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KUROKAWA-MIZUMOTO CONGRUENCES AND DEGREE-8 L-VALUES

NEIL DUMMIGAN, BERNHARD HEIM, AND ANGELO RENDINA

Abstract. Let f be a Hecke eigenform of weight k, level 1, genus 1. Let $E^k_{2,1}(f)$ be its genus-2 Klingen-Eisenstein series. Let F be a genus-2 cusp form whose Hecke eigenvalues are congruent modulo $\mathfrak{q}$ to those of $E^k_{2,1}(f)$, where $\mathfrak{q}$ is a "large" prime divisor of the algebraic part of the rightmost critical value of the symmetric square L-function of f. We explain how the Bloch-Kato conjecture leads one to believe that $\mathfrak{q}$ should also appear in the denominator of the "algebraic part" of the rightmost critical value of the tensor product L-function $L(s, f \otimes F)$, i.e. in an algebraic ratio obtained from the quotient of this with another critical value. Using pullback of a genus-5 Siegel-Eisenstein series, we prove this, under weak conditions.

1. Introduction

The situation described in the abstract is analogous to the following. The large prime 691, which divides the numerator of $\frac{\zeta(12)}{\pi^{12}}$, is the modulus of Ramanujan's congruence between the Hecke eigenvalues of the weight-12 cusp form Δ and the weight-12 Eisenstein series. But it also occurs in the denominator of the "algebraic part" of the rightmost critical value $L(11, \Delta)$. (A discussion and proof of this, in a slightly more general setting, may be found in [Du2].) In terms of the Bloch-Kato conjecture on special values of motivic L-functions, the 691 in the numerator is the order of an element in a Selmer group, while the 691 in the denominator is the order of an element in a global torsion group. Here we replace $\zeta(12)$ by $L(2k-2, \mathrm{Sym}^2 f)$, 691 by $\mathfrak{q}$, and $L(11, \Delta)$ by $L(2k-3, f \otimes F)$.

After introducing some notation in the remainder of this introduction, and some basic notions about critical values of tensor product L-functions in §2, §3 gives a rough reason to expect the $\mathfrak{q}$ in the denominator, while §4 explains it as a consequence of the Bloch-Kato conjecture. In §5 we apply a pullback formula of the second-named author, to obtain an expression which we can show to have $\mathfrak{q}$ in the denominator, but is also a product of $L(2k-3, f \otimes F)$ and other factors. A holomorphic Siegel-Eisenstein series of genus 5 is restricted to $\mathfrak{H}_2 \times \mathfrak{H}_1 \times \mathfrak{H}_2$, and the formula involves f, F and also a certain Saito-Kurokawa lift of weight k and genus 2. The analysis for this rightmost critical value is complicated by the presence of non-cuspidal terms, though the appearance of the factor $\mathfrak{q}$ in the denominator is a consequence of the presence of the non-cuspidal term $E^k_{2,1}(f)$: Mizumoto's formula for its Fourier coefficients has a factor $L(2k-2, \mathrm{Sym}^2 f)$ in the denominator. In §6 we apply the pullback formula again, obtaining an expression that is integral at $\mathfrak{q}$ and includes $L(2k-7, f \otimes F)$, with other factors. This time the Eisenstein series is

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non-holomorphic. This introduces some technicalities concerning nearly holomorphic modular forms, Shimura-Maass operators and holomorphic projection, which are dealt with in §7. We follow the same steps in outline as Böcherer and Heim [BH2], but here we are concerned with integrality rather than just algebraicity. The main theorem is proved in §6, by dividing one expression by the other to cancel unwanted factors and isolate the ratio $\frac{L(2k-3,f\otimes F)}{\pi^{16}L(2k-7,f\otimes F)}$. It is followed by a numerical example, to demonstrate that the conditions of the theorem are not prohibitively strong, and can be checked in principle.

1.1. **Definitions and notation.** Let $\mathfrak{H}_n$ be the Siegel upper half space of n-by-n complex symmetric matrices with positive-definite imaginary part. Let $\Gamma^n := \mathrm{Sp}(n,\mathbb{Z}) = \mathrm{Sp}_{2n}(\mathbb{Z}) = \{M \in \mathrm{GL}_{2n}(\mathbb{Z}) : {}^tMJM = J\}$, where $J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. For $\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^n$ and $Z \in \mathfrak{H}_n$, let $\gamma(Z) := (AZ+B)(CZ+D)^{-1}$ and $j(\gamma, Z) := \det(CZ+D)$. A holomorphic function $F : \mathfrak{H}_n \to \mathbb{C}$ is said to belong to the space M_n^k of Siegel modular forms of genus n and (scalar) weight k, for Γ^n, if

$$F(\gamma(Z)) = j(\gamma, Z)^k F(Z) \quad \forall \gamma \in \Gamma^n,\ Z \in \mathfrak{H}_n.$$

In other words, $F|\gamma = F$ for all $\gamma \in \Gamma^n$, where $(F|\gamma)(Z) := j(\gamma, Z)^{-k}F(\gamma(Z))$ for $\gamma \in \Gamma^n$. Such an F has a Fourier expansion

$$F(Z) = \sum_{S \geq 0} a(S)\mathbf{e}(\mathrm{Tr}(SZ)) = \sum_{S\geq 0} a(S,F)\mathbf{e}(\mathrm{Tr}(SZ)),$$

where the sum is over all positive semi-definite half-integral matrices, and $\mathbf{e}(z) := e^{2\pi i z}$. We define $\overline{F}(Z) := \sum_{S\geq 0} \overline{a(S)}\mathbf{e}(\mathrm{Tr}(SZ))$.

Denote by S_n^k the subspace of cusp forms, those killed by the Siegel operator $\Phi_n := \Phi_{n,n-1}$ (see below). They are also characterised by the condition that $a(S,F) = 0$ unless S is positive-definite. The Petersson inner product on S_n^k is given by

$$\langle F, G\rangle := \int_{\Gamma^n\backslash\mathfrak{H}_n} F(Z)\overline{G(Z)}(\det(Y))^{k-(n+1)}\, dX\, dY,$$

where $Z = X + iY$, $dX = \prod_{j\leq l} X_{jl}$ and $dY = \prod_{j\leq l} Y_{jl}$.

For $0 \leq r \leq n$, given $Z \in \mathfrak{H}_n$, let $Z_* \in \mathfrak{H}_r$ be its bottom right r-by-r block, and $P_{n,r}$ the parabolic subgroup of Γ^n comprising elements of the form $\begin{pmatrix} * & * \\ 0_{n+r,n-r} & * \end{pmatrix}$. For $F \in S_r^k$, define its Klingen-Eisenstein series

$$E_{n,r}^k(F)(Z) := \sum_{\gamma\in P_{n,r}\backslash\Gamma^n} F((\gamma(Z))_*)j(\gamma, Z)^{-k}.$$

Then for $k > n+r+1$ the series converges absolutely to a holomorphic function $E_{n,r}^k(F) \in M_n^k$, and $\Phi_{n,r}(E_{n,r}^k(F)) = F$, where $\Phi_{n,r}$ is the Siegel operator, given by $\Phi_{n,r}(G)(W) = \lim_{t\to\infty} G(\mathrm{diag}(W, itI_{n-r}))$, with I_m the m-by-m identity matrix.

For $r = n$, $E_{n,n}^k(F) = F$, and for $r = 0, F = 1$ we get the holomorphic Siegel Eisenstein series

$$E_n^k(Z) = \sum_{\gamma\in P_{n,0}\backslash\Gamma^n} j(\gamma, Z)^{-k}.$$

More generally, for $2\Re(s)+k>n+1$ we have

$$E_n^k(Z,s):=\det(Y)^s\sum_{\gamma\in P_{n,0}\backslash\Gamma^n} j(\gamma,Z)^{-k}|j(\gamma,Z)|^{-2s},$$

in general non-holomorphic.

2. $\mathrm{GL}_2\times\mathrm{GSp}_2$ L-functions

Let $f\in S_1^k$ be a normalised cuspidal Hecke eigenform. Then $f(\tau)=\sum_{n=1}^{\infty}a_n(f)q^n$, with $q=e^{2\pi i\tau}$ and $a_1(f)=1$. The Fourier coefficients are also the eigenvalues of Hecke operators. The L-function

$$L(s,f):=\prod_{p\,\mathrm{prime}}(1-a_p(f)p^{-s}+p^{k-1-2s})^{-1}.$$

Let $1-a_p(f)X+p^{k-1}X^2=:(1-\alpha_{p,1}X)(1-\alpha_{p,2}X)$.

Let $F\in S_2^k$ (same weight) be a cuspidal Hecke eigenform. Let the elements $T(p),T(p^2)$ of the genus-2 Hecke algebra be as in [vdG, §16] (with the scaling as following Definition 8). Let $\lambda_F(p),\lambda_F(p^2)$ be the respective eigenvalues for these operators acting on F. The spinor L-function of F is

$$L(s,F,\mathrm{Spin})=\prod_{p\,\mathrm{prime}}L_p(s,F,\mathrm{Spin}),$$

where $L_p(s,F,\mathrm{Spin})^{-1}$

$$=1-\lambda_F(p)p^{-s}+(\lambda_F(p)^2-\lambda_F(p^2)-p^{2k-4})p^{-2s}-\lambda_F(p)p^{2k-3-3s}+p^{4k-6-4s}.$$

Let $L_p(s,F,\mathrm{Spin})=:\prod_{j=1}^4(1-\beta_{p,j}p^{-s})$.

