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Robust controller design for attitude dynamics subjected to time-delayed state measurements

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***Abstract*—Attitude control and time-delay systems are well-developed fields in control theory, but only a modicum of papers have explored control systems that fall within the intersection of the two. Indeed, combining kinematics and dynamics nonlinearities with sensor and actuator delays reinvigorates the original attitude control problem, typically leading to involved stability arguments based on nonlinear analysis techniques. This paper instead proposes solving the attitude stabilizer design problem by formulating it as a linear matrix inequality feasibility problem. The proposed approach simplifies the stability arguments, without loosing generality; the obtained conditions cope with the general case of rigid bodies that suffer from unknown, heterogeneous, time-varying state measurement delays, and have inertia uncertainties. This methodology is particularly well suited to resource-limited applications, because controllers can be designed *offline* using computationally efficient tools. Although simple, numerical evidence shows the stability criterion derived in this paper largely outperforms previous results.**

I. Introduction

Attitude control of rigid bodies subjected to time-delayed measurements represents a largely unexplored problem that falls both into the categories of attitude control and (nonlinear) time-delay systems. Separately, each of them represents well-developed areas of control theory, the former attracting the control community's attention for decades now [26], [24], [14], [15], whereas the second has experienced a surge of results since the turn of the century [7], [19], [8], [6].

Applications of attitude control are vast, ranging from aircraft, spacecraft, and satellite stabilization and maneuvering [9], [20], [13], to robotic rigid manipulator orientation control and coordination [4], [5]. On the other hand, it is well known that applications are prone to interaction with time-delayed dynamics introduced by sensors or actuators. For instance, valve circuits have electromechanical delays which affect gas jet control systems [25]. Magnetometers, which must be turned off in the presence of magnetic torques, delaying access to attitude measurements [3], are an example of sensors that can induce closed-loop delays. Low-rate sensors can contribute with delays as well, as in the case of star trackers, which may need up to ten seconds to identify stars [21]. Global Positioning System (GPS) also causes sensing delays due to data latency and momentary outages while evaluating satellite position in orbit [12], [10].

Time-delays can have multiple effects on closed-loop behavior, depending on how and on which subsystem they occur [7], [6]. In general, however, delays are detrimental to performance, and are even capable of causing unstable behavior [19], [8]. Thus, the original problem of attitude control is reinvigorated by considering delay phenomena and calls for specialized time-delay analysis techniques. In fact, kinematics and dynamics nonlinearities impede directly employing linear time-delay methods—which are more numerous than their nonlinear counterparts in time-delayed systems analysis theory.

Most of the scarce papers that have so far dealt with the attitude control problem subjected to time-delays have considered constant delays. For example, in [1], the problem was addressed assuming known, constant and sufficiently small delays, using modified Rodrigues parameters to represent attitude. In [3], rotation matrices were used to describe attitude, and delays were also considered constant and known. An algorithm to obtain controller parameters was later devised in [2], but also restricting initial orientations. Using quaternion representation, [16] considered both attitude and angular velocity subjected to constant delays; stability conditions rely only on initial velocities and delay magnitude. In [17], angular velocity measurements are discarded, and stability conditions are derived with only attitude measurements, as in [1]. Sufficient conditions that simultaneously guarantee stability and H_∞ performance were given in [23], which also addressed time-varying delays. In fact, to the best of the authors' knowledge, [2] and [23] are the only literature to develop attitude stability analysis under time-varying delays. Nevertheless, the stability analysis for time-varying delays in [2] depends on a proper estimation of the time-delay itself which considerably reduces its applicability, while [23] focused only on the kinematic case, exploiting the structure of the underlying quaternion manifold to derive conditions in form of linear matrix inequalities (LMIs).

In this paper, to the best of authors' knowledge, existence conditions for quaternion based stabilizing controller are given for the first time for the problem of dynamic attitude control subjected to time-varying time-delays in the closed loop. We adopt an approach fundamentally different from most works of the literature on attitude stabilization, seeking for reduction of design conditions to a linear form, more specifically, in terms of LMIs feasibility tests.

