective tool in mathematics learning, enabling the exploration of mathematical problems and giving meaning to mathematical concepts and the relationship between them. Visualisation “allows for reducing complexity when dealing with a multitude of information” (p. 458).

Arcavi (2003) strongly advocated placing visualisation at the service of problem solving, claiming that it “may play a central role to inspire a whole solution, beyond the merely procedural” (p. 224). Duval (2014) contends that visualisation “contributes to the development of imagination and creativity not only in mathematics but also in other fields of knowledge” (p. 169). Arcavi (2003) agreed, emphasising that “the visual display of information enables us to 'see' the story, to envision some cause-effect relationships, and possibly to remember it vividly” (p. 218). Thus “helping students to become aware of the importance of making drawings in mathematics problem solving” (Csíkos et al., 2012) will enable them “to engage with concepts and meanings which can be easily bypassed by the symbolic solution of the problem” (Arcavi, 2003, p. 222). David and Tomaz (2012) provide an illustration of how drawings of geometrical figures can have a powerful role in structuring and modifying mathematical activity in the classroom (p. 413), though Gómez-Chacón (2013) cautions that, far from just drawing pictures, visualisation in teaching must be underpinned sequenced progression in the thought process.

Van Garderen, Scheuermann and Poch (2014) observe that many different representational forms exist, and many can be used to solve word problems. One strategy that is often recommended for solving mathematical word problems is to use visual (external) representations. Van Garderen et al. (2014) argue that a diagram can be “an extremely "powerful" visual representation strategy when solving word problems" *(ibid.*, p. 136). Moreover, "diagrams as a representation strategy demonstrate great versatility as they can be used for solving various types of problems for many topic areas (e.g., geometry, number and operations, probability) and at all grade levels" (van Garderen et al., 2014, p. 136). Diagrams can also "be powerful ways to facilitate communication about critical ideas in mathematics as well as provide a platform for sharing problem solving strategies with others" (*ibid.*, p. 136).

Rivera (2014) notes that "visual strategies play a mediating role in the emergence of children's sophisticated, structured and necessary understandings of mathematical objects" (p. 59). But van Garderen et al. (2014) warn that the ability to use a diagram as a tool for solving word problems is a complex task and should not be underestimated. Encoding information from a mathematical problem into a diagram requires an extensive knowledge base as it involves decoding the linguistic information and encoding it into visual information. This includes "knowledge related to the ability to select, produce and productively use a diagram as a problem-solving tool as well as the ability to critique and modify or generate a new diagram where needed within the context of a problem-solving situation" (p. 136). Tripathi (2008) cautions that "teaching mathematics using the idea of multiple representations and helping students to develop the ability to represent a mathematical idea in various forms can be a challenging task" (p. 444).

According to Edens and Potter (2007, p. 285), "visualising objects and graphically representing numerical information are important mathematical tools that help students to solve problems and to understand [mathematical] concepts". It is evident that "students understand concepts better when they study the concepts through a variety of perspectives and develop the dexterity to move among the different representations smoothly" (*ibid.*).

Bjuland (2007) postulated that when learners translate a mathematical text into a visual representation by drawing an auxiliary figure or making a modification to a figure, they employ the strategy of visualisation. I concur with Wheatley (1991) that the use of visual imagery in a mathematics task is a function of the instructional setting and that, if mathematics tasks are presented in a familiar setting, learners have a greater opportunity to use their prior experience to give meaning to the tasks.

Visualisation processes/strategies in this study are observed and analysed by making use of the five categories of visual imagery that are discussed below.

2.3.1 Visual imagery

Visual imagery (VI), according to Hegarty and Kozhevnikov (1999), refers to the ability to form mental representations of the appearance of objects and to manipulate these representations in the mind. To varying degrees, learners link their images to the formal definitions and proofs of real analysis and move flexibly between the two (Alcock & Simpson, 2004, p. 3). According to Presmeg (1986b, p. 42), a visual image is a mental scheme depicting visual or spatial information. She emphasises that this definition is deliberately broad enough to include all other kinds of imagery which depict shape, pattern or form without conforming to the "picture in the mind" notion of imagery (Clements, 1982), although imagery which attains the vividness and clarity of a picture is also included in this definition (Presmeg, 1986a, p. 297). Presmeg (1986a) notes that while “all mathematical problems involve reasoning or logic for their solution . . . the presence or absence of visual imagery as an essential part of the working determines whether the method is visual or nonvisual” (p. 42). Presmeg (1986a) distinguishes between visual and nonvisual methods in a learner’s solution as follows:

> a visual method of solution is one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed. A nonvisual method of solution is one which involves no visual imagery as an essential part of the method of solution. (p. 289)

Visual methods thus include solutions involving constructions, drawings, diagrams, tables, charts or graphs, whether written down or in the person's mind (*ibid.*). Edens and Potter (2007) acknowledge that the visual-spatial content of mathematics is broad, and includes, among others, concepts associated with geometry and spatial sense, measurement, reasoning, and statistics (p. 289). Visualising objects and graphically representing numerical information are important mathematical tools that help learners to solve problems and to understand complex mathematical concepts.

In my study, Presmeg's (1986b) categories of visual imagery are adopted to observe and analyse the visualisation processes that learners use when they solve GWP (See Table 3.1 for the analytical tool). Presmeg (1986b) asserts that these different visualisation processes are fundamental to any visualisation study in mathematics education. The five categories of visual imagery (5VIs) essentially respond to one of the aims of this study, which is to investigate the extent to which learners opt to use visualisation as a word problem solving tool. However, in this case study, the categories are not being used to identify and differentiate visualisers from non-visualisers, as Norma Presmeg intended in her many studies. They are instead used to determine the extent to which the research participants opted to use visual imagery to solve GWP. These categories are: *concrete pictorial imagery*, *pattern imagery*, *memory imagery*, *kinaesthetic imagery* and *dynamic imagery*, as discussed below.

2.3.1.1 Concrete pictorial imagery

Concrete pictorial imagery (CPI) refers to the concrete image(s) of an actual situation formulated in a person's mind. This can include a picture in the mind drawn on paper or described verbally. A learner engaged in CPI provides evidence of drawing pictures to represent a concrete situation. This includes formulating a picture in the mind while reading a word problem (Presmeg, 1986b). When reading word problems, learners formulate mental images that they can represent on paper as pictures, images or diagrams. A diagram is defined by Mesaroš (2012) as an illustrative tool of visualisation. It is a simplified picture of a certain reality (321). Hence, the construction of a mental image and its representation (Markopoulos, Potari, Boyd, Petta, & Chaseling, 2015, p. 1521) are examples of CPI.

In this study, I encouraged learners to imagine and formulate mental images when reading the given geometry word problems. As the learners sometimes did not spontaneously draw pictures, I also asked them to draw for me the picture that they saw in their minds. Wheatley (1991) advises that "when students are encouraged and given opportunities to form mental images, most readily do so, and when they are encouraged to use imagery, their mathematical power is greatly increased" (p. 35). I therefore deemed it essential in this study to encourage the learners to use visual imagery, to formulate mental pictures that they could then represent on paper.

2.3.1.2 Pattern imagery

Pattern imagery (PI) includes the generalisation of known problem-solving strategies. In their study, Yilmaz, Argun and Keskin (2009) investigate how students used visualisation in their thinking to discover generalisations and create formulas from such generalisations. They conclude that students think their way through abstract situations via visualisation (p. 136). Hence, visualisation has an essential role to play when making general relations and rules (*ibid.*). Learners applying pattern imagery to word problem solving provide evidence of making use of known patterns when drawing/sketching. Presmeg (1986a) defines pattern imagery as the type of imagery in which concrete details are disregarded and pure relationships are depicted in a visual-spatial scheme; the essential feature of pattern imagery is that it is pattern-like and stripped of concrete detail (p. 602).

According to Presmeg (1992), examples of pattern imagery include "schemes for finding the magnitude and the direction of a vector, pattern tables which highlight regularities in the trigonometric ratios for special angles, and pattern images of trigonometric formulae for compound angles" (p. 602). Markopoulos et al. (2015) assert that the extent to which visualisation processes are considered to contribute to the development of mathematical thinking depends on whether they lead to abstraction and generalisation (p. 1521). Jones (2002) reports that visual images, particularly those which can be manipulated on the computer screen, invite learners to observe and conjecture generalisations (p. 125). In this study, learners visualised a set of images in their minds, gave verbal descriptions of these images and also drew them on paper (see an example of a word problem in Section 2.4.1).

In the context of this study, pattern imagery was manifested as the learners applied diverse strategies of word problem solving. For example, there is no specific formula for calculating an unknown side of a right-angled triangle using the theorem of Pythagoras. However, patterns of formula representations are generated and used to generalise what the theorem states. For example, somebody might use $c^2 = a^2 + b^2$, while another person might represent it using different symbols such as $(PQ)^2 = (PR)^2 + (QR)^2$. Many other representations are available depending on the types of visual images used. Patterns of data, data rebuttals, arguments and a history of recurrences are among the indicators of pattern imagery incorporated in the analytical framework for the data generated for this study (see Table 3.1). Presmeg (1986a) acknowledges that all imagery types have the potential to play a functional role in mathematical problem solving, but considers pattern imagery as the most essential type, as it identifies the relational aspects of a problem and is thus arguably better suited to abstraction and generalisation in comparison to the other categories of visual imagery (p. 29).

2.3.1.3 Memory imagery

Memory imagery (MI) refers to the ability to visualise an image that one has seen somewhere before. This includes a history of recurrent occurrences. Learners who use memory imagery visualise an image which could include a picture of a formula in their mind, visualising a book or a chalkboard/whiteboard and recalling how the formula was written when they saw it (Presmeg, 1986b, p. 44). In another study, Presmeg (1986a) finds that memory imagery also served the purpose of depicting abstract procedures or processes in a concrete image (p. 304). It can involve visual recall of formulae for specific geometric figures. For example, when informed that a square and an equilateral triangle had the same perimeter, the participants visualised the formulae for calculating the areas of a triangle and a square to enable them to work out whether the two shapes would also have equal areas.

2.3.1.4 Kinaesthetic imagery

Neuroscience studies defines kinaesthetic imagery (KI), also known as motor imagery, in relation to an individual's actions. Jeannerod (1994, as cited in Bakker, De Lange, Stevens, Toni, & Bloem, 2007) and Guillot et al. (2009, p. 2158) characterise motor imagery as the mental simulation of a given action without actual execution, hence requiring a representation of the body as the generator of acting forces. Guillot et al. (2009) explained that "the first-person perspective corresponds to the representation of a movement as if the individual takes part in the action himself, thus suggesting that he/she would visualize the movement like having a camera on his/her head" (p. 2158). Hence, KI requires one to "feel the movement" and mentally perceive muscle contractions and stretching (*ibid.*).

In this study, KI involved muscular movements and gestures by participants in the process of solving GWP. A learner using KI uses his/her hands/fingers to indicate a path on drawn images (Presmeg, 1986b). It was thus essential in this study to provide the participants with a platform where they could express themselves freely and use gestures. This enabled them to have recourse to KI when solving GWP tasks during the task-based interviews (see the methodology chapter for KI indicators). Examples of kinaesthetic gestures include walking a path with fingers, drawing geometric shapes by miming them, gesturing actual simulations of given situations and nodding and shaking one's head in agreement or disagreement.

2.3.1.5 Dynamic imagery

Dynamic imagery (DI), also known as dynamic visualisation or dynamic transformation, is defined by Mesaroš (2012) as the type of visualisation whereby one picture (as in static visualisation) "is replaced by a series of several pictures connected in one smooth motion" (p. 324). Rheingans (2002) showed that dynamic visualisation can lead to a better understanding of mathematical concepts, as multiple representations are better than a single representation (p. 7). Ploetzner and Lowe (2004) point out that dynamic visualisations increasingly occur in technology-based educational materials such as multimedia learning environments. “In contrast to static depictions, dynamic visualisations can directly display changes in space over time, either incrementally or continuously” (p. 235). In this study, dynamic visualisation was not technology based as I focused my observation on what the learners used in actual classrooms in an ordinary Namibian school setting. Learners can nevertheless visualise moving images as they transform geometric figures. Dynamic imagery is used by a learner who redraws drawn or sketched figures for the purpose of extracting simple figures: e.g., extracting a right-angled triangle from a complex 3-D shape to explain and justify a solution.

From the above descriptions of the five categories of visual imagery, the pictures, images and diagrams that learners formulate in their minds, on paper or with technological tools, can be classified as static, kinetic or transformational. For the purpose of this study, CPI, PI and MI are thus classified as static images, KI as kinetic images and DI as transformational images. This categorisation is unpacked in the analysis chapter.

2.4 MATHEMATICAL WORD PROBLEM SOLVING

Mathematical word problem solving has played a prominent role in mathematics education research worldwide for decades (Greer, 1997; Grouws, 2006; Schoenfeld, 2013). A word problem refers to any mathematics exercise where significant background information on the problem is presented as text rather than in mathematical notation. Word problems are also referred to as story problems (Boonen, van Der Schoot, Wesel, de Vries, & Jolles, 2013). Debrenti (2015, p. 20) defines word problems as real-life, practical problems in which the relationship between the known and unknown quantities is provided in the form of text, but whose solutions need some kind of mathematical model (p. 20).

At the end of the Senior Secondary Phase (SSP) in Namibia, learners are expected to be able to “use mathematical language and representation as a means of solving problems relevant to everyday life and to their further education and further careers” (Namibia. Ministry of Education [MoE], 2010b, p. 23). Csíkos et al. (2012) note that because “mathematical concepts and relations are often based on visual mental representations attached to verbal information, the ability to generate, retain and manipulate abstract images is obviously important in mathematical problem solving” (pp. 49–50). Successful problem solving requires an understanding of relevant textual information and the capacity to visualise the data (*ibid.*, p. 49).

Based on his 1992 work, Schoenfeld (2013) distinguishes four elements in problem solving that are "necessary and sufficient for the analysis of the success or failure of someone's problem solving attempt" (p. 11):

a) the individual's knowledge;
b) the individual's use of problem solving strategies, known as heuristic strategies;
c) the individual's monitoring and self-regulation (an aspect of metacognition); and
d) the individual's belief system (about him- or herself, about mathematics, about problem solving) and their origins in the students' mathematical experiences. (*ibid.*)

In my experience, there is also a need for an appropriate classroom culture to exist – a classroom culture that is clear about what constitutes desirable as opposed to undesirable solutions, as there are many solutions to each problem but not all are acceptable in the mathematics classroom. I argue that the socio-mathematical norms specific to mathematics needs to be explicit and made clear to the learners. For Rasmussen, Yackel and King (2006), socio-mathematical norms relate to learners' emerging beliefs and dispositions specifically related to mathematics, whilst general social norms relate to learners emerging beliefs about their own role in the classroom (p. 151). Presmeg (2014) supported the idea that the "sociocultural climate of the classroom (whether or not visualisation is accepted, encouraged and valued)" (p.152) plays an important role in how individuals tackle problem solving in mathematics. Gravemeijer (2004) suggests four examples of such classroom norms:

- what counts as a mathematical problem
- what counts as a mathematical solution
- what counts as a different solution
- what counts as a more sophisticated solution. (p. 6)

Gravemeijer (2004) argues that by establishing these norms, the teacher provides the learners with criteria for judging appropriate arguments and solutions, which is essential to the "intellectual autonomy of students" (p. 6). Learners can therefore use such criteria to make their own evaluations and do not have to wait for the final judgment of the teacher (*ibid.*). Gravemeijer (2004) concludes that "by establishing the socio-mathematical norms; the teacher defines what mathematics is" (p. 6). As a mathematics teacher, I establish which "solutions . . . are mathematically productive" (Gravemeijer, 2004, p. 7) as well as mathematically correct.

When solving word problems, learners should apply their knowledge to a real-world situation and not merely perform a set of abstract exercises that can be solved in an algorithmic manner (Wyndhamn & Säljö, 1997). As Flores and Braker (2013) point out, "problems can be excellent ways to foster the development and understanding of particular mathematical concepts and procedures. However, students might use an alternative solution process that does not require the concept or process that the teacher wanted to emphasise [for example visualisation]" (p. 336). Thus Flores and Braker (2013) caution that mathematics teachers need to be aware that learners might find alternative solution strategies to particular word problems not involving preferred problem-solving strategies such as visualisation. In that case, teachers need "to decide at what point, and to what extent, they should [consider] those alternative approaches" (*ibid.*, p. 336).

In the process of mathematical word problem solving, learners are expected to translate word representations into mathematical representations (Ahmad et al., 2010). The problem is that learners answer word problems "with apparent scarce regard for whether the answers make sense when considered from the viewpoint of the real-world situations verbally described in those problems" (Greer, 1997, p. 294). Greer (1997) commented that there is general agreement among researchers that the characteristic features of word problems are an important factor contributing to the apparent failure of learners to take realistic aspects into account when doing mathematics (p. 297). He therefore recommends that learners should conceptualise a word problem as a written description of some situation, in terms of which their task is to understand the situation, make reasoned and reasonable assumptions, and construct a model, or more than one model, of problem solving (Greer, 1997, p. 300).

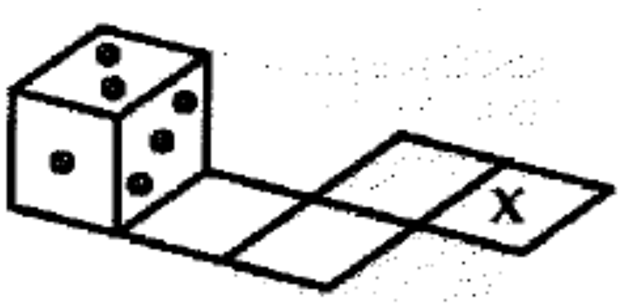
Paivio (1971, as quoted by Presmeg, 2014) suggests that "the way a task is tackled by an individual depends on the following: the task itself (in this case whether it is presented in visual form or not); instructions to do the task in a certain way; and finally, individual differences" (p. 152). Presmeg (2014) adds the "sociocultural climate of the classroom" as a factor (whether or not visualization is accepted, encouraged and valued). Mesaroš (2012) encouraged mathematics teachers to incorporate the processes of visualisation in lesson preparations and assessment. He emphasises that leading learners to acquire the habit of visualising mathematical reality will enable the teacher to gain “a powerful tool to achieve goals in the educational process, such as successful problem solving, imagination development, fighting formalism in learning and others” (p. 325). It therefore makes sense to provide learners with meaningful mathematical word problems to be collaboratively solved (Pijls, Dekker, & Van Hout-Wolters, 2007).

Grouws (2006) observes that mathematics teachers have numerous good reasons to include in their lessons tasks that embody high cognitive demands. First, using these tasks allows students to engage in “doing” mathematics and thus give them opportunities to develop the capacity to think and reason mathematically. Secondly, research evidence reveals that “even when teachers begin with high-demand tasks, the tasks have a tendency to decline in cognitive demand as they are implemented in the classroom” (p. 133). Stein et al. (1996) propose the setting of tasks that make higher cognitive demands of learners, including reasoning and sense-making. Such tasks would depend on “the existence of multiple-solution strategies, the extent to which the task lends itself to multiple representations and the extent to which the task demand explanations and/or justifications from the students” (Brodie, 2010, p. 19). Shulman (2004) claims that “the more complex and higher-order the learning, the more it depends on reflection – looking back – and collaboration – working with others” (p. 319). Brodie (2010) concurs that “tasks that support multiple voices, disagreements, and challenges also support mathematical reasoning, when used appropriately” (p. 7). These ideas are salient to the purpose of this case study, as they make a positive and effective contribution to the development of geometry word problems of the kind that the research participants should be required to solve.

2.4.1 Geometry word problems

Central to this study are GWP, which, for the purposes of this study, are referred to as Enacted Visualisation Geometric Reasoning Tasks (EVGRT). These are written descriptions of real-life situations developed in relation to the Namibian Grade 11-12 geometry curriculum that require the use of visualisation processes to solve. Along the lines proposed by Goldsmith et al. (2016), the geometry tasks for this study were developed to assess the learners’ geometric reasoning involving visual-spatial thinking, rather than their ability to recall a high level of geometry facts. The tasks are drawn primarily from South African Mathematics Olympiad (SAMO) tests. Other items were developed by the researcher herself, and yet others were taken from the Namibian national examination question papers for Grade 12 as well as from various mathematics books. These items range from problems referring to shape and space to pure word problems – problems without figures/drawings (see the methodology chapter for a justification for the inclusion of each task). Below is an example of one of these geometric tasks:

On a die the numbers on opposite faces add up to 7. The die in the diagram is rolled edge over edge along the path until it rests on the square labelled X. What is the number on top in that position?



This task is classified as a word problem with the aid of a diagram. It is realistic and requires visualisation processes to solve. The task is also complex, and learners are encouraged to use visual imagery complemented by multiple representations to solve the task. For example, a learner can work out opposite faces that add up to 7 in his/her mind, try to visualise and roll an imaginary die using mental kinaesthetic and pattern imagery until a particular number appears on the square labelled *X*. If that does not work, he/she may employ dynamic imagery (redraws the die in various orientations). The learner may also ask for an actual die to literally roll it and record the solution. These kinds of problems are developed for both EVGRT worksheets, 1 and 2. They are aimed at encouraging the use of visualisation processes while stimulating learners to talk about their problem-solving strategies (which I call mathematical reasoning – see Section 2.2).

Herbst (2006) defines a mathematical problem as “a question whose answer hinges on bringing to bear a mathematical theory within which a concept, formula, or method involved in answering the question is warranted” (p. 315). GWP are thus a form of mathematical problem that “constitute[s] a representation or an embodiment of a piece of knowledge, in that it points to some of the meaning of that piece of knowledge” (*ibid.*). To understand how learners learn from solving GWP, Herbst (2006) suggests, that it requires coming to grips not just with how the word problems embody knowledge but also with how learners’ knowing of that knowledge may emerge from the interaction between a cognising agent (i.e., the learner in the learning environment) and the problem.

Although van Hiele (1999) regarded thinking without words as not thinking in his earlier writings, he later changed his mind and conceded that visual thinking (termed nonverbal) is of great significance in mathematics education:

> Thinking without words is not thinking. In *Structure and Insight* (van Hiele, 1986), I expressed this point of view, and psychologists in the United States were not happy with it. They were right. If nonverbal thinking does not belong to real thinking, then even if we are awake, we do not think most of the time. Nonverbal thinking is of special importance; all rational thinking has its roots in nonverbal thinking, and many decisions are made with only that kind of thought. (van Hiele, 1999, p. 311)

Increasing attention is being paid to the centrality of visual thinking in learning and doing mathematics, not just for illustrative purposes but also as a key component of reasoning (Arcavi, 2003). As stated earlier, the GWP in this study are a medium through which learners are able to visualise the problem by making use of various visualisation objects and processes that eventually brings about mathematical reasoning. Herbst (2006) maintains that if word problems are to be used "to fulfil the didactical contract, we need theoretical tools to understand how the enactment of problems by students may serve an instructional purpose and be identified as such in the interactions between students and their teacher" as well as with their environment (p. 316).

In my experience, learners often find it difficult to talk about their solutions to word problems, for a number of reasons which may include complicated problem-solving strategies. In my mathematics classroom, some learners feel uncomfortable or unmotivated to talk to the teacher and/or each other in a bigger group, but tend to open up in one-on-one and small group conversations. This study thus focuses on both how the visualisation of word problems enhances mathematical reasoning and on how talking during word problem solving may enhance one's visualisation of the problem and consequently one's mathematical reasoning. The aim is also to structure a classroom culture where visualisation and mathematical reasoning are both observed processes and products of word problem solving.

2.5 THEORETICAL FRAMEWORK – ENACTIVISM

Maturana and Poerksen (2004) assert that "*the observer is the source of everything, and without the observer, there is nothing*" (p. 28). For the purposes of this study, however, I take a slightly more nuanced stance and suggest that the observer is the source of what he observes and the creator of what he sees. The following section presents some key enactivist definitions that frame the theoretical underpinnings of this study.

According to Simmt and Kieren (2015), this view of the observer has several implications. It suggests that observation is part of the conscious flow of the researcher's process; that is, the researcher is possibly and indeed likely to be impelled by his/her own observation to act afresh in the research situation and make further observations. "More generally this view of the observer suggests that the making of an observation implies a listener who can respond to, be occasioned by, extend, test … the observation being made" (p. 308). In other words, the individual knower is not simply an observer of the world but is bodily embedded in the world and is shaped both cognitively and as a whole physical organism by her interaction with the world (Kaiser & Sriraman, 2010, p. 42).

2.5.1 Overall definition of enactivism pertaining to this case study

The enactivist perspective asserts that reality exists in the eyes of observers, and any discussion and interpretation of it should begin with a description of their observations (Reid, 2014, p. 138). For Maturana, the observer is someone who sees something, affirms or denies its existence, and deals with it in ways he/she sees fit. What is said to exist independently of observers is necessarily a matter of belief and not knowledge, because to see something always requires someone who sees it (Maturana & Poerksen, 2004, p. 28). The observer is one "who arises in the act of observing and whose knowing is explained through the mechanism she describes" (Simmt & Kieren, 2015, p. 307).

Begg (2013) defines enactivism as "a way of understanding how all organisms, including human beings, organise themselves and interact with their environments" (p. 81). Towers and Martin (2015) thus characterise enactivism as a theory of cognition that has its roots in biological and evolutionary understandings and views human knowledge and meaning-making as processes that are understood and theorised from a biological and evolutionary standpoint (p. 249). Kieren, Calvert, Reid and Simmt (1995, p. 2) describe enactivism as a position on cognition that includes the concepts of structural determinism, structural coupling, bringing forth a world, observer dependence, satisficing and co-emergence. Furthermore, enactivism refers to a given situation in which we are called to position ourselves and view cognition not in terms of its products nor its mental structure, but in terms of **action,** or even better, of living in the world of significance with others.

Enactivism provides an "inclusive, expansive, apt, and fit framework" (Khan, Francis, & Davis, 2015, p. 269) for the study of cognitive processes such as visualisation and reasoning in mathematics education. Froese (2015) adds that enactivism provides a suitable interpretative framework for explaining the finding that emotional networks are among the most widely connected in the brain. It also helps us better to understand the role of "nonstandard" pathways to visual perception (p. 3). In my study, enactivism informs both the methodological and the analytical frameworks as it provides a language for discussing the links between visualisation and reasoning during word problem solving. As Varela (1999) points out, objects are not seen by the visual extraction of features, but rather by the visual guidance of action (p. 14). It is thus a concern of this study to examine how visualisation processes enhance problem-solving processes (i.e. visualisation in action), as opposed to focussing only on the visualisation of objects *per se*. I now unpack the aspects of enactivism that pertain to this study.

Fundamentally, an enactivist perspective interrogates how elements of a system work together to form that system. Maturana and Poerksen (2004) define autopoiesis as the process of living systems (re)producing themselves within their closed dynamics (p. 98). Autopoiesis is the "self-creation [of systems] and consists of Greek words auto (self) and poiein (produce, create)" (*ibid.*, p. 97). Maturana and Poerksen (2004) observe that "when we examine a living system, we find a network producing molecules that interact with others in such a way as to produce molecules that, in turn, produce the network producing molecules, and determine its boundary" (p. 98). Interactions with systems are thus crucial components of systems.

According to Reid (1995), an enactivist view of a problem solving situation is one in which "the person and the situation co-emerge through their interaction and so the reasoning employed is both determined by the structure of the person, and occasioned by the sphere of possibilities implicit in the situation" (p. 10). As the problem solver and the environment in which problem solving occurs are structurally coupled, the problem and the problem-solver co-emerge. The actions that the problem-solver brings forth when solving a problem are thus intertwined with the potential inherent in the problem itself. The one cannot exist without the other. Co-emergence is thus a key theoretical concept when I examine how my participants interact with the word problems in this case study.

An enactivist perspective views cognition as an embodied interactive process co-emergent with the environment in which the person (learner) acts. Cognition does not entail a reactive representation of a pre-given world, with its goal and success measured by its match with that world (Kieren et al., 1995). Varela (1999) concurs that the world we know is not given; it is enacted through our history, by what he calls structural coupling. According to Khan et al. (2015), the notion of structural coupling originates from a "biological perspective of organisms and environments co-adapting to or co-evolving with each other. The mutual interaction of the organism and the environment causes changes and transformations in both" (p. 275). Enactivism as a theory of cognition acknowledges the importance of the individual in the construction of a lived world. But it also emphasises that the structure of the individual co-emerges with the lived world in the course of, and as a requirement for, the continuing interaction between the individual and the situation (Kaiser & Sriraman, 2010, p. 42).

Furthermore, an enactivist viewpoint holds that "one's history of interaction and one's structure determines, and at the same time is determined by, how one acts in a given setting and under various perturbations" (Simmt, 1995). Maturana and Poerksen (2004) posit that when we are faced with new knowledge that seems to emerge out of nowhere, we create history and a domain of connections. In this way, its sudden emergence out of nowhere loses its frightening strangeness (p. 32). For example, when faced with a new and unfamiliar word problem, learners are faced with a novel situation that can be tackled from various angles. They can make use of their accumulated experiences with other previous problems and employ their repertoire of reasoning, justifications and argumentation skills to solve the new problem, or/and they can rely on an interactive process of co-emergence to come up with new and untested strategies. The point is that the problem and the problem solver constitute an intertwined system that has its own structures, and, from an enactivist perspective, these structures will determine the nature and the outcome of the problem solving process (Reid & Mgombelo, 2015, p. 173).

Damiano (2012) maintains that "co-emergence is the best notion to define the dynamical interaction between an autopoietic system and its environment, which Maturana and Varela call 'structural coupling'" (p. 285). In the context of this study, the autopoietic system is the problem solver in the act of confronting the problem. Damiano (2012) argues that "it [the interaction] is a symmetric relation of reciprocal perturbations and compensations which implies the correlated emergence, in the living system and its environment, of compatible self-determined patterns of self-production" (*ibid.*). The perturbation in this study comprises the inherent challenges of the mathematical problem at hand.

With regard to mathematical reasoning, Reid (1995) insists that it is essential to remain aware of the role reasoning plays in the co-emergence of learners and their situations, i.e.

the mathematical problems they solve. It is the structure of the learners which makes their reasoning inductively, deductively, or in some other way possible. At the same time, it is the structure of the mathematical problem in which they find themselves that occasions the reasoning they execute. Both the learner's structure and the structure of the problem are changed by the reasoning that takes place, so that the learner and the problem co-emerge through reasoning and at the same time "the reasoning co-emergences" (p. 13). Thus, one's understanding of the world comes from the organisation and history of one's system, which enables an individual to endow a given situation with meaning (Lee, 2014, p. 19).

But why opt for enactivism (at the expense of, say, constructivism)? "Enactivism emphasises knowing rather than knowledge. This contrasts with constructivism, where knowledge is interpreted as a human construct and evaluated in terms of its fit with the knower's experience" (Begg, 2013, p. 84). Khan et al. (2015) add that:

> enactivist theories of human learning attend explicitly and deliberately to action, feedback, and discernment. They emphasise the bodily basis of meaning, distinguishing it from most accounts of constructivism, which, while not denying the body as ground and mediator of meaning, have not focused so intensely on the physicality of knowing and being. (p. 272)

The enactivist view of knowledge conception is essentially performative in contrast to constructivism's concerns with conceptual understanding, propositional knowledge, and webs of association (Khan et al., 2015, p. 272). While constructivism can also be interpreted as performative, "the focus is on the outcome of actions rather than the process of interactions as in enactivism" (*ibid.*). Khan et al. (2015) described radical constructivism as:

> a theory of how people assemble ideas, with its central metaphor being an organism undergoing evolution and continually 'fitting' its cognitive schemas to the environment ... It is thus relatively silent on teaching practices, such as grading or distinguishing student interpretations as right or wrong, noting only that while learning may be dependent on such teaching acts, it is not determined by them. (pp. 272–273)

However, from an enactivist perspective, living creatures are not seen as "mere compounds of parts selected by evolution, but as whole agents individuated from their environment in terms of their internal structure" (Heras-Escribano, Noble, & De Pinedo, 2013, p. 665). Enactivism "is attentive to the many feedback structures in a greater-than-the-individual-learner system, more so than attention given to individual cognitive structures in radical constructivism. It is the organism as a whole, together with its environment, which co-evolves in enactivism" (Khan et al., 2015, p. 273). Furthermore, enactivism suggests that we understand that the observer is not neutral; his or her observations bring forth worlds of significance that intersect with the worlds of others. Hence "when the observer's world changes, the environment of the other is altered; the other has an altered environment to select from as she brings forth her world of significance" (Simmt & Kieren, 2015). Proulx (Proulx, 2008a) observes that the enactivist perspective's emphasis on co-determination and on bringing forth a world of meaning is precisely what disassociates the enactivist standpoint from any form of constructivism.

Another key concept that enactivism argues for is the inseparability of body, mind and the environment. In this study the links between the environment (the geometry word problem) and the body/mind (the problem solver) are the notions of visualisation and reasoning. It is natural for humans to form a mental image when reading something, which gives rise to the visual imagery and reasoning which are at the heart of the inseparability of the problem solver and the problem. Li et al. (2010) maintain that "a key idea of enactivism is that living systems adjust to their exceedingly complex surroundings in an autopoietic manner" (p. 411). This means that the world of meaning is not in us, nor in the physical world around us, but in the interaction between the learner (in this case the problem solver) and learning environment (in this case the problem) in a mutually affective relationship (Proulx, 2008a). It is therefore with and within the structure that we make sense of and give meaning to the physical world and bring forth a world of significance (*ibid*., p. 21).

Khan et al. (2015, p. 278) believed that enactivist perspectives offer an appropriate framework for investigating and interpreting what it means to weave one's embodied and knowing self through the world. The notion of embodied cognition further helps to explicate the role and meaning of enactivism in this case study.

2.5.2 Embodied cognition

The advocates of embodied cognition take as their theoretical starting point not a mind working on abstract problems, but a body that requires a mind to make it function (Wilson, 2009, p. 625). There is a growing commitment to the idea that the mind must be understood in the context of its relationship to a physical body that interacts with the world – i.e. the setting in which the person operates (*ibid*.). Antle (2009, p. 27) agrees that an embodied perspective on human cognition foregrounds the role of the body, physical activity, and lived experience in cognition. She defines embodiment as how a living entity's cognition is shaped by the form of its physical manifestation in the world. That is, embodied cognition emphasises how the structure of the human body acting in complex physical, social and cultural environments determines perceptual and cognitive structures, processes, and operations. In contrast to traditional views of cognition, an embodied cognition perspective suggests that humans should be considered first and foremost as active agents rather than as disembodied symbol processors.

The notion of using one's body to solve word problems both individually and in a social setting is significant in the study of visualisation and mathematical reasoning. Alibali and Nathan (2012) describe the activity of using one's body in problem-solving in terms of "gestures". They assert that "gestures are often taken as evidence that the body is involved in thinking and speaking about the ideas expressed in those gestures. That is, gestures are taken as evidence that the knowledge itself is embodied" (p. 248). Antel (2009) claims that giving consideration to the ways in which cognition is rooted in bodily actions will contribute to learners' successful development into active, thinking adults (p. 30). Although there is yet no unified theory of embodiment, Alibali and Nathan (2012) report that scholars of embodied cognition commonly agree that mental processes – i.e. thinking and reasoning – "are mediated by bodily based systems, including body shape, movement, and scale; motor systems including the neural systems engaged in action planning; and the system involved in sensation and perception" (p. 248).

In her study, Antel (2009) used the embodied cognition perspective as an analytical lens to examine users' interactions with existing products and systems. She defines embodied cognition in terms of how "the nature of the living entity's cognition is shaped by the form of its physical manifestation in the world" (p. 27). Wilson (2009, p. 626) similarly regards cognitive activity as an exercise that takes place in the context of a real-world environment and inherently involves perception and action. She adds that: "…while cognitive process is being carried out, perceptual information continues to come in that affects processing, and motor activity is executed that affects the environment in task-relevant ways" (*ibid.*). In the context of this case study, reading a word problem while trying to imagine how a sketch should look is perceived as an example of a cognitive activity that is situated in a problem-solving context.

Like many other enactivist researchers in the field of mathematics education, Alibali and Nathan (2012, p. 248) view mathematical cognition as embodied in two senses: (1) it is based in perception and action, and (2) it is grounded in the physical environment. Khan et al. (2015) add that "our potential for action (walking) and goal (destination) is depended on the perception and selection of sensory information from the physical world" (p. 272). The first view of cognition is expounded by enactivist researchers in terms of the phrase "perceptually guided action" (Brown, 2015; Khan et al., 2015). According to Brown (2015) perceptually guided action refers to the "means of coping with what the world throws at us and of us making changes in our environment, 'bringing forth a world' within a medium of effective action" (p. 188). Alibali and Nathan (2012) conclude that "an appreciation of the embodied nature of mathematical cognition will help one to understand why certain types of mathematical problems are more difficult than others, to identify suitable assessment methods that accurately gauge mathematical knowledge, to design more effective learning environments" (p. 248).

The second view regards embodied cognition as grounded in the physical environment is known by the phrase 'bringing forth a world (of significance/meaning)' among enactivist researchers (Simmt, 2000; Proulx, 2008a; Simmt & Kieren, 2015). This is discussed in the next section.

2.5.3 Enactivism and social interaction

One of the fundamental ideas of an enactivist perspective, according to Simmt (2000), is that in interaction there is potential for both the individual and the environment to change (or learn). "In enactivist terms, interaction brings forth worlds of significance which include both knower and known and those worlds of significance intersect the worlds brought forth by others" (p. 157). Because we are social beings, adds Simmt (2000, pp. 157–158), this view of the enactivist perspective on interaction has ethical implications: because the worlds we bring forth are entangled with the worlds of others, so that when we act, our actions have the potential to alter the worlds and possibilities of others.

Di Paolo et al.'s (2010) exploration led them towards "a middle way between individualistic and holistic views of social interaction and to highlighting the central role played by the temporality of social engagements in generating and transforming social understanding at different time scales through joint participation" (p. 36). They used the notion of social interaction to discuss "social understanding along enactive lines" (p. 61). To fully understand how meaning comes about in social understanding, they argue that there is a need to focus not only on the embodiment of interactors (i.e. research participants solving problems in a group setting), but also on the interaction processes that take place between them. This interaction is understood as "the coupling between an agent and a specific aspect of its world: another agent. Interaction is the mutual interdependence (or bidirectional, coregulated coupling) of the behaviours of two social agents" (pp. 61–62). They define social interaction as follows:

> Social interaction is the regulated coupling between at least two autonomous agents, where the regulation concerns aspects of the coupling itself and constitutes an emergent autonomous organisation in the domain of relational dynamics, without destroying in the process the autonomy of the agents involved (though the latter's scope can be augmented or reduced). (p. 70)

Social interaction in the context of this case study resulted from the mutual interdependence of the research participants when they solved word problems in small collaborative argumentative groups (cf. Nussbaum, 2008, p. 348). In their small groups, the participants learned to attempt problems together and arrive at a collective solution. They made use of their bodies through gestural utterances to help them find solutions to the word problems. Visualisation includes embodied actions performed when the participants are actively involved in problem solving. Khan et al. (2015) argue that "a renewed study of visualisation in mathematics education must necessarily attend to the significant role of the body in space with other bodies" (p. 278) – a role that, following the precedent of other writers, is here termed social interaction. During collaborative argumentations, suggests Yackel (2001), learners working in a group are expected to develop personally-meaningful solutions to problems, to explain and justify their thinking and solutions to others, "to listen to and attempt to make sense of each other's interpretations of and solutions to problems, and to ask questions and raise challenges in situations of misunderstanding or disagreement" (p. 13). From an enactivist perspective, Simmt (2000) puts it thus:

> …when we humans engage in mathematical activity, that activity intersects with our personal, social and cultural domains of our lives. In action, we bring forth a world of significance, which in this case is called mathematics and, in doing so we bring forth ourselves. In each act of bringing forth a world of significance and our "selves", we anticipate the future as our spheres of behavioural possibilities expand making possible our next utterance, movement, action, and thought. Further, because we bring forth worlds of significance with others, what we do, what we say, and what we know makes a difference, not only for ourselves, but for the other. (p. 158)

In their study, Di Paolo et al. (2010) posit that although social skills depend on a "rhythmic capacity", it is not so much an individual capacity as one that comes about in interaction and is changed by both the interactional process and the individuals involved in the interaction.

Therefore, this central capacity of social cognition is defined as the ability to coordinate our interactions with another person. This capacity, Di Paolo et al. (2010) stressed, “is crucially dependent both on the individual interactors and on the process of engagement that ensues between them in every interaction” (p. 70). Khan et al. (2015, p. 275) concurs that the enactivist theory of cognition does not apply solely to individual learning, in the sense that collectives of individuals learn together, making use of resources available in the environment. Therefore, “as we cannot particularly tell who oversees the process of interaction” (Di Paolo et al., 2010, p. 70), social interactions need to be studied as wholes, together with their histories, as social meaning generation relies on the coordination of individual sense-making. It relies on coordination as a process and not as a product. That is, a precise mutual coordination of sense-making is not necessarily the goal of interacting; rather, it is the process of coordination between the actions involved in sense-making that contributes to people understanding each other (*ibid.*). In this way, social understanding is explained in terms of its roots in the dynamics of interaction between the cognitive agent and the environment, as “something that is enacted – co-constructed – in the interaction” (Di Paolo et al., 2010, p. 72).

Simmt and Kieren (2015) observe that in enactivist research, it is essential to “recognise the relationship between the learner and the environment in which the learner is seen to bring forth a world” (p. 310). They modelled the interaction in which a learner is seen to bring forth a world of significance in the manner portrayed in Figure 2.1.

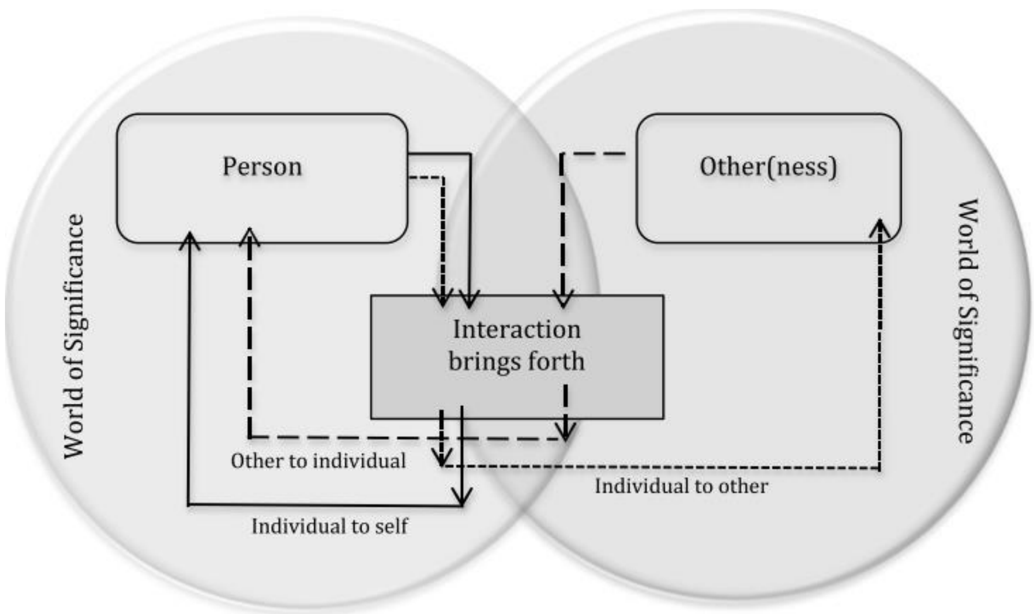


Figure 2.1 Simmt and Kieren's model of interaction that brings forth a world of significance[1]

[1] Simmt and Kieren (2015)

Figure 2.1 illustrates how the interaction between an individual and others brings forth a world of significance. The physical world is essential for an individual to perceive and make sense of, for "without a physical world or a subjective knower, there is no meaning that can emerge" (Proulx, 2008a). If the world of meaning is not in us or in the physical world, then it lies in the interaction of individuals who are "in a mutually affective relationship" (*ibid.*). When learners solve word problems in small groups, they are in a relationship with each other. They understand each other's moves, language and are able, to a certain extent, read each other's minds. As Davis (1995) puts it, "the students' language and action are not outward manifestations of their inner workings, they are visible aspects of these students' embodied (enacted) understandings" (p. 4).

A crucial and immediate implication of this conception of personal understanding is that each learner's knowledge is entwined with every other learner's knowledge, and "understanding in this frame, then, cannot be thought of in strictly subjective terms; collective knowledge and individual understanding are dynamically co-emergent phenomena. One might thus say that the mathematical knowledge is located in the activity or, perhaps more descriptively, in the inter-activity, of learners" (Davis, 1995, p. 4). As the learners interact with the physical world, they collectively and collaboratively bring forth a world of meaning. As they bring forth this world, they emerge and co-emerge with it in the sense that they are within their descriptions of that world; it is after all their structures that allow them to bring the world forth (Proulx, 2008a). In his study, Davis (1995) concludes that "as the events of the lesson are re-traced, it becomes apparent that it was not so much the possibility for individual action as it was the opportunity for interaction that contributed to the flow of the mathematics" (Davis, 1995, p. 4).

In the context of this case study, the participating learners were seen to bring forth a world of significance with others when they solved word problems in small groups (this is analysed and discussed in Chapter 4 of the study). That is, they interacted with the environment (the word problems that they solved) and each other (during problem solving in small groups), in the physical world. My role as researcher was that of an observer trying to make sense and give meaning to that physical world. Although my relationship with the participants was that of an observer-participant, "it is my structure that allows me to 'see' or perceive things in the physical world, and so my structure allows me to give meaning to the attributes of the physical world. I – my structure – allow the physical world to be brought forth" (Proulx, 2008a). Nevertheless, it must be noted that although the participants were observed to bring forth a world of meaning with others when they worked in their respective small groups, there was usually room for individual growth when they uncoupled from their groups and concentrated on individual problem solving. This coupling and uncoupling of participants from social interaction is discussed in the next section, as well as in the analysis chapter.

2.5.4 Enacted visualisation and the study of mathematical reasoning

Proulx (2008b, pp. 21-22) reiterated that as humans, we bring forth the physical world's attributes when we give meaning to it. We acknowledge their physical presence by bringing them forth; if we do not bring them forth, the physical world's attributes will still be there although they will remain unnoticed, unobserved and unstructured – not made sense of and kept "in the dark". It is therefore in this sense that the physical attributes themselves are brought forth by the interactions we have with them through the process of sense-making.

Mathematical reasoning is viewed as a means to sense-making of and in a mathematical activity (cf. Brodie, 2010, p. 59). In this study, learners employed visualisation processes to make sense of the GWP. Sense-making in an enactivist perspective refers to the process by which an organism brings a meaningful point of view to bear on the world, which always contains a mixture of sensation, pre-reflective interpretation, and valuation (Froese, 2015, p. 2). Di Paolo et al. (2010) view sense-making as the creation and appreciation of meaning. They concur with Proulx (2008b) that living organisms do not passively receive information from their environments, which they then translate into internal representations. Rather, "they participate in the generation of meaning through their bodies and action, often engaging in transformational and not merely informational interactions; they enact a world" (Di Paolo et al., 2010, p. 39).

In this study, enactivism is treated as a mediating perspective to bring visualisation and reasoning processes together. It is the lens through which the co-emergence of visualisation and reasoning processes are observed, analysed and discussed. Khan et al. (2015, p. 272) assert that enactivism is attentive to the coupling of organisms and their environments, to action as cognition, as well as to sensorimotor coordination. Since it is the potential for action in the world that focuses attention and drives learning, enactivism is concerned with "*learning in action*" as opposed to embodied cognition's "*learning from action*" (*ibid.*, p. 272). Enacted visualisation is therefore defined in this case study as 'visualisation in action'.

In an enacted visualisation context, the key enactivist concepts of structural coupling and co-emergence are what enable a discussion of the links between reasoning and visualisation emerging during word problem solving. In the GWP setting, the participant's structure[2] coupled with that of the environment and the other participants' structures through a history of recurrent interactions (Maturana & Varela, 1998). I made use of the five categories of visual imagery (discussed in Section 2.3.1) and the reasoning processes (discussed in Section 2.2.1) to analyse the co-emergence of visualisation and reasoning processes in the data collected for this study (see Chapter 3). It was therefore the participants' structures which determined the type of visualisation processes to employ and to which problem, which in turn determined and was determined by the reasoning processes that co-emerged with it. In short, the learner's structure becomes structurally coupled with that of the context in which he/she operates.

As mentioned earlier, structural coupling and co-emergence are key enactivist concepts that describe the manifestation of enactivism in this study of mathematical reasoning in an enacted visualisation context. It was one of the central aims of this study to analyse how enacted visualisation processes co-emerge with mathematical reasoning processes during collaborative argumentations. It is therefore essential to define certain aspects of the enactivist perspectives that pertain to this study.

