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Structuralist Neologicism

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Abstract

Neofregeanism and structuralism are among the most promising recent approaches to the philosophy of mathematics. Yet both have serious costs. We develop a view, *structuralist neologicism*, which retains the central advantages of each while avoiding their more serious costs. The key to our approach is using arbitrary reference to explicate how mathematical terms, introduced by abstraction principles like Hume's, refer. Focusing on numerical terms, we argue that this allows us to treat abstraction principles as implicit definitions serving to determine all (known) properties of the numbers, achieving a key neofregean advantage, while preserving the key structuralist advantage that which objects play the number role doesn't matter.

1 Overview

Neofregeanism and structuralism about mathematical objects both vindicate many intuitions about the mathematical domain. Among the most attractive are:

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(1) mathematical objects are the referents of syntactically singular terms;

(2) truths about them are conceptual truths;

(3) the only properties they have are *mathematical* properties.

While the neofregean wants to vindicate (1) and (2), the structuralist aims at vindicating (1) and (3).[1] Yet these intuitions often conflict in practice, tempting neofregeans and structuralists into controversial commitments which weaken the initial attractiveness of their views.

In light of this, it's inevitable that someone would synthesize these views in an attempt to salvage all three intuitions. We bravely volunteer to take on this role. *Structuralist neologicism*, the view we develop here, can seamlessly satisfy these desiderata. All that's needed is a bit of ideology—which we assure you is a reasonable cost.

We start by exploring a serious problem for the neofregean approach, the *Caesar problem*.[2] Neofregeans treat syntactically singular terms introduced by abstraction as functioning semantically like proper names. But this requires abstraction principles to single out particular referents. The Caesar problem—HUME'S PRINCIPLE's silence on the question of whether numbers are Roman emperors—shows that there's no simple route to paradigmatic reference for the neofregean, which threatens (1).

The problem can be resolved, but only at significant cost. In particular, we can invert the Caesar problem to show that any solution to it conflicts with the privileged epistemic status of abstraction principles, threatening (2). If the Caesar problem is solved, then HP implies truths about non-mathematical objects, which is an unwelcome feature for an allegedly analytic or logical principle such as HP. This in turn jeopardizes (3). See §3 below for details.

[1] The *ante rem* structuralist, anyways. *In re* structuralism abandons (1). We discuss both these structuralist views in §4 below.

[2] The Caesar problem originates with Frege's *Grundlagen der Arithmetik*, §66. See e.g. Wright [1983] and Hale and Wright [2001] for discussion in the neofregean context.

For structuralists, mathematical objects like natural numbers are picked out by their roles within the natural number structure. Correspondingly, in order for numerals to refer in the usual way, appropriate individuals need to be so specified for each. As abstract structures have many realizations, the only way to do this treats structures as self-subsisting objects which our mathematical terms latch onto. Yet some mathematical terms, like i, cannot be so treated. Structuralism thus struggles to rescue the intuition that all syntactically singular terms of mathematics are semantically singular even granting the existence of abstract structures. See §4 for details.

Our view, *structuralist neologicism*, avoids both problems by interpreting mathematical singular terms in terms of *arbitrary* reference. Mathematical singular terms, as devices of arbitrary reference, function in all essential ways as singular terms; they are devices of genuine, if non-standard, singular reference.[3] So interpreting mathematical singular terms thus permits us to view abstraction principles like Hume's as sufficiently individuating of mathematical objects as to secure reference, satisfying (1).[4]

Moreover, our arbitrary interpretation of number-terms treats certain abstraction principles as both implicit definitions and a type of logical truth, vindicating a strong form of (2), the neofregean insight that abstraction principles like HP have privileged epistemic status. Yet we treat only the arithmetical properties—those following from facts about the equivalence relation of equinumerosity—as properties of the numbers, retaining (3), the central insight of the structuralist program.[5]

[3]Fine [1998, 2002] also suggests indeterminate or variable interpretations of mathematical objects—especially those induced by abstraction. His account differs from ours, but it's clearly in the same spirit.

[4]We discuss the genuine referentiality of arbitrary terms introduced by HUME'S PRINCIPLE in §6.

[5]For an important caveat, see fn. 27 below.

2 Abstraction Principles

Neofregeans ground mathematical concepts in abstraction principles. These are principles of the form:

$$\S e_1 = \S e_2 \leftrightarrow E(e_1, e_2)$$

where § is an operator mapping some entities e_1, e_2 into entities of a (possibly) different sort, and E is an equivalence relation holding between e_1, e_2. Roughly, they say that the abstract of e_1 is identical with the abstract of e_2 just in case e_1 and e_2 stand in the equivalence relation E. The idea is to "transfer", in some sense, the content expressed by the equivalence relation on the right to the complex syntactically singular terms flanking the identity sign on the left.

Abstraction principles originate with Frege, who singled out three of particular interest:

(1) HUME'S PRINCIPLE: $\#F = \#G \leftrightarrow F \approx G$
which says that the number of the Fs and the Gs are identical iff the Fs and the Gs are in one-to-one correspondence.[6]

(2) BASIC LAW V: $\{x : Fx\} = \{x : Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx)$
which says that the extensions of the concepts F and G are identical iff F and G are co-extensive.[7]

(3) DIRECTION PRINCIPLE: $d(a) = d(b) \leftrightarrow a//b$
which says that the directions of line a and b are identical iff a is parallel to b.[8]

Of these, HUME'S PRINCIPLE (HP) has been the focus of significant interest since it interprets full second-order Peano arithmetic and, unlike BASIC

[6] See Frege's *Grundlagen der Arithmetik*, §62 and Mancosu [2015] for a survey of early principles about equality of numerosities.

[7] See Frege's *Grundgesetze der Arithmetik*, vol. I §20.

[8] See Frege's *Grundlagen der Arithmetik*, §65.