Now we define $L(s,f\otimes F):=\prod_{p\,\mathrm{prime}}L_p(s,f\otimes F)$, where $L_p(s,f\otimes F)^{-1}:=\prod_{i=1}^2\prod_{j=1}^4(1-\alpha_{p,i}\beta_{p,j}p^{-s})$. To understand the conjectured functional equation and critical values for this L-function, it is convenient to introduce the motive M_f attached to f, and the conjectured motive M_F attached to F, of ranks 2 and 4 respectively. The Betti realisations have Hodge decompositions $M_{f,B}\otimes\mathbb{C}\simeq H^{0,k-1}\oplus H^{k-1,0}$ and $M_{F,B}\otimes\mathbb{C}\simeq H^{0,2k-3}\oplus H^{2k-3,0}\oplus H^{k-2,k-1}\oplus H^{k-1,k-2}$, with each $H^{p,q}$ 1-dimensional. The L-functions associated to (q-adic realisations of) M_f and M_F are $L(s,f)$ and $L(s,F,\mathrm{Spin})$ respectively. The L-function $L(s,f\otimes F)$ is associated to the rank-8 motive $M:=M_f\otimes M_F$, which has Hodge decomposition $M_B\otimes\mathbb{C}\simeq\oplus(H^{p,q}\oplus H^{q,p})$, where $p+q=3k-4$ and $p=0,k-2,k-1,k-1=:p_1,p_2,p_3,p_4$. According to [De1, Table 5.3], each (p,q) contributes i^{q-p+1} to the sign in the conjectural functional equation, and one checks easily that the sign should be $+1$. Following the recipe in [Se] (or see again [De1, Table 5.3]), the product of gamma factors is $\gamma(s)=\prod_{i=1}^4\Gamma_{\mathbb{C}}(s-p_i)$, where $\Gamma_{\mathbb{C}}(s):=(2\pi)^{-s}\Gamma(s)$. Note that, following [BH1, Remark 6.2], it makes no difference to replace any p_i by $q_i=3k-4-p_i$. Anyway, the conjectured functional equation is $\Lambda(3k-3-s)=\Lambda(s)$, where $\Lambda(s):=\gamma(s)L(s,f\otimes F)$. The meromorphic continuation and functional equation have been proved by Furusawa [Fu], and extended to the case of unequal weights by Böcherer and Heim [BH1].

The critical values are $L(t,f\otimes F)$ for integers t such that neither $\gamma(s)$ nor $\gamma(3k-3-s)$ has a pole at $s=t$, which is for $k-1<t\leq 2k-3$. Let us suppose for convenience that the coefficient field of M_f and M_F (hence of M) is $\mathbb{Q}$. (Then M_B and M_{dR} are $\mathbb{Q}$-vector spaces.) For each critical t, there is a Deligne period

$c^+(M(t))$ defined as in [De1], up to $\mathbb{Q}^\times$ multiples. (It is the determinant, with respect to bases of 4-dimensional $\mathbb{Q}$-vector spaces $M_B(t)^+$ and $M_{\mathrm{dR}}(t)/\mathrm{Fil}^0$, of an isomorphism between $M_B(t)^+\otimes\mathbb{C}$ and $(M_{\mathrm{dR}}(t)/\mathrm{Fil}^0)\otimes\mathbb{C}$.) Deligne's conjecture (in this instance) is that $L(s,f\otimes F)/c^+(M(t))\in\mathbb{Q}^\times$. Later we shall sometimes make a special choice of $c^+(M(t))$, and define $L_{\mathrm{alg}}(t,f\otimes F)=L(t,f\otimes F)/c^+(M(t))$. If t,t' are critical points with $t\equiv t' \pmod 2$, then $c^+(M(t'))=(2\pi i)^{4(t'-t)}c^+(M(t))$, because $M_B(t')=M_B(t)(2\pi i)^{t'-t}$ while $M_{\mathrm{dR}}(t)/\mathrm{Fil}^0$ does not change for t within the critical range. So the ratio $\frac{L_{\mathrm{alg}}(t',f\otimes F)}{L_{\mathrm{alg}}(t,f\otimes F)}=\frac{L(t',f\otimes F)}{(2\pi i)^{4(t'-t)}L(t,f\otimes F)}$, which should be a rational number, is independent of any choices.

Remark. Yoshida has shown that in fact (up to $\mathbb{Q}^\times$), $c^+(M(t))$ would be $(2\pi i)^{4t+4-3k}\langle f,f\rangle\,\langle F,F\rangle$, independent of the parity of t. See [Y, (4.14)]. Moreover, Böcherer and Heim have proved (assuming non-vanishing of the first Fourier-Jacobi coefficient of F) that $\frac{L(t,f\otimes F)}{(2\pi i)^{4t+4-3k}\langle f,f\rangle\,\langle F,F\rangle}$ is algebraic [BH2, Theorem 5.1].

3. Expected consequences of Kurokawa-Mizumoto type congruences: the rough version

Let $f\in S_1^k$ be as above. Sometimes it is possible to prove a congruence (mod q) of Hecke eigenvalues between the Klingen-Eisenstein series $E^k_{2,1}(f)\in M^k_2$ and some cuspidal Hecke eigenform $F\in S^k_2$. Here $q>2k$ is a prime divisor of the numerator of $L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2 f)$, which we can take to be $L(2k-2,\mathrm{Sym}^2 f)/\pi^{3k-3}\langle f,f\rangle$, where $L(s,\mathrm{Sym}^2 f)=\prod_{p\text{ prime}}((1-\alpha_{1,p}^2p^{-s})(1-\alpha_{1,p}\alpha_{2,p}p^{-s})(1-\alpha_{2,p}^2p^{-s}))^{-1}$. The first examples were proved by Kurokawa and Mizumoto [K1, Mi1], and they can be viewed as instances of Eisenstein congruences for the Klingen parabolic subgroup of GSp_4 [BD, §6].

Note that $E^k_{2,1}(f)$ *is* a Hecke eigenform, and the eigenvalue of $T(p)$ is $a_p(f)(1+p^{k-2})$, in fact its spinor L-function (defined in terms of Hecke eigenvalues just as for the cuspidal case) is $L(s,E^k_{2,1}(f),\mathrm{Spin})=L(s,f)L(s-(k-2),f)$. Then $L(s,f\otimes E^k_{2,1}(f))=L(s,f\otimes f)L(s-(k-2),f\otimes f)$. Since $L(s,f\otimes f)=\zeta(s-(k-1))L(s,\mathrm{Sym}^2 f)$, we find that

$$L(s,f\otimes E^k_{2,1}(f))=\zeta(s-(k-1))\zeta(s-(2k-3))L(s,\mathrm{Sym}^2 f)L(s-(k-2),\mathrm{Sym}^2 f).$$

Because of the factor $\zeta(s-(k-1))$ on the right hand side, $L(s,f\otimes E^k_{2,1}(f))$ has a pole at $s=k$. The mod q congruence of Hecke eigenvalues between $E^k_{2,1}(f)$ and F, hence between coefficients of the Dirichlet series for $L(s,f\otimes E^k_{2,1}(f))$ and $L(s,f\otimes F)$, might lead one roughly to expect that the pole of $L(s,f\otimes E^k_{2,1}(f))$ at the leftmost critical point $s=k$ (for $L(s,f\otimes F)$) should cause a pole *mod q* of $L_{\mathrm{alg}}(k,f\otimes F)$, i.e. a factor of q in its denominator, hence by the functional equation also in the denominator of the rightmost critical value $L_{\mathrm{alg}}(2k-3,f\otimes F)$. As noted already, the exact meaning of algebraic part is ambiguous, but we should detect the factor q in the denominator of $\frac{L(2k-3,f\otimes F)}{\pi^8L(2k-5,f\otimes F)}$.

The observant reader may have noticed that strictly-speaking, the claim in the previous paragraph about $\mathrm{ord}_{s=k}L(s,E^k_{2,1}(f))$ is incorrect. At $s=k$ the factor $L(s-(k-2),\mathrm{Sym}^2 f)$ actually has a "trivial" zero (because 2 is even, in the range $1\leq t\leq k-1$, not a critical point), which cancels the pole of $\zeta(s-(k-1))$ in the product expression for $L(s,E^k_{2,1}(f),\mathrm{Spin})$. But at $s=k+4$, $\zeta(s-(k-1))$ no longer has a pole, while $L(s-(k-2),\mathrm{Sym}^2 f)$ still has a zero to cancel the one at

$s = k$ (when dividing $L(k, f \otimes F)$ by $L(k+4, f \otimes F)$), so we can still argue that maybe we should see a factor q in the denominator of $\frac{L(2k-3, f\otimes F)}{\pi^{16} L(2k-7, f\otimes F)}$. (We consider $L(2k-7, f\otimes F)$ rather than $L(2k-5, f\otimes F)$ for a technical reason that will emerge in §7.)