2.5.4.1 Structural coupling

Structural coupling is one of the key enactivist concepts that frame this study. According to Rossi et al. (2013), structural coupling occurs as a result of the interaction between the organism (with his/her living and active body) and the environment. This interaction creates co-emergences and in return produces the "structural coupling" (p. 38). Enactivist researchers talk about structural coupling "whenever there is a history of recurrent interactions leading to structural congruence between two (or more) systems" (Brown, 2015, p. 189). Maturana and Poerksen (2004) referred to structural coupling as the "dynamics of structural congruence that takes place between an organism and a medium" (p. 86). In their interview, Maturana unpacked structural congruence by using a metaphor: "If you want to enter a locked room without breaking the door open or destroying the lock, you will need the right kind of key to gain access to the new domain. I would say therefore, that a lock and key must have a congruent structure" (p. 86). Furthermore, "structural coupling arises if the

[2] Structure denotes the components and relations that actually constitute a particular unity and make its organisation real.
Organisation denotes those relations that must exist among the components of a system for it to be a member of a specific class (Maturana & Varela, 1998, p. 47).

structures of two structurally plastic systems change through continual interaction without [changing] the identity of the interacting systems" (Maturana & Poerksen, 2004, p. 85).

Proulx (2009, p. 4) maintains that structural coupling means that both learners and teacher co-evolve and co-adapt in the learning process or teaching dynamic. The very notion of structural coupling is derived from a biological context in which organisms and environments co-adapt or co-evolve with each other: "The mutual interaction of the organism and the environment causes changes and transformations in both" (Khan et al., 2015, p. 275). For example, in the teaching/learning environment, the teacher influences what is learned by interacting and coupling with the learners. The teacher is therefore coupled within the learners' knowledge and cognitive acts, and through this structural coupling, "the teacher influences and orients the learning that happens, hence is seen as complicit in this knowledge production" (Proulx, 2009, p. 5).

Moreover, as the environment and the organism interact with one another they experience a mutual history of evolutionary changes and transformations. They both undergo changes in their structure in the process of co-evolution, which makes them "adapted" and compatible with each other (Proulx, 2008b, p. 147). The environment does not usually act as a selector but as a trigger for the learners to co-evolve. Although events and changes are occasioned by the environment, they are explicitly determined by the learners' knowledge/structure. The environment is there as a trigger, but what the learner learns is determined by who they are and what they know (Proulx, 2008b, 2009). Towers and Martin (2015) agree that species and the environment co-adapt to each other, which means that each influences the other in the course of co-evolution – via the process known as structural coupling (p. 249).

Proulx (2009, p. 2) observes that the problems we encounter and the questions we attempt to answer are as much a part of us as they are part of the environment – they emerge from our structural coupling with the environment. As humans, we interpret events as issues to address or problems to solve. We do not act on pre-existing situations as our co-determination and interaction with the environment creates, enables and specifies the possible situations to act towards (*ibid.*). Therefore, "the problems we solve are then implicitly relevant for us and are part of our structure. Our structural determinism allows these to be problems for us, as the environment 'triggers' them in us" (Proulx, 2009, p. 2). Kieren et al. (1995) confirm that neither the person nor the environment is privileged; the construction of knowledge is seen to occur in the interaction. It is in the structural coupling between the structure of the person and the structure of the environment.

Figure 2.2 illustrates my general understanding of structural coupling. I perceive structural coupling as occurring when two or three learners are busy working on a word problem together. Each learner thinks about the problem, starts working on it. They look at each other (gaze into each other's eyes looking for inspiration and courage to proceed); then continue working on the problem, first on an individual level and then as a group, so that the structures of individuals are coupled with those of other members of the group and the learning environment. They read each other's minds and sense each other's next moves. They explain their methods, work more as a group and reach an understanding of what each of them brings forth. They reach a consensus and arrive at a collective solution. This interactive process is repeated over and over until the learners' structures become congruent. In the tree of knowledge, Maturana and Varela (1998) spoke of structural coupling as occurring "whenever there is a history of recurrent interactions leading to the structural congruence between two (or more) systems" (p. 75).

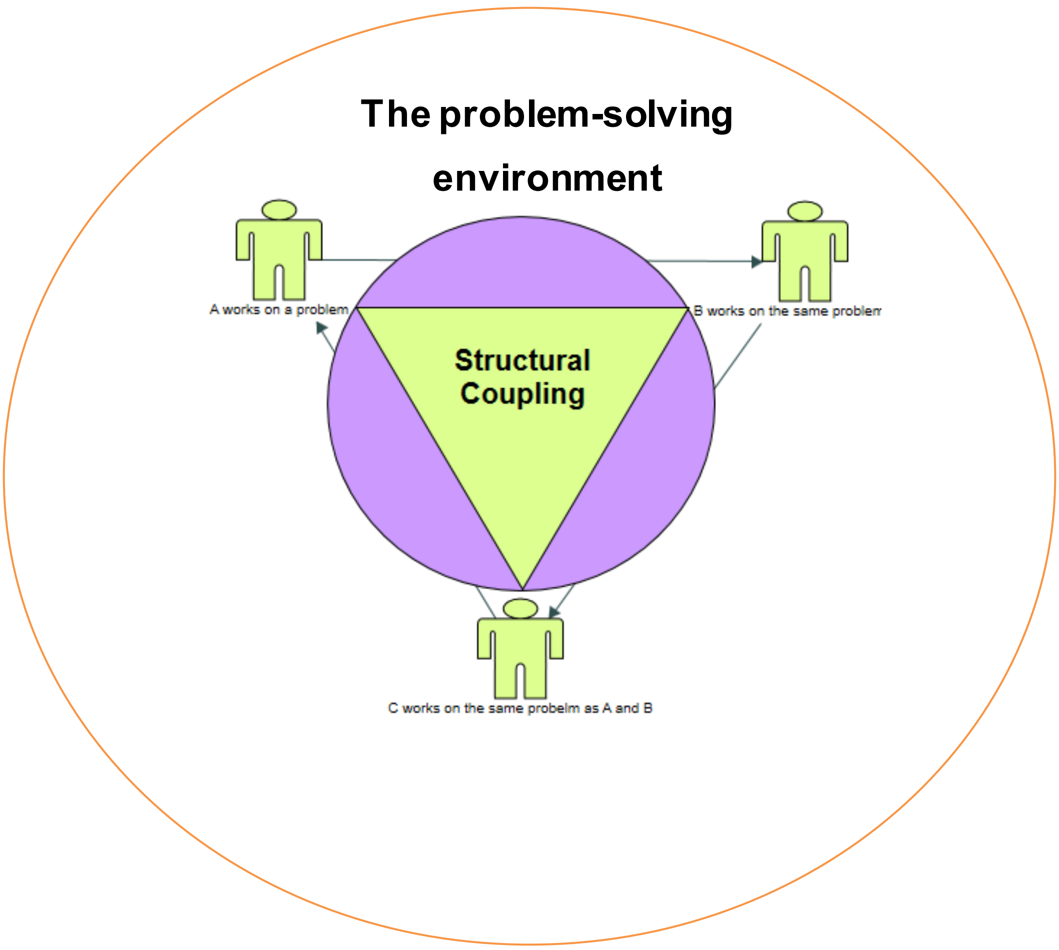


Figure 2.2 My general view of structural coupling

The changes that result from the interaction between the living being and its environment, brought about by disturbing agents but determined by the structure of the disturbed system, are evidence of what Maturana and Varela call structural determinism (Proulx, 2008b, p. 147). Proulx (2008b) emphasises that it is the structure of the organism that allows for changes to occur. These changes are “triggered” by the interaction of the organism with its environment. The “triggers” from the environment are essential but they do not determine the changes that are enacted. The “changes in the organism are dependent on, but not determined by, the environment” (p. 147).

When the organism (e.g. a learner) interacts with its environment, changes in the structure of the organism occur. As structure-determined beings, our structure orients the sense we can make of a situation (Proulx, 2008b, p. 147). Proulx (2009) adds that within the concepts of structural coupling and structural determinism, learning is not seen as a causal event determined by an external stimulus. Rather, “learning arises from the learner’s own structure as it interacts with its environment” (p. 1). Thus, the tasks developed for this case study do not include reasoning cues that learners are required to explicate; rather, mathematical reasoning is determined by the learners' own structures as they interact with the word problems, the environment and each other (see Chapter 4).

Structural determinism in this study refers to how learners’ experience in a particular situation allows or encourages them to react in a certain way in a similar situation in the future. Learners generalise a past situation so as to cope with situations that arise in similar environments or contexts. In this study, structural determinism comes into play when the learners are encouraged to consult their pre-knowledge of geometry to get a general understanding of the word problems presented to them. It also applies to the interaction between the subject (the learner) and the object (the word problem), or between the two subjects (two learners in collaborative argumentation). Rivera, Steinbring and Arcavi (2014) claim that “classroom interactions that are rooted in sociocultural practices, for instance, can effectively provide grounding for meaningful and purposeful visualisation, encouraging learners to engage in visual thinking and reasoning in well-defined structural contexts” (p. 1). It is thus an ongoing challenge to relate school topics to the everyday life of the learners, and to foreground questions that are of relevance to them (Maturana & Poerksen, 2004, p. 130).

2.5.4.2 Co-emergence

Co-emergence is the overall process of structural coupling. Li et al. (2010, p. 407) maintain that co-emergence is the central idea of enactivism and focuses on the idea that change in either a living system or its surrounding environment depends on the interaction between this system and the environment. When a system and an environment interact, they are structurally coupled, and they co-emerge (p. 407). Li et al. (2010) caution that while co-emergence suggests that the system and the environment interact, it does not guarantee greater or lesser adaptation on the part of either to each other (p. 407). Learners bring forth a world; they emerge with it, but it is their structures that bring them forth (Proulx, 2008a, p. 22).

What does co-emergence mean for my study? Co-emergence manifested itself in this study as the learners constructed and appreciated the meaning of the GWP through action and interaction with each other and the enacted environment (see the Methodology and Analysis chapters). Di Paolo et al. (2010) assert that the "finding" of meaning is always an activity with formative traces, and must always be enacted via a concrete and specific reduction of the dimensions that the organism-environment system affords, along the axis of relevance for autonomy. This should never be seen as merely about the innocent extraction of information "as if this was already present to a fully realised (and thus inert) agent" (pp. 39-40). Thus, the co-emergence of visualisation and reasoning processes are observed when participants attempt to make sense of the GWP and share their understanding of each problem with each other, justifying their problem-solving strategies through arguing about these strategies with each other. In this sense, their structures are enabled to co-emerge.

Di Paolo et al. (2010, p. 40) define emergence as the development of a new process or idea as a result of the interaction of different existing processes or events. Proulx (2004, p. 116) suggests that it is the internal dynamics of the agent that enable the changes to occur, as the agent has to recognise the potentialities of change in the environment from its interaction with it. Therefore, meaning is not to be found in elements belonging to the environment or in the internal dynamics of the cognisant agent, but to the interactive domain established between the two (Di Paolo et al., 2010, p. 40).

Begg (2013) claims that in the enactivist perspective, humans and the world are inseparable: they co-emerge. Cognition (learning) cannot be separated from being (living), and knowledge is the domain of possibilities that emerges as we respond to and cause changes within our world (p. 82). Thus, one cannot separate knowing from doing and from the body. Knowing is doing and in the end is inseparable from self-identity or being (*ibid.*, p. 83): who we are and what we believe cannot be separated from what we have done (Brown, 2015, p. 185). "We are co-emergent and where there is a coordination of actions, like in a classroom, or a collaborative group in a research project, a culture of practices emerges that is good-enough (effective action) to get done what needs to be done" (Brown, 2015, p. 188).

Perceptually guided action is embedded within our individual selves to enable us to cope with what the world throws at us and to make changes in our environment by "bringing forth a world" within a medium of effective action (*ibid.*).

2.5.5 Critiquing enactivism

Enactivism is critiqued in many studies in relation to what the constructivist perspective can afford. In a critical review of enactivism from different theoretical perspectives, Reid (2014, pp. 150–152) highlights a number of crucial points that I took into consideration when developing items for the two EVGRT Worksheets. Reid disagreed with Ernest's (2010, p. 43) contention that enactivism does not foreground social interaction. Reid (2014, p. 151) believed that Ernest might not be referring to enactivism but perhaps to embodied mathematics, when he claims that enactivism subordinates the social and interpersonal dimension and that language and other persons are not central to enactivism. Reid (2014) offered further clarification:

> Radical constructivists have adopted the concept of a consensual domain, and the concept of embodied cognition has been employed by researchers interested in gesture, but neither group has actually adopted enactivism as a theoretical frame. This does not mean that enactivism itself is insufficient, however, only that the way it has been employed by radical constructivist and embodied mathematics researchers is insufficient. (p. 151)

I concur with Reid's (2014) insistence that enactivism considers other persons as important, given that this study, which is underpinned by an enactivist perspective was conducted in a social interactive setting. Further, when my research participants worked in small collaborative argumentative groups, they socially interacted with each other. This does not mean that enactivism is exempt from criticism. On the contrary, there are some acceptable critiques of enactivism that the enactivist spectrum of researchers have accepted and are presumably working on.

With regard to social systems, Reid (2014, p. 151) stressed that while enactivism undeniably addresses the social interaction and language use of living systems, it could be criticised for being unable to address the functioning of non-living social systems. He adds that there have been efforts to apply Maturana and Varela's concepts to these social systems, which are like living systems in some ways. However, "unless care is taken to establish the nature of social systems first, there is a danger of misapplying enactivist concepts" (Reid, 2014, p. 153).

On the aspect of coherence, Reid (2014, pp. 152–153) notes that enactivism provides a grand theory that is sufficient to address both the individual and the social in mathematics education. However, there remain aspects of mathematics education that enactivism does not address, most notably the nature and growth of mathematics itself. Hence, other theories must be used to address this aspect. In agreement, Simmt and Kieren (2015) reiterated that "enactivism as a methodological frame for mathematics education research is a form of research that is occasionally and multiversally incomplete, which implies that there is necessarily always more to be said and different grounds for the saying about the phenomena under investigation" (p. 316). Reid (2014) cautions that "with the ability to address a wide range of phenomena with a single framework comes the risk of incoherence…" (p. 153). Thus, in this study, although enactivism provides the main framework, it is complemented by other frameworks – visualisation, mathematical reasoning and embodied cognition – in order to minimise the risk of incoherence.

2.6 CONCLUDING REMARKS

The purpose of this chapter was to provide a contextual background to the study and to establish its theoretical framework. I have surveyed the literature pertaining to the three principal conceptual pillars of the study: mathematical reasoning, visualisation and enactivism. I have also discussed geometry word problems as these are used in this study as a vehicle through which mathematical reasoning is observed. I chose an enactivist perspective to inform my study as it focuses intensely on action and emphasises the bodily basis of meaning (embodied cognition). The chapter concludes with a consideration of some criticisms that have been made of enactivism. The next chapter on research methodology describes the research methods and discusses the data collection and analysis of the data for the case study.

CHAPTER 3

RESEARCH METHODOLOGY

This chapter reports on the methodological and analytical frameworks that structured the processes of data collection and analysis in this case study. As is explained in this chapter, elements of an enactivist perspective informed both the methodology and theoretical framework of the study. The data comprised one-on-one and focus group interviews and whole cohort reflections.

3.1. ORIENTATION OF THE STUDY

In this study I wished to understand how learners made sense of geometry word problems by examining how they employed visualisation processes to transform the written and textual representations into visual mathematical representations; and how these processes related to their mathematical reasoning. In order to understand the co-emergence of visualisation and reasoning processes when learners engaged with the EVGRT worksheets, efforts were made to understand their world of significance based on their experiences, interactions and sense-making processes. From an enactivist perspective, meaning-making is not to be found in "elements belonging to the environment or in the internal dynamics of the agent, but belongs to the relational domain established between the two" (Di Paolo et al., 2010, p. 40). In his seminal paper, Begg (2000) reflected on the enactivist elements of knowing and "coming to know". He observes that:

> In enactivism, instead of seeing learning as "coming to know", one envisages the learner and the learned, the knower and the known, the self and the other, as co-evolving and being co-implicated. In this situation context is neither the setting for a learning activity, nor the place where the student is; the student is literally part of the context. (p. 8)

Varela (1999) claims that the world we know is not given to us; it is enacted through our history. Brown (2015) adds that the world is enacted through a history of recurrent interactions leading to structural congruence between two or more systems (p. 189). Engaging with the EVGRT worksheets, the participants in this study went through the enactivist processes of co-evolvement, co-thinking and co-learning. These are deeply embedded within the enactivist concepts of structural coupling and co-emergence. They created their own learning context through recurrent interactions with both the tasks that they were solving and with each other. The participants' thinking, and reasoning coupled with the thinking and reasoning of other participants as they deliberated collective solutions to word problems in small collaborative argumentation groups (see Chapter 4).

With the aim of examining, analysing and interpreting how visualisation processes are integral to word problem solving, and studying their co-emergence with reasoning processes during geometry word problem solving, the study is located within the interpretive paradigm. The interpretive paradigm, according to Cohen, Manion and Morrison (2011), is characterised by a concern with the individual. The central endeavour in the interpretive paradigm is to understand the subjective world of human experience. In an attempt to retain the integrity of the phenomena under study, "efforts are made to get inside the person and understand from within" (Cohen et al., 2011, p. 17). Bertram and Christiansen (2014) emphasise that interpretive researchers do not desire to predict what people will do but rather aim to describe and understand how people make sense of their worlds, and how they make meaning of their particular actions (p. 26). The interpretive paradigm fits an enactivist study as the interpretivists purpose to understand the meaning that informs human behaviour and to make "interpretations with the purpose of understanding human agency, behaviour, attitudes, beliefs and perceptions" (Bertram & Christiansen, 2014, p. 26). This aligns well with the enactivist notion of co-emergence.

3.2. METHODOLOGY

3.2.1 Qualitative case study

My research purposed to gain an in-depth understanding of the co-emergence of visualisation and reasoning processes when learners solved geometry word problems. The use of visualisation as a word problem solving strategy in this study was based on the premise that visualisation can assume a wide range of forms other than just the obviously visual field of geometry. Arcavi (2003) suggests that spatial visualisation forms the base on which mathematical problem solving relies heavily (pp. 216–217). Mathematical reasoning is equally at the heart of mathematical understanding and application, as a means of sense-making of and in a mathematical activity (cf. Brodie, 2010).

Qualitative research aims to understand the meaning of phenomena as well as the relationships among naturally occurring variables (Ross & Onwuegbuzie, 2012, p. 86). Rule and John (2011) assert that what drives qualitative research in the social sciences and humanities is the desire to understand behaviour and experiences, often from the point of view of the research participants (p. 60). In the case study methodology, the researcher does not attempt to exert control or influence on the case under investigation, as is sometimes the case with quantitative research. A case study researcher attempts to understand the case in its natural state and context (Rule & John, 2011, p. 61). Furthermore, the criterion *of fit for purpose* should be what guides researchers on the question of which approach to adopt (*ibid.*).

Cohen et al. (2011) define case study research as a specific and single instance of a bounded system such as a child, a class, or a school, frequently designed to illustrate a phenomenon in a particular context. Bell (1993) believed that "a case study approach is particularly appropriate for individual researchers because it gives an opportunity for one aspect of a problem to be studied in some depth within a limited time scale" (p. 8). Cohen et al. (2011) suggest that good case study research requires in-depth data, a researcher with the ability to gather data that addresses fitness for purpose, and skill in probing beneath the surface of phenomena. This implies that a case study researcher "must be an effective questioner, listener (through many sources), prober, able to make informed inferences and adaptable to changing and emerging situations" (p. 296). Yin (2012) concurs that selecting a case for a case study should not simply be a matter of finding the most convenient or accessible case or site from which data can be obtained. The case selection should be based on a clear and substantive rationale (p. 33). Most importantly, the case should be screened beforehand by collecting sufficient data to determine whether the case meets the pre-established criteria (Yin, 2012, p. 33). In this study, I piloted both the data collection instruments and the analytical frameworks to ensure their reliability before the actual data collection (see Section 3.5.1).

From an enactivist perspective, attention was given to the "on-going co-constructed interaction among bodily actions [using various visualisation processes], cognition and the environment [the word problems]" (Khan et al., 2015, p. 272). These permit the structural coupling of the participant (learner) and his/her environment (word problems) through enacted learning (visualisation) that brought forth their co-emergence. During the first phase of data analysis, attention was given to the embodied processes evidenced by individual learners engaging in geometry word problem solving. During the second phase of data analysis, close attention was given to the relationship between visualisation and reasoning, and how these two processes co-emerged. My case thus comprised a cohort of Grade 11 learners engaging with geometry word problems, whilst the units of analysis were the observed visualisation and reasoning processes of the selected learners as they solved these problems in a classroom setting.

3.2.2 Participant Selection

I initially conveniently selected a cohort of 17 mixed-gender and mixed-ability Grade 11 learners to participate in my case study. The reason I opted for a mixed ability group was the presumption that the use of visualisation processes is generic and not specific to a particular ability group. I opted for Grade 11 learners to participate in this study because they are familiar with various concepts of geometry and had time to participate in the study, as opposed to Grade 12 learners whose lives are dominated by the end-of-year national examinations. I opted for convenience sampling as these participants were also the nearest available and accessible individuals who could participate (Cohen et al., 2011, pp. 155–156). Cohen et al. (2011) suggest that convenience sampling may be used for a case study or a series of case studies, whereby the researcher simply choses the sample from those to whom he/she has easy access (p. 156).

Likewise, the research site (the school that all the participating learners attend) was also conveniently selected on the basis of its availability and accessibility (Cohen et al., 2011). I have a good relationship with the management and the teachers at the school, which facilitated a smooth recruiting and buy-in process for the participants and their parents.

The sampling was also purposeful in that the participants needed to be able to express themselves. The research site promotes a culture of freedom of expression among teachers and learners alike. Through the school pledge, learners are encouraged to be proud of and develop a sense of belongingness to the school, town and country. They are also encouraged to embrace their cultures and to speak their various mother tongues with pride, given that they interact in a multicultural environment. The culture of freedom of expression is also encouraged at the school via a variety of sports and cultural activities, which includes job shadowing. The learners participate in the town and regional Junior Council activities, where they can be voted into positions such as Junior Mayor and regional chairpersons. The explicit promotion of this freedom to express themselves assisted me in my selection of participants. After involving the 17 participants mentioned above, I then selected eight from this group to complete the study. See details of the selection in Section 3.4.1.1.

3.2.3 Enacted Visualisation Geometric Reasoning Tasks (EVGRT)

The Enacted Visualisation Geometric Reasoning Tasks (EVGRT) formed two worksheets (EVGRT W1 and the EVGRT W2), each consisting of a set of problem-solving tasks that the research participants were asked to complete in my presence. The tasks were accompanied by task-based interviews (Koichu & Harel, 2007), which happened during and after the EVGRT tasks were solved. A total of 15 items were sourced for the EVGRT worksheets; ten items for EVGRT W1 and five items for EVGRT W2. The problems posed in these worksheets (that I am referring to as 'tasks') were unusual and interesting problems which could be solved through numerous possible solution strategies. The format of the tasks encouraged the participants to discover their own methods and visual representations that also necessitated some sort of mathematical reasoning. Brodie (2010) cautions that simply posing open-ended mathematical problems that require mathematical reasoning is insufficient to help learners learn to reason mathematically. Merely asking learners to explain their thinking also does not satisfy their quest to reason (pp. 19-20). According to Brodie, the product of a reasoning process is a text, either spoken or written (Brodie, 2010, p. 7), which represents a conclusion that is acceptable within the community producing the argument (*ibid.*). This consideration formed the basis for where and how these items were either developed by myself or adapted from items published elsewhere. The items were adapted to suit the purpose of this case study: they reflected a Namibian context and provoked visualisation and reasoning processes when learners engaged with them. Before actual data was collected for the case study, the EVGRT items were piloted through a number of cycles with Grade 11 and Grade 12 learners from two schools, one of which is the research site (see Section 3.5.1 for clarification on piloting). The rationale behind the choice and inclusion of each item is supplied with the description of two phases of data collection (see Sections 3.3.3 and 3.3.4).

3.3. DATA COLLECTION

3.3.1 Three Phases of the Case Study

Data for this case study was collected in three phases. In the first phase, data was collected through the EVGRT W1 individual task-based interviews with 17 participants. The interviews and the problem-solving processes were both audio and video recorded.

In the second phase, eight selected participants solved five items in the EVGRT W2 in small collaborative argumentative groups. The purpose of this phase was for the participants to converse and deliberate at length with each other (as opposed to only with me) as they solved each of the word problems. Data for this phase was collected in the same manner as in the first phase.

Phase three of this case study was essentially for reflection purposes. I had a semi-structured reflective interview with the second-phase participants in which they were asked to reflect on their experiences with the EVGRT Worksheets and on what they had learned (see Appendices for interview questions). The data collection process is expounded in detail hereunder.

3.3.2 Data Collection Methods

I made use of the following data collection techniques:

- Interviews:
  - One-on-one task-based semi-structured interviews
  - Focus group task-based semi-structured interviews
  - Semi-structured reflective interview
- Audio and video recording

Semi-structured interviews

Three types of semi-structured interviews were used to collect data. First, as alluded to earlier in this chapter, I administered one-on-one task-based semi-structured interviews as the research participants solved the ten items of EVGRT W1 in my presence. In these interviews I encouraged the participants to talk through their problem-solving strategies as they solved each of the tasks. Secondly, I conducted focus group interviews that were also task-based and semi-structured. They differed from the first interviews in that, instead of the participants explaining their methods to me, they were encouraged to talk to each other in the group. The semi-structured interview format allowed me to ask probing questions and to discover, among other things, aspects of the participants' mental and imaginative processes that may not have been verbally uttered. My role was thus that of an observer-facilitator (see Appendix 3 for follow-up questions in the EVGRT interactions). Thirdly, the reflective interview was conducted with the eight participants to learn about their overall experience of the whole research process.

Hancock and Algozzine (2006) note that interviews are a very common form of data collection in case study research as they enable the researcher to obtain rich, personalised information from the individuals or groups. The purpose of the task-based interviews (see Section 3.2.3, above) was to prompt the participants to use visualisation processes to solve the EVGRT Worksheets. The participants were encouraged to use both verbal and nonverbal modes of communicating answers to the questions. Cohen et al. (2011) observe that "the order of the [semi-structured] interview may be controlled whilst still giving space for spontaneity, and the interviewer can press not only for complete answers but for responses about complex and deep issues" (p. 409). According to Bertram and Christiansen (2014), the interview method is extensively used in interpretive research, with its particular aim of exploring and describing participants' perceptions and understandings that might be unique to them (p. 82). The interview method also allows researchers to ask probing questions, to clarify questions and to discuss participants' understandings with them (*ibid.*). Cohen et al. (2011) concur that a semi-structured interview can keep a conversation going, motivating the participants to discuss their thoughts, feelings and experiences (p. 422).

Focus groups

A focus group is a form of group interview where the focus is on interaction within the group in response to a specific topic introduced by the researcher. What emerges is a collective rather than an individual view of the topic. During focus group interviews, "the participants interact with each other rather than with the interviewer such that the views of the participants can emerge – the participants' rather than the researcher's agenda can predominate. It is from the interaction of the group that the data emerge" (Cohen et al., 2011, p. 436). In case study research, the specific unit of analysis in these focus group interviews is the interaction (both verbal and nonverbal) between the participants, in terms of the broader unit of analysis, i.e. the co-emergence of visualisation and reasoning processes. Rule and John (2011) point out that focus groups are useful for gaining a sense of the range and diversity of participants' views and opinions. The focus group interviews were both audio and video recorded.

3.3.3 Data Collection Phase 1

In this phase data was collected from a series of task-based interviews. Each of the 17 research participants was individually interviewed while solving the 10 tasks of the EVGRT W1. This data was collected by means of audio and video recording as well as the participants' written work. The video recordings were used to capture the participants' gestures, their drawings or sketches, and their subtle body movements that could not be evident in the audio recordings. As mentioned in Chapter 2, Alibali and Nathan (2012, p. 248) argue that mathematical cognition is embodied in perception and action, and grounded in the physical environment. "Gestures are often taken as evidence that the body is involved in thinking and speaking about the ideas expressed in those gestures", since mental processes are mediated by body-based systems including body shape, movement and scale (*ibid.*). The purpose of the task-based interviews was also to talk the participants through their problem-solving strategies, to clarify unclear actions and methods as well as to provide the necessary support without intentionally leading them to the solutions. Abrahamson and Lindgren (2014, p. 7) suggest that learners often need guidance in taking action, moving their bodies in ways that stimulate spatial relations, and in articulating their strategies for interacting with materials in the environment.

The task-based interviews had two purposes. The first was to determine the visualisation processes that were evident when the research participants solved EVGRT W1 through task-based interviews. The second was to enable me to select appropriate participants for Phase 2 of the case study – that is, those participants who preferred the use of visual methods to solve word problems, even when analytical and algebraic methods were accessible. The analytical tool employed to execute this assignment is discussed in Section 3.4.1.

The research participants who agreed to participate in Phase 2 of the case study (see Section 3.4.1.1 for the selection criteria) were each given a short invitation letter congratulating them and informing their parents/guardians of their progress to the next phase of data collection (their participation in all phases remained voluntary). The participants were also informed through the invitation letter with whom they would be working in the small group work. I then asked each group to commit to at least one two-and-a-half-hour session for completing the EVGRT W2, at a time that suited each group (see Appendix 9 for a copy of the invitation letter).

EVGRT Worksheet 1 (EVGRT W1)

The EVGRT Worksheet 1 consisted of 10 geometry word problems. The reasons for including each task in this worksheet are set out below.

Task 1 (EVGRT W1 T1) Imagine a 10m ladder leaning against a half-painted wall behind you. The bottom of the ladder is 6m from where the wall meets the ground. a) What special name is given to the geometrical shape formed between the ladder, the wall and the ground? b) At what height from the ground does the top of the ladder lean against the wall?

This task was developed by the researcher in alignment with the concepts inherent in the theorem of Pythagoras. The task is strongly visual in nature and was designed to encourage the learners to use visualisation processes in their minds or on paper to find a solution. It is not too time consuming and I decided to use it as an “icebreaker” for the participants. Pilot participants did well with this task as they both managed to visualise in detail and were able to justify their solution strategies based on their sketches and formulae used. Figure 3.1 below shows the solution strategy of one of the pilot participants. (See Appendix 12 for pilot analysis of the first EVGRT worksheet.)

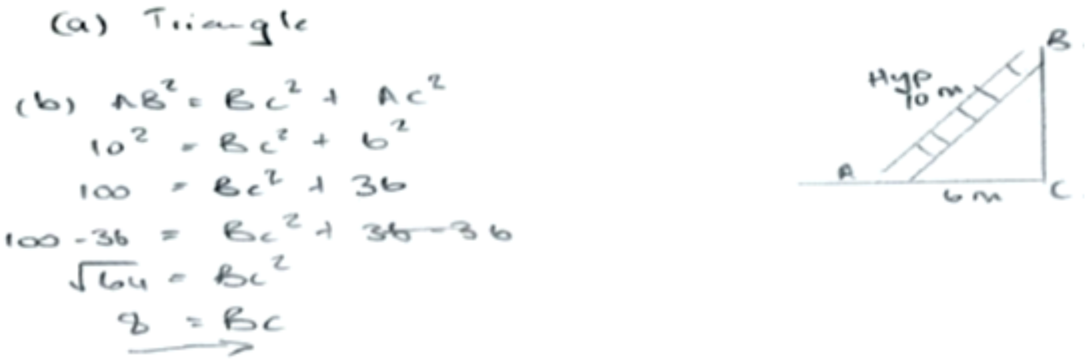


Figure 3.1 Carl’s visual representation and solution for EVGRT W1 T1

Task 2 (EVGRT W1 T2) Dalia wants to design traffic sign boards for her school play. Her mathematics teacher instructed her to ensure that all her sign boards have equal perimeters. Dalia designed a square board of side 12cm and an equilateral triangular board of the same perimeter. a) What was the side of Dalia’s equilateral triangular board? b) Are the areas of the two boards equal? Explain.

This two-dimensional task was adapted from a library book and designed to suit a real-life classroom situation. The context of the question has been modified to suit the nature of a Namibian mathematics classroom, in the hope that the learners will use their imagination and think ‘outside the box’ when solving it. The task was also piloted to ensure reliability. Figure 3.2 shows the solution strategy of one of the pilot participants, which affirmed the task’s inclusion in EVGRT W1. The solution shows that the task is doable for the intended purpose of eliciting visualisation and reasoning processes. (See Appendix X for analysis of this task.)

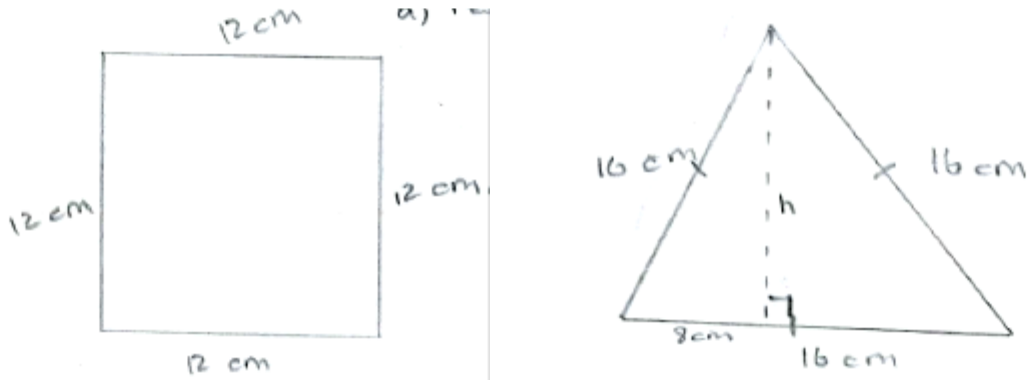
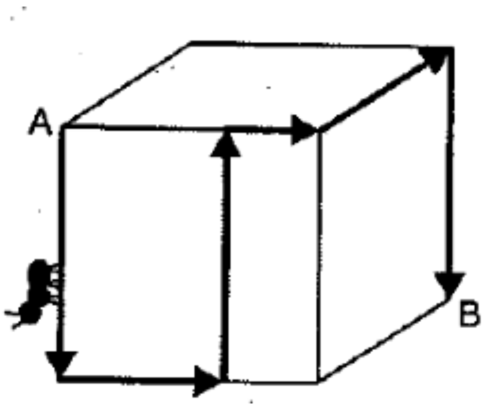


Figure 3.2 Natalia’s visual representation of the word problem for EVGRT W1 T2

Task 3 (EVGRT W1 T3)

The edges of a cube are 12 cm long. An ant moves on the cube surface from point A to point B along the path shown.



a) What is the length of the ant’s path?

b) What is the shortest possible distance that the ant can move from A to B?

This task was adapted from the South African Mathematics Olympiad (SAMO). With this task, I wanted to see what the participants might do if I gave them an already drawn figure. The anticipation was that the learners would unpack the visual aspects of the given figure to elicit enough information to help them reach a solution. I piloted this task with one learner during the second cycle of piloting EVGRT W1 (see Section 3.5.1). The learner’s ability to visualise and talk about her visualisation processes and how the real aspect of the question helped her are some of the reasons why I decided to include the task in EVGRT W1. The task is both two dimensional and three dimensional and can be solved in various ways.

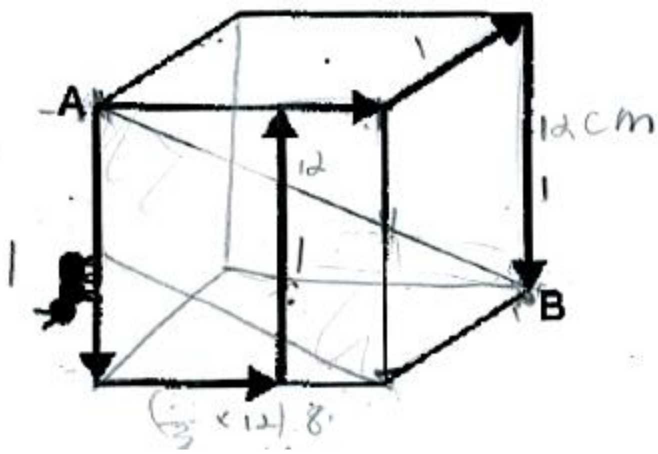


Figure 3.3 Rauna's visual evidence for solving EVGRT W1 T3

Task 4 (EVGRT W1 T4)
a) Imagine a clock with hands, on the wall in front of you. The long hand is pointing to 4. The short hand is pointing between 11 and 12. What time is it? b) Now imagine the clock is behind you and you can see it in the mirror. There are dots instead of numbers. The hands look as though they are saying twenty-five to three. What time is it really? What is the size of the angle formed between the two hands? **NB**: There are dots instead of numbers on this clock.

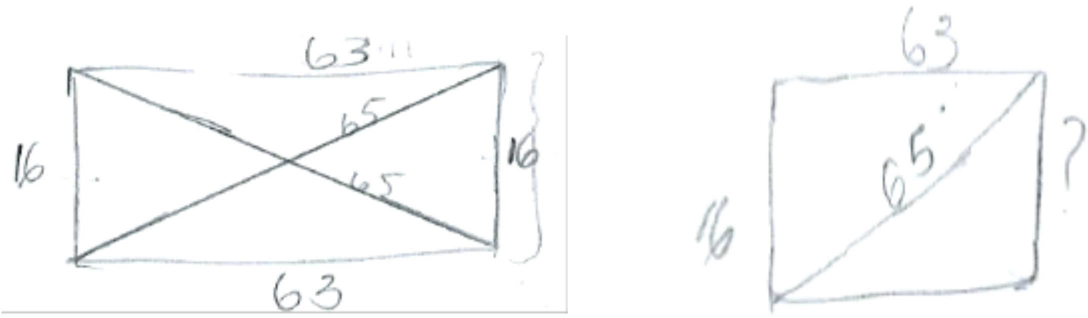
This task was also adapted from SAMO. SAMO's questions have gone through thorough reviews and can be trusted in the assessment of learners' problem-solving skills. The SAMO questions also promote creative problem-solving skills as these are necessary and very marketable in today's technically-oriented market place[3]. Part b) of this question was added to enrich the task and to facilitate the learners' thinking, visualisation and reasoning.

[3] See the South Africa Mathematics Olympiad website at http://www.samf.ac.za/sa-mathematics-olympiad

Figure 3.4 Dynamic visual aspects of Rauna’s problem solving strategy for EVGRT W1 T4

Task 5 (EVGRT W1 T5)
The longer side of a rectangle has a length of 63 cm and the diagonals both have a length of 65 cm. Calculate the width of the rectangle (in cm).

This two-dimensional task was developed by the researcher with the intention of encouraging the participants to represent the word problem visually. The task is geometric in nature and purposes to analyse how the participants incorporate categories of visual imagery during problem solving. The more the pilot participants used visual imagery, the more they could reason. This task is therefore crucial for the purposes of this study, as it not only appears to facilitate the analysis of visualisation in geometry word problem solving, but also creates rich opportunities for the participants to articulate and reason during the problem-solving process. Figure 3.5 shows the solution process of one of the pilot participants (snapshots arranged Left to Right in a clockwise direction).



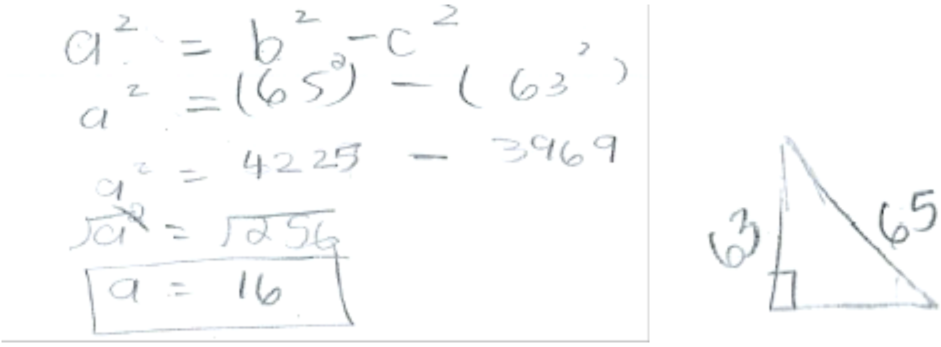


Figure 3.5 Rauna's processes of visual imagery during problem solving for EVGRT W1 T5

Task 6 (EVGRT W1 T6) How would you explain to a Grade 5 learner how to calculate the sum of the interior angles of a 5 sided polygon? (**Hint**: Grade 5 learners hardly understand formula)

This two-dimensional task was also developed by the researcher in relation to the Grade 10 mathematics syllabus. In Grade 10, learners are introduced to the geometry concept of the interior angles of polygons and are taught how to calculate both the sum of the interior angles and the size of one interior angle. I noticed over my years of teaching the Grade 11-12 mathematics curriculum that these learners are mostly taught using formulae, and often never get around to understanding the concepts behind them. Therefore, I decided to include this task in EVGRT W1 to analyse the use of visualisation processes in solving it. All pilot participants started off with the formula for calculating the sum of the interior angles of polygons: $S_n = 180(n-2)$, despite the hint in the question that Grade 5 learners did not understand formula. The following is an extract from a pilot transcript:

***Carl**: If it was a Grade 5 learner, first I'll try to explain it like this: it has 5 sides and one of these... then I'll just draw a little picture like this [learner draws a pentagon]. So, I'd explain it like this, because there's 5 sides and because if you extend it like this [exterior angle] then it's 180. Each one lies on a straight line then you just take the 5 sides minus 2...*
***Beata**: Why are you subtracting 2?*
***Carl**: Why am I subtracting 2...? What is that reason...? Because um... you have to sub... why am I subtracting 2?*

This was one of the reasons why I decided to include this task, as it afforded me the space to probe and encourage the learners to visualise and reason at length. I discouraged the learners from using formulae and encouraged them to incorporate visual imagery to solve it.

The participants eventually performed well in this task and enjoyed and appreciated what they learned. Thornton (2001) said of visualisation that it can often “provide simple, elegant and powerful approaches to developing mathematical results and solving problems” (p. 251). Figure 3.6 show’s Carl’s impressive visual representation and solution.

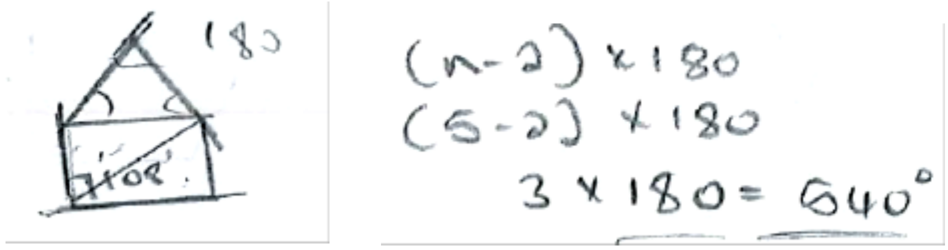
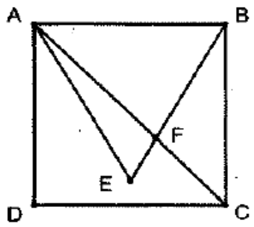


Figure 3.6 Carl’s visualisation and solution strategy for EVGRT W1 T6

Task 7 (EVGRT W1 T7)

If ABCD is a square and ABE is an equilateral triangle, then angle BFC calculated in degrees equals…



This task was also adapted from the SAMO. After piloting, I realised that it would be beneficial to include word problems with sketches in the EVGRT W1 as these too are essential visualisation stimulators. Providing sketches helped the learners to visualise and talk about their visualisation processes. They sometimes deconstructed the diagrams or drew their own as they solved the problem. EVGRT W1 T7 in particular turned out to be an excellent task, as it has a clear relation to the geometry the learners encounter in high school mathematics. Although the pilot participant did not express much visualisation on paper (see Figure 3.7), she reasoned well about each of the angles that she filled in in the given figure. The task thus fitted the purpose of analysing the visualisation processes that learners employ when solving geometry word problems.

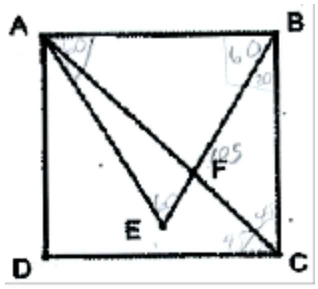
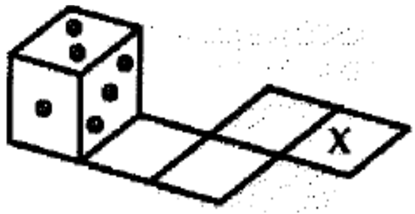


Figure 3.7 Rauna's work on EVGRT W1 T7

Task 8 (EVGRT W1 T8)

On a die the numbers on opposite faces add up to 7. The die in the diagram is rolled edge over edge along the path until it rests on the square labelled X. What is the number on top in that position?

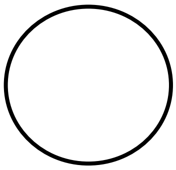


This task was also adapted from the SAMO. Its nature required a combination of the categories of visual imagery to solve, ranging from moving pictures in the mind to actual dice rolling. I found this real-life task to be very visual and effective to motivate the learners to solve it as it is practical and appears to be easy to solve. Rauna enjoyed solving this task during the pilot study, despite facing a few challenges (Figure 3.8). She eventually used an eraser to represent a die after she had exhausted rolling an imaginary die in her mind. She rolled the eraser on her sketched surface which helped to clear her confusion and successfully solve the task.

Figure 3.8 Rauna's visualisation process that led to the solution for EVGRT W1 T8

Task 9 (EVGRT W1 T9)

Show and explain how you would find the centre of this circle.



This task was suggested by my PhD supervisor, in the expectation that it would be interesting to see how learners responded. I observed from the first piloting cycle that it was a very rich task and could be solved in a variety of ways. The pilot participants enjoyed it and provided rich evidence of employing a host of visualisation processes - so I included it in the EVGRT W1. Figure 3.9 below shows some of the visualisation processes employed by the two pilot participants.

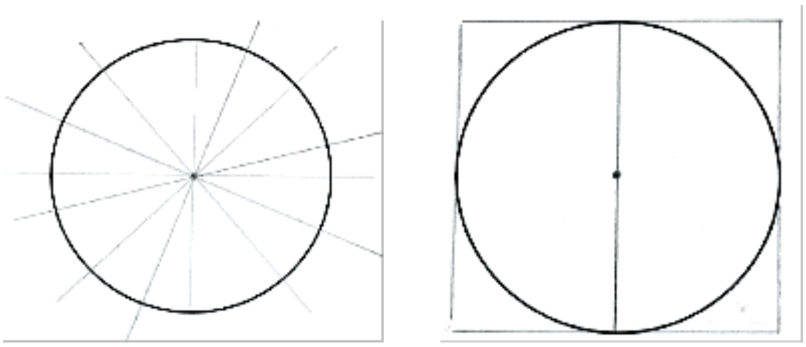


Figure 3.9 Natalia’s (left) and Carl’s (right) visualisation processes for EVGRT W1 T9

Task 10 (EVGRT W1 T10)

Mr. Mauno constructs a triangle of perimeter 30cm for his mathematics lesson preparation. During the lesson, he asks his learners to find the length of the shortest side of the triangle if two sides of that triangle were each twice as long as the shortest side. Suppose you are Mr. Mauno’s learner, what will be your answer?

This task was adopted from a mathematics book. Since the task sounds more algebraic than geometric, I decided to include it to observe which participants preferred to use visual methods when algebraic methods were eminently possible. Figure 3.10 illustrates how the pilot participants managed to incorporate both algebraic and geometric methods to solve the task. Although there was not much evidence of variety in their problem solving and visualisation strategies, I still favoured the task for its algebraic-geometric relation.

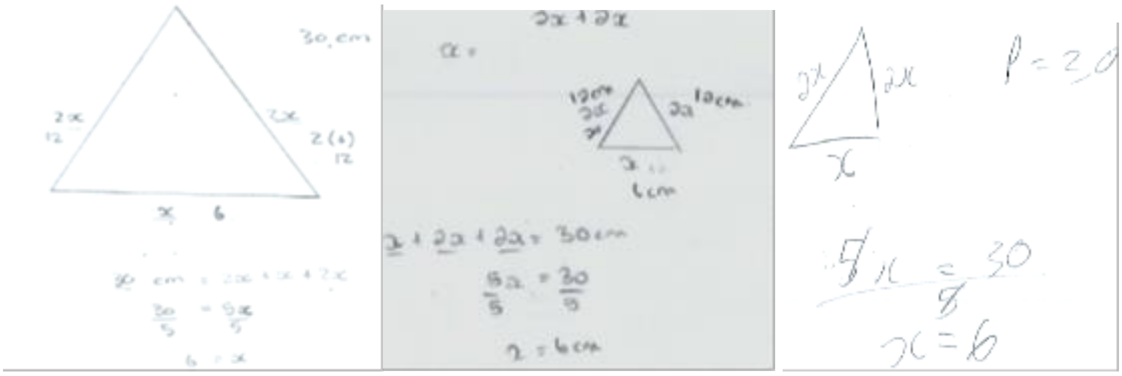


Figure 3.10 Natalia's (left) Carl's (centre) and Rauna's (right) visualisation processes and solution of EVGRT W1 T10

3.3.4 Data Collection Phase 2

The second phase of data collection was intended to answer research sub-question two, which aimed at analysing the relationship between visualisation and reasoning processes and how these co-emerged when learners solved EVGRT W2 in small collaborative groups.