LAW V, it's consistent. Neofregeans have also pointed out that HP looks conceptually true of cardinality (or, alternatively, implicitly definitional of cardinal number).[9] Moreover, they claim on the basis of the SYNTACTIC PRIORITY thesis:

> SYNTACTIC PRIORITY: The truth of claims involving numerical terms in singular term position suffices to guarantee that they refer to objects. [Wright 1983]

that HP, being true, not only specifies our conception of cardinal number, but guarantees the existence of cardinals as the referents of expressions like '$\#F$'. How, though, is this supposed to work?

2.1 Individuation, Abstraction, and Caesar

The answer is that abstraction principles are supposed to provide identity criteria for *individuating* abstract objects. Then the identity and distinctness claims on the left-hand side of abstraction principles let us identify which objects '#F' refers to, wherever it appears [Linnebo 2009]. This is a nice start, but what's meant by individuation isn't obvious. We see two possible disambiguations.

First, there's a *metaphysical* interpretation of individuation: 'whatever it is that makes [an object] the single object that it is' [Lowe 2003, p. 75]. Second, there's an *epistemic* interpretation where we 'single out' an entity as a 'distinct object of perception, thought, or linguistic reference' (*op cit.*). So abstraction principles individuate either by singling out objects (the epistemic interpretation) or by making them the sort of object that they are (the metaphysical interpretation).

But abstraction principles are far from perfect individuators on either interpretation. This is the basis of the Caesar problem, one of the most

[9] The claimed epistemic status of HP has shifted over the years in light of criticism by Boolos [1998] and others.

difficult issues for neofregeans.[10] The problem is that HP doesn't decide the truth value of "mixed" identity statements like:

$$\#\text{F} = \mathit{Caesar}$$

However, it's basic to our understanding of cardinal numbers that they don't conquer Gaul, much less do so with Caesarian aplomb. Of course, HP entails $\#F = \#G$ when F and G are equinumerous and $\#F \neq \#G$ when they're not. But, as Frege complained, HP doesn't tell us whether the number 2 is or isn't a Roman conqueror.

Many, including us, think that HP should deliver some such verdict on mixed identities:

> If [Frege's] platonist conception [of numbers as self-subsistent objects] is to be legitimate, there surely have to be facts of the matter about *which* objects the numbers are. The concept of number must possess 'sharp limits to its application' — it cannot just be indeterminate whether the putative referents of the terms which HUME'S PRINCIPLE enables us to introduce coincide with any previously understood or independently intelligible kind of objects. Hence if there really are such referents, a satisfactory conception of the objects in question must somehow implicitly settle whether they are or are not trees, tigers, persons or countries. [Hale and Wright, 2001, p. 341].

More generally, as Frege noted, solving the Caesar problem is crucial to the epistemological grounds of his project:

> If we are to use a symbol '*a*' to signify an object, we must have a criterion for deciding in *all cases* whether *b* is the same as *a* (...). [emphasis ours] (*Grundlagen der Arithmetik* §62)

[10]See Tennant, Neil [2013], *Logicism and Neologicism*, Stanford Encyclopedia of Philosophy for useful survey of other extant issues.

It's similarly crucial for neofregeans. HP is supposed to provide epistemic access to cardinal numbers, but in so doing, it should capture what cardinal numbers intrinsically are and aren't and regurgitate things we know about numbers like their ineligibility to rule.

If the Caesar problem isn't solvable, we should admit that HP fails both to settle the intrinsic nature of numbers as well as failing to explain how we single them out. Correspondingly, it neither epistemically individuates the numbers from other objects nor metaphysically individuates them. We thus claim that HP fails to secure reference to the numbers exactly because fixing of reference, as standardly understood, presupposes individuation. Since the relationship between reference and individuation is essential to our criticism of neofregeanism, we'll dwell a bit on it in the next section.[11]

2.2 What's Really Wrong with Caesar?

First a caveat: we aren't concerned with the semantic notion of reference found in a representational semantics, but rather the metasemantic explanation of how reference by singular terms can be achieved at all. Standard metasemantic accounts of reference, like those proposed for proper names in natural language, presuppose individuation of the referent from other alternatives in order to fix reference. Here are three rather central examples (the point holds generally):

- Causal chain theories require initial tagging of an object by ostension, when the baptism takes place *in præsentia*, otherwise by description. [Kripke, 1980, p. 96, fn. 42.] Securing a referent thus requires that some object be singled out, by description or ostension, from the salient alternatives.

- Descriptivist theories fix the reference of proper names via definite

[11] We think these considerations undermine also Hale and Wright [2001]'s solution to the Caesar problem. See §3.1 below.

descriptions. The entire function of the descriptive material is to single out an unique object as the referent of a singular term.

- Radical Interpretation accounts fix reference via a procedure of assigning the most reasonable referent from among the salient alternatives. This requires prior ability to individuate the potential assignees.

Fixing the referents of ordinary singular terms involves, in one way or another, singling out unique referents. Since the referent needs to be unique, we can call this *perfect individuation*:

PERFECT INDIVIDUATION An individual i is perfectly individuated just in case it's uniquely singled out, in some way, from among candidate referents for i.

We'll call the form of reference fixed via perfect individuation *canonical reference*:

CANONICAL REFERENCE A singular term 't' refers canonically just in case the referent of t is perfectly individuated.

Abstraction terms like '$\#F$' are supposed to refer analogously to ordinary singular terms. It thus seems reasonable to require that abstraction principles explain how terms like '$\#F$' canonically refer—which seems to require perfect individuation. Or, anyways, this seems reasonable unless there's an alternative form of reference on offer.