4. Expected consequences of congruences revisited: the Bloch-Kato conjecture

4.1. **Statement of the conjecture.** Recall the rank-8 motive $M = M_f \otimes M_F$ such that $L(M, s) = L(f \otimes F, s)$. (We shall assume at least the existence of a premotivic structure comprising realisations and comparison isomorphisms, as defined in [DFG, 1.1.1].) For simplicity suppose that the coefficient field is $\mathbb{Q}$. Let $q > 3k-3$ be a prime number. Choose a $\mathbb{Z}_{(q)}$-lattice T_B in the Betti realisation M_B in such a way that $T_q := T_B \otimes \mathbb{Z}_q$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant lattice in the q-adic realisation M_q. Then choose a $\mathbb{Z}_{(q)}$-lattice T_{dR} in the de Rham realisation M_{dR} in such a way that

$$\mathbb{V}(T_{\mathrm{dR}} \otimes \mathbb{Z}_q) = T_q$$

as $\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)$-representations, where $\mathbb{V}$ is the version of the Fontaine-Lafaille functor used in [DFG]. Since $\mathbb{V}$ only applies to filtered ϕ-modules, where ϕ is the crystalline Frobenius, T_{dR} must be ϕ-stable. Anyway, this choice ensures that the q-part of the Tamagawa factor at q is trivial (by [BK, Theorem 4.1(iii)]), thus simplifying the Bloch-Kato conjecture below. The condition $q > 3k-3$ ensures that the condition (*) in [BK, Theorem 4.1(iii)] holds.

Let t be a critical point at which we evaluate the L-function. Let $M(t)$ be the corresponding Tate twist of the motive. Let $\Omega(t)$ be a Deligne period scaled according to the above choice, i.e. the determinant of the isomorphism

$$M(t)_B^+ \otimes \mathbb{C} \simeq (M(t)_{\mathrm{dR}}/\mathrm{Fil}^0) \otimes \mathbb{C},$$

calculated with respect to bases of $(2\pi i)^t T_B^{(-1)^t}$ and $T_{\mathrm{dR}}/\mathrm{Fil}^t$, so well-defined up to $\mathbb{Z}_{(q)}^\times$.

The following formulation of the (q-part of the) Bloch-Kato conjecture, as applied to this situation, is based on [DFG, (59)] (where however there is a non-empty finite set Σ of "bad" primes), using the exact sequence in their Lemma 2.1.

Conjecture 4.1 (Bloch-Kato)**.**

$$\mathrm{ord}_q\left(\frac{L(M,t)}{\Omega(t)}\right)$$

$$= \mathrm{ord}_q\left(\frac{\# H^1_f(\mathbb{Q}, T_q^*(1-t)\otimes(\mathbb{Q}_q/\mathbb{Z}_q))}{\# H^0(\mathbb{Q}, T_q^*(1-t)\otimes(\mathbb{Q}_q/\mathbb{Z}_q)) \# H^0(\mathbb{Q}, T_q(t)\otimes(\mathbb{Q}_q/\mathbb{Z}_q))}\right).$$

Here, $T_q^* = \mathrm{Hom}_{\mathbb{Z}_q}(T_q, \mathbb{Z}_q)$, with the dual action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This is an invariant $\mathbb{Z}_q$-lattice in $M_q^* \simeq M_q(3k-4)$, so $T_q^*(1-t)$ is a lattice in $M_q(3k-3-t)$. On the right hand side, in the numerator, is a Bloch-Kato Selmer group, the subscript "f" denoting conditions on the local restrictions to $H^1(\mathbb{Q}_p, T_q^*(1-t)\otimes(\mathbb{Q}_q/\mathbb{Z}_q))$ (unramified at $p \neq q$, crystalline at $p = q$) for *all* finite primes p (since for us Σ is empty).

4.2. **Global torsion and Kurokawa-Mizumoto type congruences.** We revisit the situation of §3. Recall that $\lambda_F(p)$ denotes the eigenvalue of the genus-2 Hecke operator $T(p)$ acting on the cuspidal eigenform F. The q-adic realisations $M_{f,q}$ and $M_{F,q}$ should be 2-and 4-dimensional $\mathbb{Q}_q$ vector spaces with continuous linear actions ρ_f, ρ_F of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, crystalline at q, unramified at all primes $p \neq q$. For primes $p \neq q$, we should have

$$a_p(f) = \mathrm{Tr}(\rho_f(\mathrm{Frob}_p^{-1})) \quad \text{and} \quad \lambda_F(p) = \mathrm{Tr}(\rho_F(\mathrm{Frob}_p^{-1})).$$

Galois representations with these properties are known to exist, by theorems of Deligne and Weissauer [De2, We]. By Poincaré duality, $M_{f,q}^* \simeq M_{f,q}(k-1)$ and $M_{F,q}^* \simeq M_{F,q}(2k-3)$. Choosing $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant $\mathbb{Z}_q$-lattices in $M_{f,q}$ and $M_{F,q}$, then reducing mod q, we obtain residual representations $\overline{\rho}_f$ and $\overline{\rho}_F$. We suppose that (as in Example 1) $\overline{\rho}_f$ is irreducible, in which case it is independent of the choice of lattice in $M_{f,q}$. The congruence

$$\lambda_F(p) \equiv a_p(f)(1 + p^{k-2}) \pmod{q},$$

interpreted as a congruence of traces of Frobenius, implies that the composition factors of $\overline{\rho}_F$ are $\overline{\rho}_f$ and $\overline{\rho}_f(2-k)$. Which is a submodule and which is a quotient will depend on the choice of lattice in $M_{F,q}$.

Looking at the denominator of the Bloch-Kato formula, with T_q the tensor product of the $\mathbb{Z}_q$-lattices referred to above, on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by $\rho_f \otimes \rho_F \simeq \rho_f^*(1-k) \otimes \rho_F$, the q-torsion in $H^0(\mathbb{Q}, T_q(t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$ is $(\overline{\rho}_f^* \otimes \overline{\rho}_F(t+1-k))^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$, which is $\mathrm{Hom}_{\mathbb{F}_q[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]}(\overline{\rho}_f, \overline{\rho}_F(t+1-k))$. This is the same as $\mathrm{Hom}_{\mathbb{F}_q[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]}(\overline{\rho}_f(2-k), \overline{\rho}_F(t+3-2k))$. This can be non-trivial only for $t \equiv k-1 \pmod{q-1}$ (if ρ_f is a submodule of ρ_F) or for $t \equiv 2k-3 \pmod{q-1}$ (if $\rho_f(2-k)$ is a submodule of ρ_F). The only such t in the critical range $k \leq t \leq 2k-3$ (using $q > 2k$) is $t = 2k-3$. So, with a suitable choice of lattice, and $t = 2k-3$, we can have a factor of q in the denominator of the conjectural formula for $\frac{L(M,t)}{\Omega(t)}$, which leads us to expect a q in the denominator of $\frac{L(2k-3, f\otimes F)}{\pi^{16} L(2k-7, f\otimes F)}$.

This expectation is based upon the supposition that there is "no particular reason" for $H^1_f(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$ to be non-trivial when $t = 2k-3$. One might ask, what if we had chosen an *unsuitable* lattice, so that ρ_f (rather than $\rho_f(2-k)$) is a submodule of ρ_F? To account for the q in the denominator of $\frac{L(2k-3, f\otimes F)}{\pi^{16} L(2k-7, f\otimes F)}$, we had better have some particular reason for $H^1_f(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$ to be non-trivial when $t = 2k-7$. Indeed there is one. Since $\overline{\rho}_f \otimes \overline{\rho}_F(k-1) \simeq \mathrm{Hom}_{\mathbb{F}_q}(\overline{\rho}_f, \overline{\rho}_F)$ has a trivial submodule $\mathbb{F}_q$, $\overline{\rho}_f \otimes \overline{\rho}_F(k+4)$ has a submodule isomorphic to $\mathbb{F}_q(5)$. Now $H^1_f(\mathbb{Q}, \mathbb{Z}_q(5))$ is non-trivial, more-or-less by a construction of Soulé [So]. The same would be true with q replaced by any other prime, and 5 by any odd integer greater than 1. By reduction mod q we get an element of $H^1(\mathbb{Q}, \mathbb{F}_q(5))$, which maps to an element of $H^1(\mathbb{Q}, \overline{\rho}_f \otimes \overline{\rho}_F(k+4))$. But $\overline{\rho}_f \otimes \overline{\rho}_F(k+4)$ is the q-torsion in $T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q)$ for $t = 2k-7$ (since $(k+4) + (2k-7) = 3k-3$), and one ought thereby to get a non-zero element of $H^1_f(\mathbb{Q}, T_q^*(1-t) \otimes (\mathbb{Q}_q/\mathbb{Z}_q))$.

5. Pullback of a genus-5 Siegel Eisenstein series

Let $d_n^k := \dim(S_n^k)$ and let $\{h_1, \dots, h_{d_1^k}\}$, $\{H_1, \dots, H_{d_2^k}\}$ be orthogonal bases of Hecke eigenforms for S_1^k and S_2^k respectively, the h_j normalised. For basics on the Saito-Kurokawa lift, see [vdG, §21].

Lemma 5.1. *Let $G \in S_2^k$ be a Saito-Kurokawa lift of $g \in S_1^{2k-2}$ (with k even). For $(\tau, \tau') \in \mathfrak{H}_1 \times \mathfrak{H}_1$, let $G_\tau(\tau') := G(\mathrm{diag}(\tau, \tau'))$, viewed as an element of S_1^k depending on the parameter τ. Then for $Z' \in \mathfrak{H}_2$,*

$$E_{2,1}^k(G_\tau)(Z') = \sum_{j=1}^{d_1^k} \chi_j \, h_j(\tau) E_{2,1}^k(h_j)(Z'),$$

where the χ_j are defined by

$$G(\mathrm{diag}(\tau, \tau')) = \sum_{j=1}^{d_1^k} \chi_j \, h_j(\tau) h_j(\tau').$$

Proof. That $G \mid_{\mathfrak{H}_1 \times \mathfrak{H}_1}$ has to be of the form $\sum \chi_j \, h_j \otimes h_j$, with no $h_i \otimes h_j$ terms for $i \neq j$, follows from [He2, Theorem 1.3]. Now one simply applies the Klingen-Eisenstein lifting to both sides of $G_\tau(\tau') = \sum_{j=1}^{d_1^k} \chi_j \, h_j(\tau) h_j(\tau')$ as functions of τ'. □

Remark. A theorem of Ichino [I, Theorem 2.1] expresses the central value of the L-function $L(\mathrm{Sym}^2(h_j) \otimes g, s)$ in terms of χ_j, as predicted by Gross and Prasad.