After Phase 1 of data collection I identified and purposively selected eight participants from the initial 17 (see Section 3.4.1.1) to proceed to the second phase of data collection – specifically those participants who seemingly preferred to use visual methods to solve word problems, and were able to converse well, articulate clearly and express themselves richly. The eight participants were divided into small groups (the grouping criterion is also discussed in Section 3.4.1.1). Rule and John (2011) define purposeful or purposive sampling as a method in terms of which the people selected as research participants are deliberately chosen because of their suitability in advancing the purposes of the research (p. 64).

As discussed earlier, data for this phase was collected by means of focus group semi-structured task-based interviews. Each group was provided with a single booklet of EVGRT W2 and tasked with solving the word problems as a group to reach a collective solution. They were also provided with blank pages in case they needed more space to work. The data consists of audio and video recordings, interview transcripts, completed EVGRT worksheets and the extra sheets of paper that the focus groups worked on.

EVGRT Worksheet 2 (EVGRT W2)

The purpose of this worksheet was to investigate the relationship between visualisation processes and mathematical reasoning processes through problem solving in small groups, in order to address research sub-question two. At this stage, I had already identified the cohort for this worksheet based on the findings of EVGRT W1. The selected participants were assigned to work in small collaborative argumentative groups to solve the five items in EVGRT W2. Bell and Walters (2014, p. 183) underscored the importance of focus groups in problem solving. The intention is that participants interact with each other, are willing to listen to all views, perhaps to reach consensus about some aspects of the topic or disagree about others, and to give a good airing to the issue which seems to be interesting or essential to them. The researcher becomes less of an interviewer and more of a moderator or facilitator.

Akin to the items in EVGRT W1, the items on this worksheet were drawn from different sources and also aligned with the Grade 11 mathematics syllabus. The rationale for including each of the five tasks is discussed hereunder.

Task 1 (EVGRT W2 T1)

Marina's backyard is a square with a side length of twenty meters. In her backyard is a circular garden that extends to each side of her yard. In the centre of the garden is a square patch of spinach so big that each corner of the square touches a side of the garden. Marina really likes spinach! How much area of Marina's garden is being used to grow spinach?

The research for this study was conducted in a town where almost every house has a garden. I thus thought it would be appropriate for my data collection to include a context that the participants could immediately relate to. There are also gardens in the school grounds that learners could relate to. I adapted this task from a mathematics book. Inherent in the task is the mathematical concept of mensuration and the theorem of Pythagoras that the learners are familiar with, and which they could discover with effective visualisation. In terms of reasoning, there were many strategies that could be employed by the participants to solve this task, thus creating room for argumentation in the group as each member justifies his/her methods. I anticipated collecting rich data from this task: as Reid and Mgombelo (2015) assert, "when student knowing is seen as a dynamic, local and emergent mathematical activity the process of collecting and analysing data is changed in fundamental ways" (p. 172). Figure 3.11 below shows a series of visual representations that the pilot participants sketched when they solved the task. Although this only shows visual representations, lots of verbal reasoning was observed during piloting which motivated my decision to include the task in this worksheet.

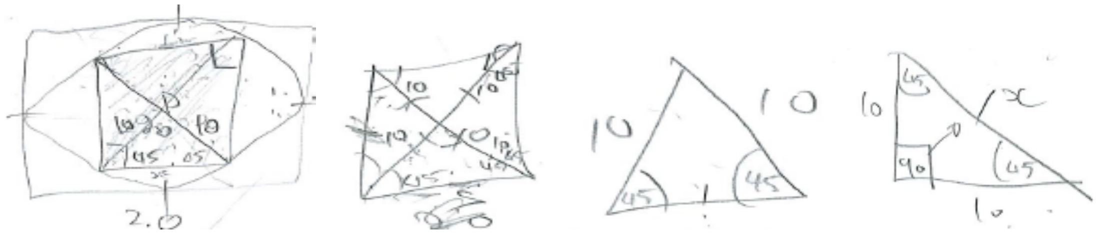


Figure 3.11 Pilot group's visual representation for EVGRT W2 T1

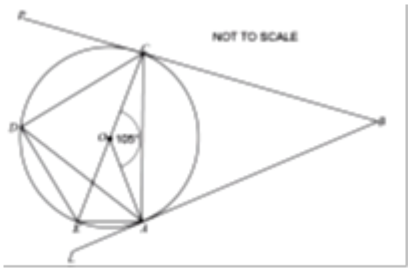
Task 2 (EVGRT W2 T2)

Mr. Onesmus has a plan for his sick goat. He tied the goat to a tree with a 7m rope in such a way that the goat is able to move freely around the tree for it to graze. If the goat moves a complete revolution with the maximum length of the rope, what is the total possible area that the goat would graze?

I found this task very interesting as I could relate to its context. I grew up on a farm and we would tie our sick and naughty goats to a tree with a rope long enough for them to graze in addition to the food we provided them with. The rope needed to be loose enough for the goat to move freely. My siblings and I would wonder and argue about what area the goat would graze. The mathematical idea behind the task is embedded in the geometrical concept of a locus, which in this case is defined as set of points equidistant from a fixed position. Other mathematical concepts that are central to this problem are notions of area and pi (π). Not only was this task rich in terms of visual representations, it also created an environment for the learners to explain their methods, justify their strategies and support their arguments. Figure 3.12, below, shows one of the visual representations and a solution strategy from the pilot study.

Figure 3.12 Pilot participants' visual representation and solution for EVGRT W2 T2

Task 3 (EVGRT W2 T3)



Find the following angles, giving a reason for each answer.

a) ADC b) OCP c) EAC d) AEC e) ABC f) ACE

I adapted this task from the 2015 Namibian mathematics national examination paper for Grade 12. On the one hand, I expected that the participants would engage reasoning processes to solve the task, and on the other hand, I anticipated that the participants would employ various categories of visual imagery to find the answer. The pilot participants did fairly well (see Figure 3.13), but the onus was on me to ensure that effective probing elicited the co-emergence of visualisation and reasoning processes during actual data collection. I therefore found this task suitable for inclusion in the worksheet, anticipating that it would generate rich data.

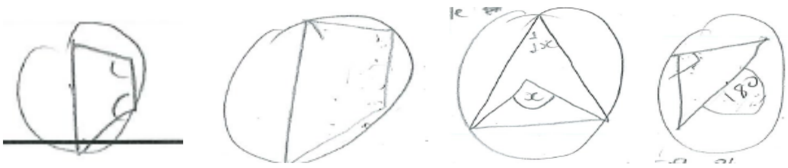


Figure 3.13 Additional sketches that the pilot participants used to solve EVGRT W2 T3

Task 4 (EVGRT W2 T4)

A pack of 52 cards is dealt out to 10 people seated around a circular table in such a way that the first person gets the 1^{st} card, the fourth person gets the 2^{nd} card, the seventh person gets the 3^{rd} card, the tenth person gets the 4^{th} card, and the third person gets the 5^{th} card and so on.

a) Which person gets the last card?

b) If the cards were 72 instead, which person gets the last card?

c) What about 96 cards? Who gets the last card?

This task was adapted from the SAMO papers. It is very rich in terms of Presmeg's (1986b) categories of visual imagery and requires the participants to communicate throughout the problem-solving process. The task also promotes teamwork, as the participants engage and interact with each other. Dejarnette and González (2013) argue that "a teacher's implementation of tasks that allow students the autonomy to work productively and promote student discussions of a problem can provide an avenue through which students in algebra [in my case geometry] may develop their reasoning and sense making skills" (p. 3). Parts b) and c) were incorporated into the task to test the participants' generalisation skills in solving the task. Figure 3.14 shows the pilot participants' visual representation as part of their problem-solving strategy.

Figure 3.14 Pilot participants' visual representation for EVGRT W2 T4

Task 5 (EVGRT W2 T5)

In a cube with sides of length 10cm, denote one vertex by the letter *V*. Find the sum of the shortest possible total distances from *V* to each of the other vertices of the cube.

This task was initially a part of the EVGRT W1, but after the pilot analysis of that worksheet, I realised that it was a good task for group work as it required the problem solvers to sketch the problem in multiple ways (both two and three dimensional). I therefore moved it from the individual worksheet to the group worksheet, anticipating interesting interaction among the research participants. Walkup (1965) promotes the use of a similar problem, called the *painted cube*, for studying the use of visual imagery in science. He reported considerable individual differences in the ability to visualise the cube and its sequential transformations in solving the problem. Since this task had the potential for different solution strategies, I anticipated when I opted to include this problem that the participants would go beyond visualisation and mental arithmetic, using pencil and paper as well as manipulatives as they saw fit. Figure 3.15 shows visual representations of the pilot participants' working on the task. It does, however, have the potential for different solution strategies.

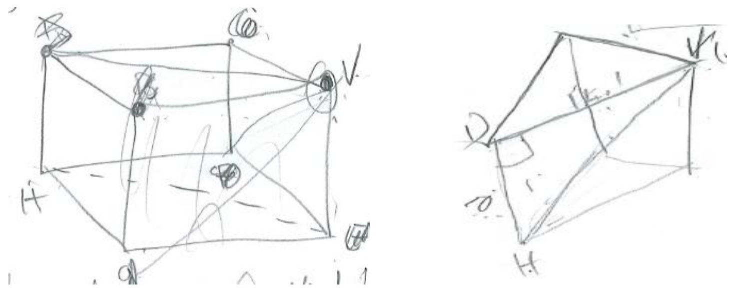


Figure 3.15 Pilot group participants' visual representation for EVGRT W2 T5

3.3.5 Data Collection Phase 3

The third phase of data collection was primarily for reflective purposes. The cohort from the second phase of data collection participated in a reflective interview whose aim was to provide feedback about their experience of the whole process, from the first phase through to the second. Data was collected in the form of a short semi-structured interview with five open-ended questions (see Appendix 11).

3.4. DATA ANALYSIS

For this case study, there were three datasets that needed analysing: the one-on-one semi-structured task-based interviews with my original 17 participants, focus-group semi-structured task-based interviews with my eight participants, and the semi-structured reflective interview with the same eight participants. The data was coded using themes and categories from the literature, as well as themes that emerged from the data generated, as discussed in detail below.

The categories and observable indicators used in the analytical tools are descriptions of underlying frameworks. As I explored appropriate theoretical framings for my study, I took up the challenge to synthesise the frameworks of mathematical reasoning, visualisation, enactivism and embodied cognition, in order to develop an amalgamation of analytical tools. This amalgamation formed the basis for a comprehensive narrative about the co-emergence of visualisation and mathematical reasoning processes.

Cohen et al. (2011) assert that "there is no one single or correct way to analyse and present qualitative data; how one does it should abide by the issue of *fitness for purpose*" (p. 537). A fitness for purpose approach for this case study means that the frameworks for data analysis should align with the kind of data that the researcher collected. This data was informed by the research questions and the research design (Bertram & Christiansen, 2014, p. 41).

In an effort to make sense of the data collected, I allocated codes to patterns and categories mentioned in the literature as well as to emergent themes in the data itself. Cohen et al. (2011) define coding as the ascription to a piece of data of a category or label that is either decided in advance or responds to the nature of the data itself (p. 559). Kawulich (2017) characterises data coding as the process through which researchers begin to make sense of qualitative data; it involves labelling and organising the data to facilitate interpretation (p. 769). Cohen et al. (2011) point out that the same segment of text may have more than one code ascribed to it, depending on the richness and content of that segment. The process of coding enables the researcher to identify similar information as well as to search and retrieve items of data that bear the same code (p. 559). Section 3.4.1 provides further information on data coding.

Data coding in the transcripts

Once the audio recordings were transcribed, I read through each transcript to ensure the accuracy of the transcriptions as well as to add to the transcripts the participants' nonverbal gestures which may be important for data analysis. Kawulich (2017) highlights the importance of revisiting transcripts prior to data analysis. She wrote that the quality of the transcript is an essential aspect of one's ability to analyse data appropriately. "Once audio recordings or video recordings are converted to text, it is the responsibility of the researcher to read through transcriptions along with the media source to ensure that the transcriptions are correct, prior to beginning analysis" (Kawulich, 2017, p. 773). In the transcripts of both Phase 1 and 2 of the task-based interviews, I used *italics* to emphasize and identify the participant's visible 'mood' that came with having to solve the tasks (for instance, when the participant laughed, sighed or exclaimed). I adopted Nemirovsky and Ferrara's (2009, p. 162) use of the term "utterance" to encompass all types of bodily activity that played a part in a given conversational turn or transaction, including multimodal aspects such as facial expression, gesture, tone of voice, sound production, eye motion, body poise, gaze, and so forth. I used square brackets [ ] to denote when a participant "uttered" something without using words.

3.4.1 Data analysis phase 1

The data analysed for this phase served two purposes: first, to answer the research sub-question 1 which sought to examine the visualisation processes that were evident when the participants solved geometry word problems; and secondly, to select the participants who took part in the second phase of data collection.

The exploratory provisional coding method (Saldaña, 2009) was used to analyse the transcripts of the task-based interviews. First, all transcripts were read while the video recordings were being viewed, which enabled the participants' actions not recorded on audio to be added to the transcripts. As stated earlier, these were added in square brackets to ensure that all visualisation processes were included in the transcripts and could be coded for the purpose of analysis. Arcavi (2003) observes that visualisation occurs in different forms and at different levels, and that a visual solution to a word problem may enable us to engage with concepts and meanings that could have been bypassed by a verbal or symbolic solution to the problem (p. 222). Samson and Schäfer (2011, pp. 42–43) argue that mathematics teachers need to have both the visualisation and algebraic capacity to verify the learners' general expressions. There is also a need for teachers to critically engage, at an embodied level, with the learners' explanations of their generalisation process.

Saldaña (2009) proposes that provisional coding begins with a "start list" of researcher-generated codes based on what preparatory investigation suggests might appear in the data before it is analysed (p. 118). In this study, the provisional list of observable indicators was generated from the literature review, the study's conceptual framework and research questions, previous research findings, pilot study fieldwork, the researcher's previous knowledge and experience, and researcher-formulated hypotheses and hunches. As qualitative data is collected, coded and analysed, these provisional codes can be revised, modified, deleted, or expanded to include new codes (Saldaña, 2009, pp. 120–121).

Saldaña's (2009) provisional coding is akin to what Kawulich (2017) calls deductive coding and defines as follows:

> When researchers identify codes and themes in the data that derive from their preconceived ideas, from previous studies on the topic, or from their philosophical and theoretical framework of the study, they are using a deductive approach to analysis (from the general to the specific), a top-down approach to coding data, as the preconceived codes are applied to the data collected. (p. 771)

I had a provisional analytical tool populated with deductive codes and observable indicators derived from previous research and my own hunches on anticipated outcomes prior to my Phase 1 analysis. This provisional coding list consisted initially of 15 observable indicators. The list grew from 15 to 20 observable indicators as more emerged from the first round of pilot data analysis. I then conducted the second round of pilot study, analysis of whose results increased the provisional coding list to 23 observable indicators. However, once in the field to collect the actual data, I realised that some of the observable indicators overlapped. It became necessary to combine the overlapping indicators and eliminate those that did not emerge in the participants' responses as initially anticipated. This enabled me to modify the indicators, so that at the end of the first phase of data analysis I ended up with a final list of 15 observable indicators (see Table 3.1), three for each of the visual imagery categories. Saldaña (2009) indeed suggests that interviews in the actual fieldwork may yield a more relevant set of themes for incorporation into the provisional list of coding (p. 122). The observable indicators in Table 3.1 were adapted from various visualisation readings which included, *inter alia*, Arcavi (2003), Mesaroš (2012), Presmeg (1986a, 1986b), Wheatley (1991) and Yilmaz et al. (2009).

All 10 tasks of EVGRT W1 as addressed by each of the 17 participants were analysed for visual imagery, as per Table 3.1, below. The participants' responses were analysed in terms of the five categories of visual imagery (see Section 2.3.1), making use of a Computer Assisted Qualitative Data Analysis Software (CAQDAS) called the NVivo 11 Pro Software.[4] The software was used to code the participants' responses in accordance with the five categories of the visual imagery analytical tool's observable indicators (see Table 3.1). The results were presented visually using charts and tables to simplify the data without losing its meaning (Adu, 2016). Saldaña (2009, p. 123) believes that CAQDAS programs such as the NVivo software allow for the development of provisional codes in the code management system. As documents are reviewed, a pre-established code from the list can be directly assigned to a selected portion of data. Yin (2012) cautions that irrespective of whether or not computer software is used to help with coding and support the analysis, "you will be the one who must define the codes to be used and the procedures for logically piecing together the coded evidence into broader themes – in essence creating your own unique algorithm befitting your particular case study" (p. 15).

[4] NVivo 11 Pro Software is referred to simply as NVivo elsewhere in this case study.

Table 3.1, below, shows the final list of provisional indicators that were used to code the participants' responses using NVivo. Each sentence of the participants' responses, their sketches, gestures and subtle body movements were analysed using the observable indicators in each of the categories of visual imagery. This was done to examine the visualisation processes that are evident when the participants solved geometry word problems, and to determine the participants who preferred using visual modes when solving the word problems.

Table 3.1 Analytical Framework 1 - Visual Imagery

Category of visual imagery	**Code**	**Definition** (See elaboration of these definitions in Section 2.3.1)	**Observable Indicators** A learner:
Concrete pictorial imagery	**CPI**	Concrete images of an actual situation formulated in a person's mind; picture in the mind drawn on paper or described verbally	**CPI1**: formulates a picture in the mind (PIM) (while reading/rereading a word problem); draws/sketches to represent a mental image or a concrete situation **CPI2**: concentrates silently (after a question is posed) – the thinking process involves imagination and mind pictures. **CPI3**: clarifies the structure of the problem; gives explanations/suggestions based on imagination and the formulated PIM
Pattern imagery	**PI**	This refers to the type of imagery in which concrete details are disregarded and pure relationships are depicted in a visual-spatial scheme. PI's essential feature is that it is pattern-like and stripped of concrete detail. PI embodies the essence of structure without detail.	**PI1**: formulates/uses patterns with the purpose of depicting/communicating information. For example, patterns of the theorem of Pythagoras i.e. $c^2 = a^2 + b^2$) **PI2**: engages patterns of data and arguments. **PI3**: uses visualisation to discover generalisations and to derive nonobvious concepts/formulae from such generalisations.
Memory imagery	**MI**	This refers to the ability to visualise an image of a formula that one has seen somewhere before or have previously learned.	**MI1**: formulates a mental image of a book/ board and depicts how a formula/concept was written (visualises something previously learned) **MI2**: sees a specific formula/method in mind that is needed to solve the problem (he/she may give description of the problem-solving strategy) **MI3**: recalls from memory; uses previous knowledge; an act of remembrance
Kinaesthetic imagery	**KI**	This is imagery that involves muscular activity. A kinaesthetic visualiser wants to feel and touch.	**KI1**: patterns of movement and body engagement as part of problem solving. **KI2**: walks/traces a path with fingers/hand/pencil to illustrate an image of something. **KI3**: mimics/imitates/traces shapes without placing the pencil on paper.
Dynamic imagery	**DI**	This category involves the processes of transforming shapes i.e. redrawing given or initially own drawn figures with the aim of solving the problem.	**DI1**: redraws given or own drawn diagrams with a purpose of extracting simple figures from complex figures, or to divide figures with lines to form other figures **DI2**: visualises a series of several images connected in one smooth motion, in mind and/or paper. **DI3**: transforms/changes the orientation of picture/shapes/concrete objects.

3.4.1.1 Selecting Phase 2 participants

The second purpose of data analysis Phase 1 was to identify appropriate research participants for Phase 2 of the case study – i.e. the eight participants who showed a preference for using visual imagery during EVGRT W1 task-based interviews. This preference was measured by the percentage of visual imagery in the participants' transcripts. To ensure that a fair and credible decision prevailed, I employed the NVivo 11 Pro Software (NVivo) to code and perform a word count as well as to work out the percentage of visual coverage for each research participant, as shown in Figure 3.16. There were 11 participants who scored a coverage of more than 30%. Of this cohort, eight participants agreed to take part in Phase 2 of the case study – Millie, Denz, Ethray, Ellena, Jordan, Meagan, Nate and Rauna.

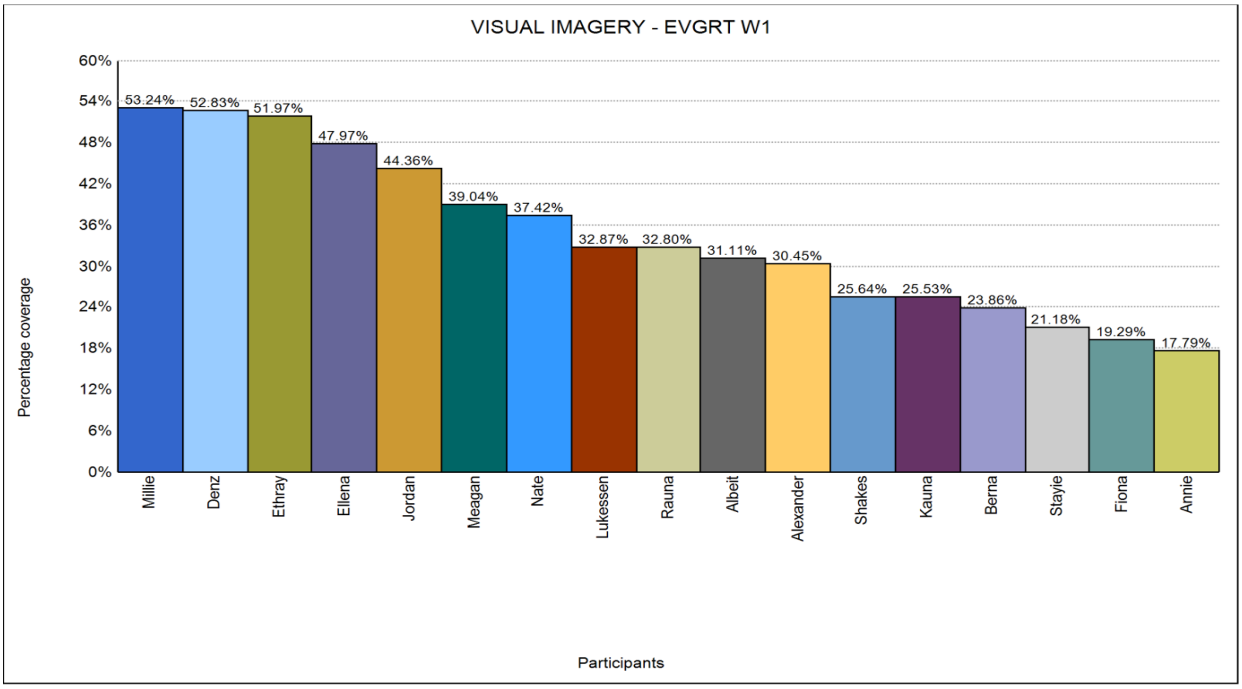


Figure 3.16 Overall percentage of the use of the 5VIs in the EVGRT W1 for all participants.

3.4.1.2 Colleagues' involvement in grouping the eight participants

As data collection Phase 2 required that participants work in small groups, three groups were formed from the eight participants. These comprised of three girls; two boys, one girl; and two boys. The groups consisted of a varied combination of participants in terms of their individual usage of visual imagery in EVGRT W1. Each of the top three users of visual imagery, Millie, Denz and Ethray (Figure 3.17) was grouped with one or two other participants who scored less than 50% coverage. I thought that the top three participants might be able to lead the others towards visualisation and direct them in the use of it, should the group hesitate or come to a halt in the process of solving the given word problems in EVGRT W2.

In addition to these criteria, I also received input from my teacher colleagues on how to best group these participants, as I did not know them as well as some of my colleagues did. I had only taught some of the participants for less than a year prior the data collection process. Therefore, I approached six colleagues with a little worksheet (See Appendix 10), requesting them to help me to place the eight participants in three groups. The ideal scenario for me at the time would have been to group/pair the participants according to their use of the 5VIs in EVGRT W1. However, the teacher colleagues thought that it would be a better idea to consider their likeness in terms of 'freedom of expression', 'temperament' and 'similar types of visual preferences', among others, when deciding upon the grouping. I weighed up each contribution and made the final grouping according to both my colleagues' reasoning and my own observations during EVGRT W1. This exercise helped to validate my data collection process. The final groups consisted of three girls: Millie, Meagan and Rauna; two boys: Denz and Jordan; and two boys and one girl: Ethray, Nate and Ellena.

Figure 3.17 Phase 2 selected participants' use of the 5VIs in EVGRT W1

3.4.2 Data analysis phase 2

The purpose of this phase was to analyse the co-emergence of visualisation and reasoning processes that were observed when the participants solved the EVGRT W2 in small groups. Similarly to Phase 1 data analysis, provisional coding (Saldaña, 2009) was used for the reasoning processes analytical tool. The initial coding and observable indicators for the analytical framework were adapted from studies on visualisation, reasoning, enactivism and collaborative argumentation. These *inter alia* include Webb (1991), Yackel (2001), Dove (2009), Brodie (2010), Staples et al. (2012), Conner et al. (2014) and Otte et al. (2015). The analytical tool was progressively modified as the observable indicators were refined through intensive literature study and as a result of the pilot data. At least four drafts were developed before the analytical tool was finalised (see Table 3.2). In his coding manual for qualitative researchers, Saldaña (2009, p. 122) emphasises that a small but vital investment of time and energy will go toward the development of provisional codes. Preparatory pilot study through participant observation and interviews at the actual fieldwork site may yield a more relevant and accurate set of provisional codes than previously published research (*ibid.*).

Table 3.2 Analytical Framework 2 – Reasoning Processes template

Reasoning Processes (RP)	**Code**	**Definition** (See elaboration of these definitions in Section 2.2.1)	**Observable Indicators** A learner:
Explanation	**RPE**	Mathematical explanation refers to the classification aspects of one's mathematical thinking that one thinks might not be readily apparent to others.	**RPE1**: makes sense of the problem and establishes a claim e.g. explains what the problem entails in simple terms and suggests known concepts/procedures. **RPE2**: explicates his/her own thinking processes (to produce meaning – includes reasoning without words) **RPE3**: suggests and defines problem solving strategies
Justification	**RPJ**	Mathematical justification refers to an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning.	**RPJ1**: provides proofs to validate claims and arguments **RPJ2**: provides acceptable reason for action (asks for clarification from others) **RPJ3**: promotes understanding among those engaged in justification e.g. does something to answer another person's concerns and lessen their worries.
Argumentation	**RPA**	An argument is a verbal, social and rational activity aimed at convincing a reasonable critic of the acceptability of a conclusion by foregrounding a constellation of propositions justifying or refuting the proposition expressed in the conclusion.	**RPA1**: provides support for explanations and justifications (this includes insisting on accuracy of their own and others' claims) **RPA2**: convinces/persuades others via verbal/visual activity of the truth of their claims and appropriateness of their reasoning (or is convinced and persuaded by others – i.e. when they accept the truth of each other's claims and explanations) **RPA3**: accepts/refutes truth of others' claims that they may agree/disagree with
Generalisation	**RPG**	To generalise a problem situation is to identify the operators and the sequence of operations that are common among specific cases and to extend them to the general case.	**RPG1**: elaborates the problem further to try to learn more from the result by relating the problem to similar situations. **RPG2**: uses visualisation to demonstrate how the problem can be solved in a different way

Adapted from the work of Webb (1991); Yackel (2001); Dove (2009); Brodie (2010); Staples, Bartlo and Thanheiser (2012); Conner, Singletary, Smith, Wagner and Francisco (2014), and Otte, Mendonça and de Barros (2015)

Having already analysed the data using Presmeg's (1986b) categories of visual imagery, I then interweaved the four processes of mathematical reasoning with the five categories of visual imagery. This enabled me to complete the analysis by discussing the co-emergence of visualisation and reasoning processes that had been observed when learners solved the geometry word problems in small groups (EVGRT W2). The enactivist themes of structural coupling and co-emergence thus made it possible to code the participants' responses in a combined (and coupled) matrix, as represented in Table 3.3 below. A completed matrix coding is included in the second phase of the data analysis discussion in Chapter 4.

Table 3.3 Data Analysis Matrix for the co-emergence of visualisation and reasoning processes

		Reasoning Processes			
	Co-emergence	**Explanation** Clarifying aspects of mathematical thinking	**Justification** Validation of claims to provide insight into the phenomenon	**Argumentation** Acceptability or refutability of a conclusion	**Generalisation** Extension of identified common operators to the general case
Categories of visual imagery	**Concrete Pictorial Imagery** Picture in the mind drawn on paper or described verbally				
	Pattern Imagery Concrete details are disregarded, and pure relationships are depicted in a visual-spatial scheme.				
	Memory Imager The ability to visualise an image of a formula that one has seen somewhere before or has previouslylearned.				
	Kinaesthetic Imagery Involves muscular activity; patterns of movement and body engagement				
	Dynamic Imagery Involves the processes of transforming shapes i.e. redrawing given or initiallyown drawn figures				

3.4.3 Data analysis Phase 3

In this phase of data analysis, the research participants' reflective responses to semi-structured interview questions were analysed to further inform the data collected and analysed in the first two phases.

Table 3.4 summarises all the data collection and analysis processes in the case study.

Table 3.4 Summary of data collection and analysis

Method	Purpose	Data	Analysis
EVGRT W1: One-on-one semi-structured task-based interviews	➢ To observe and analyse the extent to which all selected participants preferred visual methods to solve EVGRT ➢ To select the cohort for EVGRT W2	17 worksheets 17 transcripts Field notes	Presmeg's categories of visual imagery
EVGRT W2: Focus group semi-structured task-based interviews	➢ To analyse the co-emergence of visualisation and reasoning process when participants worked in small collaborative groups	3 worksheets 3 transcripts	Reasoning Processes template
Semi-structured reflective interview with the whole cohort	➢ To get overall feedback about the participants' experiences with EVGRT Worksheets	1 transcript	Summarise the analysis process

3.5. VALIDITY

To ensure validity, I triangulated my data by means of different data collection methods: video recordings, voice recordings, worksheets, interviews, a researcher's journal (memos) and reflective interviews. Rule and John (2011) define triangulation as "the process of using multiple sources and methods to support propositions or findings generated in a case study" (p. 109). Triangulation is viewed as a vehicle for achieving high quality, rigorous and respectable research, as a multiplicity or diversity of sources, methods and other aspects of a study serve to strengthen the validity of the assertion or finding by eliminating the inaccuracy or bias possibly introduced by reliance on a single source or method (Rule & John, 2011, pp. 108–109).

3.5.1 Piloting

To ensure validity, reliability and rigour, data collection instruments and analytical frameworks were piloted and refined over a number of cycles. The first cycle of piloting came immediately after the first EVGRT worksheet was developed.

Piloting Cycle One – Validating EVGRT W1 and Analytical Tool 1

The purpose of this cycle was to measure the reliability of the items developed for EVGRT W1. The items were piloted for language, mathematical accuracy and their ability to elicit visualisation when being solved by learners. After the items were developed, I piloted them with two Grade 11 learners at a state school that was not part of the actual data collection. After this cycle of piloting, I realised that six out of the initial 11 tasks were not appropriate for the worksheet. The tasks either required too much from an individual learner or were too complex mathematically for the purposes of the study. These items were either adjusted or removed from the worksheets. Following the results of cycle one analysis, the second draft of EVGRT W1 was developed and piloted with a learner at one of the private schools where the research was conducted. Data from audio and video recording was transcribed and analysed to validate the tasks for final drafting for the study. The pilot analysis also revealed minor language issues that were refined for the final draft of EVGRT W1.

Piloting Cycle Two – Validating EVGRT W2 and Analytical Tool 2

This cycle purposed to validate the tasks planned for data collection phase two. The piloting processes of rectifying and refining EVGRT W1 enabled me to refine the items developed for EVGRT W2 even before I piloted them. I initially set 10 items for this worksheet, but after piloting them with three learners from a state school I realised that there were too many of them: it took about three hours for the learners to complete the worksheet. I also initially adapted most of the tasks from various mathematics books, but after piloting them with the learners, I decided to replace some with SAMO questions. This worked very well as the SAMO tasks were thoroughly reviewed for their initial purposes. They were also free of language and numeracy errors. After the EVGRT W2 was refined, I piloted it again with three Grade 12 learners to ensure that the tasks were indeed doable and reliable. I also wanted to ensure that each task enabled the participants to both visualise and reason as they worked together in small collaborative groups.

The items on the two EVGRT Worksheets and the analytical tools were further piloted for validity and reliability at the Namibian National Mathematics Congress in Swakopmund in 2016, the Southern African Association of Mathematics, Science and Technology Education (SAARMSTE) conference in Pretoria in 2016 and in Bloemfontein in 2017, and the International Congress of Mathematics Education in Hamburg in 2016. This was done in form of oral and poster presentations. During these sessions, the conference delegates were asked for input on the validity and reliability of the test items and the analytical tools. The feedback provided during the questions and discussions slots helped to inform my research design as I liaised with more knowledgeable others and benefitted from their advice and direction.

3.5.2 Member checking

Some of the participants indicated during EVGRT W1 that they were uncomfortable with having their faces appear in video recordings, while others did not have a problem with it. While these preferences could not serve as criteria when I grouped the participants for EVGRT W2, care was taken to ensure that there was as little as possible coverage of the sensitive participants' faces. This made it difficult for me to know who was speaking when I read through the group interview transcripts, as I could not always see their faces and some of their voices were similar. Cohen et al. (2011) warn that researchers using interviews have to be aware that ensuring anonymity may be difficult (p. 409). To ensure accuracy and increase validity I asked the group members to verify the content of the interview transcripts. This technique is referred to as member checking (Rule & John, 2011, p. 108). Cho and Trent (2006) note that member checking can occur throughout a research project, and "is a process in which collected data is 'played back' to the informant to check for perceived accuracy and reactions" (p. 322). The verification exercise provided space for participants to correct data which they felt was inaccurate (Rule & John, 2011, p. 108).

3.6. ETHICAL CONSIDERATIONS

The participants in this case study were informed about the ethical implications as per Rhodes University's ethical guidelines. I received signed consent from the school principal, the deputy principal and all the participants' parents/guardians to make audio and video recordings of the participants during data collection, and view these with my supervisor and two or three other researchers who would assist me with data analysis. I also received signed consent to use video frames of the participants in my data analysis and for presentation at educational conferences (see Appendix 7). The participants were informed about voluntary participation and were notified that they were free to withdraw at any time (see Appendix 4). They were also informed that their performance in the EVGRT worksheets would not be made public at school and that their performance would in no way compromise their position at school.

It was stated in the consent letters that the learners who agreed to participate in this research would benefit a great deal in terms of mathematical knowledge, as the research aimed to expose them to various problem-solving strategies, including mathematical reasoning through visualisation. The participants could apply problem-solving strategies learned during the task-based interviews to their school mathematics. For example, when they attempted Task 6 of EVGRT W1, the participants recalled how they had always used a formula to find the sum of the interior angles of polygons. However, after participating in this study, they discovered visual ways of solving the problem and even commented on how they would apply these visual strategies in activities for assessment. The other benefits of participating in this study included gaining experience of being involved in research (which could be beneficial for their future studies), and a stipend which was offered as a token of appreciation for their time and input.

In this case study, the participants' interests were prioritised and respected above those of the researcher. Bell and Walters (2014) posit that "people who agree to be interviewed deserve consideration and so you will need to fit in with their plans, however inconvenient that may be for you" (187). Cohen et al. (2011) concur that the welfare of the participants should be kept in mind even if it involves compromising the impact of the research. Furthermore, "researchers should never lose sight of the obligations they owe to those who are helping, and should constantly be alerted to alternative techniques should the ones they are employing at the time prove controversial" (p. 86).The practice of non-maleficence was an integral part of the research design of this case study. Cohen et al. (2011, pp. 85) insist that the research should not in any way damage the participants. They also claim that "it is a golden rule that the research must ensure that participants are no worse off at the end of the research than they were at the beginning of the research" (p. 85). In this case study, for example, the participants who were unhappy with their word problem solving and mathematical reasoning skills at the beginning of the research project were motivated to employ visualisation processes to assist them with solving the tasks. They were encouraged to remain positive throughout the research process even when they struggled to find an appropriate method or answer.

3.6.1 Positionality

The participants in this case study were Grade 11 learners from the school at which I teach. I explained to them prior to each task-based interview that I was not judging them as their teacher and that I was not interested in the accuracy or correctness of their responses to the tasks as much as their use of visualisation processes. I wanted them to see me as a researcher rather than a mathematics teacher. Rule and John (2011) advise that it is important for researchers to be constantly aware of how they are positioned in relation to the study context and participants, and how such positioning may influence the study and its overall quality. Being transparent about one's positionality and its possible effects contributes to the credibility and confirmability of the study (p. 113).

It was not always easy for the participating learners to completely ignore the fact that I was indeed a mathematics teacher at the school. They sometimes felt uncomfortable with certain tasks and would inevitably look to me as a teacher. I however constantly reminded them that I was simply a researcher with a deep interest in what they were doing with each task, and not whether or not they were getting it "right." Rule and John (2011, p. 113) warn that while researchers may foreground their identities and roles as researchers, participants in the study may respond to them as teachers or principals, as the case may be. They believe that the researcher's status and authority may influence the data to be generated.

3.7. CONCLUDING REMARKS

This chapter presented an in-depth discussion of the methodology implemented to execute this qualitative case study. In the chapter, I discussed the methods used to collect data for the case study as well as the frameworks for analysing the data in each of its three phases. I also discussed measures that were put in place to ensure the validity and reliability of the research findings, as well as ethical measures to ensure that the research participants' rights were protected. The results of the study are presented in Chapter 4.

CHAPTER 4

ANALYSIS AND DISCUSSION OF RESULTS

Qualitative analysis is about reducing data
without losing its meaning ~ Philip Adu (2016)

4.1 INTRODUCTION

This chapter analyses data collected for this case study in three distinct phases. In Phase 1, responses from individual task-based interviews were analysed using an adapted version of Presmeg's categories of visual imagery, as presented and discussed in Section 3.4.1. The purpose of this phase was to investigate the extent to which research participants used visual imagery to solve geometry word problems.

The results of Phase 2 are presented, analysed and discussed in Section 4.3 of this chapter. This phase focused on the participants' reactions and responses to EVGRT W2, which was intended to highlight the relationship between visualisation processes and reasoning processes when participants solved geometry word problems in small groups. Data from this phase was analysed using analytical tools presented and discussed in Section 3.4.2.

Section 4.4 of this chapter discusses the research participants' reflections on the entire research process. The participants' views and experiences were analysed from the point of view of visualisation as a geometry problem-solving tool and a route to mathematical reasoning.

4.2 DATA ANALYSIS PHASE 1

The purpose of analysing the data from this phase was twofold: first, to answer the sub-research question 1 as outlined in Chapter One of this case study (Section 1.2): "What visualisation processes are evident in all the selected Grade 11 participants when they solve geometry word problems?"; and secondly, to identify those participants whose method of word problem solving was dominated by the use of visual methods, as opposed to those who preferred nonvisual methods of solving word problems (see Section 2.3.1 for a discussion on visual imagery and Table 3.1 for observable indicators).

Commentary on the data analysis tool

Data was collected from individual task-based interviews in respect of EVGRT W1 using video and audio recordings as well as the participants' written work. The recordings were transcribed, and the participants' responses were analysed according to the criteria of the visual imagery analytical tool (see Table 3.1). There were three observable indicators for each of the five categories of visual imagery that resulted from a revised, modified, deleted/expanded "start list" of researcher-generated provisional codes (Saldaña, 2009). Each of the 17 transcripts was coded according to these observable indicators, with the help of NVivo software. In the case of the NVivo software, there are terms whose meaning may not be apparent and familiar to readers. These terms are defined as per the NVivo beginner's guide (NVivo 11 Pro for Windows: Getting started guide, 2017, pp. 24-26).

Nodes

Nodes represent themes, topics, concepts, ideas, opinions or experiences. For example, I created the five nodes of VIs as I explored my sources (documents, journal articles, datasets, audio, video and pictures), and under each VI, I created 'child nodes' (observable indicators – see Table 3.1). I coded all references to a node, gathering, say, all CPI1 at a particular node.

Child nodes

A child node is a type of a sub-node. Using Table 3.1 to illustrate the meaning of a child node, we could say that the five categories of visual imagery (CPI, PI, MI, KI & DI) are each a node. Their observable indicators (e.g. CPI1, CPI2, PI2, MI3, etc.) are child nodes.

Cases

Cases represent units of observation – a case might be a person, place, site, organisation or any other entity. Cases are a special type of node because you can classify them and then assign attributes (variables) to them, such as age, gender or location.

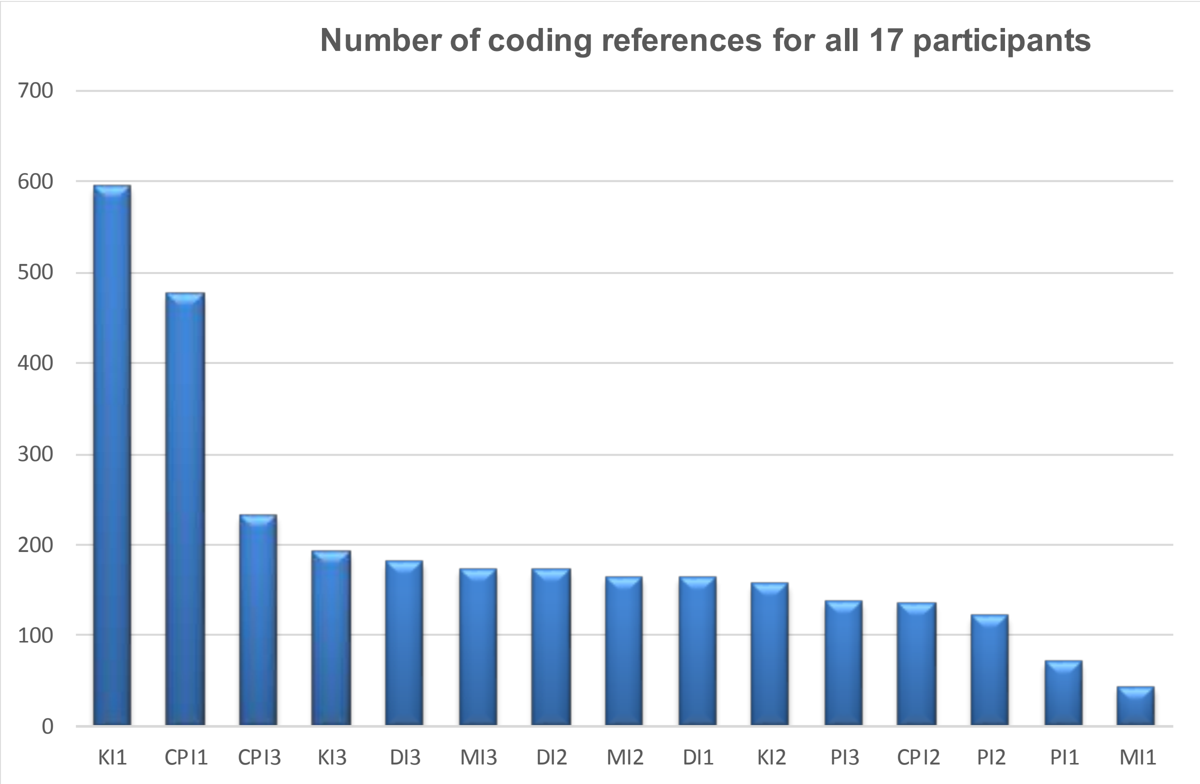
Coding references

The reference tab in NVivo shows all the text content coded at the node. Hence, coding references register the number of times that the text content has been coded at the node.

NVivo queries

NVivo queries can be used to find and analyse the words or phrases in sources, theme nodes, cases and relationships. This enables one to find specific words or words that occur most frequently. Queries also enables one to ask questions and find patterns based on the coding, check for coding consistency among team members, and review the process.

After the coding process had been completed, the data was visualised using charts and tables for easy access and interpretation.



Observable indicators	Number of coding references	percentage of coding references	Number of participants coded
CPI1	477	15,83	17
CPI2	136	4,55	17
CPI3	232	7,70	17
PI1	71	2,36	17
PI2	121	4,01	17
PI3	137	4,55	17
MI1	43	1,43	15
MI2	164	5,44	17
MI3	173	5,74	17
KI1	595	19,74	17
KI2	156	5,18	17
KI3	192	6,37	17
DI1	164	5,44	17
DI2	172	5,71	17
DI3	181	6,01	17
Total	**3014**	**100**	

Figure 4.1 Global overview of the observable indicators' coding references for all 17 participants – a general response to sub-research question 1

It is shown in Figure 4.1 that all five categories of visual imagery (5VIs) were evident in the research participants' solutions to the word problems. There was a total of 3014 coding references across the transcripts of the 17 research participants. The percentage of coding references provides a rich representation of how each indicator was coded as a percentage of the total number of coding references. Kinaesthetic Imagery (KI) stood out as the most used category of visual imagery, with 31.29% of the coding references. Concrete-Pictorial Imagery (CPI) was the second most used VI by the participants, at 28.08%; Dynamic Imagery (DI) comes in third position with 17.16%, while Memory Imagery (MI) and Pattern Imagery (PI) were the least coded observable indicators with 12.61% and 10.92% of the coding references, respectively. Figure 4.1 also shows that KI1 was the most coded observable indicator while MI1 was the least coded indicator. This means that the research participants were 19.74% more active in terms of movement and body engagement (KI1) than they were in terms of recalling from memory (MI1). They only managed to formulate 1.43% mental images of something they had previously seen/learned. In fact, 2 out of the 17 research participants' responses to task-based interviews were not coded for MI1 observable indicators, as indicated in Figure 4.1.

I now discuss each indicator as per the 5Vis as presented in Table 3.1.

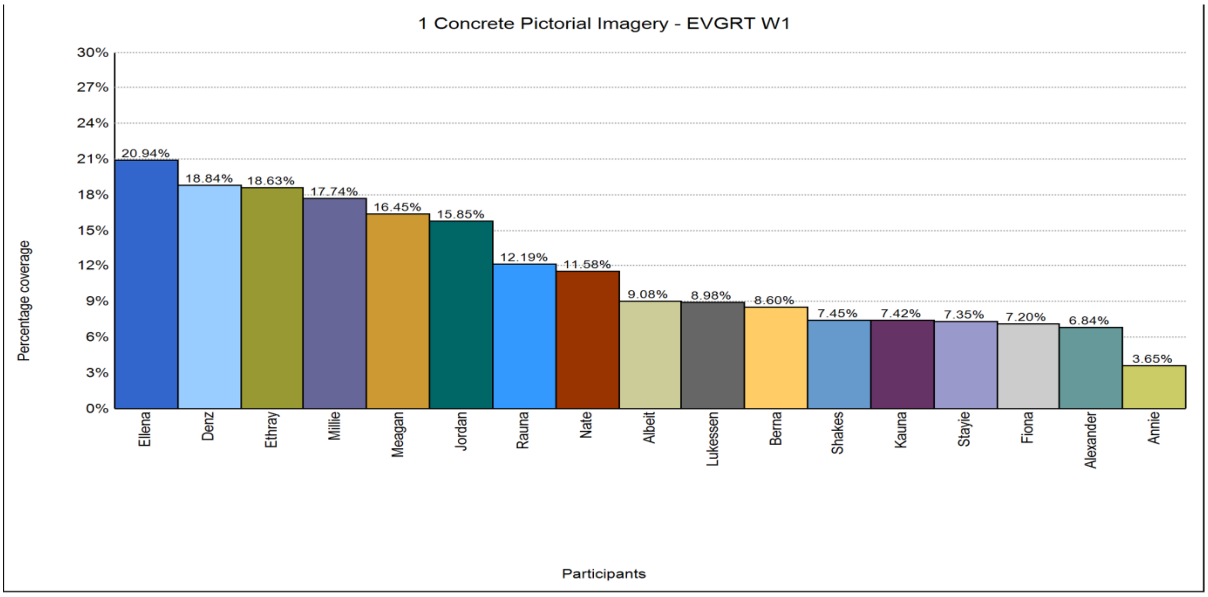
4.2.1 Concrete Pictorial Imagery

Concrete pictorial imagery (CPI) includes concrete images of an actual situation formulated in a person's mind; a picture in the mind drawn on paper or described verbally (see Section 2.3.1.1). The responses to the ten tasks of EVGRT W1 from 17 participants were analysed for CPI by making use of its observable indicators. The following extract from Table 3.1 reminds the reader of CPI's observable indicators.

CPI1: formulates a picture in the mind (PIM) (while reading/rereading a word problem); draws/sketches to represent a mental image or a concrete situation
CPI2: concentrates silently (after a question is posed) – the thinking process involves imagination and mind pictures.
CPI3: clarifies the structure of the problem; gives explanations/suggestions based on imagination and the formulated PIM.