Yet the Caesar problem shows that certain abstraction principles fail to perfectly individuate their abstracts from other objects. As canonical reference requires perfect individuation, the problem spreads to a failure to provide canonical referents. Of course, when we can't single out a unique referent just by something like HP, we might let some outside criterion fill the gap. For instance, we might use the relative *naturalness*—in Lewis's sense—of salient objects to decide which lucky one is referred to by '$\#F$'. Some interpretativist views likewise treat being the most reasonable referent as an external factual matter—not necessarily something available to someone actually interpreting.

But this patch, however attractive elsewhere, looks problematic here. Why would any of many possible referents of $\#F$ be more "natural" or reasonable than any other? In cases of reference to concrete objects, it sometimes seems reasonable to let external criteria partially determine what we refer to. But reference to mathematical objects by abstraction seems different. If our attempted conceptual analysis or implicit definition of cardinal number doesn't single out a particular object for '$\#F$' to refer to, what materials external to our definition could reasonably close the gap?[12]

The obvious move is invoking pre-existing substantive knowledge of the nature of numbers to more finely individuate potential referents. But then it's HP *and* whatever background materials are so invoked which succeed at securing canonical reference, not merely HP, contra neofregeanism. And this suggests another worry.

3 Inverting Caesar: The Raseac Problem

We don't believe the Caesar problem can be solved satisfyingly without abandoning canonical reference. But, even if we're wrong, solving it requires giving up on (2), that HP has a privileged epistemic status. Why? Because in showing that HP determines that 2 isn't a Roman, we'd learn something about the nature of *both* numbers and Roman generals. If HP entails that '$\#F$', for any F, isn't Caesar, it also entails Caesar isn't a number. But HP's supposed to be some form of logical or conceptual truth and these

[12]We don't deny the world might sometimes help secure canonical reference—consider analytic functionalism about mental properties. Our point, rather, is that the world seems unable to play that role for the case of '$\#F$'. In a way, this is the real lesson of Benacerraf [1965]—extra-arithmetical properties distinguishing between potential referents of numerals simply seem irrelevant to their role, so choosing which is referred to on this basis seems, at absolute best, *ad hoc*. Thanks to John Divers for discussion.

aren't supposed to inform us about the underlying nature of the objects they applied to. Especially not Romans.

These intuitions are nowadays explicated in terms of *topic-neutrality*. This is represented formally in terms of invariance of the meaning of logical terms under operations like permuting the domain.[13] Variance of the meaning of an expression under a permutation demonstrates that it:

> somehow discriminate[s] among individuals in the domain, and any consideration which discriminates among individuals lies beyond the reach of logic, whose concerns are entirely general. [McGee, 1996, p. 567].

Permutation invariance is widely accepted as a necessary condition on logicality; it's underlying motivation, that logical, conceptual, and definitional truths shouldn't have substantive implications for the underlying objects they apply to is accepted even more widely. These "special" sorts of claims shouldn't convey meaning on an expression where that meaning carries substantive individual discriminating content.[14] Yet if we can solve the Caesar problem, we can show that $\#F$ isn't Caesar, clearly discriminating between the two. In short, solving the Caesar problem seems to undermine the claim that HP is a conceptual, logical, or definitional truth.

Limited attention has been paid to this problem. Three distinct invariance relations on abstraction principles have been investigated: invariance of the equivalence relation; invariance of the abstraction principle itself; and invariance of the induced abstraction operator [Antonelli 2010]. The literature on how invariance relates to abstraction, such as the work of Fine [2002], Cook [2016], and Antonelli [2010], has focused on invariance of the equivalence relation on the righthand side of abstraction principles. The details of this notion of invariance and invariance of abstraction principles themselves won't concern us here given our focus on singular terms.

[13] The Raseac Problem is novel to us, but inspired by Antonelli [2010].

[14] Of course, we could debate what 'substantive' implications are. Here, we won't.

For our purposes, the important type is the third. An abstraction operator § is invariant in this sense just in case $\S(\pi(X)) = \pi(\S(X))$ for any permutation π of the underlying domain. This type of invariance for abstraction operators hasn't been seriously investigated since it looks nearly trivial. #, standardly understood, has the semantic type of a function from subsets of the domain into the domain; any object of that semantic type is invariant only on domains containing a single object.

#, as standardly understood, isn't permutation invariant and thus fails the most widely accepted and plausible necessary condition on logicality.[15] But, if '#', with the meaning conveyed on it by HP, fails a widely accepted criterion for being a logical operation, how could HP be a logical or conceptual truth? It's a worrying feature of a logical or conceptual truth if its function is to generate operations which are themselves non-logical. At best, the resulting claim that HP is a logical, conceptual, or definitional truth would be much weakened. So solving Caesar seems to undermine one of the central motivations for the neofregean program.

To drive this point home, suppose we can know, by means of HP and additional materials, that, no matter what F is, $\#F \neq Caesar$. We can always find a permutation π of the actual domain where $\pi(\#F) = Caesar$.[16] If # were permutation invariant, then $\#(\pi(F)) = \pi(\#(F)) = Caesar$, contradicting our assumption. So any interpretation of # which solved Caesar would *eo ipso* be permutation variant, violating a necessary condition on logicality, and thus run straight into the Raseac problem.

3.1 Hale and Wright on Caesar

Hale and Wright [2001] appeal to a theory of categories to respond to the Caesar problem. Their approach demands discussion since it also potentially

[15] On permutation invariance as a necessary condition on logicality, see e.g. McGee [1996]. See also the authors mentioned in §7 below.

[16] Obviously we're assuming Caesar is in our actual domain. If you're feeling pedantic, rerun the example with your favorite personality.

skirts the Raseac problem. A category can be understood as a collection of objects which share a common identity criterion—e.g. *abstract object*, *person*, etc. Categories are distinguished by the identity criteria associated with objects falling under them. The identity criterion given by HP, for instance, is distinct from identity criterion for persons, whatever that is. Persons and numbers are supposed to *essentially* belong to different categories; on this basis, Hale and Wright claim cross-category identity claims are uniformly false. While both of us agree that that 'Caesar isn't $\#F$' is conceptually flawed, we believe our view has some significant advantages.