Lemma 5.2. *Let G and $E_{2,1}^k(G_\tau)$ be as in Lemma 5.1. Then for $(\tau, Z') \in \mathfrak{H}_1 \times \mathfrak{H}_2$, $E_{3,2}^k(G)(\mathrm{diag}(\tau, Z'))$*

$$= E_1^k(\tau) G(Z') + \sum_{j=1}^{d_1^k} \chi_j \, h_j(\tau) E_{2,1}^k(h_j)(Z') + \sum_{j=1}^{d_1^k} \sum_{i=1}^{d_2^k} \mu_{ij} h_j(\tau) H_i(Z'),$$

for certain coefficients μ_{ij}.

Proof. From [He1, Theorem 2.3] we get

$$E_{3,2}^k(G)(\mathrm{diag}(\tau, Z')) = E_1^k(\tau) G(Z') + E_{2,1}^k(G_\tau)(Z') + \sum_{j=1}^{d_1^k} \sum_{i=1}^{d_2^k} \mu_{ij} h_j(\tau) H_i(Z').$$

Now substitute for $E_{2,1}^k(G_\tau)(Z')$, using Lemma 5.1. □

Lemma 5.3. *For $Z \in \mathfrak{H}_2$, $W \in \mathfrak{H}_3$, $E_5^k(\mathrm{diag}(Z, W)) = E_2^k(Z) E_3^k(W)$*

$$+ C_{k,1} \sum_{j=1}^{d_1^k} \Lambda(h_j) E_{2,1}^k(h_j)(Z) E_{3,1}(h_j)(W) + C_{k,2} \sum_{i=1}^{d_2^k} \Lambda(H_i) H_i(Z) E_{3,2}^k(\overline{H_i})(W),$$

where $\Lambda(h_j) = \frac{L(2k-2, \mathrm{Sym}^2 h_j)}{\zeta(k)\zeta(2k-2)\langle h_j, h_j \rangle}$, $\Lambda(H_i) = \frac{L(k-2, H_i, \mathrm{st})}{\zeta(k)\zeta(2k-2)\zeta(2k-4)\langle H_i, H_i \rangle}$, $C_{k,1} = \frac{2^{3-k} i^k \pi}{(k-1)}$, $C_{k,2} = \frac{2^{8-2k}\pi^3}{(k-1)}$, and $\overline{H_i}$ is obtained from H_i by conjugating the Fourier coefficients.

This is due to Garrett and Böcherer. We use [G, §5 Theorem], getting the coefficients (constants and "standard" L-values) from [BSY, Proposition 4.4] by setting $l = 0$.

Lemma 5.4. *Let g, G be as in Lemma 5.1.*

$$\langle\langle\langle E_5^k(\mathrm{diag}(Z,\tau,Z')), G(Z)\rangle h_j(\tau)\rangle H_i(Z')\rangle$$
$$= \frac{2^{14-6k}\pi^{6-2k}((k-3)!)^2}{(k-1)(2k-3)}\langle \Phi_1^G, \Phi_1^{H_i}\rangle_J \frac{L(2k-4,g)L(2k-3,h_j\otimes H_i)}{\zeta(k)\zeta(2k-2)\zeta(2k-4)},$$

where Φ_1^G and $\Phi_1^{H_i}$ are first Fourier-Jacobi coefficients, and $\langle \Phi_1^G, \Phi_1^{H_i}\rangle_J$ is their inner product as Jacobi forms.

This is [He1, Theorem 5.1], with $s = 0$. Note that the condition $k > 6$ is necessarily satisfied.

Lemma 5.5. *In Lemma 5.2,*

$$\mu_{ij} = \frac{2^{6-4k}\pi^{3-2k}((k-3)!)^2\langle \Phi_1^G, \Phi_1^{H_i}\rangle_J\, L(2k-3,h_j\otimes H_i)}{(2k-3)\zeta(k-2)L(2k-3,g)\langle h_j,h_j\rangle\,\langle H_i,H_i\rangle}.$$

Proof. Using Lemma 5.3,

$$\langle E_5^k(\mathrm{diag}(Z,W)), G(Z)\rangle = C_{k,2}\Lambda(G)\langle G,G\rangle E_{3,2}^k(G)(W)$$

(using $\overline{G} = G$, which follows from the fact that the Fourier coefficients of g are real). Using $L(s,G,\mathrm{st}) = \zeta(s)L(s+(k-1),g)L(s+(k-2),g)$, this becomes

$$\langle E_5^k(\mathrm{diag}(Z,W)), G(Z)\rangle = C_{k,2}\frac{\zeta(k-2)L(2k-3,g)L(2k-4,g)}{\zeta(k)\zeta(2k-2)\zeta(2k-4)}E_{3,2}^k(G)(W).$$

Now restricting W to $(\tau,Z') \in \mathfrak{H}_1\times\mathfrak{H}_2$, using Lemma 5.2 to substitute for $E_{3,2}^k(G)(\mathrm{diag}(\tau,Z'))$, and plugging it all into Lemma 5.4, we get

$$C_{k,2}\frac{\zeta(k-2)L(2k-3,g)L(2k-4,g)}{\zeta(k)\zeta(2k-2)\zeta(2k-4)}\,\mu_{ij}\langle h_j,h_j\rangle\,\langle H_i,H_i\rangle$$
$$= \frac{2^{14-6k}\pi^{6-2k}((k-3)!)^2}{(k-1)(2k-3)}\langle \Phi_1^G, \Phi_1^{H_i}\rangle_J \frac{L(2k-4,g)L(2k-3,h_j\otimes H_i)}{\zeta(k)\zeta(2k-2)\zeta(2k-4)}.$$

Recalling that $C_{k,2} = \frac{2^{8-2k}\pi^3}{(k-1)}$ gives the desired result. □

Let $\mathbb{T}$ be the algebra generated over $\mathbb{Z}$ by all the operators $T(p)$ and $T(p^2)$ on M_2^k. Given $T \in \mathbb{T}$ and a Hecke eigenform $F \in M_2^k$, let $\lambda_F(T)$ be the eigenvalue of T acting on F.

Lemma 5.6. *Suppose that $q > 2k$ is a prime number, $\mathfrak{q}$ a divisor of q in a sufficiently large coefficient field. Suppose that*

(1) $q \nmid B_k B_{2k-2} B_{2k-4}$;
(2) *there is no Hecke eigenform $H_i \neq G$ in S_2^k such that $\lambda_{H_i}(T) \equiv \lambda_G(T) \pmod{\mathfrak{q}}$ for all $T \in \mathbb{T}$;*
(3) *there exist Hecke eigenforms $h \in S_1^k, H \in S_2^k$ such that $\lambda_{E_{2,1}^k(h)}(T) \equiv \lambda_H(T) \pmod{\mathfrak{q}}$ for all $T \in \mathbb{T}$;*
(4) *there is no Hecke eigenform $h_j \neq h$ in S_1^k such that $a_p(h_j) \equiv a_p(h) \pmod{\mathfrak{q}}$ for all primes p;*
(5) *there is no Hecke eigenform $H_i \neq H$ in S_2^k such that $\lambda_{H_i}(T) \equiv \lambda_H(T) \pmod{\mathfrak{q}}$ for all $T \in \mathbb{T}$.*

Then the Fourier coefficients of $\Lambda(G)G(Z)(\chi\, h(\tau)E_{2,1}(h)(Z') + \mu h(\tau)H(Z'))$ are (up to a power of π) integral at $\mathfrak{q}$, where if $h = h_{j_0}$ and $H = H_{i_0}$ then $\chi := \chi_{j_0}$ and $\mu := \mu_{i_0 j_0}$.

Proof. Start with $E_5^k(\mathrm{diag}(Z,W))$

$$= C_{k,1}\sum_{j=1}^{d_1^k}\Lambda(h_j)E_{2,1}^k(h_j)(Z)E_{3,1}(h_j)(W) + C_{k,2}\sum_{i=1}^{d_2^k}\Lambda(H_i)H_i(Z)E_{3,2}^k(\overline{H_i})(W),$$

from Lemma 5.3. It follows from [Ha, Theorem 4.14] and from [B, (5.3)–(5.5), Proposition 3.4], given $q>2k$ and $q\nmid B_kB_{2k-2}B_{2k-4}$, that the Fourier coefficients of E_5^k (which Siegel proved are rational) are integral at q. This remains true after restriction to the diagonal.