Figure 4.2 shows the coding references for the participants' responses that were coded for CPI. The chart shows the percentage coverage of each participant's transcript content that

was coded for CPI; for example, in Ellena's transcript, 20.94% of the content was coded for CPI while Annie's transcript contained only 3.65% of her total CPI coding references, which means a discrepancy of no less than 17.29% between the most CPI-coded participant and he least CPI-coded participant. The table below the chart in
Figure 4.2 shows the number of coding references that were assigned to each of the observable indicators of CPI. It also shows the percentage of such coding and the number of transcripts coded for the observable indicators. A total of 845 references were coded for the CPI category of visual imagery.



Observable indicators	Number of coding references	Percentage of coding references	Number of participants coded
CPI1	477	56,45	17
CPI2	136	16,09	17
CPI3	232	27,46	17
Total	**845**	**100**	

Figure 4.2 Overall indication of the use of CPI in EVGRT W1

CPI1 was often used at the beginning of each task, as the participants sketched diagrams to represent their mental images.
Figure 4.2 reveals that all the research participants formulated pictures in their minds while reading the word problems, as indicated by 56.45% of all their CPI coding references combined. More than 50% of the participants' transcripts coded more than 10% of total CPI coding references. To learn about the types of CPI applied to the tasks, I asked the participants to explain what they had in their minds. This was intended to assist them to clarify the structure of the problem and to ensure that the language used in the tasks did not limit them from visualising and expressing themselves fully. Thorton (2001, p. 254) argues

that the role of CPI in word problem solving is to motivate the learners, help them clarify the structure of the problem and assess the reasonableness of their results.

CPI2 was the least coded observable indicator in the CPI category with 136 coded references and 16.09% of total CPI coding references. In terms of the criteria of the observable indicators as stipulated in Table 3.1, the participants' less frequent use of CPI2 is an indication that they concentrated less when probing questions were posed in comparison to how they sketched (CPI1).

Figure 4.2 also illustrates that the participant clarified the structures of the word problems as well as the way they explained their imaginations (CPI3) better than they have concentrated on the task after probing questions were posed (CPI2). Hence, CPI1 and CPI3 both have more percentage coding references in comparison to CPI2.

4.2.2 Pattern Imagery

Pattern imagery (PI) is imagery in which concrete details are disregarded and pure relationships are depicted in a visual-spatial scheme (see Section 2.3.1.2). The following extract from Table 3.1 serves as a reminder of the observable indicators used for PI analysis.

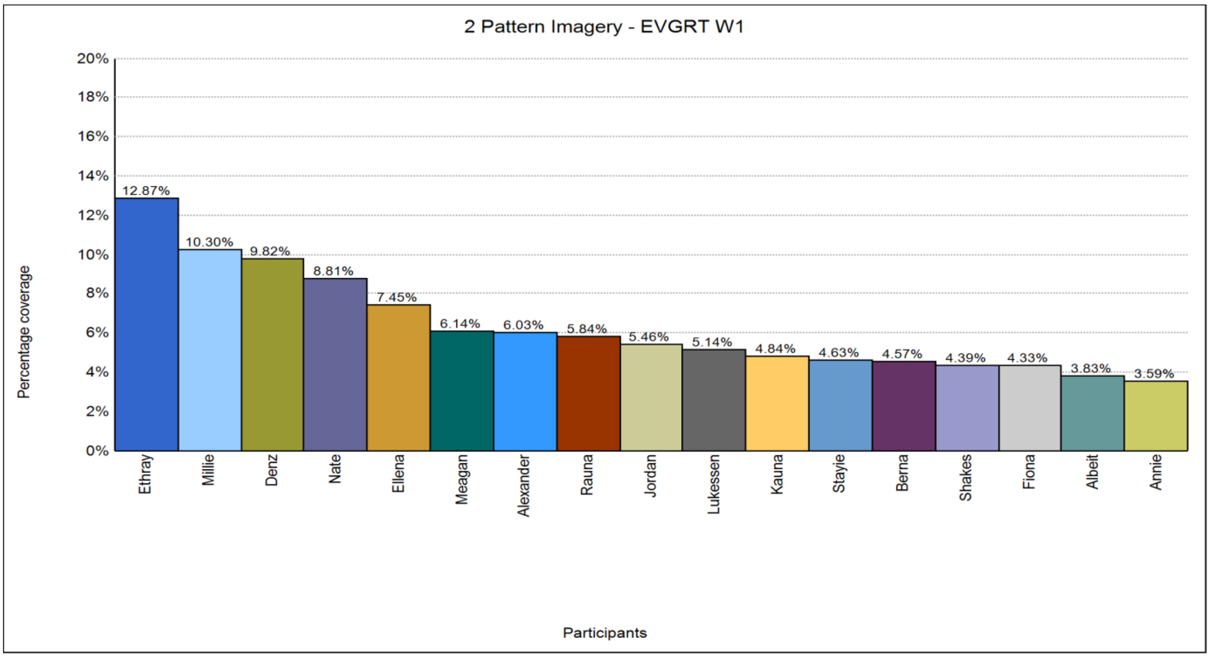
PI1: formulates/uses patterns with the purpose of depicting/communicating information. For example, patterns of the theorem of Pythagoras i.e. $c^2 = a^2 + b^2$)

PI2: engages patterns of data and arguments.

PI3: uses visualisation to discover generalisations and to derive nonobvious concepts/formulae from such generalisations.

Across all the 5VIs, PI recorded the least number of total coding references, 329 out of 3014 and a percentage coverage of 10.92%, as indicated in Figure 4.1.

Figure 4.3 shows how PI was employed by the research participants as a visual method during the EVGRT W1.



Observable indicators	Number of coding references	Percentage of coding references	Number of participants coded
PI1	71	21,58	17
PI2	121	36,78	17
PI3	137	41,64	17
Total	**329**	**100**	

Figure 4.3 Overall indication of the use of PI in EVGRT W1

Figure 4.3 shows that two out the 17 participants' transcripts managed to score more than 10% in terms of total PI category in EVGRT W1, with a range of 9.3% between the most coded and the least coded participant. PI1 recorded the least number of coding references, with 21.58% of all PI references. It is also the overall second least-coded indicator in the entire VI analytical framework, with a percentage coverage of 2.36%. PI2, which had to do with patterns of data and arguments in the participants' responses, had a total number of 121 coding references which equals 36.78% of PI's coding references while PI3 recorded the highest number references in the PI category, 137, amounting to 41.64%. This means that there was less usage of known patterns (PI1) during EVGRT W1 task completion than the participants' engagement in generating new patterns of data and argument (PI2). The participants often used visualisation to discover unknown or forgotten concepts as they generalised the solutions of the word problems (PI3). Looking at the chart in Figure 4.3, one can see that although there is a considerable range between the most coded and the least coded participant in the PI category, the participants' manner of applying pattern imagery to word problems was generally similar. As one moves from Meagan down to Annie,

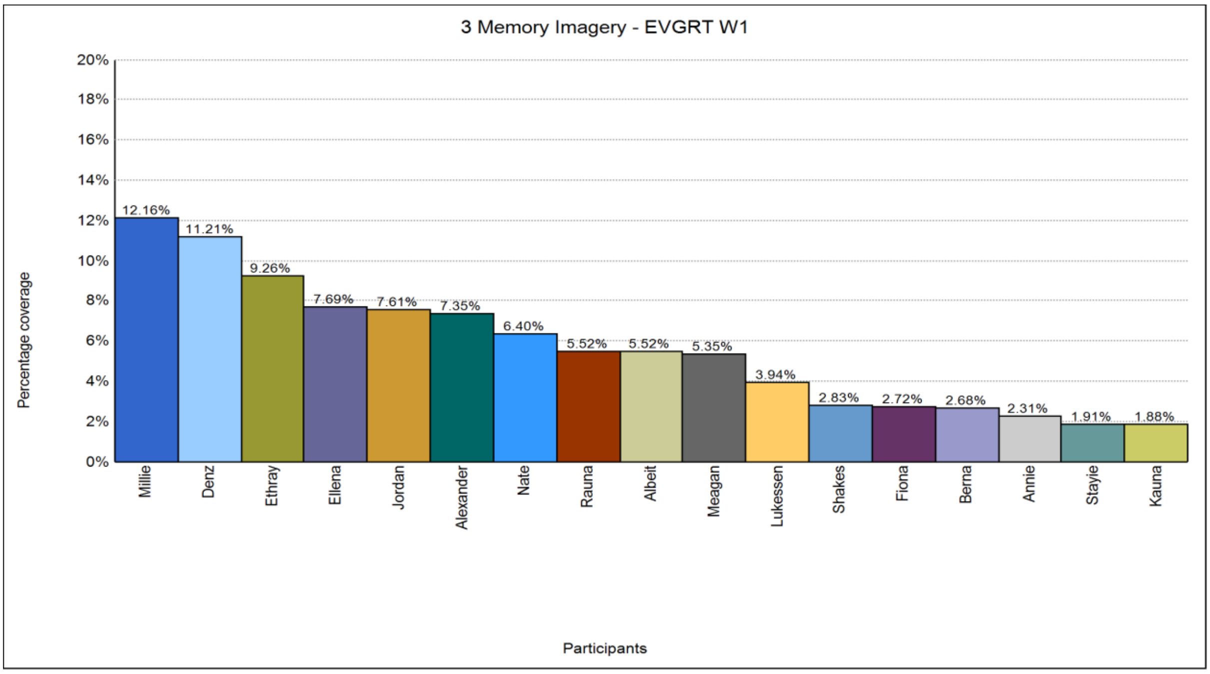
Figure 4.3 illustrates a noticeable gradated pattern in terms of how the research participants' use PI in EVGRT W1.

4.2.3 Memory Imagery

Memory imagery (MI) refers to the ability to visualise an image of a formula that one has seen somewhere before or has previously learned (see Section 2.3.1.3). The following extract from Table 3.1 serves as a reminder of the observable indicators used for MI analysis.

MI1: formulates a mental image of a book/ board and depicts how a formula/concept was written (visualises something previously learned)
MI2: sees a specific formula/method in mind that is needed to solve the problem (he/she may give description of the problem-solving strategy)
MI3: recalls from memory; uses previous knowledge; an act of remembrance

Figure 4.4 shows that a total number of 380 coding references were assigned to the MI category. This represents only 12.61% (Figure 4.1) of the total number of coded references within the VI analytical framework across all research participants, making MI the second least coded category.



Observable indicators	Number of coding references	Percentage of coding references	Number of participants coded
MI1	43	11,32	15
MI2	164	43,16	17
MI3	173	45,53	17
Total	**380**	**100**	

Figure 4.4 Overall indication of the use of MI in EVGRT W1

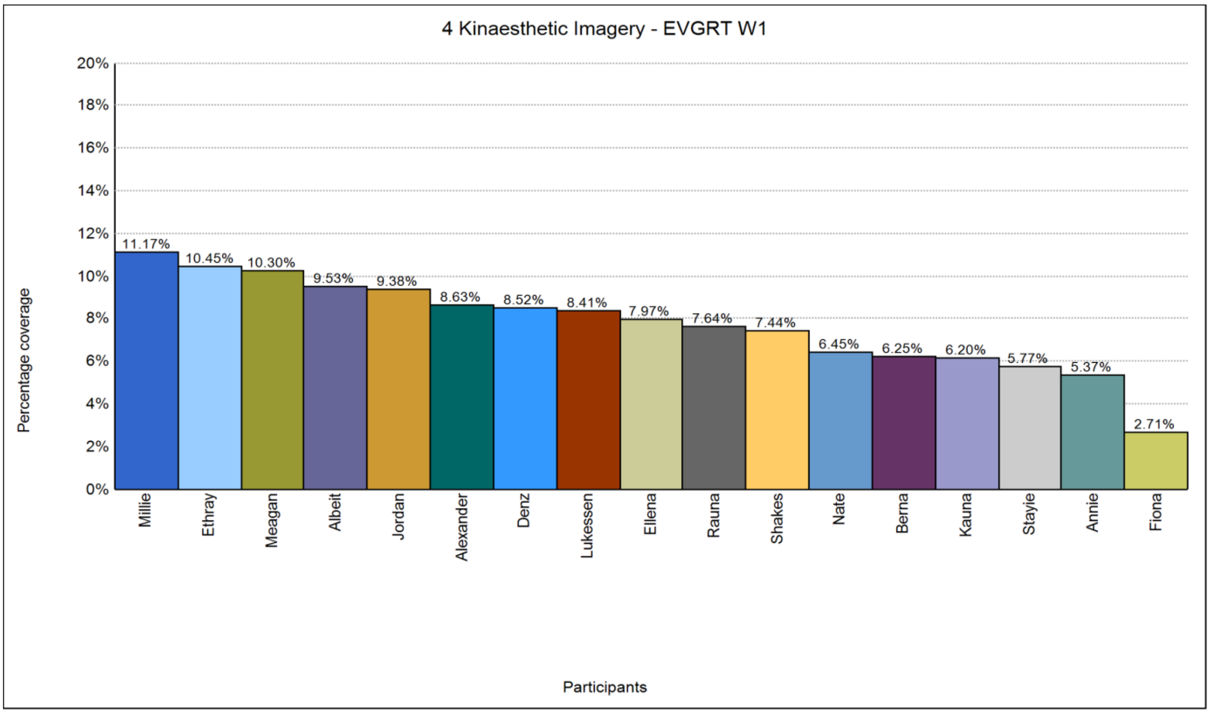
Two participants managed a score of more than 10% of coding references in the MI category and seven participants scored less than 5%. This resulted in a vast range of 10.28% between Millie, the most coded participant, and Kauna, the least coded participant. MI1 was the least coded observable indicator in the entire framework, with only 43 out of a total of 3014 coding references. This represents 11.32% of the MI coding references and a mere 1.43% of the entire VI analytical framework. Figure 4.4. reveals that two out of the 17 participants did not use MI1 during EVGRT W1 task-based interviews. This, according to the MI criteria, means that those two participants did not formulate mental images of their books or the board to see how a formula or a concept was written (see Section 2.3.1.3). However, all participants employed MI2 and MI3 in their problem-solving strategies, as illustrated in Figure 4.4. The MI2 and MI3 observable indicators were closely related in terms of their coded references, with MI2 covering 43.16% and MI3 45.53% of the total references coded for the MI category.

4.2.4 Kinaesthetic Imagery

Kinaesthetic imagery (KI) is the type of imagery that involves muscular activity (see Section 2.3.1.4). Below is an extract from Table 3.1 that serves as a reminder of KI's observable indicators.

KI1: patterns of movement and body engagement as part of problem solving.
KI2: walks/traces a path with fingers/hand/pencil to illustrate an image of something.
KI3: mimics/imitates/traces shapes without placing the pencil on paper.

KI is by far the most coded category with a total number of 943, references as shown in Figure 4.5.



Observable indicators	Number of coding references	Percentage of coding references	Number of participants coded
KI1	595	63,10	17
KI2	156	16,54	17
KI3	192	20,36	17
Total	943	100	

Figure 4.5 Overall indication of the use of KI in EVGRT W1

It is clear from Figure 4.5 that KI was employed to a similar degree by all participants except for Fiona, whose percentage coverage of 2.71% falls out of the common KI range of 5% to 11%. Three participants scored more than 10% in terms of percentage coverage of coding references. Eight participants are in the middle range of 7% to 9%, five participants in the lower range of 5% to 6%, followed by Fiona with the lowest percentage coverage at the far right of the chart. The range between Millie, the most coded participant, and Fiona, the least coded participant, is 8.46%.

KI1 recorded the most number of references of all the observable indicators across the VI analytical tool with 595 out 943 KI coding references. This represents 63.10% coverage in the participants' transcripts of the KI category (Figure 4.5) and 19.74% coverage of the VI analytical tool (Figure 4.1). This indicates that most references were coded for patterns of movement and body engagement within the participants' problem-solving strategies during EVGRT W1 task-based interviews. KI2 was the least coded observable indicator with 16.54% coded references for the category and 5.18% of coded references across the VI framework. KI3, which is defined by mimicking, imitating or tracing shapes without placing the pencil on paper, resulted in 192 coded references, which represents 20.36% of the participants' transcripts' content.

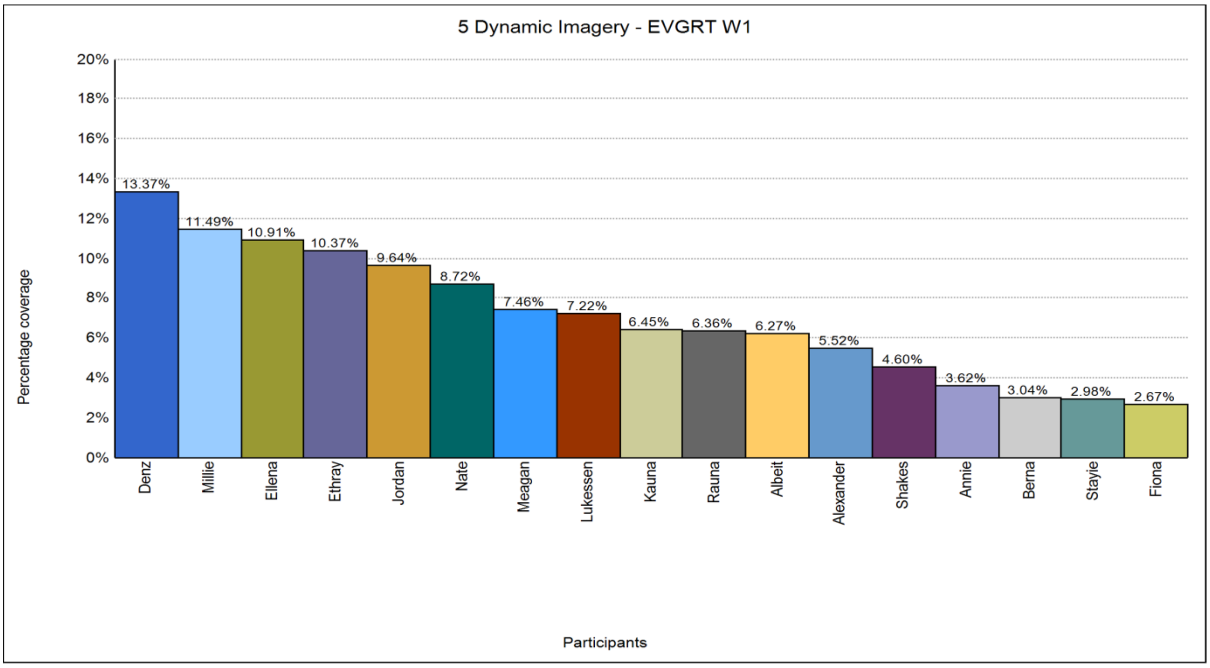
4.2.5 Dynamic Imagery

Dynamic imagery (DI) involves the processes of transforming shapes, i.e. redrawing given or one's own drawn figures with the aim of solving the problem (see Section 2.3.1.5). The following extract from Table 3.1 is a reminder of the observable indicators that were used to analyse DI in the participants' responses.

DI1: redraws given or own drawn diagrams with a purpose of extracting simple figures from complex figures, or to divide figures with lines to form other figures
DI2: visualises a series of several images connected in one smooth motion, in mind and/or paper.
DI3: transforms/changes the orientation of picture/shapes/concrete objects.

Figure 4.6 illustrates how DI was used as a visualisation process to solve the 10 tasks of EVGRT W1 by the 17 research participants. A total number of 517 references within the overall VI analytical tool (Figure 4.1) were coded to the DI category, which represents 17.16% of all coded references.

Figure 4.6 shows that despite the disparity in individual participants' use of DI as a visualisation tool, there are barely distinguishable differences in number of coding references among the three observable indicators in the DI category. DI's 1, 2, and 3 have a range of 3.29% of coding references, the narrowest range among all 5 VIs. Four research participants' DI coded references exceeded 10% coverage of their transcripts' content. These are followed by seven research participants whose number of coded references ranged between 6% and 9% of their transcripts' content, and five research participants who recorded below 6%.



Observable indicators	Number of coding references	Percentage of coding references	Number of participants coded
DI1	164	31,72	17
DI2	172	33,27	17
DI3	181	35,01	17
Total	**517**	**100**	

Figure 4.6 Overall indication of the use of DI in EVGRT W1

4.2.6 Summary of Phase 1 analysis

The five categories of visual imagery were observed and analysed for all 17 participants. Kinaesthetic imagery (KI), which involved muscular activity in the form of gestures and bodily engagement, was the most used category of visual imagery during EVGRT W1. Although only 3 out of the 17 participants obtained more than 10% of KI coverage in their transcripts' content, only one participant recorded less than 5% of KI coverage. Concrete pictorial imagery, which involved the participants' formulation of a 'picture in the mind' while reading the word problems, was the second most coded of the VI categories. The participants invariably sketched as they read the word problem, and some gave descriptions of pictures formulated in their minds when prompted to do so. Eight out of the 17 participants obtained more than 10% coverage in their transcripts of content coded to CPI, with only one scoring less than 5% of CPI coverage in their transcript content. Participants in this case study made hand movements when explaining concepts, showing stated objects, and justifying their actions. In fact, every participant employed visualisation processes of some kind during each of the 10 tasks of EVGRT W1. This is an affirmation that visualisation is both a process and a product of embodied cognition, as asserted by Wilson (2009). She argues that there is a movement under way in cognitive science to grant the body "a central role in shaping the mind. Proponents of embodied cognition take as their theoretical starting point not a mind working on abstract problems, but a body that requires a mind to make it function" (cf. Wilson, 2009, p. 625).

This overview of Phase 1 of the data analysis raised a number of interesting visualisation aspects that were briefly commented on. What follows is Phase 2 of data analysis with the eight selected participants, as discussed in the methodology chapter (see Sections 3.4.1.1 and 3.4.1.2)

4.3 DATA ANALYSIS PHASE 2

The purpose of the second phase of data analysis was to answer research sub-question 2, and consequently the main research question as outlined in Chapter 1 of the study (Section 1.2).

Sub-research question 2:

1. How do visualisation and reasoning processes co-emerge when learners solve geometry word problems in small collaborative groups?

The main research question:

- How do visualisation processes relate to mathematical reasoning processes when selected Grade 11 learners solve geometry word problems?

Data in this phase was analysed and is presented in four sections. Section 4.3.1 presents a broad vertical analysis of the mathematical reasoning processes (4PRs) that the eight research participants displayed when they completed EVGRT W2 in small groups. Section 4.3.2 also presents a broad vertical analysis of data for the relations between visualisation processes and mathematical reasoning processes for each of the eight research participants. Section 4.3.3 presents a horizontal, fine-grained analysis of the interpretation of mathematical reasoning in enacted visualisation. Section 4.3.4 summarises the insights into structural coupling and co-emergence gleaned from Phase 2 data analysis as a whole.

4.3.1 Data analysis for mathematical reasoning processes

Learners' mathematical reasoning was observed and analysed using the reasoning processes analytical tool (as presented in the methodology chapter: see Table 3.2). The analysis in this section focuses on each participant's reasoning processes as an individual within a small group. The focus was not on group reasoning as such, and the participants were grouped to prevent them from being intimidated, which may have been the case with one-on-one interviews. Each participant's use of the four reasoning processes (4RPs) during EVGRT W2 is presented. Figure 4.7 provides a global overview of Phase 2 participants' mathematical reasoning as observed during EVGRT W2.

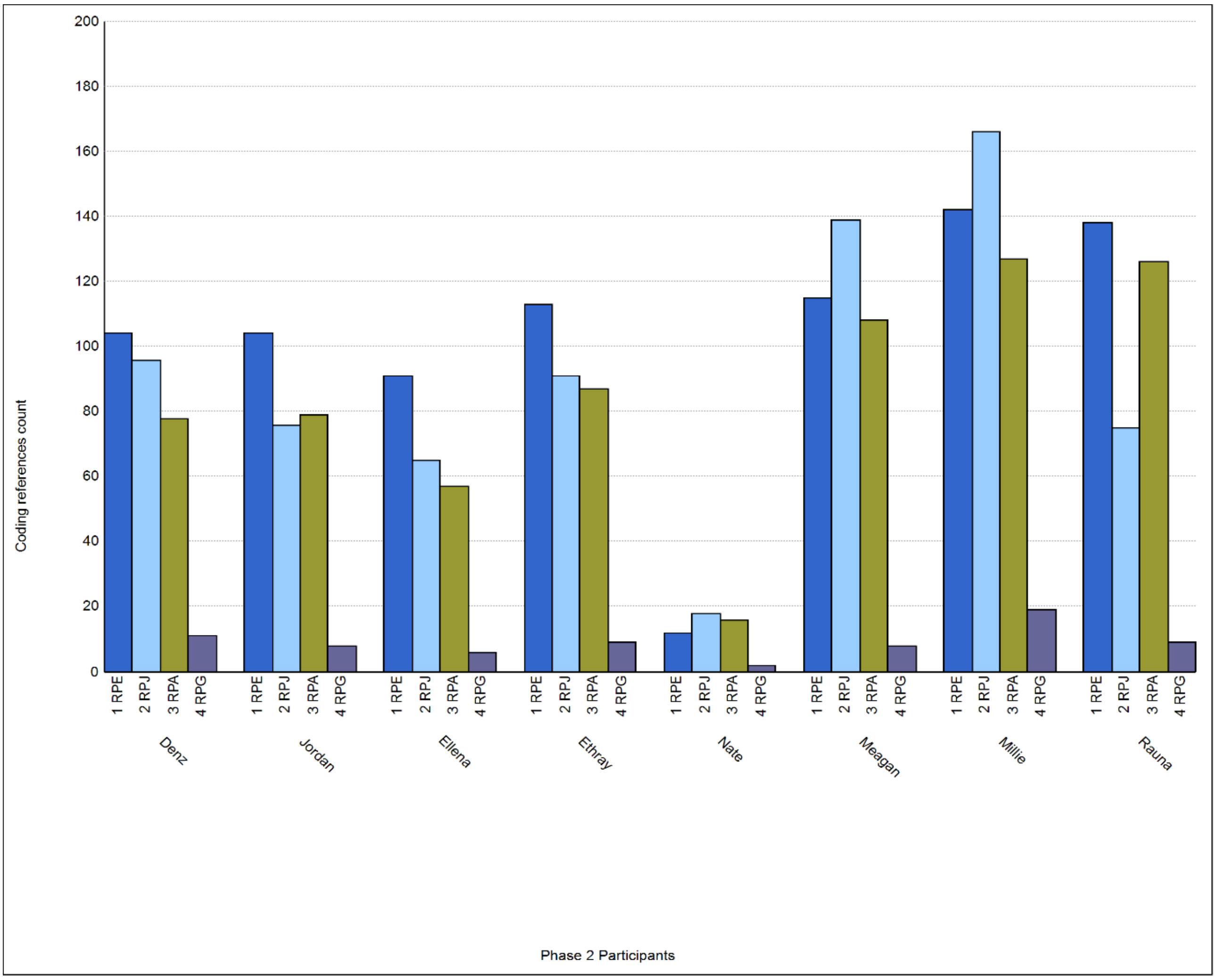


Figure 4.7 Global overview of the participants' reasoning processes

A cursory glance at Figure 4.7, above, might lead one to think that, on average, the three girls, Meagan, Millie and Rauna, reasoned more or better than the other participants. I would like to clarify this possible misapprehension prior to analysis of the mathematical reasoning processes of the individual research participants. All the participants solved the same tasks in their small focus groups and were all given sufficient time to work independently and/or with each other to solve the tasks. The first focus group interviewed consisted of the three girls, Meagan, Millie and Rauna. These girls took about two and a half hours to complete the worksheet and the transcription of their data amounted to 120 pages. The second focus group that was interviewed consisted of a girl and two boys, Ellena, Ethray and Nate. It took this group only about an hour and a half to complete the same worksheet that the first group completed. The third interviewed focus group consisted of the two boys, Denz and Jordan, who took less than an hour to complete the same worksheet. The variable length of time that the three groups took to complete the tasks resulted in a skewed graph and a possible misinterpretation of the data. As I was more interested in the quality of the co-emergence of reasoning and visualisation processes, this skewed graph should not be a concern.

Rauna

Rauna's mathematical reasoning processes and their observable indicators are presented in Figure 4.8, below.

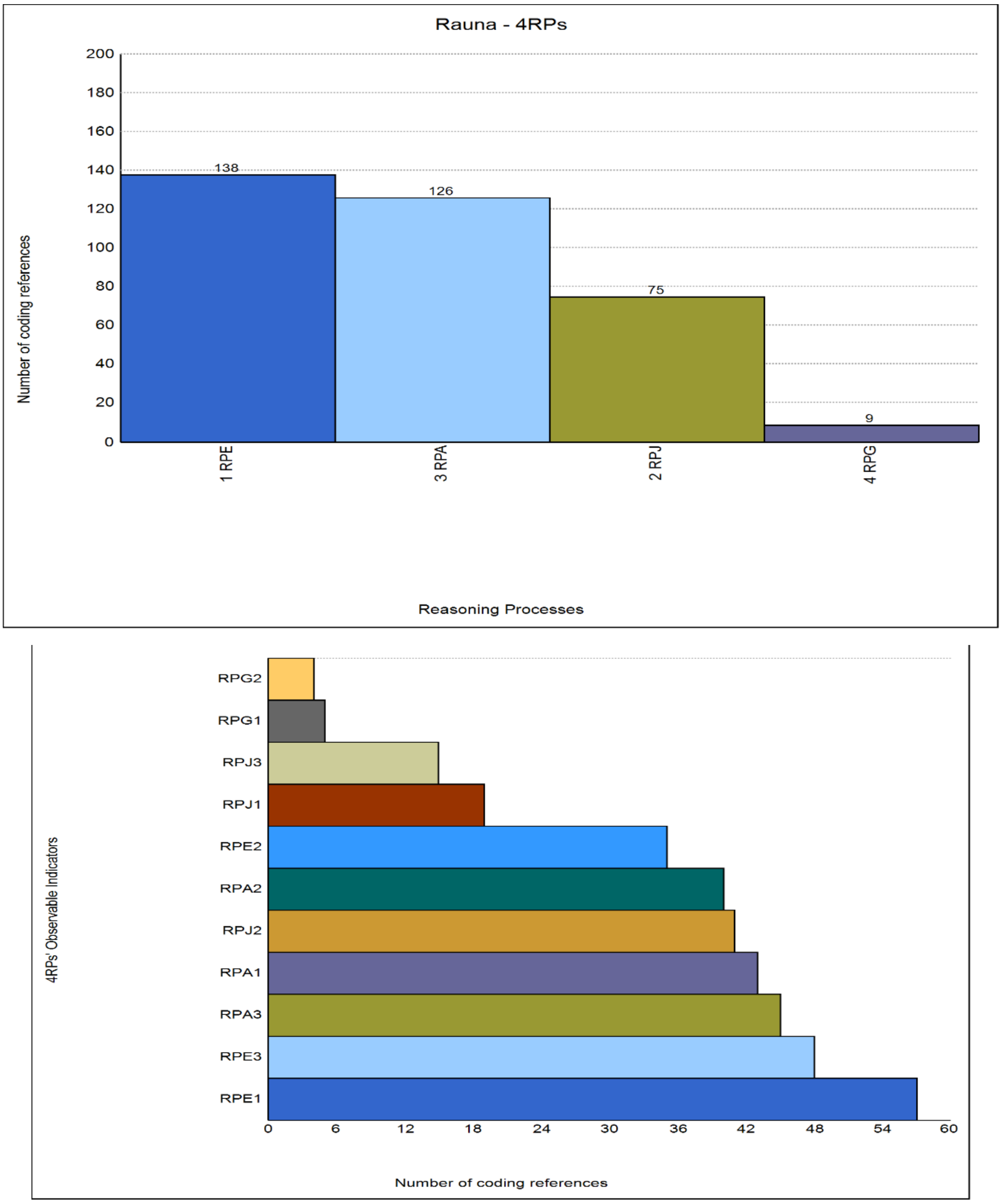
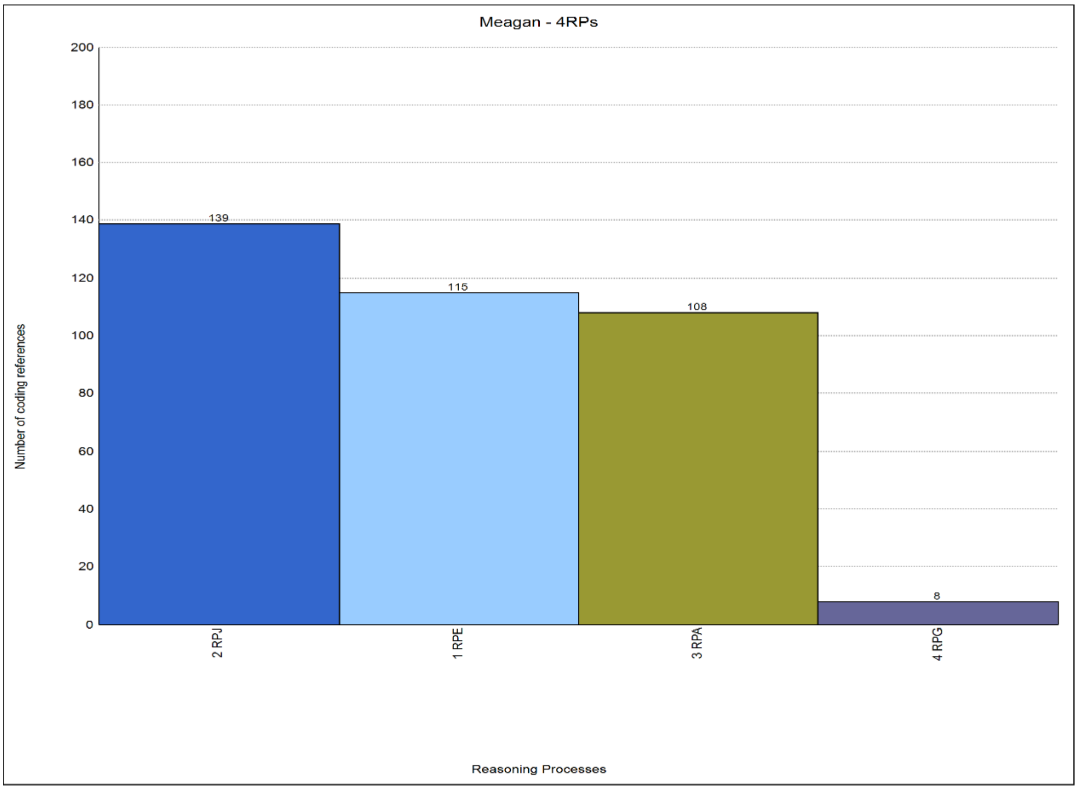


Figure 4.8 Rauna's reasoning processes

As can be observed from Figure 4.8, Rauna's reasoning processes consisted mostly of explanation and argumentation. The former has the highest total number of coding references, 138, while the latter obtained 126 coding references. She justified her claims (RPJ) up to 75 times during the focus group interview and managed to generalise her solutions and problem-solving strategies (RPG) on nine occasions. A closer observation of Figure 4.8 also reveals that Rauna spent most of her reasoning time trying to make sense of the problems and to establish what was required of her, RPE1. RPE3, which is also an observable indicator for explanation, comes in second position, with RPE2 five places later. RPJ indicators can be seen towards the far end where RPG observable indicators appear as Rauna's least coded indicators.

Meagan

Meagan's mathematical reasoning processes consisted mostly of justifications, with a total of 139 coding references as shown in Figure 4.9, below. There is evidence of an even balance between Meagan's reasoning processes of explanation and argumentation, with 116 and 108 coding references, respectively. Although the RPG node recorded the lowest number of coding references in comparison to the other nodes, I found Meagan's application of RPG to word problem solving interesting, as discussed in Section 4.3.3.



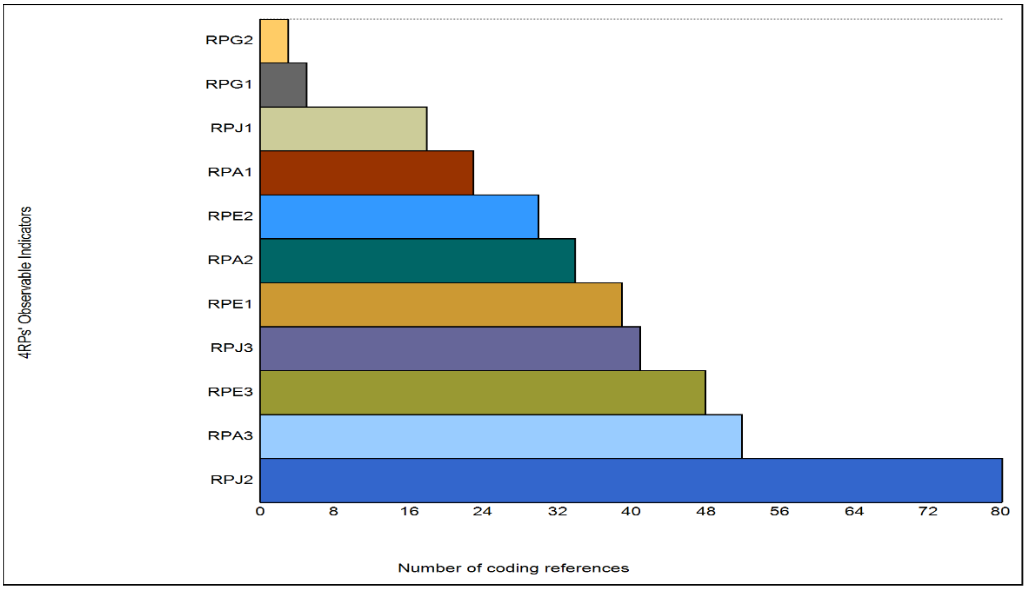


Figure 4.9 Meagan's reasoning processes

Figure 4.9 also shows that Meagan's RPJ2 was the most coded observable indicator, with up to 80 coding references. According to the reasoning processes analytical framework (Table 3.2), a learner who engages RPJ2 either "provides insight and reason for action (or inquires insight and reason for action from others)". In Meagan's case, she sought insight more than she provided it (see Section 4.3.3 for more details).

Millie

Millie's reasoning processes consisted mostly of justification, followed by explanation and argument, as presented in Figure 4.10, below. She recorded the highest number of coded references across the 4RPs in her group category (Figure 4.7).

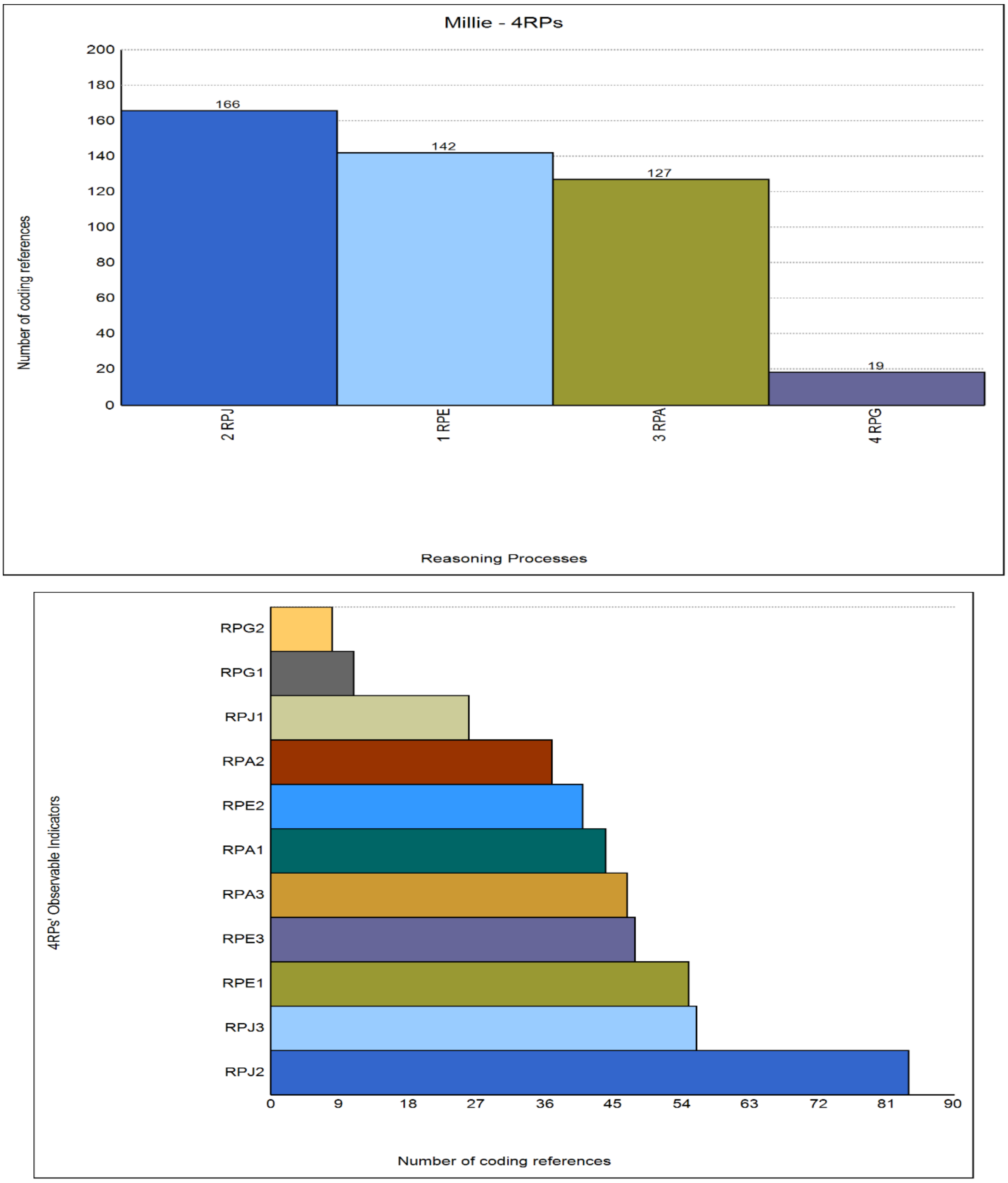


Figure 4.10 Millie's reasoning processes

As illustrated in Figure 4.10, RPJ was Millie's most coded reasoning process with two of its observable indicators, RPJ2 and RPJ3, recorded as the top two coded indicators. This shows that Millie provided insight into and reasons for her actions, mostly to Meagan who asked for justification of almost every step taken. Millie also promoted common understanding within her group (RPJ3) as she ensured that everybody operated at the same pace (see Section 4.3.3). It was intriguing to note that Millie's RPG was noticeably high, given the fact that RPG was the least coded RP across all eight participants. She recorded a staggering 19 coding references for RPG, which spoke volumes about her ability to generalise the word problems.

Denz

Denz's reasoning processes consisted mostly of explanations and justifications, with 104 and 96 coding references respectively, as illustrated in Figure 4.11, below.

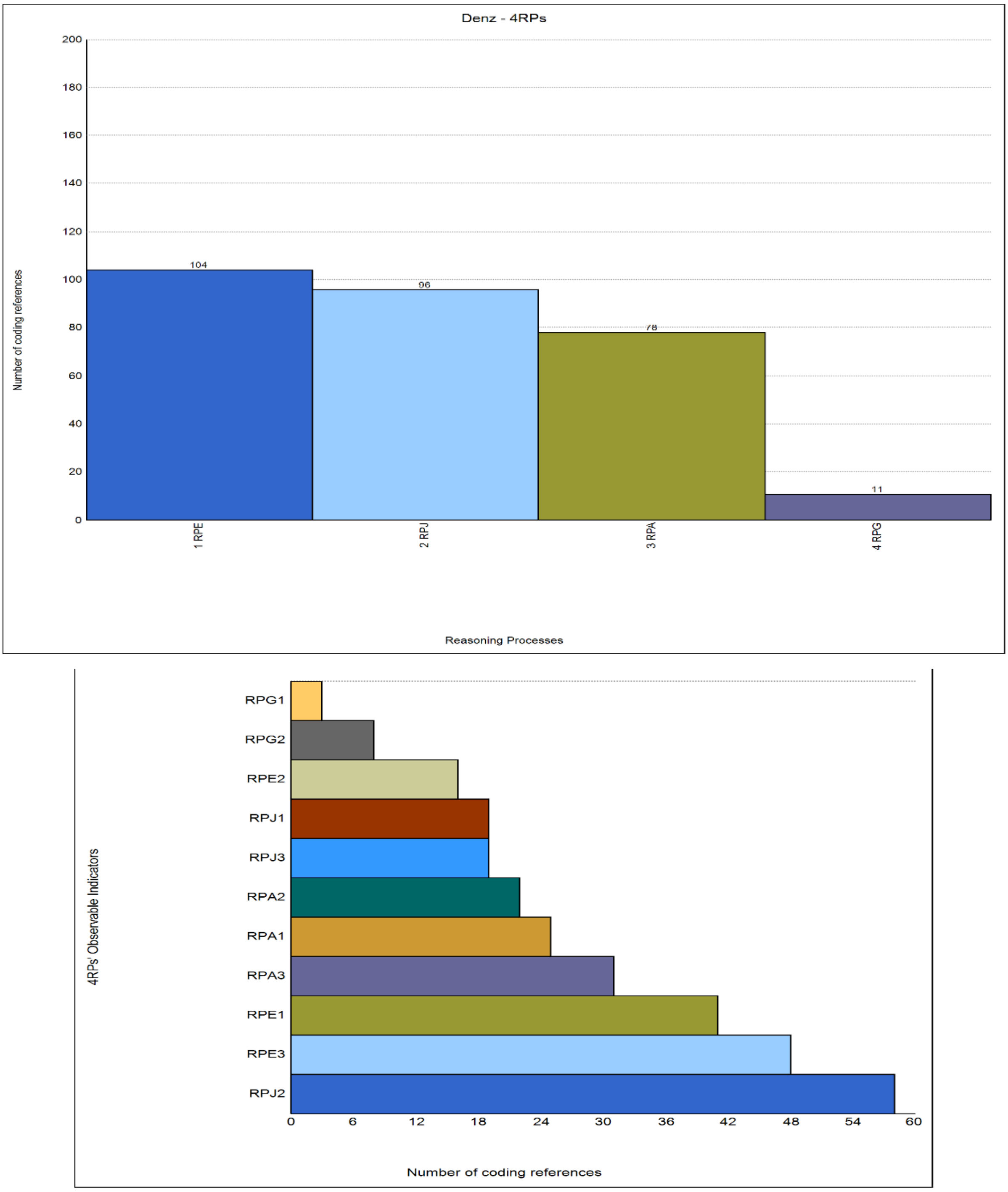


Figure 4.11 Denz's reasoning processes

As illustrated in Figure 4.11, RPJ2, RPE3 and RPE1 were Denz's most coded observable indicators, with a total number of coding references ranging between 40 and 60. His argumentation indicators all followed each other in the middle range of the chart and the two RPG indicators were the least coded. It is also worth noting that Denz's 11 coding references for RPG is relatively high given the fact that RPG was least coded across all the RPs.

Jordan

Like Denz's with whom he worked, Jordan's reasoning processes consisted mostly of explanations, as illustrated in Figure 4.12 below.

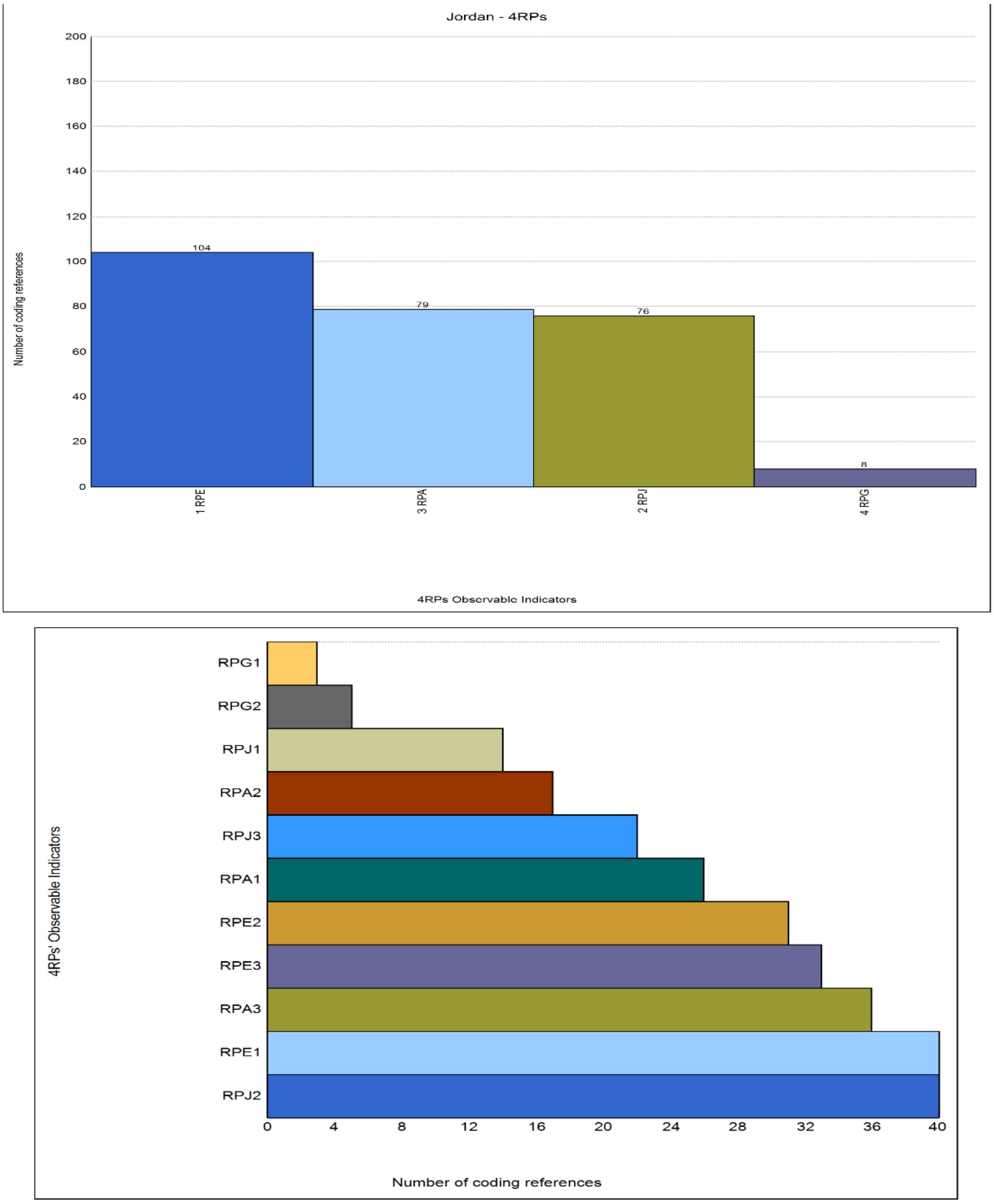


Figure 4.12 Jordan's reasoning processes

Figure 4.12 illustrates a close relationship between Jordan's arguments and justifications during the focus group task-based interview – he used RPA 79 times and RPJ 76 times. RPE1 – indicated by the learner's ability to make sense of the problem and to establish its claim – was coded 40 times. RPJ2, which involved the provision of insight into and reasoning for one's action, was also coded 40 times.