For example, consider cross-category claims like $\#F \neq Caesar$ and alleged intra-category claims like 'the natural number 2 = the real number 2'. The essence of Hale and Wright solution is that the sortal concepts for persons and numbers don't share an identity criterion. But the most plausible explanation of this difference in identity conditions invokes the fact that numbers and persons have different *intrinsic* properties.

It seems to us that the intrinsic properties of numbers should be determined by HP. But since HP doesn't settle the Caesar non-identity, how can we explain the difference in identity conditions which settles the Caesar non-identity? We worry that solving Caesar Hale and Wright's way demands that HP already provide necessary and sufficient conditions for distinguishing cardinal numbers from humans. This, though, is what invoking categories was supposed to accomplish for us.[17]

Of course, Hale and Wright could just stipulate HP as the identity criterion for cardinal numbers. But this also stumbles on inter-category claims. Both the natural number 2_n and the real number 2_r are abstract objects, so they should share a common identity criterion. If this is right, what stops us from also saying that both 2_n and Caesar are both things, so they should share a common identity criterion, so the Caesar problem still isn't resolved?

[17]See Stirton [2017] for a similar worry: Hale and Wright's solution hinges on the idea that it is not a conceptual necessity that *number* and *person* share a common identity criterion. This seems to hinge on already understanding the mixed identity statements.

Suppose that 2_n and 2_r don't share an identity condition. Then they are of different categories and so $2_n \neq 2_r$. But we think that it's problematic to say there's a determinate answer—false!—to whether or not they're identical. This can be emphasized by noticing that similar moves would mean that the cardinals and the finite cardinals aren't identical either [Linnebo forthcoming].[18]

Finally, Hale and Wright could claim that cross-category identifications simply don't make sense, but this overgenerates. It's determinately true that we're not the same as the proof that there's no largest prime. But presumably the relevant identity conditions are distinct. Likewise the co-authoring duo Francesca and Jack isn't the same as either member; but again, presumably the relevant identity criteria are different. Far better to solve Caesar our way by maintaining there's simply no determinate answer to certain mixed identity claims.

4 Structuralism – Two Types

Structuralist views also abandon at least one of our opening intuitions. In their foundationalist guise, these views comes in two main varieties differing in how they treat presumptive mathematical singular terms. *Ante rem* structuralism [Shapiro 1997, Resnik 1997], holds that mathematical singular terms canonically refer to positions in structures where structures themselves are treated as abstracta. *In re* structuralism [Chihara 2004, Hellman 1989] contextually eliminates mathematical singular terms in favor of modal generalizations over instantiated structures. Our worries about each are different, so we take them in turn.

[18]This argument presumes Hale and Wright's *Grundgedanke* (which is essential to their solution to Caesar): any object has a single associated criterion of identity. See Fine [2002, pp. 48-49] and Linnebo [forthcoming, §9] for worries about it.

4.1 Structuralism – *Ante rem*

Ante rem structuralists treat singular terms as referring canonically. Since they are devices of canonical reference, they need to pick out objects. Since there's no principled choice of set, etc. which can be reasonably taken to be *the* number 2, we need an object to be the canonical referent of the numeral '2'. Individual mathematical objects are "places in structures", where structures are abstract objects and isomorphic structures are identified.

Since there's only one natural number structure, there's just one candidate place in the structure to assign to '2', so there's a unique number 2. There are ways to push Benacerrafian worries about the intuitiveness of this solution and metaphysical worries about existence of structures themselves, but we'll mainly put these aside since there are problems more related to our present concerns.

Structural properties and relations are supposed to fully constitute positions in structures. But we can't distinguish, using just these properties and relations, between certain positions in structures admitting non-trivial automorphisms (like the complex numbers) [Shapiro 1997, Burgess 1999, Keränen 2001]. Shapiro [2008, 2012] solves this by assuming that expressions aimed at indiscernable positions, like 'i' and '$-i$', are treated as *parameters*: as not genuinely referential but only contextually so. They're assigned in a context to an arbitrary referent and canonically refer, in that context, to that referent. For Shapiro, the relevant context continues forward throughout our mathematical practice.

We like arbitrary reference, but Shapiro's solution puzzlingly imposes a double standard for reference via mathematical singular terms. Some singular terms genuinely refer, analogously to ordinary referential terms, some other do so only in specific contexts. So, in order to solve the particular issue indiscernibles poses for *ante rem* structuralism, Shapiro abandons the idea that all mathematical terms uniformly and canonically refer, seemingly undermining a key advantage of *ante rem* structuralism.[19] It would be better if this kind of disjunctive solution could be avoided since there's no surface difference between the grammar of i and 2. Indeed, why not just go all the way with arbitrary reference, treating it as a genuine form of reference, as we do below? This neatly avoids the double standard by treating the ability to uniquely refer as a relatively common special case. Once we've gone arbitrary anyways, why not just go arbitrary across the board?

4.2 Structuralism – *In re*

In re structuralists contextually eliminate mathematical singular terms wherever they appear, dropping intuition (1) entirely. This strategy is available, but costly. Consider a claim like '2 is prime'. This can be reinterpreted *á la* Hellman as:

> $\Box\forall X(X$ is simply infinite$\rightarrow$ '$\forall x\ x$ divides $2 \leftrightarrow x = 2 \vee x = 1$' holds in $X)$

but the translation pinches since the surface grammar of '2 is prime' mirrors 'Guillermo is human'. Numerals seem to intuitively mean the same things within mathematical contexts and without. Constructions like:

> 2 is prime, even, my favorite number, and an abstract object

are perfectly coherent and possibly true.