Now we apply elements of Hecke algebras to kill unwanted terms on the right while preserving integrality at q on the left. For any prime p, $\lambda_{E_{2,1}^k(h_j)}(p) = a_p(h_j)(1+p^{k-2})$. Reading this (mod $\mathfrak{q}$), it is the trace of a Frobenius element Frob_p^{-1} on a Galois representation with irreducible composition factors $\overline{\rho}_{h_j}$ and $\overline{\rho}_{h_j}(2-k)$ (both 2-dimensional), where $\overline{\rho}_{h_j}$ is the representation in characteristic q attached to h_j. This $\overline{\rho}_{h_j}$ is irreducible since $q>k$ and $q\nmid B_k$. On the other hand, $\lambda_G(p)=a_p(g)+p^{k-1}+p^{k-2}$ is a trace of Frobenius on a representation with composition factors of dimensions $2,1,1$. So we cannot have a (mod $\mathfrak{q}$) congruence of Hecke eigenvalues between G and any $E_{2,1}(h_j)$. For each j choose $T_j\in\mathbb{T}$ such that $c_j:=\lambda_G(T_j)-\lambda_{E_{2,1}^k(h_j)}(T_j)$ is not divisible by $\mathfrak{q}$. Now applying $\prod_{j=1}^{d_1^k}(T_j-\lambda_{E_{2,1}^k(h_j)}(T_j))$ to both sides (in the variable Z) maintains $\mathfrak{q}$-integrality on the left (by [An, (2.1.11)]), and kills all the terms in the first sum on the right, while multiplying the term $\Lambda(G)G(Z)E_{3,2}^k(G)$ by $\prod_{j=1}^{d_1^k}c_j$, which is not divisible by $\mathfrak{q}$. Similarly, by (2) we may choose $T_i'\in\mathbb{T}$ for each $i\neq i_0$, such that $e_i:=\lambda_G(T_i')-\lambda_{H_i}(T_i')$ is not divisible by $\mathfrak{q}$, then apply $\prod_{i=1,i\neq i_0}^{d_2^k}(T_i'-\lambda_{H_i}(T_i'))$ (in the variable Z) to kill all the $H_i(Z)$ terms for $H_i\neq G$.

What remains on the right hand side is $C_{k,2}\prod c_j\prod e_i\ \Lambda(G)G(Z)E_{3,2}^k(G)$. We may divide out the factor $C_{k,2}\prod c_j\prod e_i$ without disturbing integrality at $\mathfrak{q}$. Thus $\Lambda(G)G(Z)E_{3,2}^k(G)$ has Fourier coefficients which are (up to a power of π) integral at $\mathfrak{q}$. Lemma 5.2 says that $E_{3,2}^k(G)(\mathrm{diag}(\tau,Z'))$

$$= E_1^k(\tau)G(Z') + \sum_{j=1}^{d_1^k}\chi_j\, h_j(\tau)E_{2,1}^k(h_j)(Z') + \sum_{j=1}^{d_1^k}\sum_{i=1}^{d_2^k}\mu_{ij}h_j(\tau)H_i(Z').$$

Now similarly using (4) and (5) we may apply elements of Hecke algebras to functions of τ and functions of Z' to kill all the $h_j\neq h$ and $H_i\neq H$ terms, without introducing any factor divisible by $\mathfrak{q}$, and using $q\nmid B_k$ we know that $h\not\equiv E_k$ (mod $\mathfrak{q}$), so may likewise kill the E_1^k term. The lemma follows. □

As in [DIK], we define, for each $1\leq t\leq 2k-3$, $L_{\mathrm{alg}}(t,g)=\frac{L(t,g)}{(2\pi i)^t\Omega^{(-1)^t}}$, where $\Omega^{\pm}$ are certain carefully scaled Deligne periods. We also define $L_{\mathrm{alg}}(k-1,g,\chi_D)=\frac{L(k-1,g,\chi_D)}{\tau(\chi_D)(2\pi i)^{k-1}\Omega^-}$, where D is any negative fundamental discriminant, $\chi_D=\left(\frac{D}{\cdot}\right)$ the associated quadratic Dirichlet character, $L(s,g,\chi_D)$ the twisted L-function, and $\tau(\chi_D)$ a Gauss sum.

Proposition 5.7. *Suppose that the prime $\mathfrak{q}$ satisfies all the conditions of Lemma 5.6, and that G and H as above are scaled to have all their Fourier coefficients* *integral at $\mathfrak{q}$, but not all divisible by $\mathfrak{q}$. (These scalings determine those of χ and μ.) Suppose that then (neglecting powers of π)*

(1) $\mathrm{ord}_{\mathfrak{q}}(\chi) = 0$*;*
(2) $\mathrm{ord}_{\mathfrak{q}}(B_{k-2}) = 0$*;*
(3) $\mathrm{ord}_{\mathfrak{q}} L_{\mathrm{alg}}(t, g) \geq 0$ *for all* $1 \leq t \leq 2k-3$*;*
(4) *there exists a fundamental discriminant $D < 0$ with*

$$\mathrm{ord}_{\mathfrak{q}}(|D|^{k-1} L_{\mathrm{alg}}(k-1, g, \chi_D)) = 0;$$

(5) *there exists a fundamental discriminant $D' < 0$ with $q \nmid D'$ and*

$$\mathrm{ord}_{\mathfrak{q}}(L(2-k, \chi_{D'}) L(k-1, h) L(k-1, h, \chi_{D'})/\langle h, h\rangle) = 0;$$

(6) $\mathrm{ord}_{\mathfrak{q}}\left(\frac{L_{\mathrm{alg}}(2k-2, \mathrm{Sym}^2 h)}{\langle h,h\rangle}\right) > 0$.

Then $\mathrm{ord}_{\mathfrak{q}}(\mu) < 0$*, where*

$$\mu = \frac{2^{6-4k}\pi^{3-2k}((k-3)!)^2 \langle \Phi_1^G, \Phi_{1'}^H\rangle_J \, L(2k-3, h\otimes H)}{(2k-3)\zeta(k-2)L(2k-3, g)\langle h, h\rangle \, \langle H, H\rangle}.$$

Note in particular that this implies that $\langle \Phi_1^G, \Phi_{1'}^H\rangle_J \neq 0$, an instance of something in general not known to be true.

Proof. Using results of Kohnen, Skoruppa and Zagier [KS, KZ], as in [DIK, (5)],

$$\Lambda(G) = \frac{\pi^{4k-7}\zeta(k-2)L_{\mathrm{alg}}(2k-3, g)L_{\mathrm{alg}}(2k-4, g)\,|D|^{k-1}\, L_{\mathrm{alg}}(k-1, g, \chi_D)}{\zeta(k)\zeta(2k-2)\zeta(2k-4)c(|D|)^2\, L_{\mathrm{alg}}(k, g)\mathfrak{c}(g)}.$$

Here $\mathfrak{c}(g)$ is a certain integral "congruence ideal" which, thanks to condition (4) in Lemma 5.6, is not divisible by $\mathfrak{q}$. Though $D < 0$ could be any fundamental discriminant, we choose it as in condition (4). The coefficient $c(|D|)$ comes from $\tilde{g} = \sum c(n)q^n \in S_{k-1/2}(\Gamma_0(4))^+$, a Hecke eigenform in the Kohnen plus space, corresponding to g under the Kohnen-Shimura correspondence. This $\tilde{g}$ is only defined up to scalar multiples, and its scaling determines that of the Saito-Kurokawa lift G. As in [DIK, Lemma 6.2] we may scale it so that all the Fourier coefficients lie in the number field generated by those of g, and they are integral at $\mathfrak{q}$, with $\mathrm{ord}_{\mathfrak{q}}(c(|D|)) = 0$. Moreover then all the Fourier coefficients of G are integral at $\mathfrak{q}$, and if we choose $S = S_D$ so that $-4\det(S_D) = D$ then $\mathrm{ord}_{\mathfrak{q}}(a(S_D, G)) = 0$, so G is now scaled as in the statement of the proposition. We have used conditions (3) and (4). Using also condition (2), and condition (1) from Lemma 5.6, we have

$$\mathrm{ord}_{\mathfrak{q}}(\pi^3\Lambda(G)\, a(S_D, G)) = 0.$$

Now [Mi2, Theorem 1] tells us that $a(S_{D'}, E_{2,1}(h)) =$

$$(-1)^{k/2}\frac{(k-1)!}{(2k-2)!}(2\pi)^{k-1}\,|D'|^{k-3/2}\frac{L(k-1, \chi_{D'})}{L(2k-2, \mathrm{Sym}^2 h)}L(k-1, h)L(k-1, h, \chi_{D'}).$$

Using conditions (5) and (6), it follows that

$$\mathrm{ord}_{\mathfrak{q}}(a(S_{D'}, E_{2,1}(h))) < 0.$$

Combining this with the result of the previous paragraph, and (1), we see that the coefficient of $\mathbf{e}(\mathrm{Tr}(S_D Z) + \tau + \mathrm{Tr}(S_{D'}Z'))$ in $\Lambda(G)G(Z)\chi\, h(\tau)E_{2,1}(h)(Z')$ is not integral at $\mathfrak{q}$. It follows from Lemma 5.6 that the coefficient of $\mathbf{e}(\mathrm{Tr}(S_D Z) + \tau + \mathrm{Tr}(S_{D'}Z'))$ in $\Lambda(G)G(Z)\mu h(\tau)H(Z'))$ is not integral at $\mathfrak{q}$ either, hence that $\mathrm{ord}_{\mathfrak{q}}(\mu) < 0$, as required. ∎

We found above that $\mathrm{ord}_{\mathfrak{q}}(\pi^3\Lambda(G)) = 0$. Putting $\tilde{\mu} := \pi^3\Lambda(G)\mu$ and substituting $\Lambda(G) = \frac{\zeta(k-2)L(2k-3,g)L(2k-4,g)}{\zeta(k)\zeta(2k-2)\zeta(2k-4)\langle G,G\rangle}$, we obtain

Corollary 5.8. $\mathrm{ord}_{\mathfrak{q}}(\tilde{\mu}) < 0$, *where*

$$\tilde{\mu} = \frac{2^{6-4k}\pi^{6-2k}((k-3)!)^2\langle \Phi_1^G, \Phi_1^H\rangle_J\, L(2k-4,g)\, L(2k-3, h\otimes H)}{(2k-3)\zeta(k)\zeta(2k-2)\zeta(2k-4)\langle G,G\rangle\,\langle h,h\rangle\,\langle H,H\rangle}.$$

6. A non-rightmost critical value

Lemma 6.1. *Let g, G be as in Lemma 5.1, and $k > 14$. Then*

$$\langle\langle\langle E_5^k(\mathrm{diag}(Z,\tau,Z'),-4), G(Z)\rangle h(\tau)\rangle H(Z')\rangle$$
$$= \frac{2^{39-6k}\pi^{10-2k}((k-7)!)^2(2k-9)(2k-8)\langle \Phi_1^G, \Phi_1^H\rangle_J\, L(2k-12,g)\, L(2k-7, h\otimes H)}{\zeta(k-8)\zeta(2k-18)\zeta(2k-20)}$$

This is [He1, Theorem 5.1], with $s=-4$, and is the essential ingredient in the proof of the following proposition.