Ellena

The reasoning process of explanation dominated Ellena's reasoning during the focus group task-based interview.

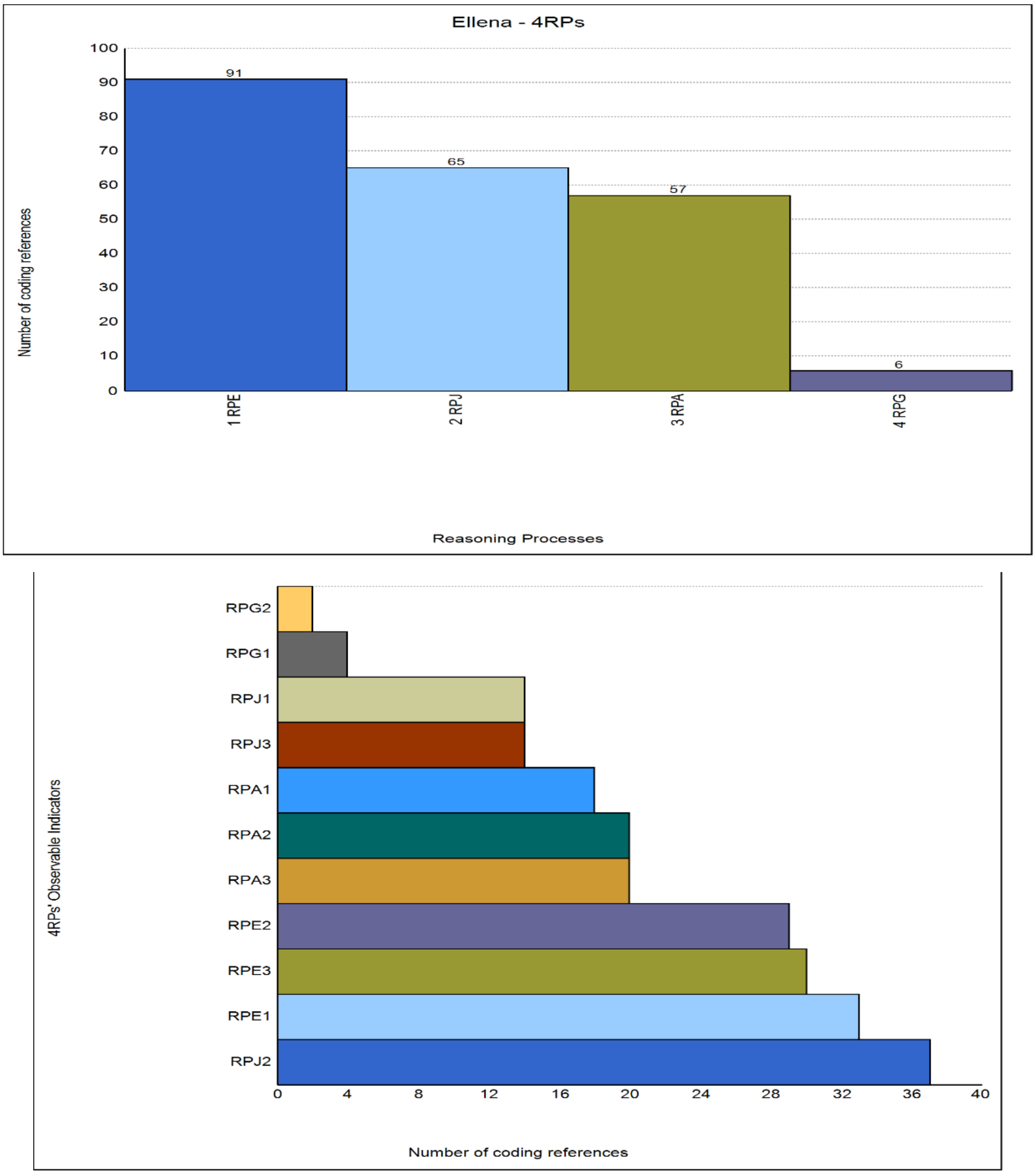
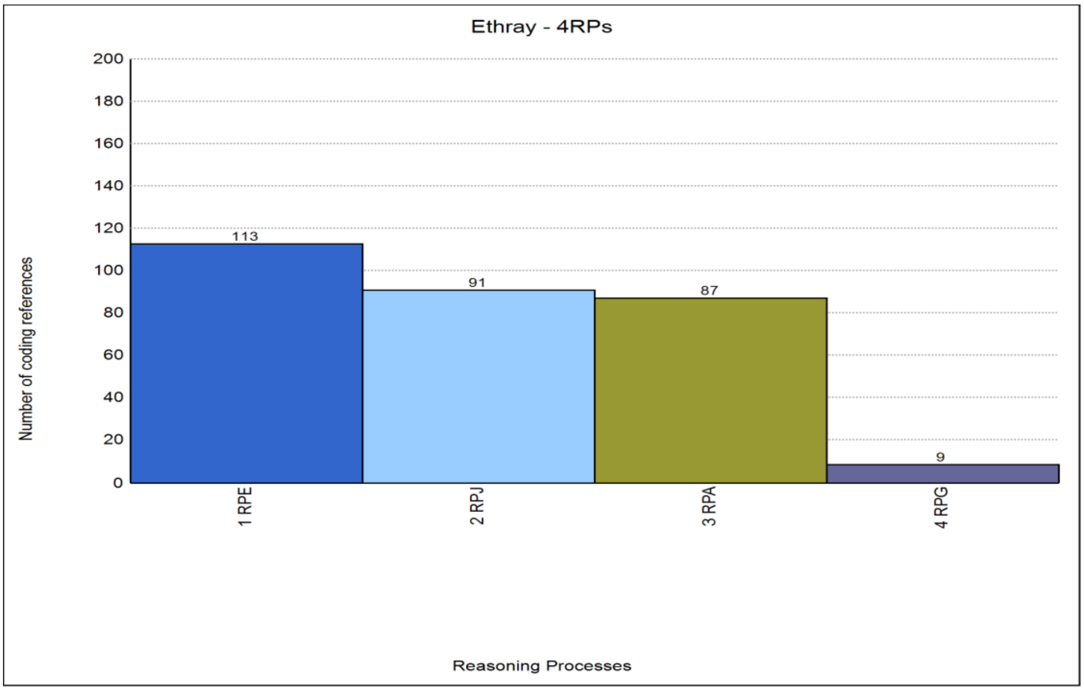


Figure 4.13 Ellena's reasoning processes

Ellena used RPE 91 times, RPJ 65 times and RPA 57 times, as indicated in Figure 4.13. RPJ2 was the most coded observable indicator with more than 36 coding references. All three RPE observable indicators were frequently coded, although RPE2 was the least coded of these with about 30 coding references. RPG was the least coded reasoning process with only 6 coding references.

Ethray

Ethray obtained the highest number of coding references for all the reasoning processes in his group category. The reasoning process of explanation dominated his reasoning, as illustrated in Figure 4.14, below.



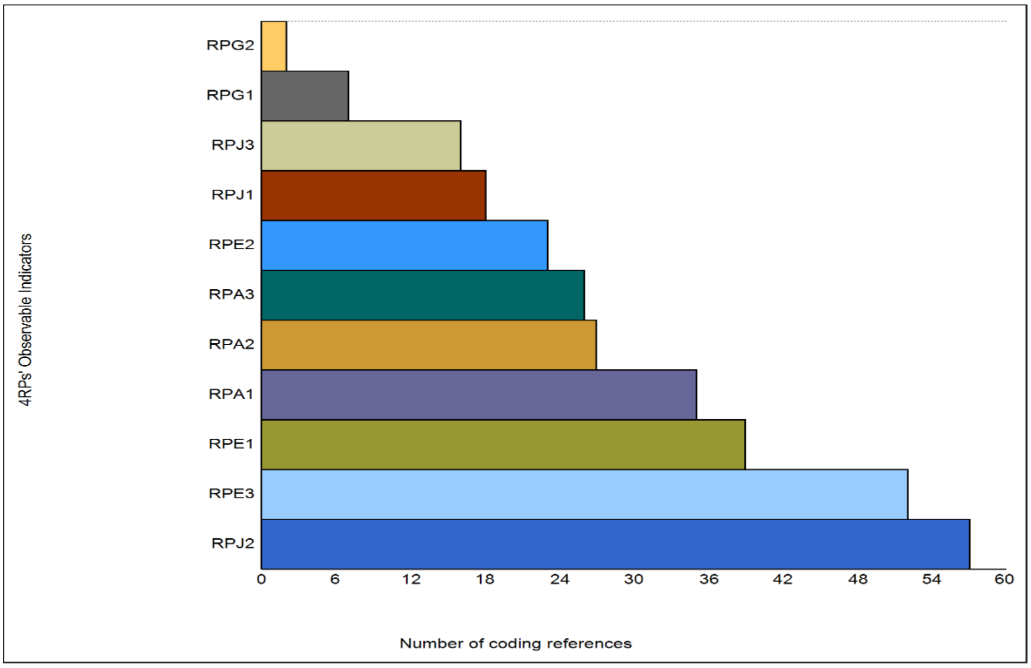


Figure 4.14 Ethray's reasoning processes

Figure 4.14 also shows that Ethray used the reasoning process of argumentation nearly as much as he used justification in his reasoning, with 91 and 87 coding references respectively. RPJ2 and RPE3 were the most coded observable indicators with over 50 coding references. RPJ1, RPJ3, RPG1 and RPG2 were the least coded observable indicators, each with less than 20 coding references.

Nate

During the task-based interview, Nate only spoke 40 times. He spent most of the time silently observing the conversation between his groupmates, Ellena and Ethray. Figure 4.15 illustrates that Nate obtained an overall number of 48 coding references. RPJ was his most coded reasoning process with 18 coding references, followed by RPA with 16 and RPE with 12. RPG was only coded twice.

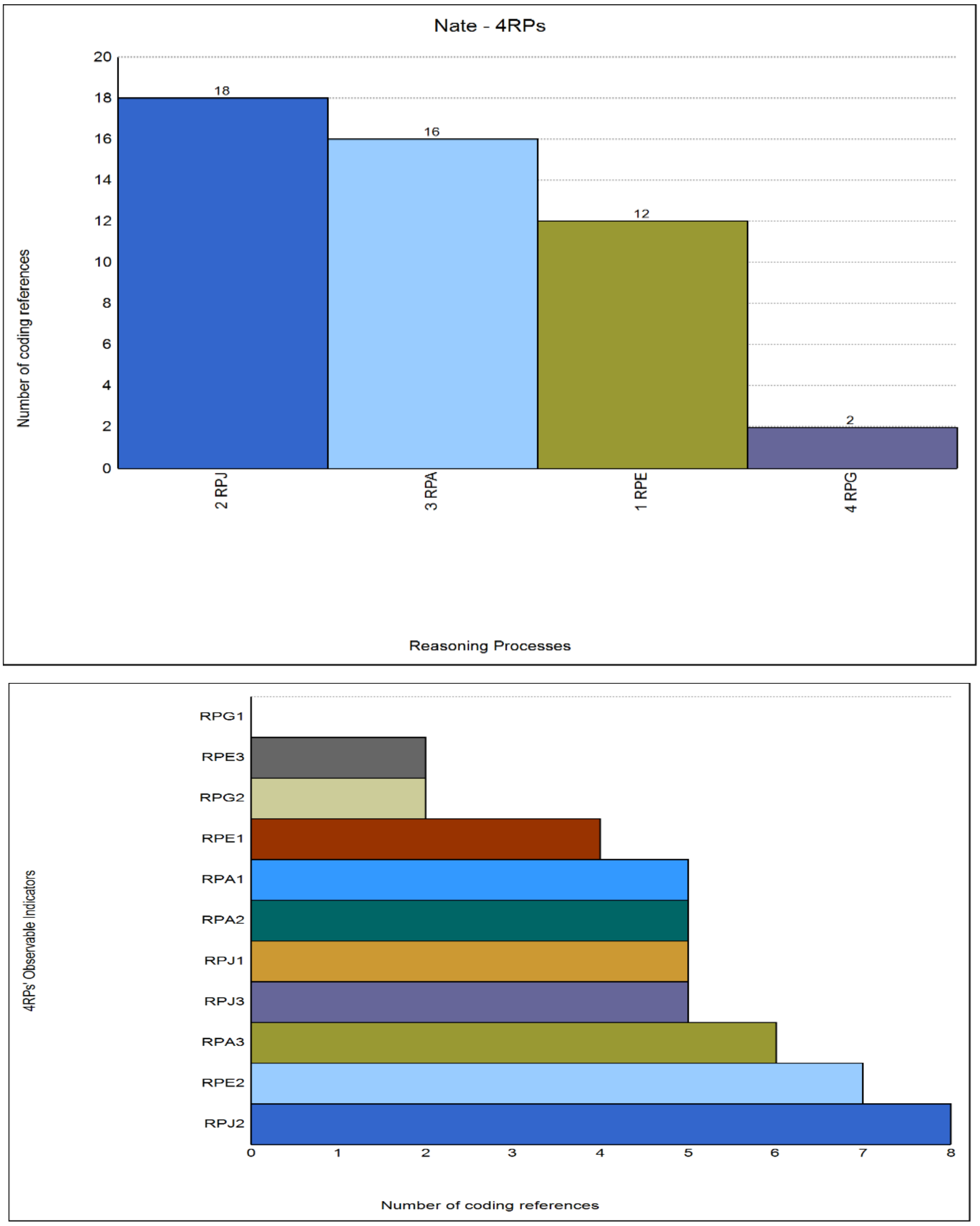


Figure 4.15 Nate's reasoning processes

Figure 4.15 also illustrates that Nate's RPJ2, which was the most coded observable indicator, was only coded 8 times. He recorded the least number of coding references across all four RPs and is the only participant with no coding reference for RPG1.

4.3.1.1 Summary of mathematical reasoning processes analysis

This section of data analysis focused on the participants' mathematical reasoning processes that were observed when they were solving the problems in EVGRT W2. During the task-based interviews, all the participants managed to contribute to the tasks in terms of the reasoning processes illustrated in Figure 4.7.

In their group, Meagan and Millie's bars for justification bars reflected the highest frequencies. This is because Millie had constantly to provide proof (RPJ1) to Meagan, who kept asking for justifications of others' actions (RPJ2). This helps to explain that Meagan's justification bar is not the highest of all her reasoning processes because she provided proofs to validate her claims (RPJ1), but rather because she repeatedly sought insight (RPJ2) from other participants in her group. Millie's RPA3 bar was the highest of all her RPA indicators as she was regularly persuaded by Rauna, via both verbal and visual cues, of the truth of her claims. This is unpacked in Section 4.3.3.

Figure 4.7 illustrates that the reason why Rauna's explanation bar turned out to be the highest was because she was typically quick to explain what the problems entailed and establish claims about the solution or solution strategies (RPE1). She also continually explained her thinking about the problems to others (RPE2) and suggested strategies for solving the problems (RPE3). Rauna's argumentation bar was also high because in addition to always providing support for her explanations and justifications (RPA1), she spent most of her argumentation time trying to convince others of the truth and accuracy of her claims (RPA2). There were several heated yet productive arguments between Rauna and Millie, in which Rauna invariably refuted and disagreed with Millie's claims, explanations and justifications (RPA3).

Denz and Jordan had a pattern of reading and sketching throughout their focus group task-based interview. With the first task, for example, Denz read while Jordan, listening to what Denz was saying, sketched a diagram (RPE1) and suggested strategies for solving the problem (RPE3). The two boys then took turns in reading and sketching throughout the task-based interview. This meant that their explanations bars reflected the highest number of coding references, as illustrated in Figure 4.7. Jordan's argumentation bar is slightly higher than Denz's because he constantly challenged Denz to provide support for his explanations and justifications (RPA1), whilst trying to convince him of the truth of his own claims (RPA2). Denz refuted most of Jordan's claims, which he deemed inaccurate (RPA3), and tried to persuade him both verbally and visually of the truth of his own claims and justifications (RPA2).

In the last conducted task-based interview produced a similar skewedness between Ellena and Ethray's reasoning processes bars and Nate's (Figure 4.7). In this group, Ethray took the liberty of both reading and explaining what the problems entailed (RPE1). He also suggested or partly suggested the problem-solving strategies for all five problems (RPE3), that the others either accepted or refuted (RPA3). Ethray also explained his thinking processes (RPE2) during the task-based interview, especially when the group struggled to make sense of the problem and to find a solution. Therefore, his RPE bar is way above 100 coding references, in contrast with his group-mates' explanation bars (Figure 4.7).

Overall, explanation was the most used reasoning process – 5 out of 8 participants recorded the highest number of coding references to RPE. Justification was the second most used form of reasoning, with 3 out of the 8 Phase 2 participants recorded the most coding references to RPJ. Generalisation was the least used reasoning process across the field of participants, its bars reflecting the lowest number of coding references for all the groups and individuals.

In this phase of data analysis, my interest was not primarily in the visualisation processes that might have emerged when the participants solved the word problems, as was the case with Phase 1. I knew that they could visualise. Nevertheless, I still concentrated on their use of visualisation processes as I needed to observe how these related to the reasoning processes. An unpacking of this relationship forms the basis of the next section.

4.3.2 Data analysis for the relationship between visualisation processes and reasoning processes

The close links between participants' use of visualisation and mathematical reasoning processes when they solved the word problems during the focus group task-based interviews were analysed and presented in this section of data analysis.

Figure 4.16 features a global overview of the relationships between observed indicators of visualisation processes and reasoning processes. I simply grouped the four RPs together, and graphically represented the five VIs. This type of relationship is called matrix coding query in NVivo. Matrix coding query enables one to compare different aspects based on search criteria. In this case, for example, I instructed the software to show me, first, the reasoning processes in terms of each kind of visual imagery for all eight participants (Figure 4.16), and then for individual participants, as illustrated in various figures below. NVivo can show the matrix coding results in a tabular or graphical form, which can make it easier to compare the relationships among different things.

The closest relationship between visualisation process and reasoning process was recorded between pattern imagery and the reasoning process of argumentation, as illustrated in Figure 4.16. This means that most of the research participants formulated patterns with the purpose of communicating information, engaged in patterns of data and argument, and used visualisation to venture generalisations (Table 3.1), providing proofs, explanations and justifications, while convincing/persuading others of the truth of their claims and accepting/refuting the truth of others' claims at the same time (Table 3.2). There was also a strong connection between kinaesthetic imagery and the reasoning process of explanation, with a matrix coding of 397 references. Pattern imagery also recorded a close relationship with the reasoning process of justification, where participants provided proofs to validate their claims, provided/sought rationales for actions taken, as well as promoted understanding among those engaged in a justification. These recorded a matrix coding of 368. These relationships continue in descending order as follows: RPA and KI with 353 matrix coding references, RPE and PI with 332, RPJ and KI with 317, RPE and CPI with 319, and so on. Apart from the matrix coding between PI and KI with various RPs, CPI/RPE was the only other matrix coding to have recorded an overall result of more than 300 coding references. The matrix least coded was between RPG and all 5 VIs, with the highest recorded being between PI and RPG at 44 coding references, as illustrated in Figure 4.16.

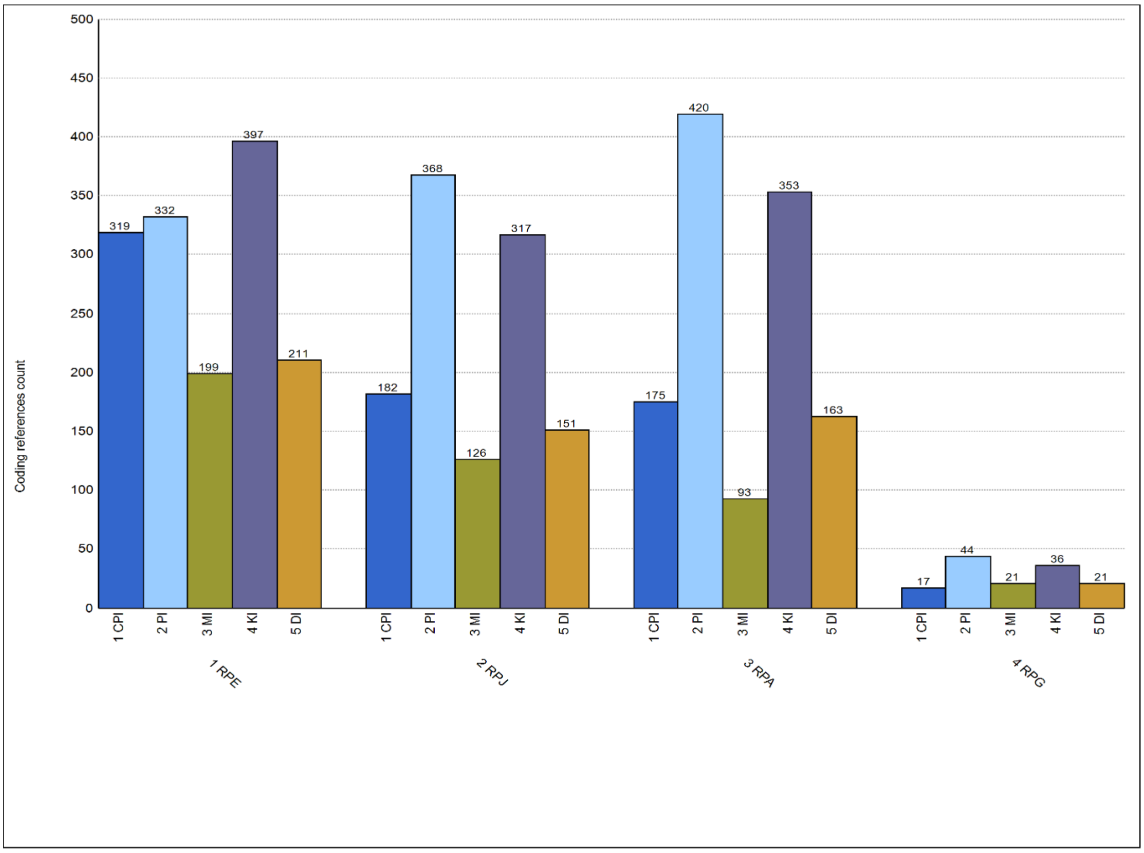


Figure 4.16 Overview of the data analysis matrix for the relations of visualisation and reasoning processes

Zooming into the matrix coding between each of the 4PRs and their respective 5VIs, Figure 4.16 shows that RPE was strongly related to KI, PI and CPI. RPJ was strongly related to PI and KI. RPA was strongly related to PI and KI. RPG was strongly related to PI and KI. These relations are observed for all the eight research participants. Each participants' matrix coding for the relations between the 5VIs and the 4RPs are analysed and presented below.

Rauna

There was a noticeably strong relationship between Rauna's pattern imagery (PI) and the reasoning process of argumentation (RPA), with more than 70 matrix coding references recorded. Figure 4.17 illustrates this matrix coding. KI and RPA came second with more than 60 matrix coding references, followed by KI and RPE with 56 matrix coding references, CPI and RPE with more than 50 coding references, and PI and RPE with more than 40 matrix coding references.

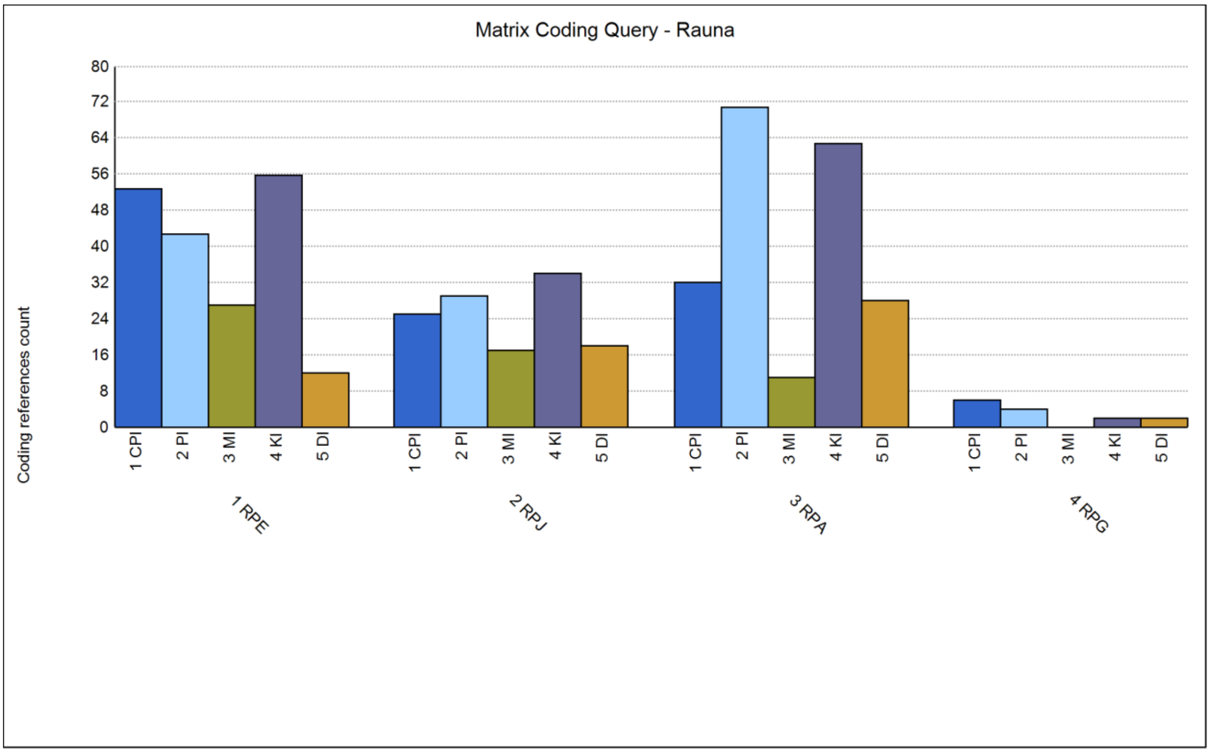


Figure 4.17 Rauna's data analysis matrix for the relations between visualisation and reasoning processes

Focusing in on the relations between each RP and its respective VIs, Figure 4.17 illustrates a strong connection between Rauna's RPE and KI, RPJ and KI, RPA and PI and RPG and CPI. There was no relation between Rauna's MI and RPG.

Meagan

The analysis of the relationship between visual imagery and reasoning processes for Meagan was dominated by the connections between KI and RPE, with almost 70 matrix coding references counted, as shown in Figure 4.18. PI and RPA with about 60 matrix coding references was Meagan's second most coded combination, followed by KI and RPJ with about 50 matrix coding references, CPI and RPE with over 40 matrix coding references, and PI and RPJ with just below 40 matrix coding references.

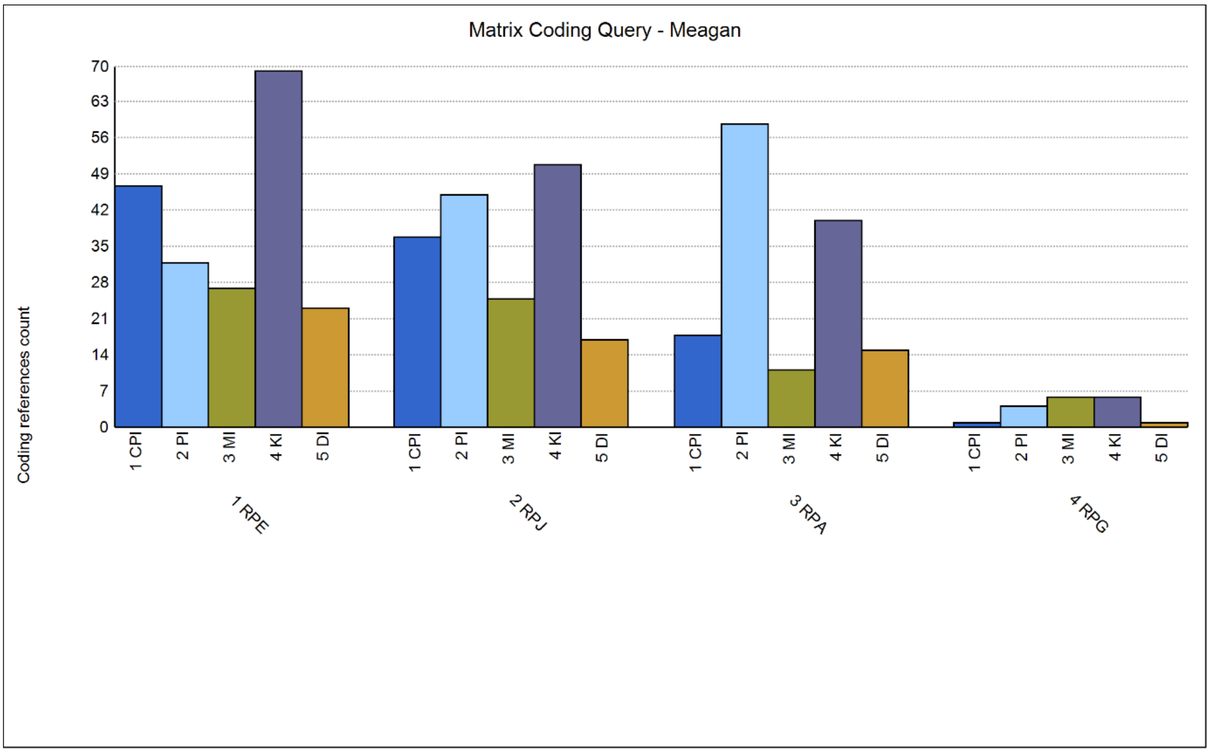


Figure 4.18 Meagan's data analysis matrix for the relations of visualisation and reasoning processes

Figure 4.18 shows that Meagan's RPE and RPJ were most closely related to KI among the various VIs. Her RPA was mostly related to PI, with a slightly weaker association with KI. The weakest correlations were between Meagan's RPG and CPI and DI, with only one coding reference observed for the matrix coding. It can also be seen that all matrix coding references between RPG and the 5 VIs were below seven.

Millie

There is a noticeably strong relationship between Millie's PI and CPI and her reasoning processes of explanation, justification and argumentation, as shown in Figure 4.19. Millie's PI and RPJ as well as PI and RPA recorded the most combined coding references of more than 100. KI and RPA followed with about 90 matrix coding references, KI and RPJ with more 80 matrix coding references, while KI and RPE counted about 80 matrix coding references combined. MI and RPG recorded the least number of combinations, as shown in Figure 4.19.

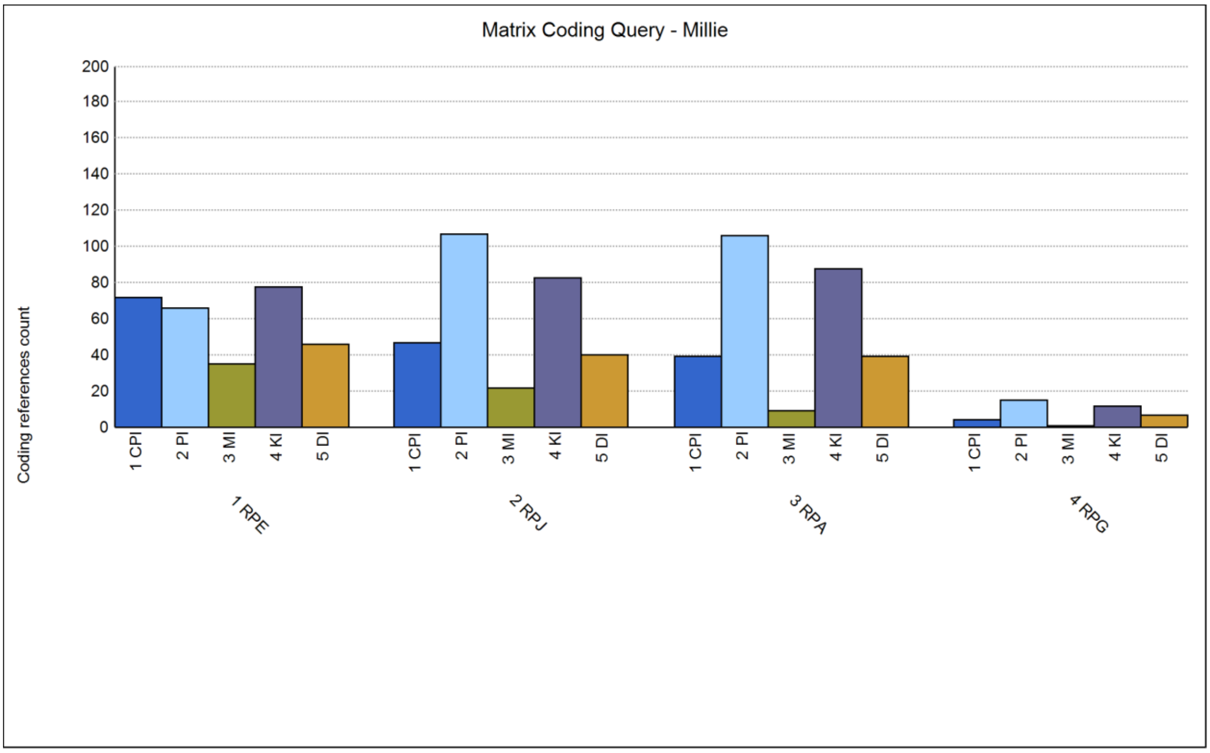


Figure 4.19 Millie's data analysis matrix for the relations of visualisation and reasoning processes

Looking at each of Millie's RPs in relation to their respective VIs, Figure 4.19 shows strong connections between her RPE and KI, RPJ and PI, RPA and PI and also RPG and PI.

Denz

Denz's VI/RP matrix is dominated by the connections between PI and each of his 4 RPs, as seen in Figure 4.20. The most coded matrix was that between PI and RPJ, with about 65 coding references. The PI and RPE matrix followed with about 60 coding references between them, and PI and RPA with more than 50 coding references. The pairs MI and RPE as well as KI and RPJ recorded about 40 matrix coding references between them. The matrix coding between CPI, DI and RPG evinced the least correlation, as illustrated in Figure 4.20, below.

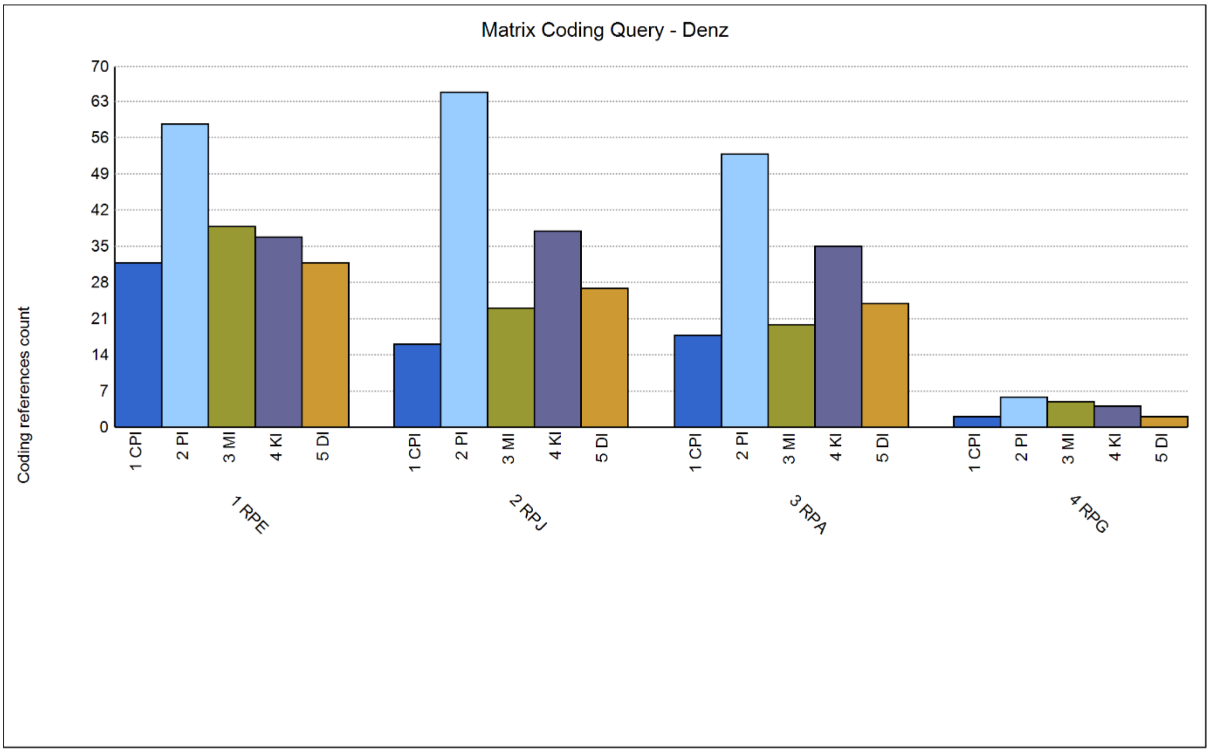


Figure 4.20 Denz's data analysis matrix for the relations of visualisation and reasoning processes

In terms of the relationship between visualisation and reasoning processes, Figure 4.20 shows that all of Denz's RPs were strongly related to his PI.

Jordan

Jordan's visualisation and reasoning matrix coding was dominated by PI with RPE and RPA, with over 45 coding references between them. Figure 4.21 illustrates these connections. KI and RPA as well as KI and RPE also displayed strong correlations, with over 40 matrix coding references. Jordan's MI and RPG was the least coded matrix, with only one coding reference.

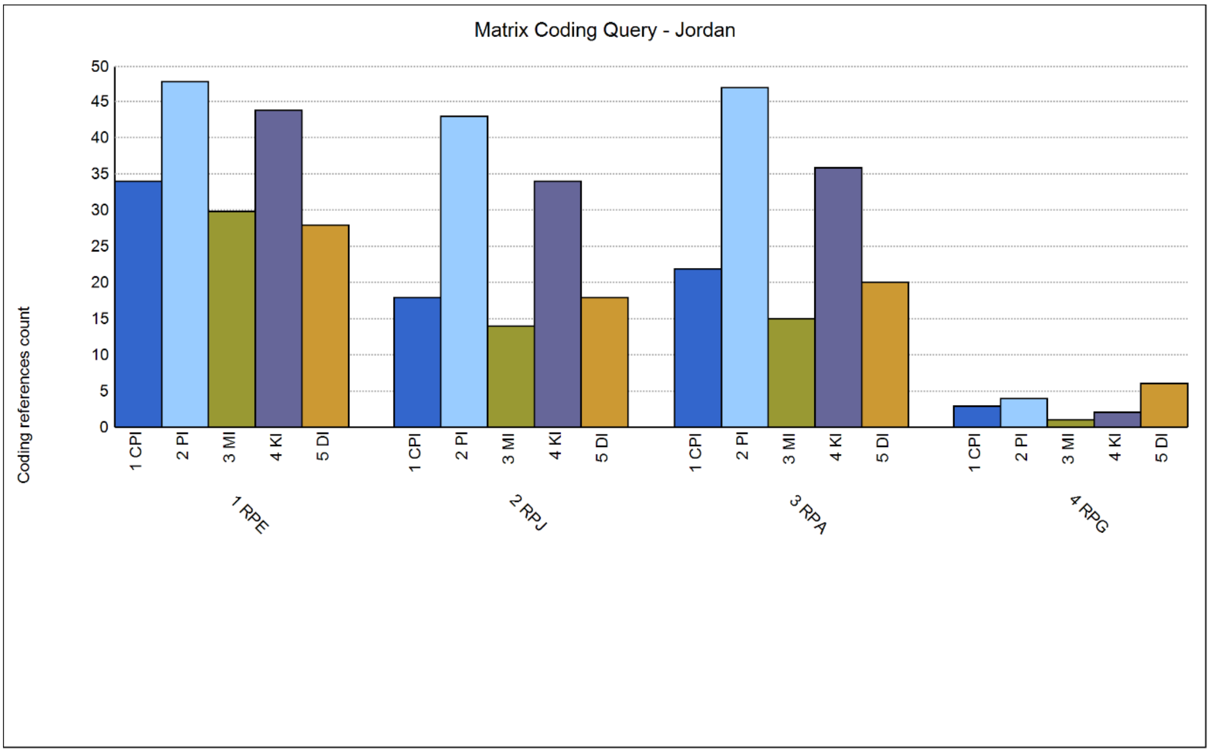


Figure 4.21 Jordan's data analysis matrix for the relations of visualisation and reasoning processes

Figure 4.21 also illustrates that Denz's RPE, RPJ and RPA all had strong connections with the PI category of visual imagery, and only RPG had more connections with DI than the other VIs.

Ellena

Analysis of Ellena's visualisation and reasoning processes indicated that in her case, the connections between visualisation and reasoning processes were dominated by PI and RPE, with over 36 matrix coding references, as illustrated in Figure 4.22. Ellena's KI and RPE was the second most coded combination with about 35 matrix coding references. Both CPI and RPE, and PI and RPJ recorded at least 30 matrix coding references between them.

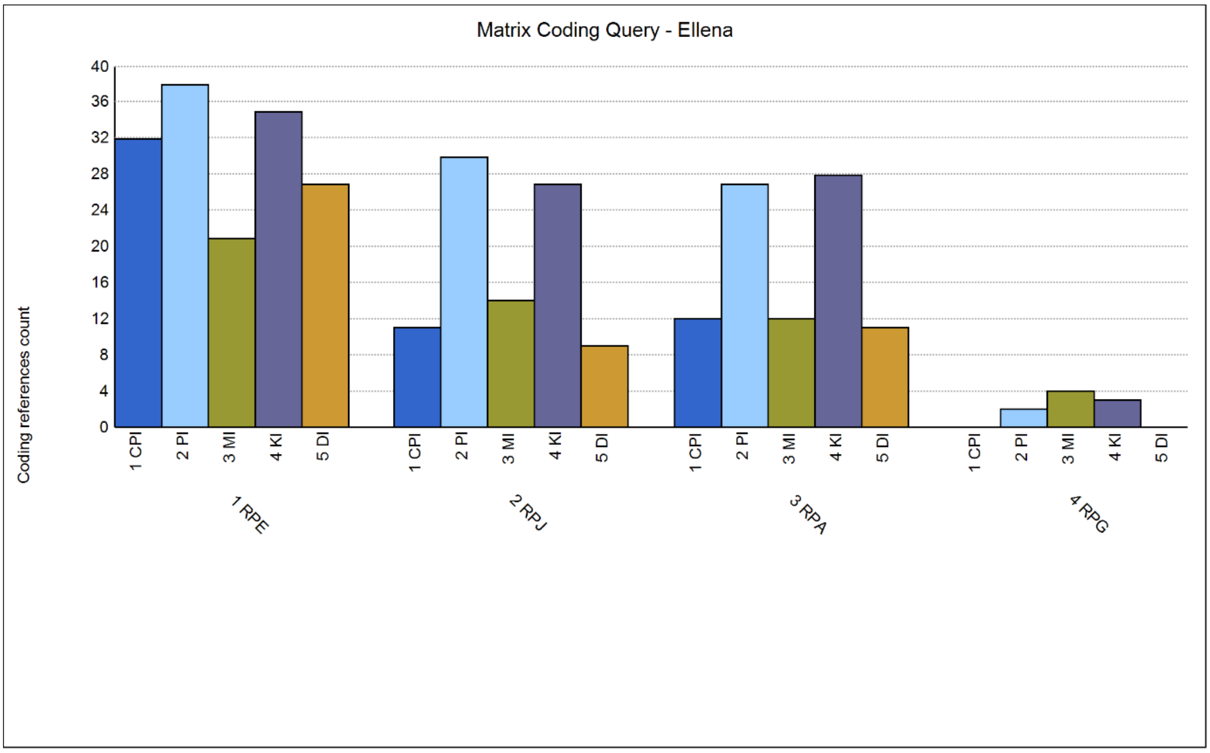


Figure 4.22 Ellena's data analysis matrix for the relations of visualisation and reasoning processes

Figure 4.22 also illustrates that there were no coding references recorded for connections between RPG and both CPI and DI. In terms of RP/VI relationships, RPE and RPJ were mostly connected to PI, RPA to KI and RPG to MI.

Ethray

Ethray's performance evinces connections between his visualisation and reasoning processes. Figure 4.23 shows that the most coded connections emerged between Ethray's KI and RPE, with over 70 coding references. The combination between KI and RPA was the second most coded with about 60 matrix coding references. This is followed by PI and RPA with almost 50 matrix coding references. The connections between KI and RPJ as well as KI and RPE were coded more than 40 times each. There were, however, no coding references to correlations between CPI and RPG and only one reference to matrix coding between DI and RPG.

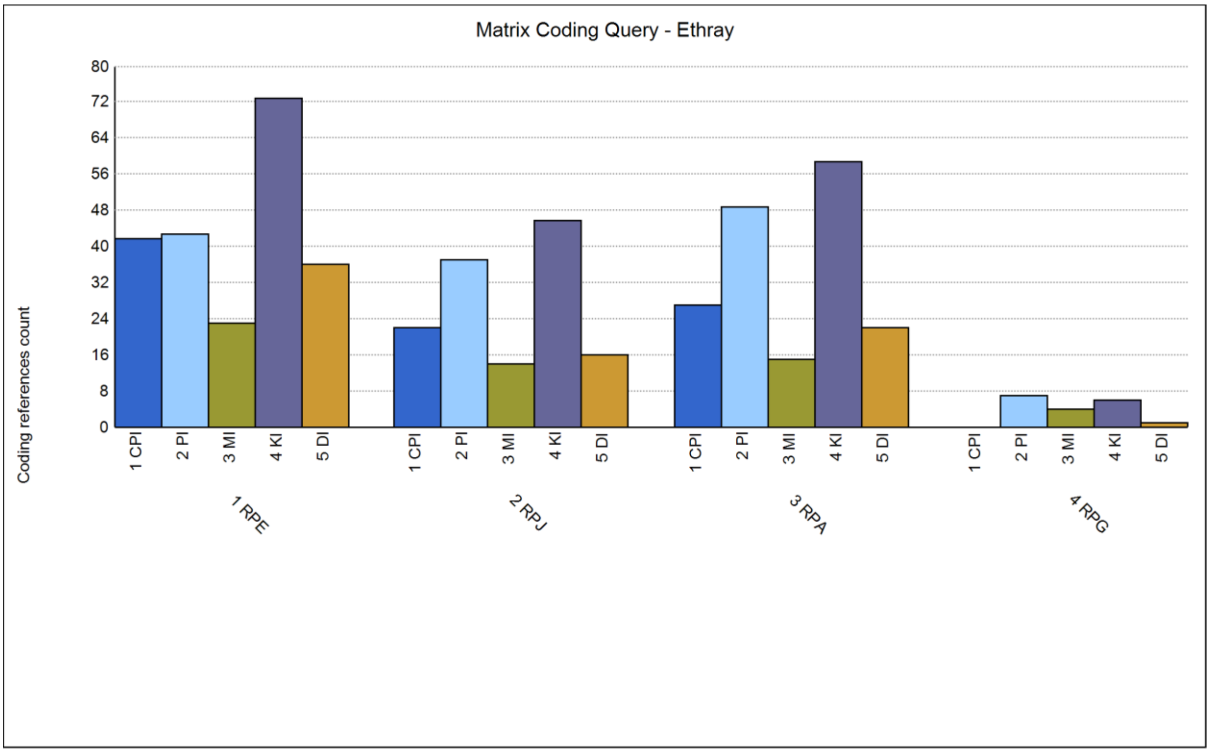


Figure 4.23 Ethray's data analysis matrix for the relations of visualisation and reasoning processes

Most of Ethray's RPs – RPE, RPJ and RPA – were interestingly dominated by coding references to KI, as illustrated in Figure 4.23. Only RPG recorded a narrow majority of relations with PI.