So *in re* structuralists either need to give contextual definitions of non-mathematical claims or posit an ambiguity about numerals. Both are unpleasant. Better to give a uniform explanation of the meaning of numerals instead of contextually, but not uniformly, eliminating it whenever it appears.

[19] See Assadian [forthcoming] and Hellman [2001] for arguments that this problem generalizes, placing in doubt the ability of *ante rem* structuralists to capture canonical reference at all.

5 A Way Forward

Summing up, none of our alternatives allows us to satisfy intuitions (1)-(3) about mathematical objects. The neofregean is caught between the Caesar and Raseac problems: either HP doesn't suffice for terms like '$\#F$' to function as uniquely designating singular terms or HP has substantial content, weakening its privileged logical, conceptual, or definitional status. Structuralist alternatives fare no better. *Ante rem* structuralism cannot capture singular reference across the board because of terms like i, *in re* structuralism entirely abandons treating the syntatically singular terms of mathematics as semantic singular terms.

Suppose, though, we can avoid the Caesar/Raseac dilemma for neofregeanism while, contra the structuralisms canvassed, salvaging the idea that all syntactically singular terms of mathematics are semantic singular terms.[20] If we can do this by rejecting canonical reference, preserving the benefits of otherwise attractive programs without abandoning their central motivations, then we should seriously consider doing so. Our primary aim in the following is showcasing the attractions of such an account.

To do so, we'll develop an account of the reference of singular terms introduced by abstraction principles which treats mathematical singular terms as referring *arbitrarily*. Arbitrary reference is a genuine alternative to canonical reference since it's a form of genuine reference without perfect individuation. It salvages intuition (1); this, in combination with other features of arbitrary reference, permits a principled solution to Caesar. We provide a semantical account of the meanings for these singular terms on which they're permutation (in fact, isomorphism) invariant. This makes it reasonable to treat

[20] An alternative which avoids both the Caesar problem and equivocal reference treats HP-numbers as higher-order properties, as we might do by just ramsifying over Peano arithmetic. This strategy though, like *in re* structuralist's, doesn't account for the surface syntactic role that expressions such as '$\#F$' seem to play. We also want to make our view appealing to both the neofregean and the *ante rem* structuralist: to this end, we want to retain intuition (1). Thanks to Volker Halbach for discussion.

them, and HP itself, as logical, vindicating a strong form of intuition (2). Thus our account solves splits the Caesar/Raseac dilemma.

The meaning of #, on our view, is indifferent to the underlying nature of the objects it refers to. We thereby additionally capture (3), the central insight of mathematical structuralism: anything can work to play the role of a number, whether tables, chairs, or beer mugs. So our view unites core features of both structuralism and neofregeanism, while abandoning much of the neofregean metaphysical outlook, i.e. that HP perfectly individuates *abstracta*. Since we treat HP as logical, we retain the logicist insight that mathematical content can be grounded in logic. The result is thus *structuralist neologicism*.

5.1 Structuralist Neologicism — First Pass

We believe that there's a non-canonical notion of reference which naturally interprets how singular terms introduced via abstraction refer. This type of reference respects the intuitive syntactic structure of mathematical expressions—conforming to the neofregean SYNTACTIC PRIORITY thesis—without the baggage of canonical reference.

Principles like HP, which function much like implicit definitions, don't specify unique meanings for #. Rather, as the Caesar problem demonstrates, they only specify a class of potential meanings. If we took HP literally as an implicit definition, and didn't assume any additional reference-fixing materials, we'd have two options: either treat HP as a failed implicit definition of # (this presumes canonical reference) or treat it as a successful implicit definition of a broader meaning for # where # refers arbitrarily over the class of potential canonical referents.[21]

[21] Implicit definitions, even when we relax the uniqueness requirement, are supposed to pick out the *weakest* possible meaning for the defined term. Since abstraction principles *can* be interpreted arbitrarily and since *any* canonical interpretation would be stronger than this interpretation, it violates the spirit of the implicit definition approach to so-interpret them [Woods 2014, §4.3].

We hold that HP lays down the exact content of number-terms and no more: HP doesn't say that $\#F$ isn't Roman, so neither should we. Treating '$\#F$' as referring arbitrarily allows us to treat the question 'Is $\#F$=Caesar?' like we treat 'Is the arbitrary triangle scalene?'. Both are terrible questions—they fundamentally misunderstand the semantic role of arbitrarily referring expression. The arbitrary term $\#F$ is treated as generally carrying only those properties that are true of *any* arbitrary choice of canonical referent which could play the $\#F$-role; asking about other properties simply misunderstands how arbitrary reference works.[22] The properties of the referent of '$\#F$' are given by equivalence relation of equinumerosity on the right-hand side of HP. So the meaning conferred on '$\#F$' generally doesn't outstrip what's explicable in terms of the equivalence relation of equinumerosity.[23]

6 Arbitrary Reference

Our move is less startling than it might appear. Several theorists have recently suggested treating certain singular terms as *parameters* or *arbitrary names*: terms behaving grammatically like singular terms, but functioning semantically like pronouns or indefinite descriptions. We've mentioned Shapiro already [2008, 2012], but Brandom [1996], Pettigrew [2008], and Woods [2014] have also recently suggested this approach. In mathematical discourse these kinds of terms are typically introduced by locutions of the form 'Let j be an F', where j is thereafter treated as functioning just like a singular term.

Pettigrew and Shapiro interpret these terms as not genuinely referential.[24] We, in contrast, take arbitrary reference to be a genuine, if non-canonical, form of singular reference. We are indifferent to whether canoni-

[22] The 'generally' here elides an issue involving what Fine calls the principle of generic attribution. See below and fn. 27.

[23] For the details of our view concerning HP, see §6.4 below.