Casting stabilizing controller existence as LMI feasibility conditions enables the designer to take advantage of the well

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developed LMI computational tools and also to cast the design of a controller with additional specific properties as a convex optimization problem. In contrast with [2] which is, in the authors' best knowledge, the only other work to address time-varying delays in the dynamic case, the proposed LMI feasibility conditions are easily verified using any LMI solver, provide directly the controller gains, and enable controller design to be fully performed *offline*. The controller proposed by [2] is more difficult to implement in practical applications since the controller gains stem from matrix differential equation solutions which must be obtained *online* in real time by some differential matrix equation solver. At each time this solver requires knowledge on the terminal conditions and the equation must be solved backwards in time. In addition, to address time-varying delays, the controller needs to estimate the delays, which can be a rather difficult task. In comparison with the only prior work on dynamic attitude stability based on quaternions [16], in addition to presenting easier linear, rather than non-linear, conditions, we verify numerically that the proposed conditions represent a drastic reduction in conservatism with respect to feasible controller gains. In particular, this enables automated design of considerably faster controllers than that can be currently obtained. The present work also sets itself apart from [23], since dealing with dynamic attitude control involves gyroscopic (Coriolis) terms that cannot be treated using the techniques presented in that work, requiring different analysis, and because we, in addition to unknown, time-varying delays, consider the more general case of modeling time-delays affecting attitude and angular velocity measurements as different phenomena. This allows heterogeneous delays, as in the case where a star tracker is subjected to considerably larger delays than an accelerometer is, but also covers the particular case where delays are the same. Moreover, the proposed criterion is robust to model uncertainties concerning the rigid body's matrix of inertia, which is assumed unknown, but in a set with known bounds.

II. Model and preliminaries

Consider a rigid body whose kinematics and dynamics are described by

$$\dot{\boldsymbol{q}}=\begin{bmatrix}\dot{\eta}\\ \dot{\boldsymbol{\zeta}}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\boldsymbol{\zeta}^T\\ \eta\mathbf{I}+[\boldsymbol{\zeta}]_\times\end{bmatrix}\boldsymbol{\omega}, \tag{1}$$

$$J\dot{\boldsymbol{\omega}}=-\left[\boldsymbol{\omega}\right]_\times J\boldsymbol{\omega}+\boldsymbol{u}, \tag{2}$$

where $\boldsymbol{q}(t)$ is such that[1]

$$\boldsymbol{q}:\mathbb{R}\rightarrow\mathcal{S}^3:=\left\{\left[\eta\,\boldsymbol{\zeta}^T\right]^T\in\mathbb{R}^4,\eta\in\mathbb{R},\boldsymbol{\zeta}\in\mathbb{R}^3:\eta^2+\boldsymbol{\zeta}^T\boldsymbol{\zeta}=1\right\},$$

and represents the rigid body's attitude. The set $\mathcal{S}^3$ forms, under multiplication, the Lie group of unit quaternions $\text{Spin}(3)$, enforcing the constraint

$$|\eta(t)|\leq 1,\ \|\boldsymbol{\zeta}(t)\|\leq 1,\quad \forall t\geq 0, \tag{3}$$

where $\|\cdot\|$ denotes Euclidean norm. The rigid body's angular velocity $\omega(t)$ in $\mathbb{R}^3$ evolves according to (2), and $[\cdot]_\times:\mathbb{R}^3\rightarrow\mathbb{R}^{3\times3}$ is an operator such that $[\boldsymbol{w}]_\times\boldsymbol{v}=\boldsymbol{w}\times\boldsymbol{v}$ for any $\boldsymbol{w},\boldsymbol{v}$ in $\mathbb{R}^3$. The rigid body's inertia is given by positive definite matrix J in $\mathbb{R}^{3\times3}$, which is assumed unknown but satisfying

$$0<m_J\leq\lambda_{\min}(J)\leq\|J\|=\lambda_{\max}(J)\leq M_J, \tag{4}$$

where m_J and M_J are positive real numbers that bound the uncertain matrix of inertia.

State measurements are assumed subjected to bounded time-varying delays $d_1(t)$ and $d_2(t)$, given by nonnegative real numbers that satisfy

$$0\leq d_1(t)\leq\nu_1,\ 0\leq d_2(t)\leq\nu_2,\ \forall t\geq 0, \tag{5}$$

where ν_i, i in $\{1,2\}$, are known quantities.

To address the system's stability, let κ_1,κ_2 be positive real numbers, and consider PD control law

$$\boldsymbol{u}(t)=-\kappa_1\boldsymbol{\zeta}_{d_1}-\kappa_2\boldsymbol{\omega}_{d_2}, \tag{6}$$

where $\boldsymbol{\zeta}_{d_1}$ and $\boldsymbol{\omega}_{d_2}$ denote $\boldsymbol{\zeta}(t-d_1(t))$ and $\boldsymbol{\omega}(t-d_2(t))$. In addition, consider the following results, which will support arguing system stability.