Proposition 6.2. *With notation and assumptions as above, suppose also that $q \nmid B_{k-8}B_{2k-18}B_{2k-20}$. Then $\mathrm{ord}_{\mathfrak{q}}(\kappa) \geq 0$, where κ*

$$= \frac{2^{39-6k}\pi^{10-2k}((k-7)!)^2(2k-9)(2k-8)\langle \Phi_1^G, \Phi_1^H\rangle_J\, L(2k-12,g)\, L(2k-7, h\otimes H)}{\zeta(k-8)\zeta(2k-18)\zeta(2k-20)\langle G,G\rangle\,\langle h,h\rangle\,\langle H,H\rangle}.$$

To actually prove the proposition is somewhat technical, since $E_5^k(Z,-4)$ is only a nearly holomorphic modular form. We will deal with this in the last section, below. Here, assuming the truth of the proposition, we deduce the main theorem of the paper.

Theorem 6.3. *Suppose that $q > 2k$ is a prime number, $k > 14$, $\mathfrak{q}$ a divisor of q in a sufficiently large coefficient field. Suppose that*

1. *there exist Hecke eigenforms $h \in S_1^k, H \in S_2^k$ such that*
$$\mathrm{ord}_{\mathfrak{q}}\left(\frac{L(2k-2, \mathrm{Sym}^2 h)}{\langle h,h\rangle}\right) > 0$$
and $\lambda_{E_{2,1}^k(h)}(T) \equiv \lambda_H(T) \pmod{\mathfrak{q}}$ for all $T \in \mathbb{T}$;
2. *there is no Hecke eigenform $h_j \neq h$ in S_1^k such that $a_p(h_j) \equiv a_p(h) \pmod{\mathfrak{q}}$ for all primes p;*
3. *there is no Hecke eigenform $H_i \neq H$ in S_2^k such that $\lambda_{H_i}(T) \equiv \lambda_H(T) \pmod{\mathfrak{q}}$ for all $T \in \mathbb{T}$;*
4. *$q \nmid B_{k-8}B_{k-2}B_kB_{2k-20}B_{2k-18}B_{2k-4}B_{2k-2}$;*
5. *there exists a fundamental discriminant $D' < 0$ with $q \nmid D'$ and*
$$\mathrm{ord}_{\mathfrak{q}}(L(2-k,\chi_{D'})L(k-1,h)L(k-1,h,\chi_{D'})/\langle h,h\rangle) = 0;$$
Suppose also that there exist a Hecke eigenform $g \in S_1^{2k-2}$ and its Saito-Kurokawa lift $G \in S_2^k$, such that
6. *there is no Hecke eigenform $H_i \neq G$ in S_2^k such that $\lambda_{H_i}(T) \equiv \lambda_G(T) \pmod{\mathfrak{q}}$ for all $T \in \mathbb{T}$;*
7. $\mathrm{ord}_{\mathfrak{q}} L_{\mathrm{alg}}(t,g) \geq 0$ *for all $1 \leq t \leq 2k-3$, with* $\mathrm{ord}_{\mathfrak{q}} L_{\mathrm{alg}}(2k-4,g) = 0$*;*
8. *there exists a fundamental discriminant $D < 0$ with*
$$\mathrm{ord}_{\mathfrak{q}}(|D|^{k-1}L_{\mathrm{alg}}(k-1,g,\chi_D)) = 0;$$

(9) $\mathrm{ord}_{\mathfrak{q}}(\chi) = 0$, *with* χ *as in Lemma 5.6 and Proposition 5.7 (ultimately from Lemma 5.1).*

Then $\mathrm{ord}_{\mathfrak{q}}\left(\frac{L(2k-3,h\otimes H)}{\pi^{16}L(2k-7,h\otimes H)}\right) < 0$.

Proof. Dividing Corollary 5.8 by Proposition 6.2 makes several unwanted factors cancel, and using $q > 2k$ to eliminate the linear factors in k leaves us with

$$\mathrm{ord}_{\mathfrak{q}}\left(\frac{2^{2k-33}\pi^{-4}\zeta(k-8)\zeta(2k-18)\zeta(2k-20)L(2k-4,g)\,L(2k-3,h\otimes H)}{\zeta(k)\zeta(2k-2)\zeta(2k-4)L(2k-12,g)\,L(2k-7,h\otimes H)}\right) < 0.$$

Now using the conditions (4) and (7), we may ignore most of the other factors too, yielding the theorem. $\square$

6.1. **A numerical example.** Let $k = 20$, so $2k-2 = 38$, and let $q = 71$. We check the conditions of Theorem 6.3. Certainly $q > 2k$ and $k > 14$.

We take $h = q + 456q^2 + 50652q^3 + \dots$, the normalised generator of the 1-dimensional space S_1^{20}. The value $\frac{L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2 h)}{\langle h,h\rangle}$ may be computed by a method of Zagier [Z], and is $\frac{2^{32}.\mathbf{71^2}}{3^{18}.5^9.7^3.11^2.13^2.17^2.19.23.29.31.37.283.617}$, as in [Du1]. Kurokawa [K2] computed a basis of Hecke eigenforms $\{\chi_{20}^{(1)},\chi_{20}^{(2)},\chi_{20}^{(3)}\}$ for S_2^{20}, where $\chi_{20}^{(1)},\chi_{20}^{(2)}$ are Saito-Kurokawa lifts. Letting $H = \chi_{20}^{(3)}$, he proved a congruence $\lambda_{E_{2,1}^{20}(h)}(T) \equiv \lambda_H(T) \pmod{71^2}$ for all $T \in \mathbb{T}$ [K1]. Condition (2) is automatically satisfied, since $\dim S_1^{20} = 1$, and (3) is easily checked using Kurokawa's computations of Hecke eigenvalues. For (4), one may check, using a computer package such as Maple, that none of $B_{12}, B_{18}, B_{20}, B_{22}, B_{36}$ or B_{38} is divisible by 71.

Using the command LRatio in the computer package Magma, $L_{\mathrm{alg}}(19,h) = \frac{2^2\cdot3^5\cdot5\cdot7^2\cdot11\cdot13\cdot17}{283\cdot617}$ and $L_{\mathrm{alg}}(19,h,\chi_{-3}) = \frac{2^6\cdot5\cdot7^3\cdot11\cdot13\cdot17\cdot19}{3}$. Computing in Maple, $L(-18,\chi_{-3}) = -B_{19,\chi_{-3}}/19 = \frac{-3^{18}}{19}\left(B_{19}(1/3) - B_{19}(2/3)\right) = \frac{-2\cdot7\cdot19\cdot7691\cdot8609}{3}$. Condition (5) follows.

The space S_1^{38} is spanned by Hecke eigenforms

$$g_1 = q + (-97200 + 48\sqrt{63737521})q^2 + \dots,$$

$$g_2 = q + (-97200 - 48\sqrt{63737521})q^2 + \dots,$$

let's say $g = g_1$. From computations in [K2], $G := \chi_{20}^{(1)}$ and $\chi_{20}^{(2)}$ are Saito-Kurokawa lifts of g_1, g_2 respectively. Condition (6) may be checked using Kurokawa's results. The odd part of the norm of $L_{\mathrm{alg}}(t,g)$, for $1 \le t \le 37$ may be computed in Magma using LRatio, hence (7) may be verified. Similarly, the odd part of the norm of $L_{\mathrm{alg}}(19,g,\chi_{-3})$ is $3^{37}\cdot5^2\cdot7\cdot11\cdot13^2\cdot17^2\cdot19^2$, hence (8). Finally $G = \chi_{20}^{(1)}$ is scaled the right way (Fourier coefficients integral and not all divisible by 71). Recalling the formula

$$G(\mathrm{diag}(\tau,\tau')) = \sum_{j=1}^{d_1^k} \chi_j\, h_j(\tau)h_j(\tau'),$$

where here $d_1^{20} = 1$, and noting that χ is then the coefficient of $e^{2\pi i\tau}e^{2\pi i\tau'}$ in $G(\mathrm{diag}(\tau,\tau'))$, we find that

$$\chi = a(A_0,G) + a(A_1,G),$$

where $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. By calculations of Katsurada [Ka, §4], if $\theta = (-2025 + \sqrt{63737521})/2$ then $a(A_0, G) = -5092 - \theta = -8159 - \sqrt{63737521}$ and $a(A_1, G) = -20(4816 + \theta) = -76070 - 10\sqrt{63737521}$. Hence $\chi = -84589 - 11\sqrt{63737521}$, an algebraic integer with norm $-2^6 \cdot 3 \cdot 5 \cdot 59 \cdot 9833$, which is not divisible by 71. Hence (9) is satisfied. We have confirmed that Theorem 6.3 is applicable to this example.