[24] Our view otherwise looks rather similar to theirs. We take this to be a good thing!

cal reference is a common special case of arbitrary reference (where there's only one candidate meaning), though we note that there's a certain elegance to this view. Our view can be developed in two ways, each of which corresponds to each author's favorite way of interpreting arbitrary reference. This is because each interpretation treats arbitrary reference as *genuine* reference.

6.1 Two Ways of Understanding Arbitrariness

The first way of understanding arbitrary reference is *epistemicist*. Roughly, we cannot *know* what an arbitrary term refers to, but it refers to a unique individual.[25] We call the second way the *supervaluational* view: arbitrary reference is a primitive form of reference which shouldn't be glossed in terms of canonical reference. We can model this kind of reference, however, in terms of a supervaluational semantics where, generally, properties had by all individual choice of referent are had by the arbitrary referent. On this view, any epistemic constraint on arbitrary reference follows from the semantic function of arbitrary terms.[26]

6.2 The Epistemicist View of Arbitrariness

The epistemicist view, championed by Breckenridge & Magidor [2012], Martino [2001], and Boccuni [2013], conceives of arbitrary reference as an epistemic phenomenon. Locutions like 'Let n be an arbitrary natural number' introduce reference to a particular arbitrary natural number n where, according to Breckenridge & Magidor [2012]:

> *Arbitrary Reference (AR)*: $[n]$ receives its ordinary kind of semantic value, though we do not know and cannot know which

[25] The name recalls Williamson's views on vagueness; the analogy is developed in detail by Breckenridge & Magidor [2012].

[26] There's also Fine's [1985] view that arbitrary terms pick out a class of special "arbitrary individuals". It's generally thought too metaphysical a solution, though it's available as another way of developing our view.

value in particular it receives. [Breckenridge and Magidor 2012, p. 377].

The epistemicist view retains classical logic while still motivating the usual restrictions imposed on the rules of introduction and elimination of quantifiers in natural deduction. In the rule of universal introduction, for example, to infer from $\phi(a)$ to $\forall x \phi x$, we require that no assumption the universal quantification depends upon contains the arbitrary name a. This is because, in order to conclude that all x are ϕ from $\phi(a)$, we need to ensure that no assumption concerning specific properties of a played a role in our *concluding* $\phi(a)$. This is can be naturally treated as explicit ignorance about the particular properties of a.

6.3 The Supervaluational View

The more radical view developed by Woods [2014] suggests taking arbitrariness as a fundamental semantical notion. The semantic function of an arbitrary expression doesn't involve it referring to an object in a canonical way. In particular, we don't model the referential nature of '$\#F$' by assigning it a particular object in a model. Rather, an arbitrary expression refers *over* the class of objects which would satisfy it if it functioned like a device of canonical reference. We then model—though we don't reduce—the truth of a statement ϕ containing arbitrary names relative to the truth of statements containing non-arbitrary names. Roughly, if every precisification of a model assigning a particular member of the domain to the arbitrary terms in ϕ agrees that ϕ is true, then ϕ is true.[27]

[27]The converse does not hold since, for certain *special properties* like $\langle \lambda x.\ x = \#F \rangle$, $\#F$ has it but no instantiation does. In terms of Fine's [2002] discussion, we need a principled restriction on the *principle of generic attribution* linking arbitrary to non-arbitrary objects. There are various plausible ways of doing this—again, see Fine [2002]—but by far the easiest notes that our reasoning concerning these objects typically needs the inference from "the arbitrary F has ϕ" to "every non-arbitrary F has ϕ" only in those clear cases of non-special properties. The supervaluational semantics in Woods [2014] vindicates

This view ties the formal semantics for arbitrary reference and the semantical function of arbitrary reference closely together. The properties the arbitrary triangle has (known or not!) are only those which any particular choice of individual referent has, not properties holding only of some of these. On the semantics sketched above claims like 'the arbitrary triangle is equilateral' and 'the arbitrary triangle is not equilateral', which predicate properties of the arbitrary triangle which do not hold of *all* triangles, come out as neither true nor false. The supervaluational view thus reflects the intended usage of the device of arbitrary reference into the semantics for it. Of course, as on any supervaluational view, we have to relinquish classical logic to obtain this advantage.

Structuralist neologicism is thus an ecumenical view, permitting several different developments. In particular, many ways of interpreting arbitrary reference serve adequately for developing the view. We offer the reader a choice as we turn to providing more detail about the arbitrary interpretation of abstraction operators.

6.4 Reading HP the Arbitrary Way

Consider again HP:(HP) $\#F = \#G \leftrightarrow F \approx G$. One (typically unintended) reading of it is as a type-lowering principle: lowering concepts to objects by assigning objects, possibly even *actual* (pre-existing) numbers, to the respective pieces of the cardinality partition on the powerset of the domain.

Viewed this way, HP gets too much out of what is intended to be an implicit definition—there's no *unique* assignment of objects to pieces of the cardinality partition which satisfies HP. How then could HP manage to pick out a particular one? Why not think that when viewed as an implicit definition, HP picks out a *class* of indexings of the cardinality partition (what Hodes [1984] calls a 'numberer'); all the indexings that would vindicate HP

this direction for non-special properties. The generally useful converse holds without restriction.

if they were the determinate meaning of #?

Once we have seen that, *qua* implicit definition, HP really indicates only a class of possible indexings, we open the way to treat terms like '$\#F$' arbitrarily. Since it doesn't matter *which* indexing plays the role of #, we can treat HP as implicitly specifying an arbitrary indexing that will do the job.[28]

Our essential disagreement with neofregeans is about whether HP singles out a unique function # such that expressions like '$\#F$' canonically refer. How could it except by presuming, as part of our background understanding of #, an intended range which excluded Caesars? But such presumptions undermine the status of HP as an implicit definition of *cardinal number*. Really, it's only HP *plus some background presumption* that specify the notion of cardinal number.