Nate

Nate recorded the least number of matrix coding references across the RPs and VIs in his group category. Figure 4.24 shows that Nate's most recorded connections were between PI and RPJ, with about 13 matrix coding references, followed by KI and RPE, and PI and RPA, each with 10 coding references. All the other coding matrices between Nate's VIs and RPs amounted to less than 10. There were no relations between Nate's MI and RPG.

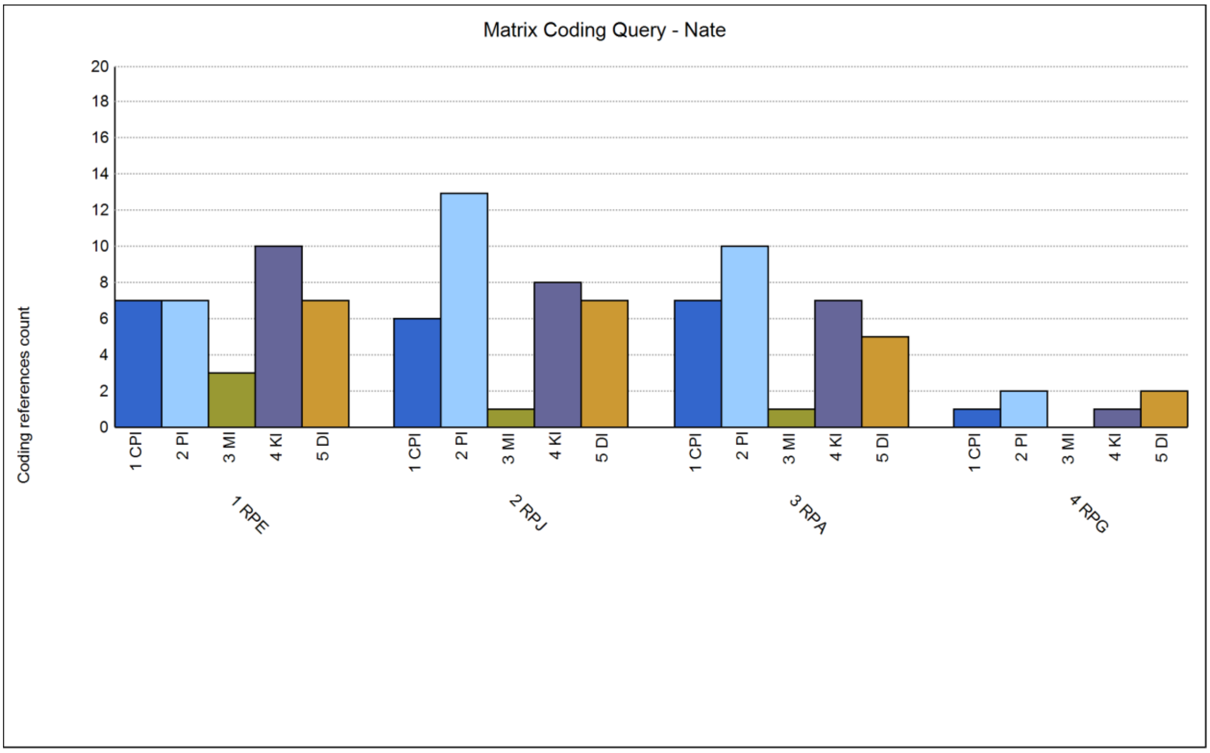


Figure 4.24 Nate's data analysis matrix for the relations of visualisation and reasoning processes

As regards the relationship between reasoning processes and kinds of visual images, Nate's RPE related mostly to KI, while RPJ, RPA and RPG all showed most connections with PI, as illustrated in Figure 4.24.

4.3.2.1 Summary of the analysis of the relationship between visualisation processes and reasoning processes

Data analysed in this section revealed that all the participants showed evidence of relationships between their visualisation processes and reasoning processes during EVGRT W2. This is illustrated by the matrix coding charts in the Figures above. Although there were close relations between the reasoning processes RPE, RPJ and RPA and certain visual imageries, the reasoning process of generalisation did not show strong connections to either of the visual imageries for all the participants (as illustrated in Figure 4.16). From the beginning, RPG started with low occurrences in comparison to RPE, RPJ and RPA. This is attributed to the participants' reluctance to generalise their solutions in fear of 'I might get it wrong if I try it in another way' syndrome. Therefore, when cross tabulated with visual imagery, the low frequencies are more a function of low frequencies in the reasoning process than in the combination of the two (reasoning processes and visual imagery).

There were a total number of 1458 matrix coding references between the observable indicators of the reasoning process of explanation (RPE) and the visual imagery observable indicators (Figure 4.16). This was the highest overall matrix coding observed between the VIs and RPs. It reveals that the research participants mostly employed visual images when they read and tried to make sense of the problems (RPE1), when they explained their own thinking to others in their small groups (RPE2) and when they suggested problem solving strategies or sought clarification from others (RPE3). Moreover, kinaesthetic imagery (KI) indicators recorded closer connections to the reasoning process of explanation than the other five categories of visual imagery. There were 397 matrix coding references to relationship between KI and RPE. This means that when they provided clarification of mathematical thinking, the participants were engaged in patterns of movement and physical engagement as part of problem solving (KI1); for instance, they walked/traced paths with fingers/hands/pencils to illustrate an image of something (KI2), and mimicked/imitated/traced shapes, without placing the pencil on paper (KI3).

The connections between the observable indicators of the reasoning process of argumentation (RPA) and the observable indicators of visual imagery made this the second closest relationship, with a total of 1204 matrix coding references. The links between the reasoning process of justification (RPJ) and the visual imagery indicators were indicated by 1144 matrix coding references, while generalisation (RPG) recorded only 139 matrix coding references alongside observable indicators of visual imagery. Furthermore, the reasoning processes of justification, argumentation and generalisation recorded close connections with pattern imagery (PI) in comparison to all the other categories of visual imagery. This connection is an indication that whenever the research participants attempted to argue, justify and generalise their solutions and problem-solving strategies, they formulated/used patterns with the purpose of depicting/communicating information (PI1), engaged with patterns of data and arguments (PI2), and used visualisation to discover generalisations and to derive non-obvious concepts/formulae from such generalisations (PI3). RPA recorded the closest relations to PI with a total number of 420 matrix coding references, in comparison to its relations with RPJ (368) and RPG (44).

The enactivist concept of co-emergence foregrounds interaction between living systems and their environment. In this section of data analysis, this interaction observed occurred between the living systems' visualisation and reasoning processes when they solved word problems in small groups. The Figures above illustrate this co-emergence in the form of a matrix as presented in Table 3.3.

In Sections 4.3.1 and 4.3.2 on Phase 2 analysis, above, several salient features of visualisation and reasoning processes surfaced and were succinctly noted. Discussion of these and other features forms the basis of the next section of data analysis.

4.3.3 Fine-grained analysis of mathematical reasoning in enacted visualisation

This section presents a horizontal analysis that consolidates the Phase 2 data analysis in the preceding sections. The discussion takes the form of eight vignettes selected from the focus groups' transcripts for each of the four reasoning processes. This selection was based on features of the NVivo graphs that were particularly striking (see the justification for the choice of each vignette, below). The analysis centred on the actions, gestures and arguments that individual research participants "uttered" when they solved geometry word problems in their small collaborative groups. Although I analysed individual participants' mathematical reasoning, I also considered other participants' influence on the analysed participant's reasoning. When, for example, I unpacked Rauna's argumentation, I did so in relation to her group's argumentative outcomes – i.e., how did Meagan and Millie react to Rauna's claims and arguments?

Reasoning process explanation (RPE)

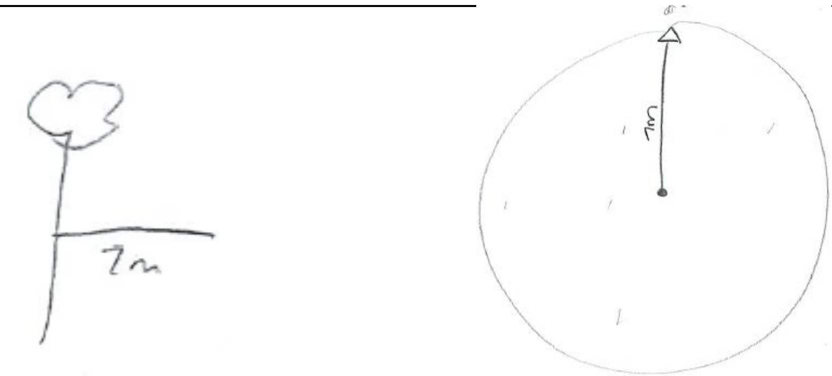
Mathematical explanations involve the classification of aspects of one's mathematical thinking that might not be clear to others (Yackel, 2001). Below is an extract from Table 3.2 to serve as a reminder of the observable indicators used to analyse data for RPE.

RPE1: makes sense of the problem and establishes a claim e.g. explains what the problem entails in simple terms and suggests known concepts/procedures.
RPE2: explicates own thinking processes (to produce meaning – includes reasoning without words)
RPE3: suggests and defines problem solving strategies

Vignette 1 – Ethray and Ellena's explanation

This vignette presents Ethray and Ellena's explanations (RPE) when they solved Task 2 and Task 4 of EVGRT W2 in their small group. I selected this vignette because both Ellena and Ethray used the reasoning process of explanation more than other reasoning processes during the task-based interview, as illustrated in Figure 4.7, above. For Task 2, Ethray was the first group member to describe the problem and the first to make a claim of sorts. As soon as he had finished reading the second task in EVGRT W2, Ethray said "*okay, so, there's a tree and there's 7m rope"*, and sketched a tree and a 7m line from its stem as illustrated in

Figure 4.25(a). Ellena, seeing a different picture in her mind; a rather more geometric picture in comparison to Ethray's, asked him: "*why don't you just do this for the tree* [marks a point], *and this for the rope* [draws a line from the point]? *Then you can just…*" [mimics drawing a circle with the tree as the centre and the rope as the radius].



(a) *"there's a tree and there's 7m rope"* (b) *"…do this for the tree…and this for the rope*"

Figure 4.25 Ethray and Ellena's visualisation of Task 2

Figure 4.25 illustrates Ethray's live image of an actual situation in his mind (a) and Ellena's mathematically transformed image of the same situation (b). Ethray accepted Ellena's suggestion of a more geometric drawing and commented:

> *"Seven metres okay. So, the goat can graze freely around it* [walks a circular path with a pencil to show the boundary where the goat will graze] *if he wants to* (…) *if the goat moves a complete revolution, what is the maximum length of the rope? What is the total possible area the goat would graze? We have got to find the area of the circle.*

Ellena and Nate both agreed that they needed to find the area of the circle. Ellena then started calculating the area of the circle while the discussion was still underway. She wrote and spoke at the same time: "$A = \pi r^2$, $\pi(7)^2$". While Ellena busied herself with elaborating the formula, Ethray worked on the calculator and provided her with the solution when she needed to write it down. The two participants effectively read each other's minds, completing each other's sentences as they reasoned their way through solving the problem. Damiano (2012) defines this type of dynamic interaction whereby an autopoietic system and its environment are structurally coupled, as co-emergence.

Ethray and Ellena's interaction further coupled and co-emerged when they interchangeably visualised and reasoned during the fourth task. Ellena started reading the task but then turned the worksheet towards Ethray to read instead. Ellena preferred to observe while somebody else read. She followed and visualised the task on paper as she listened to Ethray read. She remarked: "*so, there's two, two, two spaces* [jumps two spaces with her pencil as she realises the pattern in the distribution of cards to the people seated around an imaginary table] (…) *but there's supposed to be a formula*". Ellena recalled from memory (MI) that they could find a formula to help them answer questions dealing with sequences. She has learned this in her mathematics classroom but was unsure of how to implement it during the focus group task-based interview. Ethray explained to her that there was more than one way of getting the answer to the question. "*There should be a formula I know but, let's just see if we can get the answer then try to get the formula* [overt gestures and body movements as he speaks]. *So, this is 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29"* [Ellena and Nate observe attentively as Ethray allocates imaginary cards to his imaginary people seated around the table]. Seeing that Ethray somehow mixed up the pattern, Ellena alerted him to skip two people when allocating the cards as this was part of the worked-out pattern ("*you can't just start here, you should skip two and so there is a 1, 2, 3, 4…*"). Realising and accepting his mistake, Ethray restarted the process and together with Ellena counted all the numbers as in their imaginations they allocated cards to 10 people around the table. When I asked whether they were going to count up to 52 as they were, Ethray responded: "*yes, ma'am* [and continued to count] *30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52. The fourth person gets the last card*".

Although Ellena equally arrived at the solution that the fourth person gets the last card with Ethray, she insisted that they needed to get the formula despite the fact that they struggled to formulate it. I asked them to explain how they arrived at their solution and how their strategy could help them to answer similar questions in future. To do this, they constructed a table which Ethray alleged that they would fill in the numbers until a pattern started to form:

> *Okay, so the first person gets the 1st card* [Ellena allocates cards as Ethray speaks as illustrated in Figure 4.26]. *Fourth person gets 2nd card. The seventh person gets 3rd card. Tenth person gets a 4th card. Can I just continue? (…) So, the third person*

gets the 4th card…no, 5th…and then sixth person gets the 6th card. And the ninth person gets the 7th card. Second person gets the 10th card. Then you can see a pattern forming here (…) wait, try and draw it like (…) I mean next to each other, so we can see on the line (…) so we can see how they form some type of pattern [gestural movements with his hands as he prompts Ellena to draw lines next to each other and to write the numbers underneath each other in their respective columns].

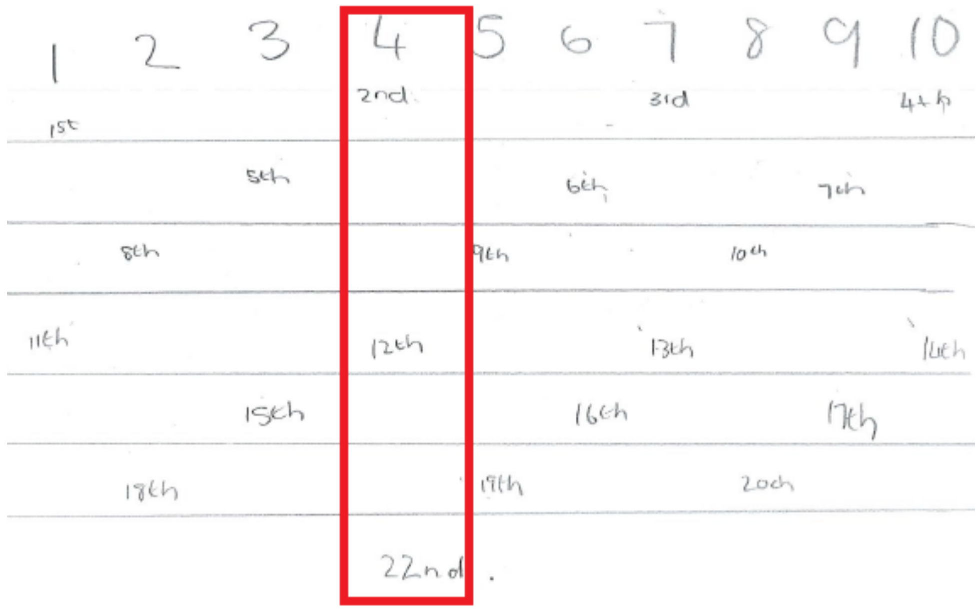


Figure 4.26 Ellena and Ethray's visualisation processes of Task 4

The two participants worked together beautifully as they allocated cards to 10 people, as illustrated in Figure 4.26 and in their conversation below:

Ethray
So, the third one gets the 5th. And the sixth one gets the…and the fifth gets the 9th. Eighth gets the 10th. And you start over with one. And 11.

Ellena
Wait… let's just do this… [counts two spaces then write the answer even before Ethray says it…she then draws lines underneath to separate the start over rounds]

Ethray
And then we just continue until we're done with this line. So, then the fourth gets 12th. 13th…yeah, and then 14 [moves his pencil to the next person as he mentions who gets which card next]. Okay, so that's…okay, so…like you can see if you add ten to get there, add ten, add ten, add ten…and then, wait…let's continue on the next line [moves his pencil as he shows the addition of 10 in each row…he searches for the pattern].

Ellena
So…

Ethray
Because if the next line…this should be 15…that's 15…that's 16, so it continues…then there's 17…yeah, that's 17…and then that's the next…and then this

is 18…19…20 [moves his pencil to show where the next number goes]. So, if you add ten

(…)

Ethray
Because we can also use this to calculate it much faster than counting. Because now we can say, add ten to the next line, ten, ten, ten, the whole time.

Beata
The answer being?

Ethray
It would still be the fourth person.

Beata
Why?

To the question why the fourth person got the last card, Ethray answered as follows:

I think the reason the fourth person gets the card is because there are 52 cards. The fourth person will receive the 2nd card, if you add ten the whole time till you reach 15…now which person got the 2nd card? You end at four because there's not enough cards (…) Like if you look at a person…if the cards were 72, which person would get it? Would it still be the fourth person? I think it still (…) I think it still will be the fourth person, because of the two.

I commented in the transcript that this was excellent observation, visualisation and reasoning from Ethray, but I still wanted to get more out of him. I wanted to ensure that his reasoning was firm, so I asked him what he meant by 'because of the two'. He said: "*because of the two at the end…what the person received*". Ellena also helped to clarify her teammate's reasoning as she commented: "*The second and the twelfth and the* …" [points down the numbers as she follows the pattern]. Ethray 'picked up' Ellena's sentence and completed it: "*and the 22nd and the 32nd, and the 42nd, and the 52nd*" [patterns of movement as he engages his hand when he speaks].

The interesting aspect of Ethray and Ellena's interactions was the way in which they each knew what to say and when to say it. It was as if they read each other's minds, often completing each other's sentences. From an enactivist point of view, there was a structural congruence between their interaction and their environment (Maturana & Varela, 1998, p. 95). Nate, on the other hand, remained quiet throughout the fourth task and did not interact with the other two participants, despite their efforts to involve him. Maturana and Varela

(1998) spoke of adaptation, that deals with the compatibility of the organism with its environment. If at any time, claim Maturana and Varela (1998), we observe a "destructive interaction between a living being and its environment, and the former disintegrates as an autopoietic system, we see the disintegrating living system as having lost its adaptation" (p. 102). Maturana and Varela (1998) further argue that the adaptation of a living organism to an environment is a necessary consequence of that organism's structurally coupling with that environment (p. 102). In this case, one might maintain that Nate lost his ability to couple with his environment as he failed to adapt by interacting with his peers during this task. On the other hand, Khan et al. (2015) assert that the ability of an organism to uncouple from its environment is also important for the organism's survival. From an educational perspective, they add, "learning is dependent on both socio-cognitive coupling and uncoupling. Coupling serves as a trigger or a perturbation and the uncoupling provides opportunities for pursuing personal interest/focus necessary for individual learning" (p. 276). Nate in this case was uncoupled from the task both in terms of his inability to adapt to the problem-solving environment and in terms of his apparent intent to focus on individual learning. I adopted this perspective on Nate's coupling and uncoupling because although he did not participate in the fourth task, he participated intensely in the last task. This is in line with what Khan et al.'s (2015) observation that "the combined socio-cognitive coupling and uncoupling can provide opportunities for learning that enable the organism to adapt learning to other environments" (276).

Vignette 2 – Denz and Jordan's explanation

I selected this vignette because, like Ellena and Ethray, Denz and Jordan's reasoning processes consisted mostly of explanation (Figure 4.7). The most striking explanations happened when they attempted to solve task 1 in EVGRT W2. When they struggled to comprehend the structure of the first task during their focus group task-based interview, I scaffolded their thinking to help them comprehend and explicate the task. This enabled them to understand what the problem required them to do and consequently to solve it accurately. When I asked them to tell me what the relationship was between the side of the largest square and the diameter of the circle, Denz suggested that the diagonal of the smaller equalled the diameter of the circle. He had this to say when Jordan refuted his claim:

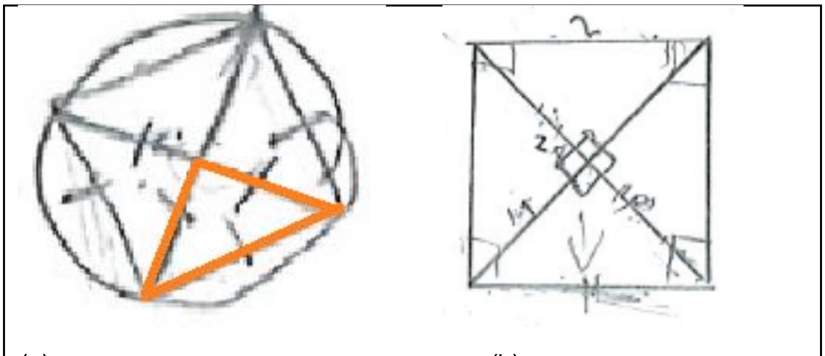
> *Oh, it's also the diameter. So, it is twenty. Because it's not through* [Mimics circular drawings as speaks] *(…) because you see here till there* [walks a path with a pencil to show the length of the square from one side to another]. *It's tog twenty, in a circle*

everywhere where it touches will be twenty. Circumference to circumference. So, it's also twenty [shows what he says with a pencil via various paths].

Denz's explanation using the five categories of visual imagery helped Jordan to learn new information as he developed a better understanding of what the problem entailed and what to do to work out a strategy to solve it. This was in line with Webb's (1991, p. 368) argument that learners have the potential to give understandable and timely explanations. As two participants shared a similar language, they were able to translate difficult vocabulary and expressions and use language that the fellow participant could understand. Webb (1991, p. 368) argues that the person receiving an explanation also has an opportunity to use the explanation to correct his or her misunderstanding or lack of understanding about the work. This was exactly what Jordan did when he listened to Denz's explanation; he corrected his misunderstanding about the word problem as follows:

> *Because anyway, just like this, it's going to be twenty, like there's going to be twenty, like there is going to be twenty* [draws other lines that are also equal to 20m]. *Even like this it's going to be twenty. So, area twenty… Twenty times twenty, that's four hundred…four hundred*.

Both boys were confused once again. They had conflated the dimensions of the sides with the diagonals of the inscribed square, as illustrated in Figure 4.27 (a) below. They claimed that if the square is divided into four triangles using its diagonals then all the four triangles are equilateral. Hence, the side of the square equals half of its diagonals in length.



(a) (b)

Figure 4.27 Denz and Jordan misconception of the diagonals and sides of the square

Denz supported Jordan's claim that if the inscribed square is divided into four triangles all the sides would be equal. He said: "*I also think so, because in a square you can divide it into four triangles. And each will be equilateral because it's of equal sides"* [makes another sketch of a square and divides it into triangles as he speaks]. Seeing that they were both convinced and deeply committed to their misconception, I encouraged them to use visualisation to see if they could alter their reasoning. I asked them whether they were convinced that their reasoning was accurate. After a long pause, Denz suggested that they draw a real square – to scale (Figure 4.27 (b)). When they measured the side of the square and its diagonals, they realised that they were not equal.

Although the boys were somewhat confused at the beginning, their engagement with both visualisation and reasoning processes helped them to interact with each other, study the patterns of arguments together and arrive at a more appropriate collective solution. Reid (2002) claims that "mathematical explanations involve more than the observation of a pattern. Although a pattern may explain some property of a sequence of values, the pattern itself requires an explanation that exposes the structure underlying it" (p. 25). In addition, the manner in which the boys arrived at the solution of this task emphasised the necessity of visualisation in word problem solving. After visualising the task further (more accurately), the boys were able to detect their misconceptions and corrected them promptly.

Reasoning process justification (RPJ)

Justification refers to an argument that demonstrates (or refutes) the truth of a claim and that uses accepted statements and mathematical forms of reasoning (see Section 2.2.1.2). The following extract from Table 3.2 is a reminder of the observable indicators used to analyse RPJ.

RPJ1: provides proofs to validate claims and arguments
RPJ2: provides acceptable reasons for action (asks for clarification from others)
RPJ3: promotes understanding among those engaged in justification e.g. does something to answer another person's concerns and lessen their worries.

Vignette 3 – Millie's justification

In this vignette, I analysed and discussed Millie's justification for the first task. I selected this vignette because of the notably strong relationship between Millie's visual imagery of PI and CPI and her reasoning processes of explanation, justification and argumentation, as illustrated in Figure 4.19, above. This indicates an embodied form of reasoning that emanated from her use of visualisation processes to galvanise her reasoning–visualisation mix with action (enacted visualisation). Another reason why I selected this vignette was to show why Millie's reasoning processes in her group work consisted mostly of justifications (Figure 4.7), as a response to the other two participants' demand that she provides proofs to validate her claims (RPJ2). The reasoning process of justification had the most coded indicators, with RPJ2 and RPJ3 in the first and second positions, respectively (Figure 4.10). The relationship between RPJ and visual images was also the strongest and therefore worthy of explication.

The excerpt below presents Millie's support for her argumentations. She started by placing a piece of tracing paper on top of the sketch, traced the inscribed square and marked off its centre. She employed visualisation processes to justify her argument as follows:

> *Okay. So, you guys are saying basically, that from here till…gosh…so this, each point of this square* [points the vertices of the square with a pencil] *is touching the circle; the circumference* [imitates drawing a circle by gestural movement with a pencil in the air]. *And, what I'm saying is from here till here,* [refers to the red lines in Figure 4.28] *it can be…it's also ten centimetres if the circle radius is ten centimetres* [walks a path with a pencil on the radius of the circle – the blue line].

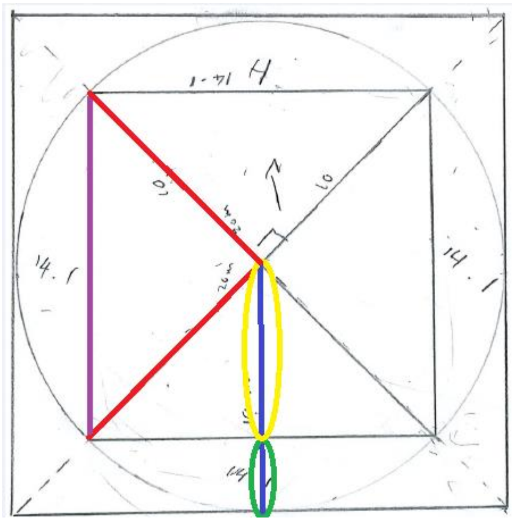


Figure 4.28 A not-to-scale sketch to represent Task 1 drawn by Millie[5]

Rauna seemed unconvinced by Millie's claim. (She exclaimed "*oh, oh!*"). Millie on the other hand continued with her explanation and provided proof for her claims. She continued:

> *Because if you turned it…if you turned it…* [holds the centre with a finger and turns the tracing paper such that the length between the centre and the vertex of the square equals the radius of the circle*] if the…because it doesn't look like this, it's not accurate, but if it was a perfect square, it would still be the same* [gestural movements with her hand as she justifies her point].

[5] The sketch is not drawn to scale hence, the red piece looks shorter than the blue piece, but they are in fact equal.

Rauna, still sceptical of Millie's claims despite visual evidence of her claims and proofs to validate them, commented: "*so, my question is, yes, it is…now we understand your argument, but what about this space that's missing here*" [refers to the green piece in Figure 4.28]? "*Because the radius from here, from this part of the circle to the middle is ten*" [the yellow piece in Figure 4.28]. In reaction to Rauna's misunderstanding, Millie exclaimed: "*No! What I'm saying is, from the middle of the square to the corner of the square* [walks a path from the centre to the vertex – the red line in Figure 4.28], *this is ten centimetres"* [draws the 10cm line]). Like Rauna, Meagan was also confused as she also disagreed with Millie. Both girls confused the radius of the circle (the red line in Figure 4.28) with a side of the inscribed square, as they believed that the diagonals of the square bisected it into equilateral triangles instead of isosceles triangles. They took the diagonals and the sides of the square to be equal in length. Attempting to improve their understanding Millie further justified her claims as follows:

> *But this isn't a side* [emphasises the radius of the square when she argues that it was not the side of the square]. *This is from the centre to the point* [walks a path from the centre to the point – the red line*]. This is from the centre to the* side [walks a path from the centre to the side – the blue line]*. So, it obviously can't be the same as this* [shows the side of the inscribed square with a pencil and smiles at Rauna as she warrants her claim].

In addition to oral justifications, Millie incorporated gestures and drawings to vividly present her arguments. This is in line with Antle's (2009) observation that the structure of the human body acting in complex physical, social and cultural environments determines perceptual and cognitive structures, processes and operations. Further, justification as a learning practice promotes understanding among those engaged in the justification – both the individual offering a justification (in this case, Millie) and the audience of that justification (Meagan and Rauna) (cf. Staples et al., 2012). However, Meagan was still confused even after Millie tried to visually unpack the problem and offer explanation to improve her understanding. Below is an excerpt from the two girls' argumentation:

> **Meagan**
> You're just confusing me more. Because this is already like ten, this ten [walks a path alongside the blue line in Figure 4.28**Error! Reference source not found.**, which is 10m].
> **Millie**
> Yes, if that's ten [traces the 10m (red) line again as she emphasises that it was 10m].
> **Meagan**

Then how is this ten? Now how is this to here, ten?

Millie

Because this is the point… [points with a pencil and smiles at Meagan; probably wondering why she does not get it].

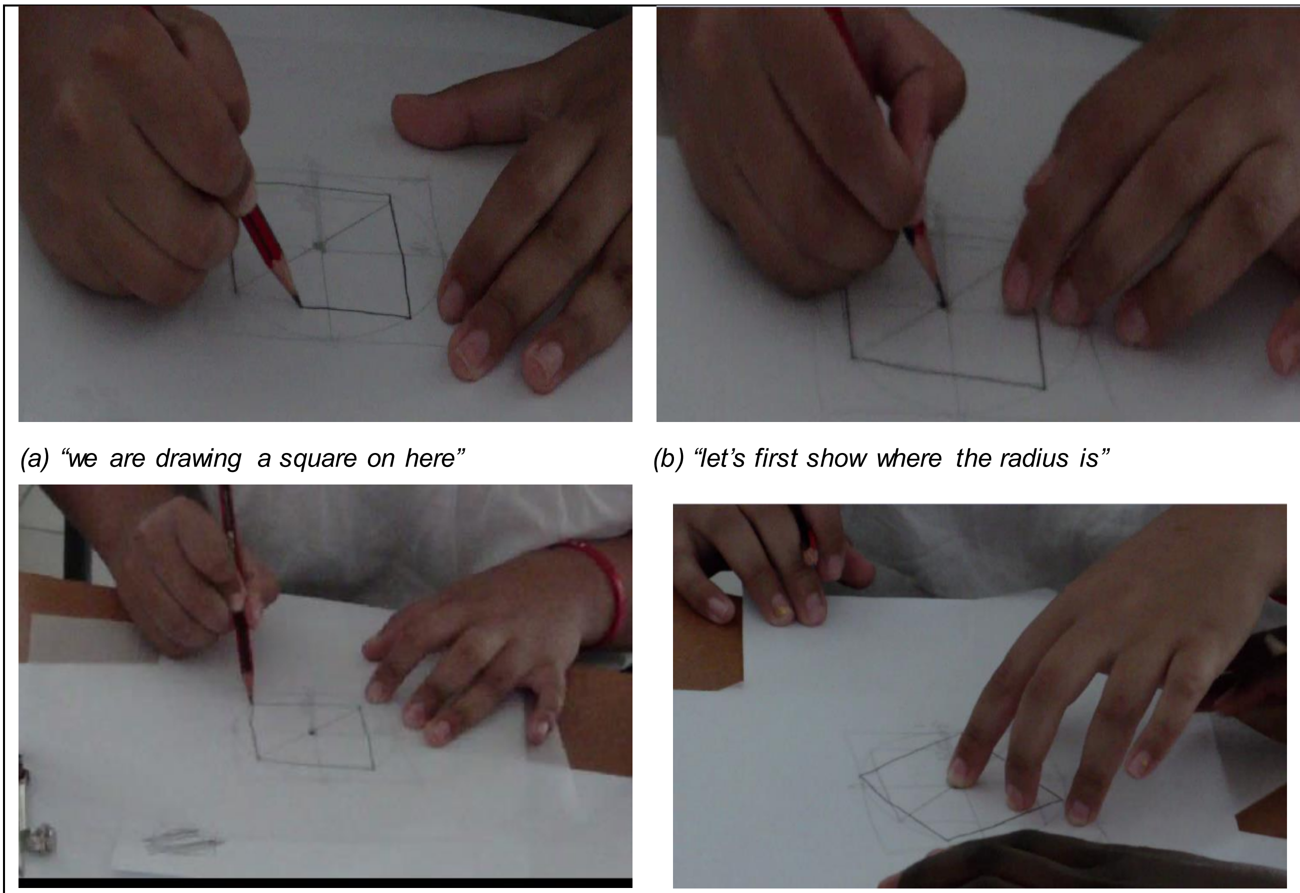
Meagan

What?

Millie

Look, now I need another piece of paper. [Tears a piece from the tracing paper and uses it to trace the radius] (…) look, [rotates the piece of tracing paper to show practically what she meant] you see?

Millie argued that half of the diagonal of the inscribed square (red in Figure 4.28**Error! Reference source not found.**) equalled the radius of the circle in which it was inscribed (blue in Figure 4.28), as they were both radii of the circle. She did this by using tracing paper to visualise the equality of the two sides, as shown in Figure 4.29, below.



(a) "we are drawing a square on here"

(b) "let's first show where the radius is"

(c) "so what I'm saying is from the centre of the square to any of its four points"

(d) "because you can rotate it…" [placed her finger at centre of rotation]

Figure 4.29 Millie's tracing paper

To help justify her solution and provide further insight, Millie opted for a more visual method as she tried to enhance her teammates' understanding. She tore off a piece of tracing paper, traced the length of the radius and rotated it on the sketch to show that the two dimensions were equal in length. But this did not convince the other two girls in her group. Both Meagan and Rauna resisted Millie's arguments as they failed to make sense of her justifications and protested that the sketch be first drawn to scale. Millie then decided to use a larger piece of paper to visually demonstrate to her teammates what she meant. In this instance, she traced both the inscribed square and half of its diagonal (Figure 4.29 **Error! Reference source not found.**(a)). She then rotated the traced figure to provide evidence of why she claimed that the two lengths precisely fitted onto each other (Figure 4.29 (d)). She used visualisation processes to help improve Meagan and Rauna's understanding of the relationship between the radius of the circle and the half of the diagonal. The more Millie justified her arguments, the better she perfected her mathematical reasoning. From an enactivist perspective, Millie's interaction with her living and active body created her structural couplings with the other two girls. This interaction further created co-emergences of Millie's and her teammates' VI and RP, which in return produces the "structural coupling" that enabled them to continue interacting (cf. Rossi et al., 2013, p. 38).

Vignette 4 – Nate's justification

Nate was silent during the second through to the fourth task, and he participated minimally in executing the first and the fifth task. I selected this vignette to illustrate that even though Nate was silent for most of the task-based interview, when he did participate he sparked some interesting visualisation and reasoning processes. So, in this vignette, I discuss the interesting way in which Nate used the reasoning process of justification (RPJ) in the course of solving two problems: first, in relation to how he justified his actions, claims and arguments during the first task, and secondly, during the fifth task in which he was more active in comparison to his participation in the previous tasks.

Nate remained silent when the group began the first task. He quietly observed his peers' lengthy deliberations and then stated: "*so, can't we use SOHCAHTOA*[6] *to find this length since we have the distance of this" [moves his pencil along the two radii, forming the half of the inscribed square's diagonals*]. When Ethray asked him how he knew that the side was 10

[6] SOHCAHTOA is a useful mnemonic for remembering the definitions of the trigonometric ratios sine, cosine, and tangent i.e., Sine equals Opposite over Hypotenuse, Cosine equals Adjacent over Hypotenuse, and Tangent equals Opposite over Adjacent.

cm he replied that it was "*because it is the diameter of the circle"* which implied that when he halved the diameter in his mind, it gave him 10cm; the length of the side in question. He was able to visualise a right angled triangle within an overall sketch and without having to visually sketch it. He reasoned:

> *I thought we're going to use SOHCAHTOA since we have the diameter,* [mimics drawing the diameter as he shows it with a pencil] *which is the diagonal of the square inside the circle* [mimics drawing the diagonal and the circle as he speaks]*. But since we… I thought it was a 90 degree triangle because it's a triangle in the semi-circle* [traces a semi-circle with a pencil].

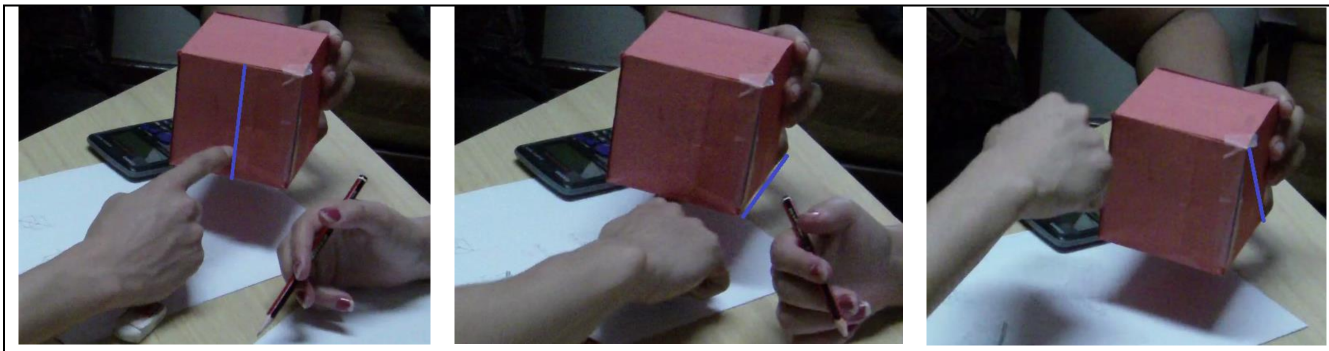
Nate initiated the use of SOHCAHTOA, provided proofs for his claims and arguments and then left the idea to his peers to complete the task. He did not talk much but he gestured and conceived pictures in his mind that he placed on paper or explained orally when prompted to do so. Nemirovsky and Ferrara (2009) argue that:

> whatever we can recognise as rational, rule-based, or inferential, is fully embedded in our bodily actions; perception and motor activity do not function as input and output for the "mental" realm; what we usually recognize as mental are inhibited and condensed perceptuomotor activities that do not reach the periphery of our nervous system. (p. 161)

As was his custom, Nate once again remained silent when his peers started to deliberate over the fifth task. He quietly made gestures with his hand and observed the others as they attempted to find the dimensions of the cube. Ethray commented: "*okay, so now we have a cube* [sketches another cube], *each side is ten. So now we're looking from there to there"* [draws a diagonal in his sketched cube]. Ellena asked him a rhetorical question: "*isn't it ten and then another ten?"* [Looks at the diagonal as if though it is made of the adjoining 10m edges of the cube]. Ethray agreed with her, saying: *"from there till there* [walks a path on the surface of the cube with a pencil; formulating a right angled triangle]*. Isn't it also this fourteen point one? Here's also fourteen point one* [mimics drawing a diagonal on the surface of the cube]". When I asked them why they believed that the length in question was 14.1 centimetres as they claimed, using Figure 4.30, Nate reasoned as follows:

> *Because this distance downwards is ten* [walks a path from top to bottom edge of the cube with a finger]. *That's ten if you go the other way* [moves his hand under the

cube to show the 10m length]. *So,* [moves his hand as if he was drawing a diagonal on the face of the cube but did not say what he was doing] *fourteen point one (…) so we basically look at the square from the side, cut it in half* [sketches a triangle to represent half of a cube].



(a) "*this distance downward is 10*" (b) "*that's 10 if you go the other way*" (c) "*fourteen point one*" (diagonal)

Figure 4.30 Visualisation and reasoning processes in action as Nate justifies his arguments

I asked Nate to show me what he meant by "cut it in half". Instead, Ethray, who was convinced by Nate's argumentation and justification, at that moment commented:

> *Cut this in half* [he placed his hand on the surface of the cube as though it were a knife cutting through the cube] *(…) if you cut this in half you'd have a triangle like that* [places his hand like a knife again and moves his finger around the surface in question]. *But then you'll have like (…) it goes that way. Something like that* [tried to sketch something like half of the cube but did not complete the sketch].

Ethray was convinced by Nate's arguments yet unsure of the shape when a cube is cut in half. Nate sketched his mind picture of the half of the cube (Figure 4.31) in an attempt to help improve and promote understanding in his group.

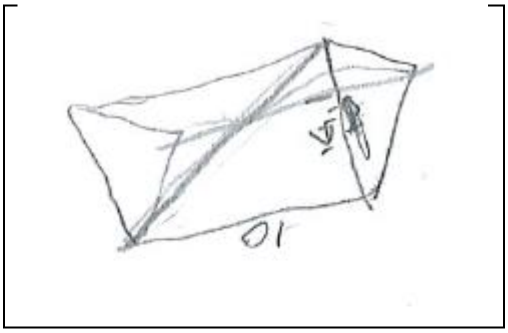


Figure 4.31 Nate visualisation of a halved cube

Figure 4.31 illustrates Nate's idea of how a cube was halved along its diagonal. Ethray finally saw the bigger picture as he placed his hands on the two opposite faces to visualise how the cube would have been cut in half. Consequently, Nate's justification of the word problem helped to promote understanding among his group members and enabled them to solve the problem.

Reasoning process argumentation (RPA)

An argument is a part of reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate and acceptable (see Section 2.2.1.3). The following extract from Table 3.2 serves as a reminder of the observable indicators used to analyse RPA.

RPA1: provides support for explanations and justifications (this includes insisting on accuracy of own and others' claims)
RPA2: convinces/persuades others via verbal/visual activity of the truth of their claims and appropriateness of their reasoning (or is convinced and persuaded by others – i.e. when they accept the truth of each other's claims and explanations)
RPA3: accepts/refutes truth of others' claims that they may agree/disagree with

Vignette 5 – Rauna's argumentation

The purpose of this vignette is to analyse and discuss Rauna's interaction with the other participants in her group and the problem-solving environment when she employed the reasoning process of argumentation (RPA) during the fifth task in EVGRT W2. I selected RPA for this vignette because of the interesting way in which Rauna argued strongly so as to validate and justify her claims as her group attempted the task. I show how she managed to convince others of the accuracy of her reasoning, and how she accepted and/or refuted the truth of others' claims and justifications.

Rauna formulated pictures in her mind when she studied the cube that she held and rotated between her fingers. She uttered her thoughts and explained to the other participants what she had in mind. As she spoke of "*going through the cube*" she argued that the diagonal of the cube equalled the length of its face, which was 10cm plus the diagonal of the face, which was 14.1cm. Simply stated, Rauna argued that the blue line in

Figure 4.32 below, plus the green line gives the same length as the red line. She clearly disregarded the 90 degrees bend at 'vertex 4' (V4[7]). All she saw was a straight line from V to V3 as the blue and the green lines look as if though they are the same length as the red line. When I inquired of her why she was passing through another vertex instead of going straight from V to V3, she argued that it was the same thing. Meagan disagreed with her ("*no, she means it's like you're going here to this one, like in the middle, like cutting through.*" *[Picks up the cube and holds it on one vertex and imitates cutting through with her hand]*). Rauna insisted that cutting through the cube (V to V3) was the same distance as passing through the V4. She refuted her groupmates' claims and disagreed with their suggestions as she instead tried to convince them of the accuracy of her own arguments.

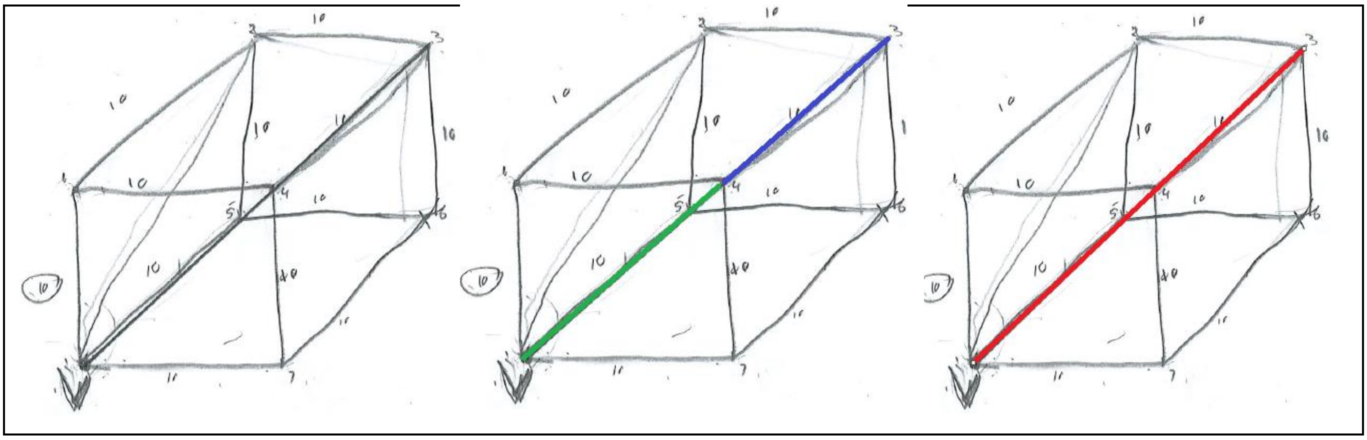


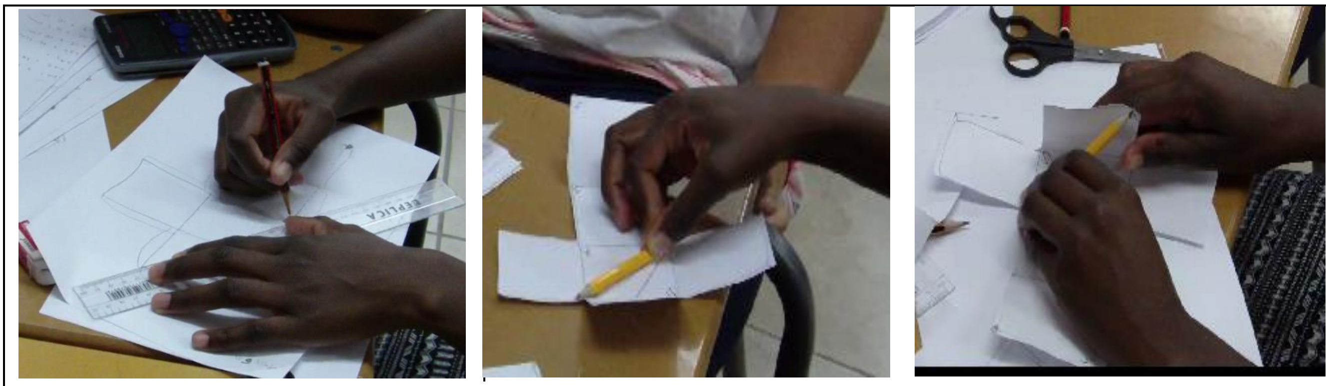
Figure 4.32 Visual representation of Task 5

I deemed it necessary to have talked the participants through their visualisation processes to enable me to grasp the meaning behind their visual images and to clear away any misconceptions. However, when I tired of Rauna's creative yet unsubstantiated arguments, I prompted her to sketch a net of a cube to enable her to see the picture from another perspective and to illustrate any misunderstandings this way. I encouraged Rauna to apply visual imagery in line with Wheatley's (1991, p. 35) observation that "when learners are encouraged and given opportunities to form mental images, most readily do so, and when they are encouraged to use imagery, their mathematical power is greatly increased" (p. 35). With her aroused mathematical power, Rauna immediately sketched a net and continued with the same argument. Drawing the net helped Rauna to visualise the picture in the mind at length and to convincingly argue her point more clearly.

[7] The girls numbered the vertices of the cube to enable them to be in accordance when they discussed the task. For example, they numbered vertices 1 to 7 after denoting the required vertex, V. I refer to these vertices as V1, V2, and so forth in this case study.

> *(…) look,* [turns the sketch to her side] *if we have to do this, it's not accurate but you guys understand (…)* [draws a diagonal from V to the vertex agreed to be 3 on the net]. *It's going to be the same thing,* [gestures by moving her hands up as if though she was folding the net] *it's the same distance* [shows the distance by holding sides of the net between her hands]*. Because it's, do you see?* [Curves up the paper] *(…) from here is mos fourteen point four, ne?* [Draws a diagonal from V across to the opposite vertex (Figure 4.33(a))] *But now, look here* [places a pencil on the diagonal (Figure 4.33 (b))] *(…) if you don't understand look* [unfolds the net then places the same pencil to cover the length of the diagonal across the same face (Figure 4.33 (c))] *(…) the distance here is fourteen point one, distance here in the middle [refers to the diagonal of a face and draws it]. Now it's the same distance as if we go through it will still look like that* [imitates going through the cube with a pencil by using the net and her famous pencil technique].