7. Proof of Proposition 6.2

7.1. Nearly holomorphic modular forms.

Definition 7.1. *A C^∞ function $f : \mathfrak{H}_n \to \mathbb{C}$ is said to be a nearly holomorphic modular form of weight k and degree d (for Γ^n) if*

(1) *f is a polynomial of degree d in the entries of Y^{-1}, with coefficients holomorphic functions on $\mathfrak{H}_n$;*
(2) *$f \mid \gamma = f$, for all $\gamma \in \Gamma^n$.*
(3) *If $n = 1$ then the Fourier expansion of f is as below, i.e. with only non-negative A occurring.*

For a fixed, n, k and d, the set of all such f is a finite-dimensional space, denoted $\mathcal{N}_n^{k,d}$. We have

$$M_n^k = \mathcal{N}_n^{k,0} \subseteq \mathcal{N}_n^{k,1} \subseteq \mathcal{N}_n^{k,2} \subseteq \dots$$

Let $\mathcal{N}_n^k := \cup_{d \geq 0} \mathcal{N}_n^{k,d}$. Any $f \in \mathcal{N}_n^{k,d}$ has a Fourier expansion

$$f(Z) = (\det(\pi Y))^{-d} \sum_{A \geq 0} p_A(\pi Y) q^A,$$

where $q^A := \mathbf{e}(\mathrm{Tr}(AZ))$, and $p_A(Y) \in \mathbb{C}[Y] := \mathbb{C}[Y_{11}, Y_{12}, \dots, Y_{nn}]$. Given $p(Y) \in \mathbb{C}[Y]$, $p(Y) = \sum_\alpha c_\alpha Y^\alpha$ where $\alpha = (\alpha_{11}, \alpha_{12}, \dots, \alpha_{nn}) \in \mathbb{N}^{n^2}$ and $Y^\alpha = \prod_{i,j} Y_{ij}^{\alpha_{ij}}$; we also use $|\alpha| = \sum_{i,j} \alpha_{ij}$.

Definition 7.2. *Given $f \in \mathcal{N}_n^{k,d}$ with Fourier expansion as above, if all the $p_A(Y) \in R[Y]$, for R some subring of $\mathbb{C}$, we write $f \in \mathcal{N}_{n,R}^{k,d}$. If q is a prime number and $R = \mathbb{Z}_{(q)}$ (localisation), we say that f is q-integral.*

7.2. Shimura-Maass operators.

Definition 7.3. *Let*

$$\frac{\partial}{\partial Z_{ab}} = \frac{1}{2}\frac{\partial}{\partial X_{ab}} - \frac{i}{2}\frac{\partial}{\partial Y_{ab}}$$
$$\frac{\partial}{\partial \overline{Z}_{ab}} = \frac{1}{2}\frac{\partial}{\partial X_{ab}} + \frac{i}{2}\frac{\partial}{\partial Y_{ab}}$$

Definition 7.4. *Let $\partial_{n,k}$ be the Shimura-Maass differential operator*

$$\partial_{n,k} = (2\pi i)^{-n} (\det(Y))^{\frac{n-1}{2} - k} \left| \frac{d}{dZ} \right| (\det(Y))^{k - \frac{n-1}{2}}$$

where $\left|\frac{d}{dZ}\right| = \det\left(\left(\frac{1+\delta_a^b}{2} \frac{\partial}{\partial Z_{ab}} \right)_{1 \leq a,b \leq n} \right)$. *Further, put*

$$\partial_{n,k}^\mu = \partial_{n,k+2\mu-2} \circ \dots \partial_{n,k+2} \circ \partial_{n,k}$$

Proposition 7.5. (1) $\partial_{n,k}^\mu(\mathcal{N}_n^{k,d}) \subseteq \mathcal{N}_n^{k+2\mu, d+n\mu}$;

(2) *If* $\mathbb{Z}[1/2] \subseteq R$ *then* $\partial^{\mu}_{n,k}(\mathcal{N}^{k,d}_{n,R}) \subseteq \mathcal{N}^{k+2\mu,d+n\mu}_{n,R}$.

Proof. (1) The statement about weights was proved by Maass[Ma, §19]. Böcherer and Heim [BH2, p.490] indicate how the statement about degrees may be deduced from results of Shimura [Sh]. Alternatively, it is a direct consequence of an explicit formula of Courtieu and Panchishkin for the action of $\partial_{n,k}$ on Fourier expansions [CP, Theorem 3.14].

(2) This may be proved by an elementary computation of the effect of each constituent $\frac{\partial}{\partial Z_{ab}}$ on a term of the form $(\det(\pi Y))^{-d-v} r_A(\pi Y) q^A$, noting that the entries of A lie in $\mathbb{Z}[1/2]$.

□

Proposition 7.6. *Let* $k > n+1$ *be even and* $0 \leq v < \frac{k}{2} - \frac{n+1}{2}$ *an integer. Then*

$$E^k_n(Z, -v) = (-4\pi)^{nv} \left(\prod_{j=1}^{v} \prod_{l=0}^{n-1} \left(k - v - j - \frac{l}{2} \right) \right)^{-1} \partial^v_{n,k-2v} E^{k-2v}_n(Z, 0).$$

This follows from work of Maass [Ma, §19]. As already noted during the proof of Lemma 5.6, if $q > 2k$ and $q \nmid B_k B_{2k-2} B_{2k-4}$ then E^k_5 is integral at q.

Corollary 7.7. *With* k, v *as above,* $\pi^{-nv} E^k_n(Z, -v) \in \mathcal{N}^{k,nv}_n$. *Suppose further that* $q \nmid B_{k-2v} B_{2k-4v-2} B_{2k-4v-4}$ *and that* $q > 2k - 4v$. *Then* $\pi^{-5v} E^k_5(Z, -v)$ *is integral at* q.

7.3. **Diagonal restriction.** The following is no doubt well-known to experts, but we include a proof.

Lemma 7.8. *Let* $n = n_1 + n_2$. *Suppose that* $f \in \mathcal{N}^{k,d}_n$. *Then, restricting to the block diagonal,*

$$f|_{\mathfrak{H}_{n_1} \times \mathfrak{H}_{n_2}} \in \mathcal{N}^{k,d}_{n_1} \otimes \mathcal{N}^{k,d}_{n_2}.$$

Proof. If we fix $W \in \mathfrak{H}_{n_2}$ then $Z \mapsto f(Z, W)$ is an element of $\mathcal{N}^{k,d}_{n_1}$. Since this space is finite dimensional, we can find a finite basis $(\phi_i)_{i \leq l}$ and

$$f(Z, W) = \sum_{i=1}^{l} c_i(W) \phi_i(Z)$$

for some coefficients $c_i(W)$ not depending on Z. Fix some $Z_1, \ldots, Z_l \in \mathfrak{H}_{n_1}$ and write the system

$$\begin{pmatrix} f(Z_1, W) \\ \vdots \\ f(Z_l, W) \end{pmatrix} = \begin{pmatrix} \phi_1(Z_1) & \ldots & \phi_l(Z_1) \\ \vdots & & \vdots \\ \phi_1(Z_l) & \ldots & \phi_l(Z_l) \end{pmatrix} \begin{pmatrix} c_1(W) \\ \vdots \\ c_l(W) \end{pmatrix}$$

If the central matrix is invertible, we can multiple by its inverse and read (from the q-th line)

$$c_q(W) = \sum_{i=1}^{l} d_i(Z_1, \ldots, Z_l) f(Z_i, W)$$

where $d_i(Z_1, \ldots, Z_l)$ are complex coefficients (depending on the choice of Z_i), showing that each $W \mapsto c_q(W)$ is an element of $\mathcal{N}^{k,d}_{n_2}$, as required.

We can prove by an inductive process that it is possible to choose $Z_1, \ldots, Z_l$ in such a way that the matrix is invertible. First, since $\phi_1 \neq 0$ we can choose Z_1 so that $\phi_1(Z_1) \neq 0$. For the last step, supposing that $\begin{pmatrix} \phi_1(Z_1) & \dots & \phi_{l-1}(Z_1) \\ \vdots & & \vdots \\ \phi_1(Z_{l-1}) & \dots & \phi_{l-1}(Z_{l-1}) \end{pmatrix}$ is invertible, its columns $\{c_1, \dots, c_{l-1}\}$ form a basis for $\mathbb{C}^{l-1}$, so

$$\begin{pmatrix} \phi_l(Z_1) \\ \vdots \\ \phi_l(Z_{l-1}) \end{pmatrix} = \sum_{j=1}^{l-1} \alpha_j c_j, \text{ for some } c_1, \dots, c_{l-1} \in \mathbb{C}.$$

Since $\{\phi_1, \dots, \phi_l\}$ is linearly independent, we may choose some Z_l such that $\phi_l(Z_l) \neq \sum_{j=1}^{l-1} \alpha_j \phi_j(Z_l)$, then the l-by-l matrix is invertible. □

7.4. **Holomorphic projection.**

Definition 7.9. *For $n \in \mathbb{N}_{>0}$ and $s \in \mathbb{C}$, let $\Gamma_n(s)$ be the generalized Γ function*

$$\Gamma_n(s) = \pi^{\frac{n(n-1)}{4}} \prod_{j=0}^{n-1} \Gamma\left(s - \frac{j}{2}\right)$$

For $n = 1$ we have $\Gamma_1(s) = \Gamma(s)$.