If we read HP the arbitrary way though, we can defend the idea that HP has a privileged status akin to an implicit definition. It fully specifies the numberers without background presumptions while also being composed *only* of logical expressions (in a sense to be specified). Moreover, unlike other accounts, the implicitly defined functor # is itself logical (isomorphism invariant) in the relevant sense!

7 Logicality

Arbitrary interpretations of HP look *logical* in a central important sense. Along with many others (Tarski, Sher, McGee, Sagi, Woods, Griffiths and Paseau), we take *invariance under isomorphism* to be a necessary, verging on sufficient, condition of logicality.[29] We'll develop this notion briefly before

[28]There are ways to eliminate the use of HP as an implicit definition by means of ascending to a slightly more powerful logic (such as Woods [2014] suggestion that we add an operation of primitive choice over functions)—but the implicit definition picture of HP is a simple and natural reading. See Hodes [1984] for useful remarks on a similar reading of HP and a quantificational way of proceeding.

[29]We'll treat it as sufficient in the forgoing, taking the usual hedges as read.

turning to its application to our project.

We treat denotations as (the graphs of) members of the type-hierarchy over a domain. Given an isomorphism (really a bijection) i from a domain D, we extend it to a function i^+ on the type-hierarchy over D by "pushing in" the bijection. For example, given a set γ of objects from D, $i^+(\gamma) = \{i(g) : g \in \gamma\}$. Standardly, an expression e is *isomorphism invariant* if, for all domains D and bijections i from D, applying i^+ to the denotation of e on D is identical to the denotation of e on the range of i. Unfortunately, any expression denoting a function from $\wp(D)$ into D, such as $\#$, will not be isomorphism invariant on any domain containing more than one object since we can simply permute its range for a counterexample.

But Woods [2014] argues that there are *two* relevant notions of invariance for explicating logicality. One notion of invariance looks at the particular denotation of an expression. In many cases, however, a domain does not determine a single object as the denotation of an expression, but a range of candidate objects of the particular semantical type. This range exhausts the semantical function of the expression; for all we care, any of the particular candidates could do the job. In such a case, it's reasonable to look at the invariance of the set of candidate objects.

When this set of candidate objects is invariant, then the semantical function of the expression is independent of the underlying nature of the objects in the domain and is thereby *formal*. This formality property is what we wanted the notion of invariance to capture in the first place, so it's reasonable to treat expressions whose set of candidate objects is invariant as thereby logical.[30]

Of course, we need to interpret this property for each of our two interpretations of arbitrariness. On the epistemicist interpretation, arbitrarily

[30] See Woods [2014] for detailed argument. It would be nice for all isomorphism-invariant abstraction operators to contain *all* candidate objects, but this fails. We could plausibly add it in as another condition on logicality without damage to our view. Thanks to Sean Ebels-Duggan for discussion.

referring expressions denote a particular member of the candidate set of objects of the appropriate semantical type. We can define the relevant sense of invariance for this interpretation of invariance—*weak invariance*—as follows:

> An expression ϕ is *weakly invariant* just in case, for all domains D, D' and bijections i from D to D', the set of candidate denotations $\phi* = \{\gamma : \gamma$ is a candidate denotation for ϕ on $D\} = i^{+}(\phi*) = \{\gamma$: γ is a candidate denotation for ϕ on $D'\}$.

Intuitively, this says that an expression is weakly invariant just in case, starting with D candidate denotations, systematically replacing the objects in D with corresponding objects in D' gives you the D' candidate denotations.

On the supervaluational view, we don't assume that ϕ has a particular object as its denotation on a domain; rather it directly denotes a *range* of candidate objects which represents the semantical function of ϕ. On this view:

> An expression ϕ is *weakly invariant* just in case, for all domains D, D' and bijections i from D to D', the denotation of ϕ on D (ϕ^D) is such that $i^{+}(\phi^D) = \phi^{D'}$.[31]

The difference between these two ways of describing weak invariance result from underlying *metasemantical* differences in understanding arbitrary reference. Either way, weak invariance captures how arbitrarily referring expressions can be independent of the underlying natures of objects in the domain. Weak invariance is an extremely plausible and discriminating necessary condition for logicality, allowing abstraction operators like #, interpreted such that no candidate denotation is excluded, a reasonable case for being taken as logical.

[31] See Woods [2014] for more details.

8 Advantages of Arbitrariness

Both approaches to referential arbitrariness solve the Caesar and the Raseac problems, albeit in slightly different ways. On the epistemicist view, '$\#F$ = Caesar' is determinately true or false, yet we do not and cannot know which one it is. Caesar may be the referent of '$\#F$', but by referential arbitrariness, none of Caesar's properties, whether essential or not, plays a part in fixing it as referent of '$\#F$'. There's thus no *point* even in asking whether Caesar *really* is the number 2. What is crucial is that, if Caesar is indeed a member of a domain D over which HP is interpreted, Caesar can be picked as the referent of the number-term '$\#F$', thus just *playing the role* of the number of the Fs. Given that neither Caesar nor any other particular object plays a role in specifying the denotation of '$\#F$', there's no threat to the special epistemic status of HP.

On the supervaluational view, since Caesar is—or would be, were he alive—a possible candidate to play the $\#F$-role, '$\#F$ = Caesar' is not determinately false. Since many other objects could also play that role, it's also not determinately true. Besides retaining a close connection between reference and truth at the expense of deviating from classical logic, the supervaluational view claims mixed identity statements like '$\#F$ = Caesar' resemble linguistic categorical mistakes—taking them to have a determinate answer manifests misunderstanding of the theoretical role of '$\#F$'. Accordingly, it and its cousins are removed from the range of statements which should follow from HP.