While Rauna presented her arguments, she tried to convince her teammates as she displayed distinct gestures that went along with her verbal utterances – noticeable relations between visualisation and reasoning processes.

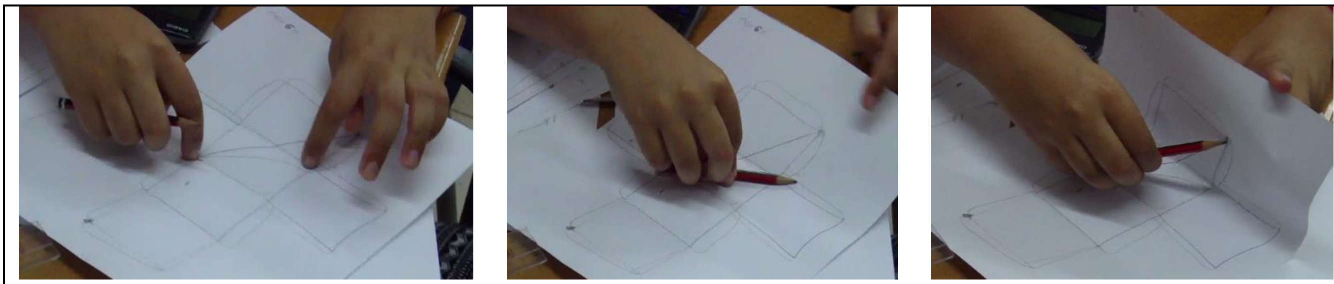


(a) "if we have to do this" *(b) "the distance here in the middle"* *(c) "is the same distance as if we go through"*

Figure 4.33 Rauna's argumentation for the fifth task

Rauna's misconception was so entrenched in her thinking that she failed to see the bigger picture. She was unable to tell the difference between the length of the diagonal of the face and that of within the cube. She justified her reasoning by inadvertently folding the net to reach for the opposite vertex as seen in Figure 4.33. Nevertheless, she argued well, and others were convinced – which is the overall focus of this case study. She displayed confidence and articulated richly even when her arguments were inaccurate – even to the point of convincing her teammates. As reviewed in literature, Rauna demonstrated strong verbal, social and rational argumentative skills aimed at convincing a reasonable critic of the acceptability of a conclusion by foregrounding a pattern of propositions by justifying or refuting the proposition expressed in the conclusion (cf. Dove, 2009, p. 139).

Figure 4.34 shows how Millie was convinced by Rauna's arguments and could imitate her beautifully.



(a) "she was saying, this distance..." *(b) "…is fourteen point one"* *(c) "if you fold it like this, it's still 14.1"*

Figure 4.34 Millie convinced by Rauna's argument

Figure 4.344 illustrates that mathematical arguments do not have to be accurate in order to convince someone – argumentative skills merged with visual imagery was all it took Rauna to convince Millie. This was evidenced by the way she imitated Rauna's argument via a display of visualisation and reasoning processes – at least for a minute or two – as displayed in Figure 4.34, above.

Vignette 6 – Millie's argumentation in relation/reaction to Rauna's argumentation

The reason why I selected this vignette was because I wanted to discuss the interesting collaborative argumentation between Millie and Rauna. Part of the reason why Millie's RPA was closely connected to her visual images (Figure 4.19) was because of her interesting interaction with Rauna.

Although initially convinced by Rauna, Millie eventually found her own way through a critical analysis of what the task entailed. She managed to unearth the misconception that led to Rauna's convincing yet inaccurate arguments. She studied the sketches quietly and commented:

Our thing is wrong! Look, this is where V has to be (…) I'm saying, this point is not this point. This is one, this is two (…) Look there, that's two, that's three. How can one be there and also be next to three? (…) it was right how you had it, you said V is three, V is here, three is here, [shows Rauna on the net] *what was wrong on your thing that you wrote, you said one is here, and three is there. But this one is two and then here is one. You see?* [Folds part of the cube as she shows where 1 is].

After numbering the vertices accurately, Millie refuted Rauna's argument to which she had initially acceded. She knew all along that it was impossible for the diagonal of the cube (red line in Figure 4.32) to equal the length of the blue and the green lines put together. She argued as follows:

If I look at it like this, ma'am, because if you just hold it like that you can tell like this…this is from this point to this point is larger [holds the cube by the opposite vertices as she explains (Figure 4.35 (a))] *(…) can't we say we cut it in half?* [Holds the cube with the sides of her hands on the opposite faces as if though she was cutting through it (Figure 4.35 (b))] *(…) It can't be the same because if you say the distance from here to here…from, say now here to here,* [shows the distance by holding up the cube between its opposite vertices] *and you cut this half, [imitates cutting the cube in half with a finger] like literally like that…* [Turns to the sketch and mimics cutting through the drawn diagonal with her finger] *(…) then it will have new measurements, won't it? Yes!*

(a) *"if you hold it like this"* (b) *"can't we say, we cut in half"* (c) "*the triangle…like this…*"

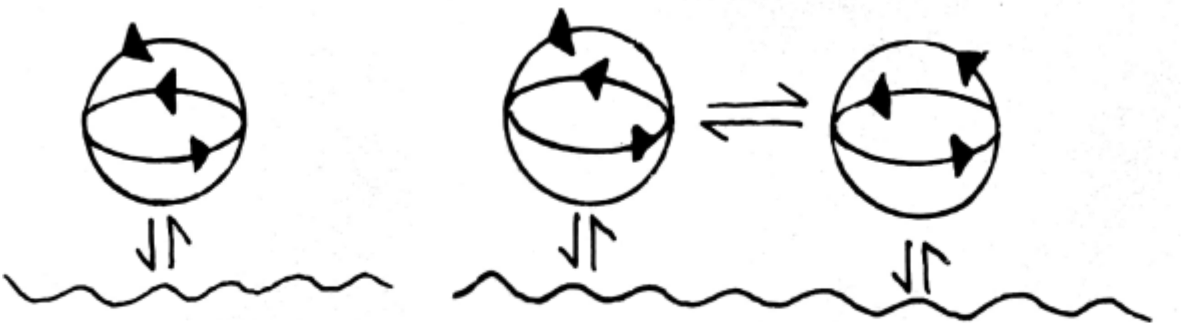
Figure 4.35 Millie's visualisation and reasoning processes in Task 5

Millie sketched a triangle that she visualised in her mind. She spoke of cutting the cube into half. She demonstrated this act (Figure 4.35 (c)) as she simultaneously perfected her justification, as follows:

> *The triangle, ma'am, like this, this, this, and there…* [Sketches a 3-D of a halved cube] *…see, it will have new measurements. Actually, this is supposed to be* wider [refers to the side that represents the diagonal of a cube] *(…) you won't find fourteen point one. You'll find a different…that's why I said this one will be larger because (…) that's what I was telling you. Look at it like* this [holds up the cube by its opposite diagonals to show Rauna (Figure 4.35 (a))].

Both Rauna and Millie employed an amalgamation of visual imageries and reasoning processes during this process of collaborative argumentation. Although Millie was initially convinced by Rauna's arguments, she eventually refuted her claims as the picture in her mind became clearer and as she visualised and reasoned more on the task. She split the cube in her mind by visualising cutting it in half (Figure 4.35 (b)) and alongside its diagonal, which was the required length. From an enactivist perspective, we conclude that Rauna, Millie and the environment have undergone transformations – they became structurally coupled (Maturana & Varela, 1998, p. 102). The offspring of this coupling was the co-emergence of visualisation and reasoning processes.

Figure 4.36 illustrates what happens when an organism enters into structural coupling with other organisms and the environment (Maturana & Varela, 1998, p. 180).



(a) organism's structural coupling with its environment

(b) organism's structural coupling with another organism and the environment

Figure 4.36 Structural coupling with the environment and other organisms[8]

[8] Adapted from Maturana and Varela (1998, p. 180)

Maturana and Varela (1998, p. 181) maintain that these couplings occur when organisms take part in recurrent interactions. It is the organism as a whole (i.e. the participant), together with its environment (i.e. the task-based interviews), which co-evolve, according to enactivist thought. Thus a classroom, a school, or two learners interacting around a shared object of interest can learn and co-emerge (Khan et al., 2015, p. 273).

Reasoning processes generalisation (RPG)

Generalisation refers to the introduction of new ideal objects, to overcome objective constraints and to identify the operators and the sequence of operations that are common among specific cases, and extend them to the general case (see Section 2.2.1.4). Below is an extract from Table 3.2 that serves as a reminder of RPG's observable indicators.

RPG1: elaborates the problem further to try to learn more from the result by relating the problem to similar situations.
RPG2: uses visualisation to demonstrate how the problem can be solved in a different way

Vignette 7 – Meagan's generalisation

Although Meagan's generalisation process was minimally used during the task-based interview, the way in which she generalised her solution to the third problem is why I opted to include this vignette. The girls visualised a triangle embedded in the overall diagram of the third task. They used this triangle to obtain 75 degrees, the solution to part (e) of the third task (see Appendix 2 for EVGRT W2 items). I asked them whether they could get 75 degrees without using a triangle as a way of scaffolding them to generalise their solution. Meagan instantly agreed that there was another way and got right into it while Rauna claimed that the quadrilateral was cyclic. Meagan initially aided Rauna's understanding that the kite in the diagram was not a cyclic quadrilateral. The excerpt below illustrates the conversation between the two girls as they unpacked the concept of cyclic quadrilaterals:

Meagan
You can use this plus… [refers to the two 90 degrees angles formed between the tangents and radii of the circle]

Rauna
Oh, it's a cyclic quadrilateral. Mm hmm. [Draws quadrilateral on paper with her finger]

Meagan

No, [draws a circle with her finger in the air] the cyclic is the one in the circle [points in the circle with a pencil.

Rauna
Is it?

Meagan
Yes.

Rauna
I'm just asking. I didn't know. I thought it was like any four pointed thing [draws a four pointed thing with a finger].

Meagan
Yeah, but then it's still the same, like…[marks 90 degrees angles repeatedly]

Rauna
It's still in the circle [points in the circle with a finger].

Meagan
You can add these ones [refers to the 90 degrees angles].

Rauna
52 point 5 plus 52 point 5…[uses the calculator]

Meagan
Yeah, but this whole thing is not in the circle [draws a circle around the thing that she talks about without placing pencil on paper] (…) she was saying that this is a (…) cyclic quadrilateral. But then I was saying it isn't because this part [the tangents] isn't in the circle.

Thereafter, Meagan proceeded to generalise the solution to show how to arrive at the solution in a different way. She commented:

> *Look, if you do this, you can also get it by, 90 plus 90 plus 105, and then you minus it from 360*. [Dances as she uses the calculator] *(…) 285…it was 285. Right? And then you can still do 360 minus 285 (…) because ma'am it will add up to 360 in this* [walks a path around the kite with fingers]*. The whole thing will add up to 360 (…) I was telling them we can use this like the 90 of the tangent and the radius* [gestures the perpendicular of the tangent and the radius with her hands] *(…) and then you add these ones* [moves between the two 90 degrees angles with a pencil].

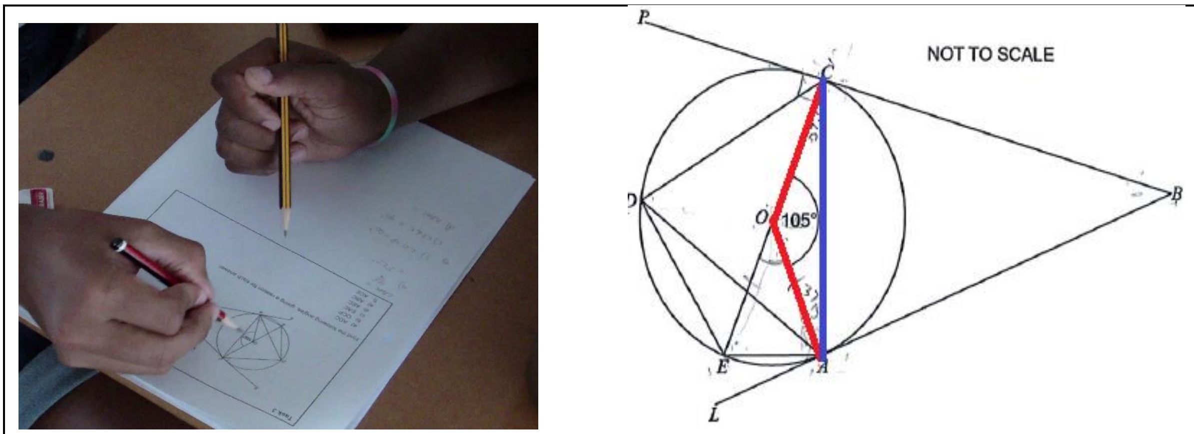
Meagan generalised well here. She used the sum of the interior angles of quadrilaterals to find the solution via a series of visual imageries. This means that her 5VIs co-emerged well with her 4PRs, which is the overarching argument of this study. She offered more than just a generalisation; she also offered justification for why she insisted that the quadrilateral in question was not cyclic. In the context of this case study I conclude that Meagan used visualisation processes to help her explicate her arguments. She mimicked drawing a circle around the sketch (KI3) ("…but this whole thing is not in the circle") to validate why she claimed that the quadrilateral was not cyclic.

Vignette 8 – Denz and Jordan's generalisation

I selected this vignette because I was interested in how Denz and Jordan interchangeably generalised their solution for the third task. When they solved part (d)[9] of the third task, Jordan and Denz elaborated the problem further to show how they could solve the task in a different way. When Denz initially looked at the problem, he visualised a right-angled triangle with an angle of 37.5 degrees. He then suggested that they subtracted the sum of 90 degrees and 37.5 degrees from 180 degrees to get the required angle. Jordan swiftly responded with a different problem-solving strategy. He reasoned as follows:

> *(…) we could have looked at it like this* [gesture movement with a pencil around the diagram as he notices the angles]. *This is also an isosceles triangle, you know? Like this is 180, this is 75* [marks off adjacent sides of isosceles triangle, supplementary angle of 105 = 75 (Figure 4.37)] *(…) It's going to be 75. Then, 180 minus 75, it's going to be this one divided by two. 105 divided by two.*

Denz also joined Jordan as they both pointed with their pencils on the diagram. They worked out the unknown angles as a team. Jordan then traced an isosceles triangle when he uttered ("*this is an isosceles triangle"*) as illustrated in Figure 4.37. He was able to visualise the triangle embedded in a complex figure. He employed both visual imagery and reasoning processes as he supported his claim.



"*wait, this is also an isosceles triangle*"

[9] See Appendix 2 for EVGRT W2

Figure 4.37 Jordan's visual processes of EVGRT W2 T3

Denz agreed with Jordan's generalisation and suggested a further strategy to see if he could obtain the same answer. He suggested that they look at the problem in a different way ("l*et's also check this way, 180 minus 90, minus 37.5 … same answer)*. Jordan then concurred with him and the two boys proceeded to the next question. The boys worked together and interacted beautifully. According to Maturana and Varela (1998, p. 75), a history of recurrent interaction between these two boys led to the structural congruence between them and the environment in which they were operating. According to Maturana and Varela (1998), this is called structural coupling.

The way that these boys solved the task vindicated my assumptions about how visualisation and mathematical reasoning relate and co-emerge. There was a noticeable pattern in how they deliberated over the task and arrived at a collective solution. Denz started by reading the question while Jordan studied the given figure to correspond with what he read. The boys chorused most of the answers to the unknown angles and often completed each other's sentences when they provided reasons for the sizes of the angles that they found. Jordan mostly worked on the calculator while Denz wrote down the answers that Jordan uttered. They were both involved with the task, at the same time. The boys also answered my second research question right there. I was so glad that I had paired them.

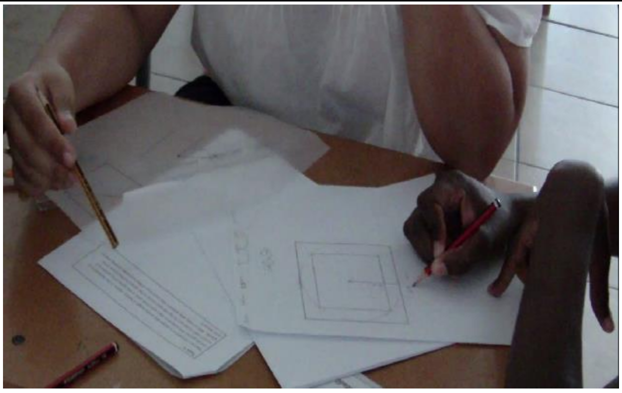
4.3.4 Structural coupling and co-emergence insights garnered from Phase 2 analysis

The purpose of this section is to tease out more of the relationship between visualisation and reasoning processes that emerged from the data when it was analysed. I focus on special cases of the close relationship between the participants and their problem-solving environments. Their actions were observed to be mostly unconsciously embodied and coupled with those of their peers as their reasoning co-emerged. Some of these actions are described hereunder.

Rauna's observed pattern

In terms of making sense of a problem as it was being read out (RPE1), Rauna always took the lead in her group. She seemed to have coupled with the other two girls in her group, as I noticed a common pattern every time they commenced a new task. Rauna drew every initial sketch for all the tasks that they solved in the group. As soon as somebody started reading out the task, Rauna invariably sketched diagrams to represent what she heard from the reader (Figure 4.38). She sometimes asked the readers to pause or repeat themselves as she sketched the details upon hearing them. This, according to enactivism, is a characteristic of structural coupling.

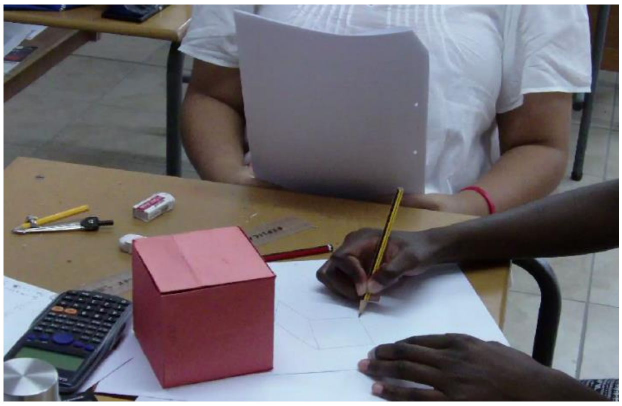
Figure 4.38, below, illustrates the following with regard to Rauna's pattern of behaviour that I observed: Tasks 1, 4, and 5 demonstrate Rauna's coupling with self, others and the environment. In Task 2, Rauna read and sketched – coupling with self and the environment – bringing forth a world of meaning. For Task 3, which did not require initial sketching as the task consisted of a given sketch, Rauna's explanations and proposals for problem solving strategies altered to suit that task's context. During this task, she suggested that they started with what they knew and got right into it ("*okay, guys, before we start…let's first see what we know. CAE is 90 degrees* [walks a path on the diagram]*. Angle in a semi-circle is 90 degrees"*) and others followed her lead. As mentioned in the literature review chapter, Simmt and Kieren (2015) observe that one of the moves in enactivist research is to "recognize the relationship between the learner and the environment in which the learner is seen to bring forth a world" (p. 310). They modelled the interaction that brings forth a world of significance, as seen in Figure 2.1. By engaging others in interaction early on in the tasks, Rauna enabled them to give and receive help. Webb (1991, 368) claims that learners working in a group have the potential to provide understandable and timely explanations to each other. In the context of this case, Rauna was able to give understandable and timely explanations to her group mates by making use of visual imagery. As a group, the girls managed to arrive at collective solutions through mathematical reasoning.



Task 1: Rauna sketched as Millie read

Task 2: Rauna first read the task and then constructed a circle as seen in her mind

Task 4: Rauna sketched as Meagan read



Task 5: Rauna sketched as Millie read

Figure 4.38 Rauna's observed pattern

Denz and Jordan's observed pattern

As previously described, Denz and Jordan completed each other's sentences throughout the task-based interview. I commented on how substantially they complemented each other when I observed them working together. I was very impressed by how they interacted with each other from the very first task. I also observed that, like Rauna, one of the pair started sketching while the other read out the task. As previously noted, Maturana and Varela (1998) define this as structural coupling. They claim that structural coupling is always mutual as both organisms and their environment undergo transformation (p. 102). Maturana and Varela (1998) also argue that "the structural changes that occur in a unity appear as 'selected' by the environment through a continuous chain of interactions" (p. 100). As a result, the environment can be seen as an ongoing "selector" of structural changes that the organism undergoes in its ontogeny (p. 102). When Denz and Jordan solved the first task, they both continually and simultaneously had their pencils pointed at the sketches. I sometimes did not know how to record their actions and exchanges, as it was often hard to decide to whom I should allocate which move or comment and when, as most movements happened concurrently.

Denz and Jordan's interaction which brought forth a number of interesting visual imageries and reasoning processes when they solved the fifth task. During this task, the boys completed not only each other's sentences but also each other's drawings. Jordan read out the task while Denz sketched as he listened. When he wanted to mark off the vertex on his sketch (see Appendix 2 for Task 5), he realised that he could not recall what it was, so he asked Jordan ("*what is a vertex?*"). Jordan used memory imagery (MI) as he replied with a rhetorical question (*"vertex, isn't it the end here?*" [marks the vertex with a point and labels it V on Denz's sketch]). When Jordan had finished reading the task, both boys commented that it was a challenging problem, but that they were up for a challenge.

Before they began to solve the task, Denz suggested that they work out the total number of vertices in a cube so that they established the problem solving starting point. Jordan picked up an actual cube and counted the number of vertices while rotating it in his hands. This was an instance of kinaesthetic imagery, involving physical activity as part of solving the problem. Denz claimed that there were eight vertices in total, but he was still confused and had little faith in his claim. To explicate the issue and to help improve his teammate's understanding (RPJ), Jordan used more kinaesthetic and also dynamic imagery to emphasise what a vertex was (*"this is a vertex. This, this, this, this*" [holds up the cube and rotates it around in his hands as he shows the vertices]). When he received this explanation, Denz was relieved and commented (*"oh, okay, so there's eight*").

It was fascinating to observe these boys' interaction from the very beginning of the task. They ensured that they were both operating at the same level before they moved on. They read the task several times before they reached a common understanding of how to work out the total distances as required. The excerpt below illustrates Denz and Jordan's interaction, and the embodied visualisation and reasoning processes that they brought to bear as a result of this interaction.

Jordan

I don't know if, ma'am, can you like move through the cube in the middle? [Lifts up the cube to show the diagonal that passes through the middle of the cube]

Denz

Yeah, the shortest way, mos.

Beata

Just the shortest way.

Denz

So how do you find…in the middle?

Beata

Good question. How do you find it in the middle? The question is from me to you.

Jordan

Okay, check here…

Denz

Find it in the middle.

Beata

Check here…? What do you want us to check?

Jordan

Move from here…through to there. It's like…no… seriously…? [Holds up the cube with his fingers on opposite vertices… he turns the cube around and observes 46:55]

Denz

It's there to…phew! [Makes a sketch that looked like half of a cube]

Jordan

That's the correct way? (47:01).

Denz

How would you get there? If you're looking at it straight, it would be what…that…

Jordan

Ten!

Denz

[Laughs] *Mm…*

Jordan

What is this? Look here…what is this…this diagonal, plus this, you know, divided by two [draws a triangle between the three diagonals with his fingers on the actual cube].

Denz

Maybe.

Beata

Can I see those two diagonals?

Jordan

It's not a diagonal, it's this diagonal, plus this side… [traces on the surface of the cube as he speaks]

Beata

Can I see them as a drawing?

Jordan

Okay. So, it's going to be… [Emphasises lines on a sketched cube to explain what he meant by triangle plus a side]

Beata

Can I see it as a separate drawing…?

Jordan

Oh, separate, there…it's going to be… [Sketches another cube]

Beata

You said this diagonal plus this side…

Jordan

Divided by two.

Beata

No, I don't want to see the divided, I'm not interested in divided by two. I'm interested in what you said. This diagonal plus this side…where is the side?

Jordan

I don't know how to draw a cube so you must forgive me.

Beata

No, only those two, I said I only want to see those two sides. The one that you spoke about on the cube. You only touched two sides and then…

Denz

These two? Here…and there…and then divided by two to find across. [Shows sides on a cube with a pencil]

Beata

No, no, no, just show me those two sides only. I don't want to see the whole cube. I only want…

Jordan

So, it's this one…from this one to this one and this one. [Shows sides on the newly sketched cube]

Beata

Can you make for me a sketch of those two sides only?

Denz

Oh, ma'am! It's Pythagoras! Like that. [sketches a triangle that is formed by the two diagonals and a side 48:38]

Beata

Which one was the diagonal? Just show me…

Denz

This would be the diagonal and then this would be…this would be there to there.

[shows corresponding sides on the cube and the sketched triangle]

Jordan

Isn't this the diagonal?

Denz

This is going from top to bottom.

Beata

Okay, top to bottom. What is the size of top to bottom?

Jordan

Oh, if you look at it like this… [Looks at the cube from a different orientation]

Denz

Yeah, and then we find the diagonal would be there, and then…

Jordan

This would be half?

Denz

The adjacent, and then this would be the opposite and hypo…

Beata

What is the length of the diagonal?

Jordan

Diagonal, that's going to be…

Denz

Let's calculate that. [Traces diagonals on the surface of the cube with a pencil]

Jordan

One diagonal…one diagonal is equal to one side.

Beata

You said that in the first task.

Denz

No, remember, it's isosceles [traces all the isosceles triangles on the surface of the cube with a pencil].

Jordan

Draw, ma'am.

Denz

Who must…Pythagoras? (laughs) *You're going to draw skew. Pythagoras?*

Jordan

Ma'am, this is to scale, right? [draws two intersecting diagonals on the face of the cube using a ruler and pencil as he speaks]

Beata

I don't know. You're looking for? (49:35)

Denz

We're just looking for the diagonal, [traces the diagonal with a pencil on the surface of the cube] so, we use Pythagoras to find the hypotenuse.

Jordan

Okay, yes, it's going to be… [measures the sides of the cube with a ruler to see if he could justify his idea of dividing by two] *okay, Yeah, you are right, sorry.*

Denz

So, diagonal would be… [traces diagonals again on the actual cube with a pencil] *diagonal squared equals that plus that. How much is that side? Ten.*

Jordan

Yeah.

Denz

Squared plus ten…again like the other one. Equals two hundred…diagonal equals…what is it, fourteen comma what, one four? [Refers to the actual cube as he writes: $d^2 = 10^2 + 10^2 = 200$, $d = \sqrt{200} = 14.14213562$]

Jordan

Yeah. We still get the same (50:15) *numbers.*

Denz

So, the diagonal is fourteen comma one four. Okay. Then you know mos how we're going to get the last one, that we can't get.

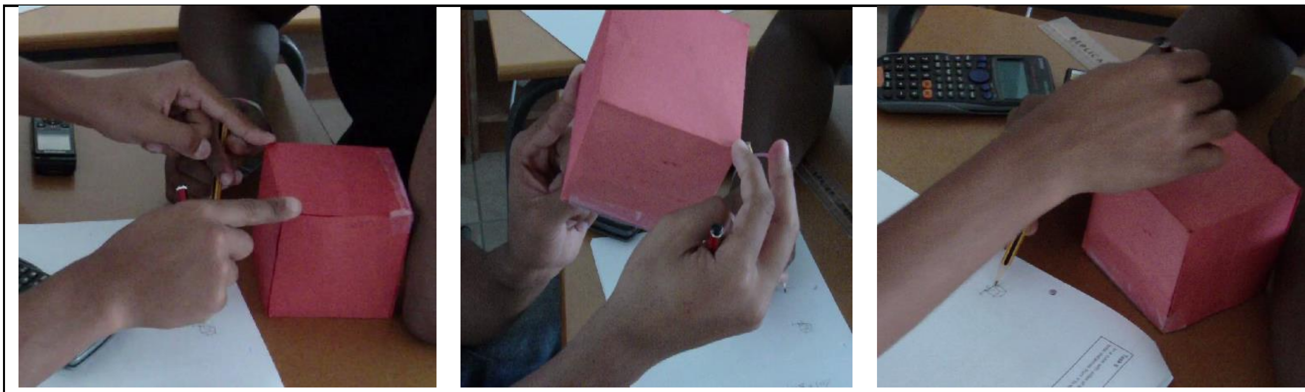
Jordan

Yeah.

Denz

It's there to there, which is diagonal, and then up and then… [Lifts up the cube and shows the diagonals using a pencil as he speaks. He walked several paths with a pencil on the surface of the cube]

Before they started to work out the total number of distances from V, Jordan used the actual cube to count the number of vertices (Figure 4.39 (a)). He flipped the cube over while counting in multiples of two. He did this to ensure that Denz was abreast with the concept of vertex so that there would be no confusion. During this process, Jordan's application of the visualisation processes (concrete pictorial, kinaesthetic and dynamic imagery) co-emerged with his reasoning processes (justification, argumentation). Thereafter, the boys deliberated about the number of distances from vertex V to all the other vertices. Once again, Jordan formulated a picture in his mind (CPI) that he explicated when he asked if he could 'move through the cube' (L1400-1403) (Figure 4.39 (b)). I observed that visualisation processes helped the learners to ask relevant and meaningful questions – even questions that they might not think they were able to ask (L1409) – and to derive justifications from such visualisations (L1421-1424, 1438-1442).



a) counting the vertices "*1, 2, 4…*" b) "*can we like move through the cube in the middle*" c) structural coupling

Figure 4.39 Denz and Jordan's special structural coupling and co-emergence

There was good collaboration between Denz and Jordan. Figure 4.39 (c) illustrates the boys' structural coupling when they worked together to determine the length of the distances from vertex V to the other vertices of the cube. At this point, while Denz traced the distances from vertex V on the sketched cube to all the other vertices, Jordan traced the same paths on the actual cube. They worked together wonderfully, even at times giving the impression that they had discussed or planned the work beforehand. They employed more than one sort of visual imagery at a time as their mathematical reasoning evolved. For example, the boys exchanged the cube between them as they each hurriedly pointed at the sides of the cube that came to their respective minds. They argued their ideas objectively. Denz exclaimed and commented ("*Oh, ma'am! It's Pythagoras! Like that* [sketches a triangle that is formed by the two diagonals and a side] (…) *this would be the diagonal and then this would be…this would be there to there*" [shows corresponding sides on the cube and the sketched triangle]).

Denz used both concrete pictorial imagery and memory imagery when he visualised a right-angled triangle on the surface of the cube. He used this to justify why he claimed that the method of problem solving involved the theorem of Pythagoras (L1494-.1497). As Arcavi (2003) puts it, “the visual display of information enables us to 'see' the story, to envision some cause-effect relationships, and possibly to remember it vividly” (p. 218).

Denz used more visual imagery to argue his point and provide proofs to validate his claims. He redrew the visualised triangle (DI) as he tried to convince Jordan, who inadvertently rebutted his argument (RPA) (L1502-1518) by claiming that the diagonal was rather one of the equal sides of the isosceles triangle. This was a fascinating example of the co-emergence of visualisation and reasoning processes. It was a back-and-forth process, as the participants visualised as much as they reasoned, and vice versa. The participants were then able to find the solution to the word problem through visualisation and reasoning processes. This was at once a verbal, social and rational activity aimed at convincing a reasonable critic of the acceptability/refutation of a conclusion (Dove, 2009, p. 139).

Maturana and Varela’s enactivist theory characterises structural coupling in nonhuman communication such as singing birds as a duet, whereby each member of a couple builds a phrase which the other continues. Each melody is peculiar to each couple and is defined during the history of their mating (1998, p. 194). Using this as an analogy for the groups in this case study, one can say that each member of the group builds a phrase which the other continues. This instance is peculiar to each group and is defined during their history of recurrent interactions from one task to another.

4.4 DATA ANALYSIS PHASE 3

This phase of the case study was conducted for reflective purposes. It was not part of my main analysis, but I needed to look back and ponder on the whole process, to round off the study and give every participant in Phase Two an opportunity to comment on their experience of participation. The eight participants were presented with five semi-structured reflective interview questions (Appendix 11), to ascertain their overall experience of involvement in the whole research process. This is discussed below.

To the first reflective interview question, the participants responded that when presented with a word problem, they first read and re-read the problem to attempt to make sense of what it entailed. According to Brodie (2010), making sense of mathematical word problems is a worthwhile everyday life activity. The participants also admitted that they drew a picture to represent given information before they attempted to solve the problem. When learners translate a mathematical text (e.g., a word problem) into a visual representation by drawing an auxiliary figure or making a modification of a figure, they employ the strategy of visualisation (cf. Bjuland, 2007, p. 3). Six out of the eight participants responded that when presented with a problem, they first ruled out other possibilities before they solved it. Millie commented that: "*I first identified the main problem, then I thought of possible solutions. After that I ruled out solutions until I found the best solution. I try to make the problem a physical thing* [by using visualisation processes] *to help me solve it better*". Meagan claimed to be "a visual person" and preferred visual methods to solve word problems. She commented that:

> *When I was presented with a problem, I read it then re-read it again with understanding. After reading it, I drew the object that I had to work with because I am more of a visual person. The reason why I decided to draw out the instructions is that I struggle to process the information fast and if I don't draw, I will confuse myself*.

There is recognition on the part of the participants that using visual methods helped them to find solutions to word problems faster, more easily and more efficiently. Denz's response was also interesting as he recalled having conversations with himself as part of his problem-solving strategy. He said:

> *Firstly, I started by evaluating what was asked. I repeatedly read the question until I understood what was asked of me before trying to solve the question. Secondly, I wrote down all formulas of the topics asked for. I mentally had conversations with myself trying to find reasons for what I am saying and eliminate options.*

The second reflective interview question asked the participants to say how they ensured that they had captured key information from written or verbal instructions before they attempted to solve the problem. The participants restated that they read and re-read the written instructions to ensure that they had captured the key information, sometimes highlighting or underlining what they deemed important. Some said that they read the problems out loud to themselves and noted the key points until such a time that they understood what they were expected to do. They all revealed that they made use of visualisation processes to represent the word problems when they read the questions and when I asked them more probing questions.

In terms of the uniqueness of the participants' methods and the satisfaction of their solutions thereof, the participants described the problems that they considered challenging and derived satisfaction from solving such problems. Below are two extracts from selected participants' responses to this reflective interview question:

> *With one problem I was struggling to visualise what they were describing. I decided to use a cube and its net to solve the problem and it made it so much easier and I found the answer faster*. (Millie)

> *I had to determine the time of the clock that was behind me. I pictured where I studied at home, there is a clock in front of me and I used this to get the answer. I got the correct answer, but it took me a while*. (Ethray)

The problems described by the participants in these extracts were solved in unique ways by the other participants as well. For the cube problem, Millie referred to Task 3 of EVGRT W1 (Appendix 1). The participants spent between 10-15 minutes solving this task and Millie, having spotted a model of a cube across the table during the task-based interview, requested to use it to help her solve the task. In addition, she drew a net of the cube and used the two visualisation objects to successfully solve the problem. Ethray described how he solved Task 4 of EVGRT W1 (Appendix 1) in a unique way. He admitted that he found the task challenging as he was not fond of reading the time on analogue clocks, being more familiar with his sports digital watch and his cellular phone's digital time. I thought his using his imagination to picture his studying place at home and thereby solving the task was both unique and interesting, in that he took long pauses during the task-based interview as he tried to imagine how to relate the clock at home to the task at hand.

To the question of whether they preferred to work individually or in groups when solving word problems, only two participants indicated that they preferred to work individually. Nate was one of those who preferred to work individually as he was more comfortable with working alone and at his own pace, believing that he could thereby achieve more. This explains why his presence was almost unfelt during the task-based interview (Figure 4.7). The rest of the participants said they preferred to work in groups as they were able to discuss the answers and share opinions with each other. Meagan felt that working in groups allowed them to work as a team and consolidate their response to the sums together. The participants indicated that they preferred group problem solving because when they were unable to figure out a solution, there was always someone in their group who could help to lead the group to an acceptable outcome. They also stated that they appreciated group work as it encouraged them to consider others' points of view and accept that people think differently from each other.

Overall, the participants claimed that they were satisfied with having participated in the research project and that it had been a pleasant experience. Meagan asserted that she had a great experience as she was exposed to different practical situations that allowed her to think outside the box and apply her knowledge of mathematics in diverse ways. She expressed satisfaction with the outcome of her solutions. Ellena expressed joy at having participated in the research project. She said that the experience had taught her how to think carefully about the problem prior to solving it. Ellena claimed that she had used visualisation processes throughout the task-based interviews, and this had helped her to better understand the instructions ("*I used visual representations to have a better understanding on what the questions have asked. The outcome was successful because it was something you can see and touch, and it allows a person to understand*"). The rest of the participants commented that it was fun, and they liked the type of mathematics in the EVGRT worksheets. They also appreciated that they had discovered more ways of solving word problems, especially ones that included visual methods. Figure 4.40, below, provides a word cloud illustration of the terms used in the research participants' responses during the reflective interview.

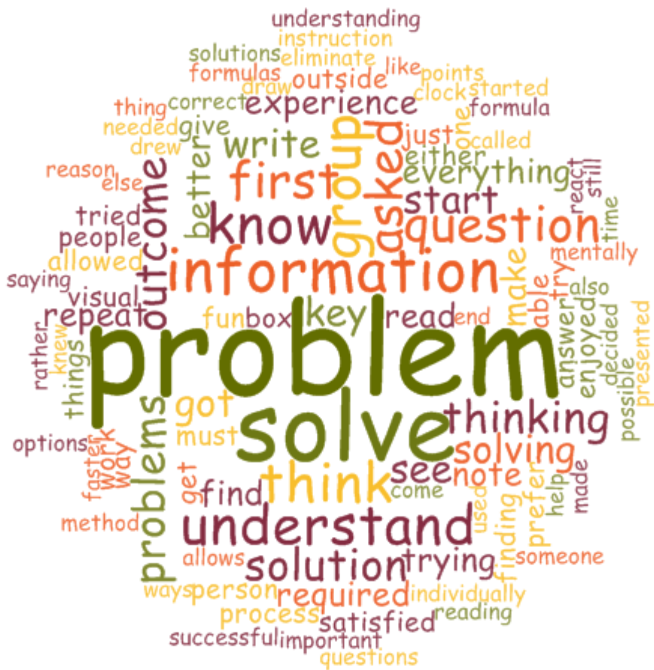


Figure 4.40 Word cloud of the terms most used by reflective interview respondents

The word cloud in Figure 4.40 shows that the participants made frequent reference to the words “problem” and “solve” during the reflective interviews. They used the word ‘problem’ when they spoke of the word problems that they were asked to ‘solve’. Interestingly, there are not many references to “image, “visual” and “draw” despite the fact that the participants practical used these during problem-solving as illustrated throughout this chapter. Perhaps they were more inclined to doing than actually saying that which they did. Nevertheless, the participants concurred that it was so much easier and faster to use visualisation processes to solve word problems. They were both surprised by and satisfied with the outcomes of their problem solving. The participants also commented that the research was beneficial to them as they learned new (especially visual) methods of solving word problems that they would be able to refer to in mathematics lessons. They also commented that they were motivated more than ever to open up in the mathematics classrooms and to air their contributions, as talking in the mathematics classroom was not such a scary thing after all.

4.5 CONCLUDING REMARKS

In this chapter I discussed and analysed the data collected in the three phases of data collection and analysis, as outlined in the methodology chapter of this case study. The NVivo Software was fed with the two analytical frameworks of visual imagery and reasoning processes and used to code the participants’ responses to the two EVGRT worksheets.

Phase 1 data analysis revealed that the research participants in this study have a notable ability to employ visualisation processes to solve geometry word problems. The results of this analysis also revealed that all learners can use visualisation processes to solve word problems, with a number of learners being able to draw upon a panoramic assortment of visual images, and a majority of the learners preferring the use of visual methods to solve word problems.

Phase 2 data analysis revealed that the research participants in this study have a remarkable ability to explain, justify, argue and generalise their solutions and problem-solving strategies in multiple ways. A fine-grained analysis of Phase 2 data, presented and discussed in the form of a series of selected vignettes, showcased the rich visualisation and reasoning processes evidenced by the research participants. This analysis was also used to determine the relationships between visualisation and reasoning processes during word problem solving.

The analysis of Phase 3 data revealed that the learners appreciated the knowledge they gained of visual methods to solve word problems. The data also reflected the learners' appreciation for working in small groups, and their feeling that it added value to both their academic knowledge and interpersonal relationships with those they worked with.

The following and final chapter of the case study consolidates the findings of this analysis, with specific reference to the three guiding research questions originally outlined in Chapter One.

CHAPTER 5

CONCLUSION AND IMPLICATIONS

5.1 INTRODUCTION

The purpose of this final chapter is to consolidate the findings of the study with reference to the original research question and sub-questions, and within the context of the theoretical and methodological frameworks. In addition, both the limitations and significance of the study are interrogated, and some recommendations for further research are made. The dissertation concludes with some personal reflections on the research project that other researchers may relate to and learn from. It is hoped that the results of this case study will help to initiate and drive a debate about how best to harness and enhance visualisation processes for problem solving in conjunction with mathematical reasoning.

5.2 REVISITING THE RESEARCH GOALS AND QUESTIONS

The goal of this study was two-fold:
1. To examine the mathematical reasoning of the selected Grade 11 learners while solving geometry word problems.
2. To analyse how enacted visualisation processes co-emerge with mathematical reasoning processes during collaborative groups.

To accomplish these goals, the case study was guided by one main research question and two sub-questions:

Main research question:
How do visualisation processes relate to mathematical reasoning processes when selected Grade 11 learners solve geometry word problems?

Research sub-questions:

2. What visualisation processes are evident in all the selected Grade 11 participants when they solve geometry word problems?
3. How do visualisation and reasoning processes co-emerge when learners solve geometry word problems in small collaborative groups?

5.3 KEY RESEARCH FINDINGS

The research findings of this case study are presented as responses to the research goals and questions.

5.3.1 Visualisation processes evidenced by Grade 11 learners when they solved EVGRT W1 individually

Five categories of visual imagery were used in this case study to describe the visualisation processes that were frequently evidenced by the selected Grade 11 learners when they solved the EVGRT W1 during one-on-one task-based interviews. Below is a synopsis of how each of these imageries was employed by the research participants.

5.3.1.1 Kinaesthetic Imagery (KI)

Using body movements is a natural way of communicating: people point, mime, trace and imitate using their hands and other body parts to convey different messages. In this case study, kinaesthetic imagery was defined as the type of visual imagery that involves muscular activity, as when the learners used gestures and subtle body movements during word problem solving. KI was the most used visual imagery, as illustrated in Figure 4.1. The findings of this case study revealed that all 17 research participants employed KI during both EVGRT W1 when they worked individually, and EVGRT W2 when the eight participants solved word problems in small collaborative groups. When the learners read the tasks, they mostly pointed with their fingers, pencils or rulers at what they were reading (KI1). When the learners read about a geometrical figure that was not provided, e.g. an equilateral triangle or a rectangular piece of land, they used their pencils/fingers to trace the figure as part of showcasing their mind-formulated pictures. The analysis of the data in this case study revealed that the participants usually traced or described paths representing given situations without placing the pencil on paper. This was observed using KI2 of the KI category's observable indicators. However, the findings of the analysed data for visualisation processes revealed that KI1 was the most coded observable indicator (Figure 4.1) during EVGRT W1. This means that the participants used overt gestures and subtle body movements when they explicated their understanding of the word problems.

5.3.1.2 Concrete Pictorial Imagery (CPI)

The findings of the data analysed for this category of visual imagery revealed that CPI was the second most used VI by all the participants during both one-on-one and focus group task-based interviews (Figure 4.1). CPI as defined by Presmeg (1986b) refers to concrete images of an actual situation in a person's mind, drawn on paper, using technological tools or described verbally. Sketching, drawing, giving descriptions of the word problems, using one's imagination and mind pictures and clarifying the structure of the problem were all CPI indicators in this case study. During the task-based interview, the participants invariably sketched while reading the word problems. Some participants gave descriptions of the pictures that they formulated in their minds as well as the ones drawn on paper. There were times when the participants paused during the task-based interviews and then blurted out the answers and reasons as though they had seen live pictures in their minds. These instances were coded and analysed as concrete pictorial imagery (CPI3) which had to do with the person's imagination (i.e. pictures in the mind).

5.3.1.3 Dynamic Imagery (DI)

Dynamic imagery, or dynamic visualisation as it is referred to by some visualisation researchers in mathematics education (Duval, 1999; Mesaroš, 2012), involves the processes of transforming geometric figures for the purpose of solving the problem. The use of DI during the task-based interviews was observed by the participants' practice of redrawing both given and own drawn diagrams with the purpose of extracting simple pictures from the complex ones. The participants also visualised a series of pictures connected in one smooth motion as they gave descriptions of moving pictures in their minds. When they solved the eighth task of EVGRT W1 (Appendix 1), each of the 17 research participants commented that they saw a picture of a moving die in their minds and used kinaesthetic imagery to clearly describe the transformation of the die from one square to another. The findings of this case study showed that DI was the third most used kind of visual imagery during EVGRT W1, with 17.16% of the coding references (as illustrated in Figure 4.1). The participants also used dynamic imagery when they envisaged the moving ant in the fourth task of EVGRT W1.

5.3.1.4 Memory Imagery (MI)

The data analysis revealed that this kind of imagery was used infrequently in the case study. MI1, in which the research participants were expected to formulate a mental image of a book/board and depict how a formula/concept was written when they saw it, was the least frequently observed MI indicator. The participants who used this visualisation process to solve word problems closed their eyes and made hand gestures as they delved into the recesses of their minds and tried to read what they saw there. During the individual task-based interview they said things like "*I can see this clearly in my mind"; "I see a picture of my book … we did this the other day in class"; "I can still see it clearly"*. This is an indication that when solving word problems, the participants visualised a method that was previously learned so as to apply it to the situation at hand. However, it was only a small number of research participants who managed to use these retrieved images to visualise the content of their books or what was written on the board so as to solve the word problems that they were presented with.

Figure 4.4 illustrates that seven out of the 17 research participants formulated a mental image of their book/board to depict how a concept was written. These seven participants only managed in this way (MI1) to recall less than 5% of the concepts concerned. But the participants had good overall recall of previous knowledge to apply in solving the problems before them. For example, when asked to solve task 10 of EVGRT W1 (Appendix 1), the participants remembered the features of an isosceles triangle and how to construct an algebraic equation from what they had learned mathematics classes (MI3).

5.3.1.5 Pattern Imagery (PI)

The frequent use of pattern imagery was observed whenever there was a visual representation of a right-angled triangle, with the participants presenting the theorem of Pythagoras in a variety of symbols/ways. The first and the fifth tasks of EVGRT W1 (Appendix 1) required the participants to sketch a right-angled triangle to find one of the unknown sides using the theorem of Pythagoras, which perhaps accounted for the frequent use of PI in those tasks. Some of the participants used the theorem of Pythagoras to find the height of the equilateral triangle in the second task while others applied the sine rule to directly calculate the area of the non-90° triangle. Where they used the theorem of Pythagoras, different participants represented the tasks in different ways. This, for instance, is how the participants represented the first and the fifth tasks: $c^2 = a^2 - b^2$, $AB^2 = AC^2 - BC^2$, $x^2 - y^2 = z^2$, $CD = AB^2 - EF^2$, $A^2 = B^2 - C^2$, etc. What I found astonishing from these solutions was that once a participant used certain symbols for the first task, say $c^2 = a^2 - b^2$, they applied those specific symbols every time they used the theorem of Pythagoras throughout the task-based interview. They reasoned that they would stick to the same symbols every time they used the theorem of Pythagoras to help themselves remember the formula and its application. They also maintained that keeping to the same symbols helped them to avoid confusion, especially over when they should add or subtract as required by the formula. However, the overall use of PI during EVGRT W1 was less frequent in comparison to other categories of visual imagery that were omnipresent across all the tasks.

The above description of the findings regarding the visualisation processes evidenced by the selected Grade 11 learners when they solved EVGRT W1 revealed that the learners employed multiple visualisation processes for one concept. For example, the learners used a different form of visual imagery to shed more light on what they were already busy visualising. The learners mostly used CPI to formulate more pictures in their minds to enable them to explain the imagery that they were busy with. With reference to the eighth task that I previously mentioned, the participants used both CPI and KI to help them describe the moving pictures in their heads. Although memory imagery and pattern imagery were used infrequently in the case study, their effectiveness may be seen in the quoted examples discussed in Sections 5.3.1.3 and 5.3.1.4.

5.3.2 Mathematical reasoning processes evidenced by Grade 11 learners when they solved EVGRT W2 in small groups

Four reasoning processes were observed when the learners solved the problems in EVGRT W2 in small collaborative argumentative groups. The findings with regard to each of these reasoning processes are discussed below.