Let Ω_n^Y be the space of positive-definite *real* symmetric n-by-n matrices, i.e. the space of imaginary parts of elements of $\mathfrak{H}_n$. Let Ω_n^X be the space of real symmetric n-by-n matrices, each of whose entries is strictly less than $\frac{1}{2}$ in absolute value.

Proposition 7.10. *The generalized Γ function admits the integral formula*

$$\Gamma_n\left(v + \frac{n+1}{2}\right) = \det(A)^{-v-\frac{n+1}{2}} \int_{\Omega_n^Y} e^{-\operatorname{tr}(AY)} \det(Y)^v dY.$$

For a proof, see [Kl, §6 Lemma 2].

Definition 7.11. *$f \in \mathcal{N}_n^k$ is said to be of* bounded growth *if, for every $\varepsilon > 0$,*

$$\int_{\Omega_n^X} \int_{\Omega_n^Y} |f(Z)| e^{-\varepsilon \operatorname{tr}(Y)} \det(Y)^{k-1-n} \, dY \, dX < \infty.$$

The following proposition is due to Sturm [St, Theorem 1].

Proposition 7.12. *Suppose that $f \in \mathcal{N}_n^k$ is of bounded growth, with $k > 2n$. If*

$$f(Z) = \sum_{A \geq 0} f_A(Y) q^A,$$

define

$$\operatorname{Hol} f := \sum_{A > 0} \tilde{f}_A q^A,$$

where

$$\tilde{f}_A = \frac{\pi^{n(k-\frac{n+1}{2})}}{\Gamma_n(k - \frac{n+1}{2})} \det(4A)^{k-\frac{n+1}{2}} \int_{\Omega_n^Y} f_A(Y) e^{-4\pi \operatorname{tr}(AY)} \det(Y)^{k-n-1} \, dY.$$

Then $\operatorname{Hol} f \in S_n^k$ *and*

$$\langle \operatorname{Hol} f, g \rangle = \langle f, g \rangle \quad \forall g \in S_n^k.$$

Lemma 7.13. *If* $f_A(Y) = (\det(\pi Y))^{-d}\sum_\alpha c(A,\alpha)(\pi Y)^\alpha$ *then* $\tilde{f}_A = \sum_\alpha c(A,\alpha)M(A,\alpha)$, *where*

$$M(A,\alpha) = \frac{\pi^{n(k-\frac{n+1}{2}-d)}}{\Gamma_n(k-\frac{n+1}{2})}\det(4A)^{k-\frac{n+1}{2}}\int_{\Omega_n^Y}(\pi Y)^\alpha e^{-4\pi \mathrm{tr}(AY)}\det(Y)^{k-n-d-1}\,dY$$

is integral at q *for any odd prime* $q > 2k-(n+4)$.

Proof. Let

$$\left(-\frac{d}{dA}\right)^\alpha = \prod_{l\le j}(-1)^{\alpha_{jl}}\frac{\partial^{\alpha_{jl}}}{\partial A_{jl}^{\alpha_{jl}}},$$

so clearly

$$\left(-\frac{d}{dA}\right)^\alpha e^{-\mathrm{tr}(AY)} = Y^\alpha e^{-\mathrm{tr}(AY)}.$$

Hence

$$\begin{aligned}\int_{\Omega_n^Y} Y^\alpha e^{-\mathrm{tr}(AY)}\det(Y)^{k-n-d-1}\,dY &= \int_{\Omega_n^Y}\left(-\frac{d}{dA}\right)^\alpha e^{-\mathrm{tr}(AY)}\det(Y)^{k-n-d-1}\,dY\\ &= \left(-\frac{d}{dA}\right)^\alpha\left(\det(A)^{k-n-d-1+\frac{n+1}{2}}\right)\Gamma_n\left(k-n-d-1+\frac{n+1}{2}\right)\\ &= c\,\Gamma_n\left(k-n-d-1+\frac{n+1}{2}\right),\ \text{ with } c\in\mathbb{Z}[1/2].\end{aligned}$$

Making a change of variable $Y\mapsto 4\pi Y$, we get

$$M(A,\alpha) = c\,4^{-|\alpha|-n(k-d-(n+1)/2)}\det(4A)^{k-\frac{n+1}{2}}\frac{\Gamma_n(k-d-(n+1)/2)}{\Gamma_n(k-(n+1)/2)},$$

which is integral at q for any odd prime $q > 2k-(n+4)$, using $\Gamma(s+1) = s\Gamma(s)$ and Definition 7.9. □

7.5. **Completion of the proof.** Putting $n=5, v=4$ in Corollary 7.7, and using $q\nmid B_{k-8}B_{2k-18}B_{2k-20}$, we find that $\pi^{-20}E_5^k(Z,-4)\in\mathcal{N}_5^{k,20}$, and is integral at q. By repeated use of Lemma 7.8,

$$\pi^{-20}E_5^k(\mathrm{diag}(Z,\tau,Z'),-4) = \sum_{r,i,j}c_{r,i,j}\phi_r(Z)\psi_i(\tau)\phi_j(Z')$$

for some coefficients $c_{r,i,j}\in\mathbb{C}$, where $\{\phi_r\}$ and $\{\psi_i\}$ are bases for $\mathcal{N}_2^{k,20}$ and $\mathcal{N}_1^{k,20}$ respectively. On the left hand side, the coefficient of each

$$(\det(\pi Y))^{-20}(\pi y)^{-20}(\det(\pi Y'))^{-20}(\pi Y)^\alpha(\pi y)^a(\pi Y'))^\beta\mathbf{e}(\mathrm{Tr}(AZ)+m\tau+\mathrm{Tr}(BZ'))$$

is some $C(A,\alpha,m,a,B,\beta)\in\mathbb{Z}_{(q)}$.

We claim that each ϕ_r and ψ_i is of bounded growth, so has a holomorphic projection. We explain the argument for the ϕ_r. Replacing each term in the series by its absolute value we find that $|E_5^k(Z,-4)|\le|H(Z,k,-4)|$, where $H(Z,k,b)$ is as in [St, Corollary 1], which gives an upper bound $|H(Z,k,b)|\le c_1\prod_{j=1}^5(\lambda_j^b+\lambda_j^{-b-k})$, with $b=-4$ and the λ_j the eigenvalues of Y. As in the proof of Lemma 7.8, each $\phi_r(Z)$ can be expressed as a linear combination of the form $\sum_{p,q}d_{p,q}E_5^k(Z,\tau_p,Z'_q,-4)$. Fixing τ and Z' amounts to fixing $\lambda_3,\lambda_4,\lambda_5$ (with a natural labelling). Hence in the integral in Definition 7.11, the absolute value of the integrand is bounded above by some $c_2\,e^{-\varepsilon(\lambda_1+\lambda_2)}\prod_{j=1}^2((\lambda_j^b+\lambda_j^{-b-k})\lambda_j^{k-1-n})$, with $b=-4$ and $n=2$. As in the proof of [St, Corollary 1], to get convergence of the integral we need both exponents of the λ_j, namely $k+b-n-1$ and $-b-n-1$, to be strictly greater than -1. For us $n=2$, which is why $b=-2$ is not enough but $b=-4$ is sufficiently far to the left, making the exponents $k-7$ and 1.

Now we know that holomorphic projection is justified, by Proposition 7.12, let $\Xi(Z,\tau,Z') := \sum_{r,i,j} c_{r,i,j}\mathrm{Hol}\,\phi_r(Z)\mathrm{Hol}\,\psi_i(\tau)\mathrm{Hol}\,\phi_j(Z')$. The coefficient of $\mathbf{e}(\mathrm{Tr}(AZ)+m\tau+\mathrm{Tr}(BZ'))$ in $\Xi(Z,\tau,Z')$ is

$$\sum_{\alpha,a,\beta} M(A,\alpha)M(m,a)M(B,\beta)C(A,\alpha,m,a,B,\beta),$$

where $M(A,\alpha)$ etc. are given by Lemma 7.13. It follows that $\Xi(Z,\tau,Z')$ is integral at q. We may now proceed as in the proof of Lemma 5.6, but it is simpler here because there are no non-cuspidal terms to deal with. We may expand $\Xi(Z,\tau,Z')$ in terms of the $H_r(Z)h_i(\tau)H_j(Z')$. Then, using elements of $\mathbb{T}$ to kill all the other terms, we see that the coefficient of $G(Z)h(\tau)H(Z')$ is integral at $\mathfrak{q}$. Since (up to a power of π) $\langle\langle\langle\Xi(\mathrm{diag}(Z,\tau,Z'),G(Z)\rangle h(\tau)\rangle H(Z')\rangle$ is the same as $\langle\langle\langle E_5^k(\mathrm{diag}(Z,\tau,Z'),-4),G(Z)\rangle h(\tau)\rangle H(Z')\rangle$, it follows from Lemma 6.1 that $\mathrm{ord}_{\mathfrak{q}}(\kappa)\geq 0$, as required.

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University of Sheffield, School of Mathematics and Statistics, Hicks Building, Hounsfield Road, Sheffield, S3 7RH, U.K.

E-mail address: `n.p.dummigan@shef.ac.uk`

E-mail address: `arendina1@sheffield.ac.uk`

German University of Technology in Oman (GUtech), Department of Applied Information Technology, PO Box 1816 Athaibah, PC 130, Sultanate of Oman, Corner of Beach Road and Wadi Athaibah Way.

E-mail address: `bernhard.heim@gutech.edu.om`