The arbitrary interpretation also shows HP (and like principles) to be logical or near enough. Operators like $\#$, taken as operators of arbitrary reference, are weakly invariant. Like $\exists x, y(x \neq y)$, it's not quite a logical truth, since the domain might be finite, but it's a truth *entirely* composed out of logical materials.[32]

[32] Not all abstraction principles will have this status. Many of them involve non-logical materials on their right-hand sides. Such principles typically induce non-logical operators,

If there's a sense in which the existence of infinitely many things is also a logically-cum-epistemically privileged fact, then HP becomes an actual logical truth. But this is unlikely since the potential infinitude of the universe is plausibly a question of metaphysics, not logic. If finite universes are possible, then alternative abstraction principles, like the so-called *Nuisance Principle*,[33] are true in these possibilities instead of HP.[34]

A word of caution. We are not suggesting that since the universe might be finite, HP is a *contingent* truth—this is an open question. Rather, we claim HP, arbitrarily interpreted, is a truth composed only of logical materials *if* the universe is infinite. *If*, contra the facts, the universe is finite, HP would be a falsehood composed only of logical materials, and the Nuisance Principle true instead. Of course, in light of this problem, there's an open and interesting question of what abstraction principles we could use if the universe was finite. But that's a question for another occasion.[35]

Both interpretations of arbitrariness have additional reasonable consequences for the conceptual status of HP:

- On the epistemicist approach, what arithmetical consequences of HP we can know are those common to any choice of referent for #.[36]

- On the supervaluational approach, the arithmetical consequences HP *actually entails* about arithmetic are, again, those common to any choice of referent. A bit of checking will show that on any reasonable

though it's possible to still defend them as implicit definitions. We hope to address this elsewhere.

[33]Wright's Nuisance Principle [1997] is satisfiable only on *finite* domains.

[34]A referee suggests that contingency is problematic for HP's privileged status as an analytic or conceptual truth. But to us that this kind of conceptual truth is *exactly* what implicit definitions that could fail should aspire to.

[35]On the view we defend, these metaphysical questions aren't for abstraction principles to answer. Desiderata (1)-(3) might be rephrased in order to allow different answers to these metaphysical questions, but as we believe in ordinary mathematics, we won't bother.

[36]Of course, by the usual limitative results, knowledge of HP doesn't entail knowledge of all arithmetical facts.

domain, subject to the hedge of fn. 27, these are just the intrinsically arithmetical properties.

In regards to Caesar, those don't want there to be any possibility that Caesar is 2 will prefer the supervaluational treatment; those content with there being no epistemic justification for holding it might rest content with the epistemic interpretation of arbitrary reference. As one might guess, we differ on this point as well as other reasons for favoring our particular interpretation of arbitrary reference. Nevertheless, we think both interpretations generate useful and interesting views which deserve serious consideration.

Both views preserve the key structuralist insight that arithmetical properties hold independently of the nature of the objects playing the appropriate structural role. The only content (or known content) given by $\#F$ is that content determined by HP, so (known) arithmetical facts are typically insensitive to the peculiar natures of the objects in a given domain; what's (knowably) true of the numbers are only those properties common to any objects which *could* serve as indices for the cardinality partition.[37]

Our view avoids the problems for structuralism mooted above. Unlike the *in re* structuralist, we treat the syntactic singular terms of mathematics as semantic singular terms, vindicating the SYNTACTIC PRIORITY thesis.[38] Unlike the *ante rem* structuralist, we treat all syntactically singular terms of mathematics alike. We do not need to single out i for special treatment. And, moreover, we avoid a commitment to substantial ontology, like a rich domain of structures. Our view is ecumenical: it's consistent with there being self-subsistent structures, but also with there being no such things. It's also consistent with independently existing natural numbers which are one among many structures which could play the number role, though again we'd prefer to avoid commitment to such.[39]

[37] Fine [1998] also points out that an indeterminate account of mathematical objects looks to be a promising and plausible way to capture key structuralist insights.

[38] Hodes's [1984] logicism also rejects SYNTACTIC PRIORITY.

[39] Thanks to James Ladyman for discussion.

9 Concluding Remarks

We've articulated our dissatisfaction with both standard structuralist and neofregean positions. Our problems with each arise from their treatment of singular terms like numerals: either they are taken to refer canonically—the *ante rem* structuralist and the neofregean—or they're contextually eliminated (the *in re* structuralist and Hodes). Our goal has been to salvage their inspirations: that principles like HP serve as conceptual analyses of terms like 'the number of' (2); that typically the only (known) properties of numbers are those which can be satisfied by any candidate structure that could serve as the numbers (3); that the syntactically singular terms of mathematics function as semantic singular terms (1). The way to do this, we think, is to drop the idea that expressions like 'the number of' refer canonically while maintaining that '$\#F$' is a genuine, though arbitrary, singular term.[40]

We went on to develop our view, structuralist neologicism, holding that numeral expressions like '$\#F$' refer *genuinely though arbitrarily* to objects which could serve to play the role of numbers. We further argued that this view delivers the key intuitions of neofregeanism:

- HP is a conceptual truth of some kind serving as an implicit definition of cardinal number;
- SYNTACTIC PRIORITY

without abandoning the key structuralist intuition that the only (known) features of the numbers are their structural relations to one another. Given all the advantages of our view and the relatively minimal cost of accepting expressions of arbitrary reference as genuine singular terms—something

[40]We haven't argued explicitly that '$\#F$' is a singular term in the philosophical sense, though it's worth remarking that it passes most of the tests for such and has the same semantic type as a function of type e—once we allow indefinite members of that type in the sense of Woods [2014]. This seems enough to claim singular termhood. We hope to give a detailed argument elsewhere.

which is strongly suggested by independent considerations anyway—there's good reason to take this view seriously.

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