5.3.2.1 Reasoning Process Explanation (RPE)

The findings reveal that the reasoning process of explanation was frequently used by the research participants during EVGRT W2 task-based interviews. Five out of the eight participants used explanation more than any other reasoning process (see Figure 4.7). This means that as soon as the participants could make sense of the problem they made claims. They established what each problem entailed in simple terms and suggested known concepts. For example, when Denz read the first task, Jordan claimed that there was a circle that touched each side of the square in which it was inscribed. Without saying a word, Rauna sketched a square of side 20cm when she listened to Millie reading the word problem. Ethray in the other group both read and paused to sketch a square and an inscribed circle. Ellena in her group claimed that the diameter of the circle was 20 meters and explained that they were required to calculate the area of the circle. In all these instances, there is enough evidence to state that after reading the word problem, the learners explicated the meaning of the problem by using both visual and verbal explanations. They clarified aspects of their mathematical thinking that were not apparent to others. Figure 4.7 further illustrates that 75% of the research participants used the reasoning process of explanation more than 100 times during the EVGRT W2 task-based interviews.

5.3.2.2 Reasoning Process Justification (RPJ)

The reasoning process of justification was in this case study the second most used reasoning process during the focus groups task-based interviews, as illustrated in Figure 4.7. It manifested itself when the research participants provided evidence for their claims and explanations to other participants in their respective small groups. This was equally observed when other participants asked for proofs and justifications from the claimants. For example, when asked by Ethray to use the tangent ratio (the trigonometric ratio) to calculate the length of an unknown side of a right-angled triangle, Nate rejected his request, pointing out that he could not use the tangent ratio because he had neither the opposite, nor the adjacent provided to him. This is classified as the reasoning process of justification in this case study. Figure 4.7 also illustrated that Meagan and Millie used RPJ very frequently in their group category. The data analysis revealed that Meagan's frequent use of RPJ came as a result of constantly requesting Millie to justify her actions and to provide proofs to validate every claim that she made. As a result, Millie's RPJ recorded the highest frequency in her group as she responded to Meagan's requests as well as trying to promote understanding among her group.

5.3.2.3 Reasoning Process Argumentation (RPA)

The analysis of the data coded for the reasoning process of argumentation revealed that the research participants provided support for their explanations and justifications or were asked to provide such support. The findings also reveal that many learners who were involved in the RPA convinced each other of the truth of their claims and were at other times also convinced by others, as discussed in Section 4.3.3. Furthermore, some arguments uttered by the participants were not mathematically accurate, yet were good enough to be accepted by the others. The participants who convinced others were very confident and effective in their arguments, providing proofs to validate their claims and explanations. The findings of this case study thus also reveal that argumentation can be an effective way to encourage talk in mathematics classrooms – which in return helps to improve the learners' mathematical reasoning.

5.3.2.4 Reasoning Process Generalisation (RPG)

The data analysed in this case study revealed that of all the reasoning processes, generalisation was the least frequently used during the focus group task-based interviews. Generalisation is a much higher level of reasoning than then the other three, hence its least application by the participants. The participants hardly elaborated the problems further to try to learn from them, and only did so when prompted. One of the groups remarked during the task-based interview that it seemed as if I wanted them to use a specific method, 'my method', when I prompted them to generalise their solutions. Generally speaking, the participants were reluctant to generalise, preferring to find an answer and proceed to the next problem rather than trying to show how to solve the current problem in other ways. That is, the participants found it difficult to identify the operators and the sequences of operations that were common among specific cases of their solutions and to extend them to general cases (cf. Swafford & Langrall, 2000). As further alluded to in the literature reviewed for the study, it is crucial that the learners stick to a problem after they have solved it. This is to encourage them to use the opportunity to elaborate the problem further and to try to learn from the result and the strategy used to solve the problem. It is deemed important for learners to investigate if they can identify related problems, and whether it is possible to generalise their result to these (cf. Fahlgren & Brunström, 2014, p. 291).

5.3.3 The co-emergence of visualisation and reasoning processes (How visualisation processes related to mathematical reasoning processes when the Grade 11 learners solved geometry word problems)

Enactivism talks of co-emergence as an overall process of structural coupling. A broad spectrum of co-emergence involving visualisation and reasoning processes was displayed by the research participants during the second phase of data collection and analysis. A fine-grained analysis of this co-emergence was presented in the form of a series of selected vignettes that were discussed in Sections 4.3.3 and 4.3.4 of the study. The findings of this fine-grained analysis revealed that the co-emergence of visualisation and reasoning processes was always observed when the learners interacted with each task, with each other and with their environment during the focus-group task-based interviews. That is, the learners' experiences with visual imagery during the preceding tasks of the EVGRT worksheets encouraged them to react in a certain way when a similar situation occurred in subsequent tasks. Moreover, the research participants' structures determined the changes that occurred in their enactment of visual imageries that determined, and in return was determined by, their mathematical reasoning processes.

There is sufficient evidence to prove that every research participant in this case study managed to use visualisation processes to solve all the geometry word problems in the EVGRT worksheets. When they read the word problems, the research participants were all able to describe the pictures formed in their minds as they imagined the scenes depicted. This type of imagery was categorised as concrete pictorial imagery. The learners also used patterns of symbols to represent the word problems and find solutions. This pattern imagery mostly arose in instances in which they needed to use the theorem of Pythagoras to find an unknown side of a right-angled triangle. Furthermore, the learners recalled certain aspects from memory to help them solve the problems. They also visualised pictures of concepts printed at the back of their minds, as it were, as they were written in their books or on the whiteboard. This was evidenced by the participants' exclamations, which *inter alia* included: "*we have done this before*", "*this is similar to what we learned in class the other day*", "*wait, I can see the formula in my book*", and so forth. This was categorised as memory imagery in this case study. The category of kinaesthetic imagery was very frequently used by the research participants during both individual and focus group task-based interviews. Scientists tell us that electrical charges originating from the brain control all the muscular activity of our bodies. There were muscular activities performed by the participants in this case study which took the form of hand gestures and subtle body movements when they explained, justified, argued and generalised their solutions and problem-solving strategies. The participants also managed to transform figures, shapes and concrete objects during word problem solving. This act of changing the orientation of shapes and objects was categorised in this case study as dynamic imagery.

From the above observations, I have enough evidence to conclude that, in the context of this case study, every learner involved in word problem solving is capable of using visualisation processes to solve such problems. The frequency with which these processes are employed is a matter of preference.

My interest was particularly drawn to those learners who preferred visual methods over algebraic methods of solving word problems. Figure 4.16 illustrates that these learners' reasoning processes were incorporated with their visualisation processes in a matrix that could be coded. The findings of the matrix coding analysis revealed that there were always connections between the 5VIs and the 4RPs when the research participants deliberated over the tasks in EVGRT W2 and discussed them with each other. Figure 4.16 also illustrates these connections and shows that there were recurrent relations between how the research participants applied pattern imagery and the reasoning process of argumentation (a bar with the highest frequency). Overall, there were connections between each process of visualisation and most reasoning processes.

In this case study, the co-emergence of visualisation and reasoning processes was defined by the coupling of specific visual imageries with specific reasoning processes at a particular instant. These imageries and reasoning processes emerged interchangeably within a single, specific action. For example, when the participants worked on EVGRT W2 in small collaborative groups their reasoning processes (4RPs) inevitably coupled with their visual imageries (5VIs). That is, the research participants employed visual imageries to enhance their reasoning processes, and in return, their reasoning processes enhanced the further use of visual imagery during the process of problem solving (Figure 5.1). This process, whereby the learners used visual imagery to develop and recreate their reasoning, which in turn was structurally coupled with and influenced their visualisation processes, is defined as co-emergence in this study. More explicitly, it was the participants' structures which determined the type of visualisation processes to employ and to which problem, which in turn determined and were determined by the reasoning processes that co-emerged with them. Thus, the learners' structure became structurally coupled with that of the context in which they operated.

The findings of this case study further reveal that if visualisation processes are incorporated into word problem solving, and learners are encouraged to talk about their solutions as well as their problem-solving strategies, then reversible navigation between the learners' use of visualisation and mathematical reasoning processes is possible. When the research participants employed visualisation processes to solve word problems in this case study, they spoke about their solutions and problem-solving strategies – i.e. they reasoned mathematically about their actions. They then employed further visualisation processes as a result of this extensive reasoning. From an enactivist point of view, the two processes incessantly co-emerged (Figure 5.1).

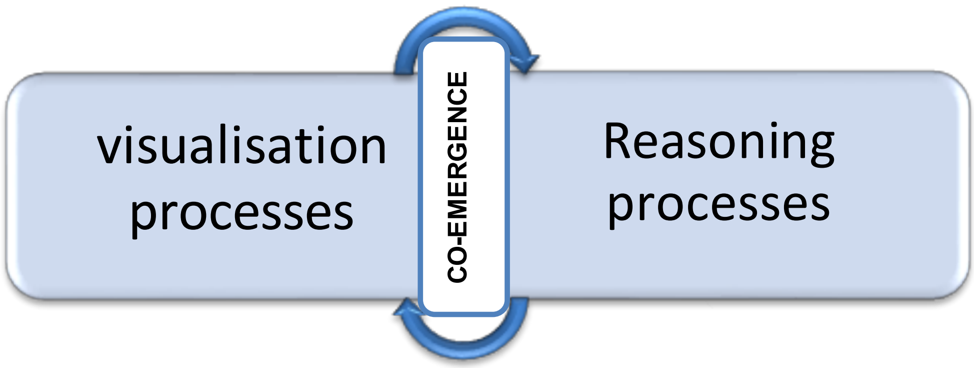


Figure 5.1 The co-emergent relationship between visualisation processes and reasoning processes

Figure 5.1 further depicts that the relationship between visualisation processes and reasoning processes in this case study was of a circular nature with no definite beginning. That is, the origin of the process is untraceable. It begins in the participants' minds, and as an observer I did not and cannot have full access to the problem solvers' minds. I thus reiterate what I mentioned in the literature review chapter, that the observer is the source of what he/she observes and the creator of what he/she sees. Thus, the design and setup of this case study permitted me some access to certain images formulated in the participants' minds. Although I was obviously unable directly to observe what the participants saw in their minds, I could rely upon what they uttered, both verbally and visually. Hence, what became of interest to me as an observer was that which the research participants and their environments permitted and gave me access to. Finally, the findings of this case study provide sufficient premise to conclude that visualisation processes and mathematical reasoning processes are closely interlinked in the process of any mathematical activity. The study also argues that the visualisation processes enacted by the participants when solving a set of word problems are inseparable from the reasoning processes that the participants brought forth; that is, they co-emerged.

5.4 LIMITATIONS OF THE STUDY

In an effort to enhance the dependability and confirmability of the study (Rule & John, 2011), I include hereunder a declaration regarding its limitations.

Generally, the scope of a case study does not permit its findings to be generalised to the entire population of cases. However, my chosen methodology was guided by the principle of *fit for purpose*. This case study research is not fit for the purpose of statistical generalisation. But it *is* fit for the purpose of generating in-depth, holistic and situated understandings of the co-emergence of visualisation and reasoning processes when the research participants solved word problems in a social setting. As alluded to in the literature review chapter, learners working in small groups are expected to develop personally meaningful solutions to problems, explain and justify their solutions and problem-solving strategies, and listen to each other's arguments (cf. Yackel, 2001). The data generated was analysed to tease out the relations between visualisation and reasoning processes in terms of the research participants' structural coupling and co-emergence that were observed when they interacted with each other and the environment. From this viewpoint, it is clear that generalisability is not the purpose of this research. I thus concur with the notion of transferability as an alternative to generalisability (Rule & John, 2011). This position holds that the researcher of a case study "understands best the phenomenon within its context" (p. 105). Thus, by providing thick descriptions of the case and its context, I allowed my findings and conclusions to gain a level of transferability.

The following choices also contributed to the limitations of this case study (John & Rule, 2011, p. 111):

- **The research site** was the only school with Grade 11s in my town. Hence, it was chosen due to its availability, convenience and accessibility.
- **For the sample,** only senior secondary learners were chosen to participate in the case study, although they were not the only cohort doing GWP in the school. The small number of participants may have limited the amount and type of data collected.
- **Practical and logistical circumstances**. The large distance and poor road to the research site at the time of data collection restricted access.
- **Personal attributes**. Firstly, my personal attributes – in the form, for instance, of limitations of language – could have influenced the data collected. Since I did not understand Afrikaans and needed to ask probing questions, I encouraged the participants to discuss in English during the focus group task-based interviews. Secondly, my position as a mathematics teacher at the research site might have limited some participants from fully expressing themselves. To minimise the impact of this limitation, I described my position as both a mathematics teacher and a researcher of this case study (Section 3.6.1). Moreover, it was impossible for me not to ask some leading questions during the task-based interviews, as some tasks would have been left unanswered by certain participants had I not somehow prompted them to visualise – e.g. by making constructions or sketches to assist them to successfully and accurately solve the problems. I observed that some of the

research participants did extremely well in some tasks that others found challenging, but then struggled with tasks that others found relatively easy to solve.

In addition to the above limitations, there was a limit to how much data I had access to as I worked with human and social beings. When it came to their visualisation and reasoning processes, I was limited to the research participants' utterances of their mind pictures, thoughts and imaginations. Moreover, the observable indicators used in the two analytical frameworks (Tables 3.1 and 3.2) are context based and study embedded. They should not be generalised to other studies, although they may be adapted to suit other contexts.

5.5 SIGNIFICANCE AND CONTRIBUTIONS OF THE STUDY

Embedded within the enactivist perspective, this study foregrounded the co-emergence of visualisation and reasoning in word problem solving among secondary school mathematics learners. The study is particularly significant in the Namibian context as it is in line with the learner-centred approach advocated by the Namibian Broad Curriculum. It is hoped that the study will be of interest to teachers, researchers, curriculum designers and authors of textbooks about the significance of visualisation processes in the teaching and learning of GWPs. The findings of this research may also inform teacher educators of the benefits of incorporating visualisation in their mathematics curriculum, to equip their student teachers with the necessary knowledge and understanding to use visualisation as a teaching tool to enhance conceptual understanding and reasoning in mathematics.

Bertram and Christeansen (2014, p. 67) maintain that research should be of benefit either directly to the participants, or more broadly to other researchers, or to society at large. This case study research was beneficial to the research participants as they were encouraged to embrace the use of visual methods to solve problems. Furthermore, the thesis contributes to knowledge of the field of mathematical reasoning, visualisation, geometry word problems, enactivism and to the mathematics education in general. This is because, the context in which the study was conducted is different from the context of the literature that has been referred to in this thesis. Moreover, close observation of student meaning making is critical for enhancing the mathematics education of students internationally. Reasoning, visualisation and problem solving are all common sites of research in mathematics education and constitute to some of the earlier work in the psychology of mathematics education. However, the use of enactivism to theorise the interaction of reasoning and visualisation is a unique contribution and it is hoped to enrich the growing enactivist discourse. Finally, the specificity of the Namibian context gives the study originality and hence the contribution to knowledge.

5.6 IMPLICATIONS

5.6.1 Implications for teaching and learning

The findings of this case study revealed that using visualisation processes to solve word problems proved successful and at the same time developed among the problem solvers a sense of pride, gave them peace of mind and enabled them to tackle problems with confidence. It is against this background that I recommend that mathematics teachers embrace and encourage the use visualisation as a problem-solving tool. By incorporating visualisation processes in their teaching, they will influence their learners to do the same in their learning. Mathematics teachers are thus encouraged to talk the learners through the problem-solving process to elicit reasoning processes that would enrich their problem-solving repertoire.

5.6.2 Implications for policy makers and curriculum developers

To enable mathematics teachers to incorporate visualisation as an integral part of their mathematics teaching, the mathematics teaching policy and curriculum should include this as an essential aspect. If policy makers and curriculum developers include the use of visualisation processes as a word problem solving tool in their planning, it will inevitably encourage educators and learners alike to embrace this method in their teaching and learning. Therefore, the curriculum should be designed so as to make enough time available for teachers to talk through their learners' problem solving during mathematics lessons and learn more about the kinds of visual images prominent in their lessons. The curriculum should also effectively and encouragingly point to visualisation and mathematical reasoning as key objectives of everyday mathematics teaching and learning. Finally, policy makers and curriculum developers should invite more researchers to participate and incorporate their research findings (especially from the Namibian mathematics education spectrum), in the development and design of the mathematics curriculum. Enough funding should be set aside for the purpose.

5.6.3 Implications for further research

This case study focused on a mixed ability group of learners who exhibited a preference for visual over algebraic methods of word problem solving. It would be interesting if a similar study could be conducted with a focus on either lower ability or higher ability groups, and perhaps groups whose preferred method of word problem solving was algebraic rather than visual. A repeat of the current study could also be conducted with respect to individual argumentation rather than collaborative argumentation. The research design would need to be altered accordingly.

This case study was part of a larger collaborative project, Visualisation Processes in Southern Africa (VIPROSA), which endeavours to promote the use of visualisation in mathematics in Southern Africa. Despite this, the linking of constructs derived from visualisation and enactivism is unique in Namibia and beyond. New lines of research and theoretical studies inspired by enactivism could further improve learners' reasoning in topics other than geometry and subjects other than mathematics. It would be equally interesting and enlightening to repeat the current study with a different Grade and age group.

5.7 PERSONAL REFLECTIONS AND CONCLUDING REMARKS

It was beneficial to this study to make the task-based interviews as interesting as possible. No matter how urgently I needed to complete the interviews within the set timeframe, I learned that giving the participants moments to relax and enjoy themselves was a good idea. The participants took pleasure in working diligently and appreciated each other and the learning environment.

When I conducted a semi-structured reflective interview with the eight participants, they commended the time they had spent on solving the tasks. Most of them recalled that they enjoyed working in small groups more than one-on-one task-based interviews. The participants in homogenous groups made jokes more than the heterogenous groups. The participants in one of the homogenous groups made more jokes than any other group, but they still managed to stay on task as they had one participant who assumed the role of a leader and consistently recalled them to their core business. This taught me that when grouping participants, one needs to consider their peculiar traits, whether innate or learned, to ensure that there is at least one group member to lead the others back to the main purpose of the task.

Another interesting thing that I learned from using focus group task-based interviews was the need for mutual understanding, respect and appreciation among research participants. I observed mutual understanding between two boys throughout the EVGRT W2. They both respectfully disagreed with each other whenever it was necessary, and when they needed to correct each other's mistakes, they did so with sensitivity. When they argued about the cube task (Task 5), for example, one boy apologised and thanked the other when he realised that he had been wrong, and the other boy was right. In a different group, when one participant remained silent for most of the time during the focus group interview, his group mates' actions towards his silence were commendable. They did not force him to talk, yet when he did, they respected and considered his opinion.

During one of the focus groups, the rate at which the participants were solving each task was slow as I required that they used more than one method to solve the tasks. As a result, a certain participant became very irritated halfway through the task-based interview. This participant minimised participation and literally stormed out of the venue upon completion of the interview. I believe that the only reason she did not withdraw during the middle of the interview was out of respect for me – being a mathematics teacher at the time of the interview. When the interview was completed, she called out a blunt 'good-bye ma'am' from halfway through the doorway. When I called all the participants back to thank them for their time and participating, she straight after rushed out of the room, calling out a 'thank you ma'am' when she was already out of sight. Since we both lived in the same town, I met her later that day; she seemed happy being out with a friend and unbothered as I expected her to be. The participant assured me that she was not at all bothered. This is exactly what Johnson (1984, as cited in Bell & Waters, 2014) warns about when he remarks: "if an interview takes two or three times as long as the interviewer said it would, the respondent(s), whose other work or social activities have been accordingly delayed, will be irritated in retrospect, however enjoyable the experience may have been at the time" (p. 189). Not every interviewer is or will be fortunate enough to reside in the same town as their participants, and hence able to get timely feedback on their interview experiences as I did. It would clearly be advantageous to keep the interviews as concise as promised and avoid irritating the respondents, which might affect their participation in future studies.

Tips on data management, data cleaning and the use of technology for future researchers

Data management was a crucial aspect of this case study. I collected video and audio data, participants' responses to EVGRT worksheets, researcher's journal and memos. To avoid 'swimming in data' during data analysis, I ensured that the data was properly managed from the time of collection. After each interview and video recording, I immediately saved the data onto the computer and two external drives that I stored in separate locations. I created a folder for each participant and subfolders bearing names such as videos, audios, scanned EVGRT, snipping tools, NVivo charts, and so forth. Saving collected and analysed data in separate folders helped me to keep track of what was done and what needed to be done. The participants' responses to the EVGRT worksheets were timeously scanned and stored in their respective folders.

I embraced the use of technology throughout this case study. I used Mendeley software to assist me with referencing. Although it was time consuming having to add and verify each file onto the database, Mendeley made it easier for me to insert the bibliography instead of to manually and traditionally add each reference into the reference list. Moreover, I imported all my word documents into Dropbox, which enabled me to save all typed materials online and to retrieve these on any available machine when I needed to work on my dissertation. Dropbox ensures the safekeeping of typed material in an event of machine malfunctions, theft or any other related causes. I also used the snipping tool application to capture special moments while watching the video recordings, and added design details using the paint application in Microsoft Office.

The NVivo software played an integral role in data analysis in this case study. After data cleaning, the transcripts were imported into NVivo for analysis. Data cleaning involved reading through the transcripts while listening to audio recordings to ensure accuracy of transcription, as well as while watching the videos to add the participants' gestures and subtle body movements that could not be picked up in audio recordings for transcription. This was done to ensure the quality of transcription prior to data analysis. Kawulich (2017, p. 773) exhorts researchers to ensure the quality of their transcriptions, as this is an important aspect of one's ability to analyse data appropriately.

In NVivo, the transcripts were read several times for different coding purposes, such as for visual imagery indicators and for reasoning processes indicators. The advantages of using NVivo for data analysis include the ability to present the data visually in charts and word clouds, as featured throughout the data analysis chapter. NVivo also allowed me to add memos and annotations to the transcripts while coding, which consequently formed part of my final data analysis. The relationship between visualisation and reasoning processes, which is the central thesis of this case study, was analysed and explored using NVivo matrix coding. This enabled me to visually present and verbally discuss the findings of this relationship as articulated in the analysis chapter and in Section 5.3.3.

From a personal perspective, I found that this research journey has left me with an eternal desire to observe more, to reflect more and to research more. It made me cognisant of all the visualisation processes that were utilised by my learners on a daily basis and that I unconsciously tended to overlook in my mathematics lessons. Travelling this terrain also made me appreciate the many variations in my learners' solutions to word problems and their related mathematical reasoning, which has not always been verbal. I concur with Antle's (2009) proposal that we ought to give consideration to the ways in which cognition is rooted in bodily actions, as this contributes to the learners' successful development into active, thinking adults. Moreover, the enactivist lens in this study made available a powerful language that enabled me to more than just engage with the notions of co-emergence and structural coupling (as discussed at some length in Chapter 4). It also informed my own observation and reasoning as a mathematics teacher and researcher. I am delighted and proud that in this research journey I truly have taken the road less travelled, and I trust that the difference it has made has been a good one!

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7. APPENDICES

7.1 APPENDIX ONE: EVGRT W1

Task 1

Imagine a 10m ladder leaning against a half painted wall behind you. The bottom of the ladder is 6m from where the wall meets the ground.

a) What special name is given to the geometrical shape formed between the ladder, the wall and the ground?

b) At what height from the ground does the top of the ladder lean against the wall?

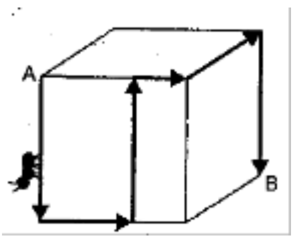
Task 2

Dalia wants to design traffic sign boards for her school play. Her mathematics teacher instructed her to ensure that all her sign boards have equal perimeters. Dalia designed a square board of side 12cm and an equilateral triangular board of the same perimeter.

a) What was the side of Dalia's equilateral triangular board?
b) Are the areas of the two boards equal? Explain.

Task 3

The edges of a cube are 12 cm long. An ant moves on the cube surface from point A to point B along the path shown.



a) What is the length of the ant's path?
b) What is the shortest possible distance that the ant can move from A to B?

Task 4

a) Imagine a clock with hands, on the wall in front of you. The long hand is pointing to 4. The short hand is pointing between 11 and 12. What time is it?
b) Now imagine the clock is behind you and you can see it in the mirror. There are dots instead of numbers. The hands look as though they are saying twenty-five to three. What time is it really? What is the size of the angle formed between the two hands? **NB**: There are dots instead of numbers on this clock.

Task 5

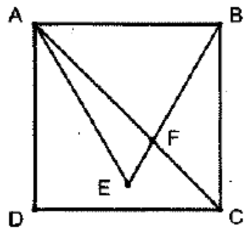
The longer side of a rectangle has a length of 63 cm and the diagonals both have a length of 65 cm. Calculate the width of the rectangle (in cm).

Task 6

How would you explain to a Grade 5 learner how to calculate the sum of the interior angles of a 5 sided polygon? (**Hint**: Grade 5 learners hardly understand formula)

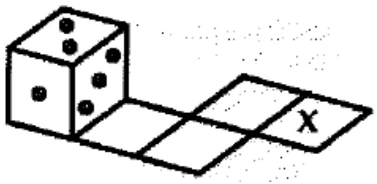
Task 7

If ABCD is a square and ABE is an equilateral triangle, then angle BFC calculated in degrees, equals…



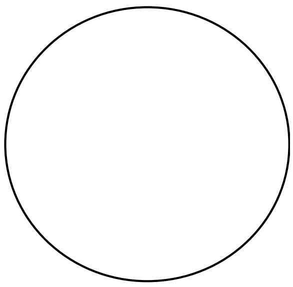
Task 8

On a die the numbers on opposite faces add up to 7. The die in the diagram is rolled edge over edge along the path until it rests on the square labelled X. What is the number on top in that position?



Task 9

Show and explain how you would find the centre of this circle.



Task 10

Mr. Mauno constructs a triangle of perimeter 30cm for his mathematics lesson preparation. During the lesson, he asked his learners to find the length of the shortest side of the triangle if two sides of that triangle were each twice as long as the shortest side. Suppose you are Mr. Mauno's learner, what will be your answer?

7.2 APPENDIX TWO: EVGRT W2

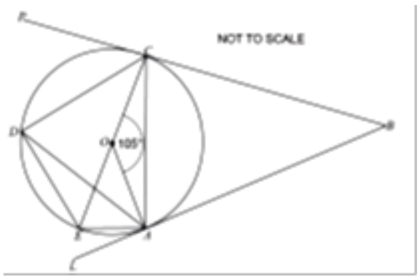
Task 1

Marina's backyard is a square with a side length of twenty meters. In her backyard is a circular garden that extends to each side of her yard. In the centre of the garden is a square patch of spinach so big that each corner of the square touches a side of the garden. Marina really likes spinach! How much area of Marina's garden is being used to grow spinach?

Task 2

Mr. Onesmus has a plan for his sick goat. He tied the goat to a tree with a 7m rope in such a way that the goat is able to move freely around the tree for it to graze. If the goat moves a complete revolution with the maximum length of the rope, what is the total possible area that the goat would graze?

Task 3



Find the following angles, giving a reason for each answer.

a) ADC
b) OCP
c) EAC
d) AEC
e) ABC
f) ACE

Task 4

A pack of 52 cards is dealt out to 10 people seated around a circular table in such a way that the first person gets the 1st card, the fourth person gets the 2nd card, the seventh person gets the 3rd card, the tenth person gets the 4th card, and the third person gets the 5th card and so on.

a) Which person gets the last card?
b) If the cards were 72 instead, which person gets the last card?
c) What about 96 cards? Who gets the last card?

Task 5

In a cube with sides of length 10cm, denote one vertex by the letter *V*. Find the sum of the shortest possible total distances from *V* to each of the other vertices of the cube.

7.3 APPENDIX THREE: FOLLOW UP QUESTIONS ON THE EVGRT INTERACTIONS

a) What are you imagining now?
b) What is going through your mind?
c) Is there a picture in your mind?
d) Can you draw the picture for me?
e) What does that picture mean to you?
f) Do you find it helpful to draw a picture?
g) How does using a picture help you solve the problem?
h) Why do you use this diagram?
i) Why do you not use a diagram?
j) How do you know that this is the final answer?

- ✓ Why are you not solving this task?
- ✓ What if you use a picture to solve the task?
- ✓ Why do you think a picture won't work?
- ✓ Can you tell me what you don't know about the problem
- ✓ Tell me what you tried here?
- ✓ Are you seeing something in your mind? What are you seeing?

    - o Does it help you to draw a picture?
    - o How does it help you?
    - o Is there any other method/way that can help you solve the problem?
    - o What is going through your mind?
    - o What are you thinking? Can you draw it for me?

7.4 APPENDIX FOUR: ETHICAL GUIDELINES TO RESEARCH PARTICIPANTS

These guidelines were attached to each participant's consent letter for them to read them with their parents/guardians.

1. Participants must be informed that they are being asked to participate in a research study,
2. Participants must be provided an explanation of the purposes of the research and the expected duration of their participation,
3. Participants must be given a description of the procedures to be followed and of any experimental procedures must be identified,
4. Participants must be given a description of any reasonably foreseeable risks or discomforts they may experience,
5. Participants must be given a description of any benefits to themselves or others that may reasonably be expected from the results of the study,
6. Appropriate alternative procedures or courses of treatment, if any, that might be advantageous to the subject of an experimental or quasi-experimental study must be disclosed
7. Participants must be given a statement describing the extent, if any, to which confidentiality of records identifying the subject/participant will be maintained
8. For research involving more than minimal risk, participants must be given an explanation about any treatments or compensation if injury occurs and, if so, what they consist of, or where further information may be obtained. (Note: A risk is considered "minimal" when the probability and magnitude of harm or discomfort anticipated in the proposed research are not greater, in and of themselves, than those ordinarily encountered in daily life or during the performance of routine physical or psychological examinations or tests).
9. Participants must be told whom to contact for answers to pertinent questions about the research and research subjects'/participants' rights, and whom to contact in the event of a research-related injury
10. Participants must be given a statement that **participation is voluntary**, refusal to participate will involve no penalty or loss of benefits to which the subject/participants is otherwise entitled, and the subject/participant may discontinue participation at any time without penalty or loss of benefits to which the subject/participant is otherwise entitled

7.5 APPENDIX FIVE: LETTER REQUESTING PERMISSION FROM SCHOOL PRINCIPAL

To: The Principal
From: Beata Dongwi

Date: September 9, 2016

Dear Sir

Re: Request for access to carry out my research project in Oranjemund Private School

This letter serves to notify the office of the Principal of Oranjemund Private School (OPS) that I, Beata Dongwi, a mathematics teacher at the school and a part-time postgraduate student at Rhodes University, South Africa, would like to carry out a research project at the school as part of my study.

My research study is titled: Examining mathematical reasoning through enacted visualisation when solving word problems. This research is part of a bigger study which seeks to research the effective use of visualisation processes in the mathematics classrooms in Namibia and Zambia, called the VISNAMZA Project. This study includes observation of visualisation and reasoning process as Grade 11 learners solve geometry word problems. It is envisaged that the study will provide some insights into how learners perceive mathematics and the nature of mathematical word problem solving which remains a challenge in mathematics education both in Namibia and beyond.

I would thus respectfully request to be officially granted access to the school, as I am seeking the participation of the Grade 11 mathematics learners to participate in the study. The learners will benefit a great deal of knowledge as the research aims to expose them to various problem solving strategies of which visualisation is the mean to mathematical reasoning. My proposal has been approved by the Rhodes University Higher Degree Committee, and I promise to carry out this study with the utmost regard to professional and ethical research standards.

Your positive prompt response in this matter will be highly appreciated.

Yours faithfully

Beata Dongwi
Student Number: 609D6388

Professor Marc Schäfer
(Supervisor)

7.6 APPENDIX SIX: LETTER REQUESTING PERMISSION FROM DEPUTY PRINCIPAL

To: The Deputy Principal
From: Beata Dongwi

Date: September 9, 2016

Dear Sir

Re: Request for access to carry out my research project in Oranjemund Private School

This letter serves to notify the office of the Principal of Oranjemund Private School (OPS) that I, Beata Dongwi, a mathematics teacher at the school and a part-time postgraduate student at Rhodes University, South Africa, would like to carry out a research project at the school as part of my study.

My research study is titled: Examining mathematical reasoning through enacted visualisation when solving word problems. This research is part of a bigger study which seeks to research the effective use of visualisation processes in the mathematics classrooms in Namibia and Zambia, called the VISNAMZA Project. This study includes observation of visualisation and reasoning process as Grade 11 learners solve geometry word problems. It is envisaged that the study will provide some insights into how learners perceive mathematics and the nature of mathematical word problem solving which remains a challenge in mathematics education both in Namibia and beyond.

I would thus respectfully request to be officially granted access to the school, as I am seeking the participation of the Grade 11 mathematics learners to participate in the study. The learners will benefit a great deal of knowledge as the research aims to expose them to various problem solving strategies of which visualisation is the mean to mathematical reasoning. My proposal has been approved by the Rhodes University Higher Degree Committee, and I promise to carry out this study with the utmost regard to professional and ethical research standards.

Your positive prompt response in this matter will be highly appreciated.

Yours faithfully

Beata Dongwi
Student Number: 609D6388

Professor Marc Schäfer
(Supervisor)

7.7 APPENDIX SEVEN: LETTER REQUESTING PERMISSION FROM PARENTS/GUARDIANS

To: The Parents/guardians
From: Beata Dongwi

Date: September 15, 2016

Dear parents/guardians

Re: Consent for voluntary participation in my research project

I, Beata Dongwi, a mathematics teacher at Oranjemund Private School (OPS) and a part-time postgraduate student at Rhodes University, South Africa, would like to carry out a research project at the school as part of my study. Learners were asked to volunteer as participants in this study and your child .... in Grade 11 volunteered to become one of the participants. This letter therefore serves to humbly request for your consent to allow your child to participate in my study.

The study is titled: "Examining mathematical reasoning through enacted visualisation when solving word problems". This research is part of a bigger study which seeks to analyse the effective use of visualisation processes in the mathematics classrooms in Namibia and Zambia, called the VISNAMZA Project. This study includes observation of visualisation and reasoning process as Grade 11 learners solve geometry word problems. It is envisaged that the study will provide some insights into how learners perceive mathematics and the nature of mathematical word problem solving which remains a challenge in mathematics education both in Namibia and beyond.

The learners who agree to participate in this study will benefit a great deal of knowledge as the research aims to expose them to various problem solving strategies of which visualisation is the mean to mathematical reasoning. My proposal has been approved by the Rhodes University Higher Degree Committee, and I promise to carry out this study with the utmost regard to professional and ethical research standards.

Please complete the attached consent form if you are willing to assist me with this research:

a) By allowing me to observe your child, make field notes and keep samples of video records of him/her participating in my research project.

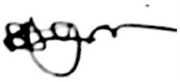
b) By allowing him/her to be video-recorded while working during task-based interviews and to use these videos as evidence in the research write up and conference presentations.

c) By allowing him/her to please come to school in the afternoon between 14H30 and 17H00 for two days only to attend the task-based interviews. The exact dates and time will be communicated to your child prior his/her participation.

You may rest assured that video tapes and field notes whereby your child is a participant will be confidentially stored and will not be viewed by anybody without your consent.

Your positive prompt response in this matter will be highly appreciated.

Yours faithfully



Beata Dongwi (bdongwi@yahoo.com)
A mathematics teacher: Oranjemund Private School.

Professor Marc Schäfer
(Supervisor, Rhodes University)

Consent Form

I ...., understand the contents of the consent letter hereby giving consent to Beata Dongwi in her research. I understand that she will be:

- Observing my child, making field notes, keeping samples or photocopies of my child's work and recording videos to use in the research project.
- Information collected will be kept confidential and permission will be sought whenever videos and notes are to be viewed for purposes other than those of the current study i.e. conference presentations.

Signed: .... Date: ....

7.8 APPENDIX EIGHT: PARTICIPANTS' SCHEDULE – PHASE 1

No.	Name	Date	Time
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7.9 APPENDIX NINE: LETTER CONGRATULATING AND INFORMING PHASE 2 PARTICIPANTS

Congratulations Lukessen, Berna and Fiona!

You have been selected to participate in the second round of task-based interviews. You will be working on 5 tasks as a group to reach a collective solution. It is anticipated that the process will take 2 – 2.5 hours to complete. Your participation time is **15H30** on Friday April 21, 2017. Please be on time.

Congratulations Rauna, Meagan and Millie!

You have been selected to participate in the second round of task-based interviews. You will be solving 5 tasks as a group to reach a collective solution. It is anticipated that the process will take 2 – 2.5 hours to complete. Please see me so that we may work out your exact time of participation.

Congratulations Ellena, Ethray and Nate!

You have been selected to participate in the second round of task-based interviews. You will be working on 5 tasks as a group to reach a collective solution. It is anticipated that the process will take 2 – 2.5 hours to complete. Please see me so that we may work out your exact time of participation.

Congratulations Denz, Jordan!

You have been selected to participate in the second round of task-based interviews. You will be working on 5 tasks as a group to reach a collective solution. It is anticipated that the process will take 2 – 2.5 hours to complete. Please see me so that we may work out your exact time of participation.

7.10 APPENDIX TEN: A NOTE FOR COLLEAGUES' INVOLVEMENT

Dear Colleague

Please help me out with my little research endeavour.

How would you group the following learners if you were asked to place them in groups of two/three in order for them to solve a critical thinking problem within the same time range?

1. Millie
2. Jordan
3. Ethray
4. Denz
5. Nate
6. Meagan
7. Rauna
8. Ellena

NB: Please provide reason/criteria for each arrangement. (Whatever thought comes to your mind is very important for me)

7.11 APPENDIX ELEVEN: REFLECTIVE INTERVIEW QUESTIONS

1. When presented with a problem, how did you go about finding a solution? What kind of thought process did you go through? What was your reasoning behind your decision?

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2. How did you ensure that you have captured key information from written or verbal instruction presented to you?

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3. Think about a problem that you have solved in a unique or unusual way. What was the outcome? Were you satisfied with it?

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4. Which did you prefer the most and why? Solving problems individually or in a group?

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5. What was your general experience as a participant in this research project?

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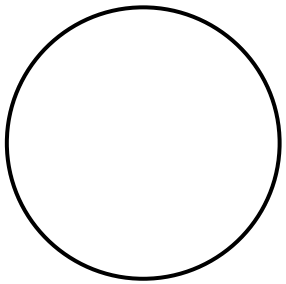
7.12 APPENDIX 12: PILOT ANALYSIS OF THE INITIAL EVGRT WORKSHEET

EVGRT	Participant 1: Carl	Participant 2: Natalia	Validity imperatives/language and mathematical accuracy/Alternations and decisions taken
Task 1 Imagine a 10m ladder leaning against a half painted wall behind you. The bottom of the ladder is 6m from where the wall meets the ground. What special name is given to the geometrical shape formed between the ladder, the wall and the ground? At what height from the ground does the top of the ladder lean against the wall?	Visualised the task at length by using all categories of visual imagery. Lots of noticeable KI in forms of gestures followed by [10]CPI with some noticeable PI and MIF. There was however little observation of the DI for this participant.	Formulated a picture while reading the question (CPI). She employed lots of KI as she justified her solutions and with some MIF and PI. There was also no DI observed for this task.	By using CPI, MIF and PI, the participants recalled and applied the theorem of Pythagoras to solve this task. They both spent reasonable time on the task which acted like an icebreaker for the worksheet. This is an indication that the task is valid enough to generate the data that it is intended to generate. Therefore, we decided to keep the task with numbering changes: Number sub-questions (a) and (b).
Task 2 Marina's backyard is a square with a side length of twenty meters. In her backyard is a circular garden that extends to each side of her yard. In the centre of the garden is a square patch of spinach so big that each corner of the square touches a side of the garden. Marina really likes spinach! How much area in Marina's garden is being used to grow spinach?	Although the participant's drawing was very small, he indeed visualised this task. He employed lots of KI in comparison to all other types of VI.	Well visualised – all 5 VI were represented although KI, CPI and DI were more outstanding than PI and MIF. The participant talked through her problem solving process.	This was a knowledge stimulating task, but it was too much for individuals to solve. Both participants took more than 25 minutes to complete the task successfully. They also got lots of hints from the researcher as the task required more than one person's input to solve. Since this is a good task that can definitely stimulate collaborative argumentation, a decision was taken to move it to EVGRT Worksheet 2.
Task 3 Dalia wants to design traffic sign boards for her school play. Her mathematics teacher told her that if she wanted her signs to be generally the same size and she must ensure that they all have equal perimeters. Dalia designed a square of side 12cm and decided to design an equilateral triangle of the same perimeter. a) What was the side of Dalia's equilateral	The participant claimed to have used a picture in this mind to calculate the dimension of the equilateral triangle. He first worked a formula but sketched a triangle to find the perpendicular height. He used DI as he	The participant first sketched a square, calculated its perimeter and found the side of the equilateral triangle. She then sketched a triangle to, used the theorem of Pythagoras to work out the height of the triangle.	This task was solved in reasonable time by both participants. It provided for good and limitless visual representations and all visual imagery categories were observed during piloting. The task stimulated the participants' thinking and created a real self-challenge and a good learning experience. The language used was appropriate and the concepts included in the task were

[10] Concrete Pictorial imagery (CPI), Pattern Imagery (PI), Memory Images of Formulae (MIF), Kinaesthetic Imagery (KI), Dynamic Imagery (DI)

triangle? b) Are their areas equal?	transformed the sketch, PI/MIF when he used Pythagoras' theorem to find the height. He made all sorts of gestures (KI) as he justified his actions.	She employed all 5 categories VI as she visualises her solution.	mathematically accurate. Therefore, a decision was taken to keep the task without alterations.
Task 4 Explain how you would help Amos to find the area of a square inscribed in a circle of circumference 18.84 cm . (Use π as 3.14)	The participant visualised the task very well and effortlessly explained his problem solving strategies. He made lots of references to how similar it was to task 2.	The task was well represented, well visualised by the participant. She made lots of cross references to task 2 and found the task less challenging to solve.	This was a good task and we wanted to keep it but after the pilot study analysis, we realised that there was already a task with an inscribed circle problem (task 2). We thought it would not be good to have similar tasks that would generally generate the same problem solving strategy. Therefore, we decided to exclude this task from the EVGRT Worksheets.
Task 5 Suppose you draw a circle of radius 6cm and sketched an equilateral triangle inside the circle and whose vertices are exactly on the circumference of the circle. What is the area of that triangle?	The task was well visualised as all 5 VI categories were well represented. It took the participant more than half an hour to solve the task successfully.	The participant enjoyed this task as she started off with a huge sketch in the middle of the page. She employed all the 5 categories of VI.	This was a very good task – well visualised by both participants. The task entails non-90° triangles which are new to Grade 11 learners in Namibia and would have covered the concepts by the time of the larger study and it would be exciting to observe them solving something they would have recently learned in class. We however observed that that task was too complex for individual learners hence; we moved it to EVGRT Worksheet 2.
Task 6 In a cube with sides of length 10cm, denote one vertex by the letter V. Find the sum of the distances from V to each of the other vertices of the cube.	The participant sketched a cube as he saw it in his mind (CPI). He employed lots of KI as he described parts of a cube. He also showed pattern movements from vertex V to other vertices. He transformed portions of the cube to clearly indicate	The participant sketched the cube as the read the question (CPI). She drew a net of a cube (DI) to show clear direction from V and the find the sum of the distances. She employed lots of KI as she explained her solution strategy.	This was an excellent task, well represented visually by both participants. The language used is appropriate although certain terms such as denote still needs to be explained to the participants. Both took too long and never got around the final answer. Like tasks 2 and 5, I believe that it would generate excellent arguments among participants working in small groups (see bigger PhD study). Therefore, we moved it to EVGRT Worksheet 2.

	dimensions from V.		
Task 7 How would you explain to a Grade 5 learner how to calculate the sum of the interior angles of a 5 sided polygon? (**Hint**: Grade 5 learners hardly understand formula)	The participant automatically wrote down the formula as soon as he read the first part of the task. He then sketched a pentagon as he explained using the formula. He later did fairly well as he formulated patterns to work out the solution.	The participant started by writing down the formula as she claimed that it would be easier to explain using the formula. She finally visualised the problem through probing and she realised that she could not have solved the task had she not sketched.	This was an excellent yet frustrating task for the participants. Both participants could not solve the task unless they made a sketch. They were both keen to use the formula and teach it to the Grade 5 learners as they claimed that there was no other way of solving it. However, the more questions they were asked, the more they visualised and generated appropriate drawings to solve the task. All the 5 VI categories were effectively employed. We decided to keep this task and only change the way the probing questions.
Task 8 Savannah's house is 4km north of her school. The local market is due east of the school and on a bearing of 120° from Savannah's house. Find: a) The bearing of the school from the store b) The distance between Savannah's house and the store.	The participant visualised the problem situation making a sketch with directions on to locate the positions of the subjects presented in the question. He employed lots of KI as he presents his arguments. All the other VI categories were also employed during the problem solving process.	The participant sketched the situation presented in the question as she read it. She marked all the necessary dimensions as she employed lots of KI as the participant explained and justified her solution. She used a 'SOHCAHTOA' pattern to determine the formula of the ration that she used to answer the question. She employed all 5 VI categories in her solution strategy.	This task was particularly easy to solve for both participants as they each used totally different visualisation processes to solve it. We strongly believe that there are still many ways to represent this task and more arguments could emerge from the many representations if more than one learner worked together to solve it. Alternation: 'store' was changed to 'local market' Therefore, we moved this task to EVGRT Worksheet 2.
Task 9 The surface of a study table consists of a square of 1 m per side and two semicircles attached on either	The participant made a sketch of a table and labelled 1m on each side. He was however not keen	The participant made a neat sketch of the table as she read the question. She sketched a	Both participants struggled with this task. They did not enjoy it as they did most tasks. They were not as keen to solve it in comparison to other tasks as I only observed CI

opposite end. Find the total surface area of the table.	to continue with the task.	circle and divides it to visualise the two semicircles on the table. She was however unable do anything else.	with some DI from one of the participants. They also asked for lots of clarifications but could still not extensively visualise it further. We therefore I decided to reject this task.
Task 10 Mr. Mino constructs a triangle of perimeter 30cm for his mathematics lesson preparation. During the lesson, he asked his learners to find the length of the shortest side of the triangle if two sides of that triangle were each twice as long as the shortest side. Suppose you are Mr. Mino's learner, what is your answer?	The participant read the task several times before he solved the task. He then constructed an isosceles triangle with x as the shortest side and the adjacent angle each equals to $2x$. He constructed an algebraic equation and solved for x. The participant interchangeably employed all 5 categories of VI as he solved the task.	The participant immediately sketched an equilateral triangle when she read the first sentence of the task (CPI). She then labelled the dimension of the triangle as she continued reading (MIF, DI). She then used PI when she employed algebraic method to solve for the unknown x plus lots of KI as she explicates her arguments.	This was an excellent task – algebra combined with geometry. Both participants represented the task in exact same way and substituted the solution in the same way – made a sketch, named x, the shortest side and constructed an algebraic equation to solve for x. We decided to keep this task and place it at the very end of the revised worksheet as a way of motivating the participants to complete the worksheet during the data collection of the bigger study.
Task 11 Show and explain how you would find the centre of this circle. 	The participant immediately drew lines around the circle – to form a square. He then measured the side of a square, divided it by two to get the dimension of the radius of the circle. There was noticeably lots of KI, CPI and also DI, and PI.	The participant used a ruler to approximate the centre of the circle and then drew in lots of diagonals to prove that she was accurate with her approximation. She saw a circle with many crossing lines in her mind before she could visualise it on paper. VI: CPI, PI, KI, DI	This was one of the excellent tasks and it was diversely represented by the two participants and in most interesting ways. Although the task has diversity, we are not convinced that it would be appropriate for EVGRT Worksheet 2. We believe that it is better for it to be solved individually as it may not create enough room for collaborative argumentations.

7.13 APPENDIX TWELVE: PROPOSAL AND ETHICS APPROVAL CERTIFICATE

RHODES UNIVERSITY

Grahamstown • 6140 • South Africa

EDUCATION FACULTY • PO Box 94, Grahamstown, 6140
Tel: (046) 603 8385 / (046) 603 8393 • Fax: (046) 622 8028 • e-mail: d.wilmot@ru.ac.za

04 February 2016

To Whom It May Concern

Re: Proposal and Ethics approval for Beata Lididimikeni Dongwi (09D6388)

The minutes of the EHDC meeting of 28 January 2016 reflect the following:

2016.01.1 CLASS A RESTRICTED MATTERS
DOCTOR OF PHILOSOPHY RESEARCH PROPOSALS

To consider the following research proposals for the degree of Doctor of Philosophy in the Faculty of Education:

Beata Lididimikeni Dongwi: 609D6388
Title: Examining mathematical reasoning through enacted visualisation when solving word problems
Supervisor: Prof Marc Schäfer
Decision: Approved

This letter confirms the approval of the above proposal at a meeting of the Faculty of Education Higher Degrees' Committee on 28 January 2016.

In the event that the proposal demonstrates an awareness of ethical responsibilities and a commitment to ethical research processes, the approval of the proposal by the committee constitutes ethical clearance. This was the case with this proposal and the committee thus approved ethical clearance.

Yours truly

Prof. Mellony Graven
Chair of the EHDC, Rhodes University