Lemma 1. *Let $P\in\mathbb{R}^{n\times n}$ be a positive definite matrix. Then, for all nonzero $\boldsymbol{x}\in\mathbb{R}^n$,*

$$0<\lambda_{\min}(P)\,\boldsymbol{x}^T\boldsymbol{x}\leq\boldsymbol{x}^TP\boldsymbol{x}\leq\lambda_{\max}(P)\,\boldsymbol{x}^T\boldsymbol{x}$$

holds, where $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the largest and smallest eigenvalues of P.

Lemma 2. *[18] Given positive definite matrix P in $\mathbb{R}^{n\times n}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$, then*

$$2\boldsymbol{x}^T\boldsymbol{y}\leq\boldsymbol{x}^TP\boldsymbol{x}+\boldsymbol{y}^TP^{-1}\boldsymbol{y}$$

holds.

Lemma 3. *Barbalat's Lemma [11]*

Let $f:\mathbb{R}_+\rightarrow\mathbb{R}$ be a uniformly continuous map on $[0,+\infty)$, and suppose $\lim_{t\rightarrow+\infty}\int_0^t f(s)\,ds$ exists and is finite. Then,

$$\lim_{t\rightarrow+\infty}f(t)=0.$$

III. Stability

Stability will be proven using a Barbalat's Lemma argument and a nonnegative function that will be taken as the following functional

$$V=V_1+V_2, \tag{7}$$

with

$$V_1=2\left[\boldsymbol{\zeta}^T\boldsymbol{\zeta}+(1-\eta)^2\right]\mathfrak{a}+\boldsymbol{\omega}^TJ\boldsymbol{\omega}\mathfrak{b}+2\boldsymbol{\zeta}^TJ\boldsymbol{\omega}\mathfrak{c}, \tag{8}$$

$$\begin{aligned}V_2=&\,\nu_1\mathfrak{p}_1\int_{-\nu_1}^{0}\int_{t+l}^{t}\dot{\boldsymbol{\zeta}}(s)^T\dot{\boldsymbol{\zeta}}(s)\,dsdl\\ &+\nu_2\mathfrak{p}_2\int_{-\nu_2}^{0}\int_{t+l}^{t}\dot{\boldsymbol{\omega}}(s)^T\dot{\boldsymbol{\omega}}(s)\,dsdl,\end{aligned} \tag{9}$$

and real numbers $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{p}_1$ and $\mathfrak{p}_2$. For V to be positive definite, cross-term $\mathfrak{c}\boldsymbol{\zeta}^TJ\boldsymbol{\omega}$ requires extra constraints. Indeed, from Lemma 1's quadratic inequality, cross-term inequality of Lemma 2, and $\lambda_{\max}(J)$ bound M_J (4),

$$\begin{aligned}V_1=&\,2\left[\boldsymbol{\zeta}^T\boldsymbol{\zeta}+(1-\eta)^2\right]\mathfrak{a}+\boldsymbol{\omega}^TJ\boldsymbol{\omega}\mathfrak{b}+2\boldsymbol{\zeta}^TJ\boldsymbol{\omega}\mathfrak{c}\\ \geq&\,\frac{2}{\lambda_{\max}(J)}\boldsymbol{\zeta}^TJ\boldsymbol{\zeta}\mathfrak{a}+\boldsymbol{\omega}^TJ\boldsymbol{\omega}\mathfrak{b}-\boldsymbol{\zeta}^TJ\boldsymbol{\zeta}\mathfrak{c}-\boldsymbol{\omega}^TJ\boldsymbol{\omega}\mathfrak{c}\end{aligned}$$

[1] Time arguments will be omitted to simplify notation whenever the context allows.

$$\geq \frac{2}{M_J}\boldsymbol{\zeta}^T J\boldsymbol{\zeta}\mathfrak{a} + \boldsymbol{\omega}^T J\boldsymbol{\omega}\mathfrak{b} - \boldsymbol{\zeta}^T J\boldsymbol{\zeta}\mathfrak{c} - \boldsymbol{\omega}^T J\boldsymbol{\omega}\mathfrak{c}$$
$$= \frac{1}{M_J}\left(2\mathfrak{a} - M_J\mathfrak{c}\right)\boldsymbol{\zeta}^T J\boldsymbol{\zeta} + \boldsymbol{\omega}^T J\boldsymbol{\omega}\left(\mathfrak{b} - \mathfrak{c}\right)$$

holds. This means that V is positive definite if constraints

$$\mathfrak{a}>0, 2\mathfrak{a}>M_J\mathfrak{c}, \mathfrak{b}>0, \mathfrak{b}>\mathfrak{c}, \mathfrak{c}>0, \mathfrak{p}_1>0, \mathfrak{p}_2>0, \tag{10}$$

are satisfied.

Theorem 4. *Let ν_1 and ν_2 be nonnegative delay bounds (5), let M_ω be a positive real number such that $\|\boldsymbol{\omega}(t)\|$ is less than or equal to M_ω for all t in $[-2\nu, 0]$, where ν is given by $\max\{\nu_1, \nu_2\}$, and consider a parameter $M_{\mathfrak{p}_2}$, a positive real number. Given positive real numbers κ_1 and κ_2, if there exist real numbers $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{m}$ such that (10),*

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ * & * & \Omega_{33} & \Omega_{34} \\ * & * & * & \Omega_{44} \end{bmatrix} < 0, \tag{11}$$

and

$$m_J^{-1} < \mathfrak{b}, \quad \mathfrak{p}_2 < M_{\mathfrak{p}_2}, \quad M_V < \mathfrak{m}, \tag{12}$$

hold, where

$$\Omega_{11} = -\mathfrak{p}_1\mathbf{I}, \qquad \Omega_{22} = \left(2m_J^{-2}\kappa_1^2\nu_2^2\mathfrak{p}_2 - \mathfrak{p}_1\right)\mathbf{I},$$
$$\Omega_{12} = \left(\mathfrak{p}_1 - \kappa_1\mathfrak{c}\right)\mathbf{I}, \qquad \Omega_{33} = \frac{\nu_1^2}{4}\mathfrak{p}_1\mathbf{I} + 2M_J\mathfrak{c}\mathbf{I} - \mathfrak{p}_2\mathbf{I},$$
$$\Omega_{13} = \mathfrak{a}\mathbf{I}, \qquad + 3\frac{\nu_2^2}{m_J^2}\left(M_J^2 - m_J^2\right)M_{\mathfrak{p}_2}\mathfrak{m}\mathbf{I}$$
$$\Omega_{14} = -\kappa_2\mathfrak{c}\mathbf{I}, \qquad \Omega_{34} = \left(\mathfrak{p}_2 - \kappa_2\mathfrak{b}\right)\mathbf{I},$$
$$\Omega_{23} = -\kappa_1\mathfrak{b}\mathbf{I}, \qquad \Omega_{44} = \left(2m_J^{-2}\kappa_2^2\nu_2^2 - 1\right)\mathfrak{p}_2\mathbf{I},$$
$$\Omega_{24} = m_J^{-2}\nu_2^2\kappa_1\kappa_2\mathfrak{p}_2\mathbf{I}, \qquad M_u = \kappa_1 + \kappa_2 M_\omega,$$

$$M_V = 8\mathfrak{a} + M_J M_\omega^2\mathfrak{b} + 2M_J M_\omega\mathfrak{c} + \frac{\nu_1^3}{8}M_\omega^2\mathfrak{p}_1 + \frac{\nu_2^3}{2}m_J^{-2}\left(M_J M_\omega^2 + M_u\right)^2\mathfrak{p}_2, \tag{13}$$

then the closed-loop system (1)-(6) is asymptotically stable.

Proof: The proof is a two-step argument. First, using Barbalat's Lemma, a conditional proof of asymptotic stability is given depending on an upper bound of V that is obtained in the second step. The aggregate requirements form the conditions stated by the theorem.

Take V_{1a} and V_{1b}, such that $V_1 = V_{1a} + V_{1b}$, where

$$V_{1a} = 2\left[\boldsymbol{\zeta}^T\boldsymbol{\zeta} + (1-\eta)^2\right]\mathfrak{a} + \boldsymbol{\omega}^T J\boldsymbol{\omega}\mathfrak{b},$$
$$V_{1b} = 2\boldsymbol{\zeta}^T J\boldsymbol{\omega}\mathfrak{c}.$$

Using cross-term bound from Lemma 2 and quadratic upper bound from Lemma 1, and substituting (1) for $\dot{\boldsymbol{\zeta}}$ and (2) for $\dot{\boldsymbol{\omega}}$, results in

$$\dot{V}_{1a} = \frac{d}{dt}\left\{2\mathfrak{a}\left[\boldsymbol{\zeta}^T\boldsymbol{\zeta} + (1-\eta)^2\right] + \mathfrak{b}\boldsymbol{\omega}^T J\boldsymbol{\omega}\right\}$$
$$= \frac{d}{dt}\left\{4\mathfrak{a}(1-\eta)\right\} + 2\mathfrak{b}\boldsymbol{\omega}^T\left(-[\boldsymbol{\omega}]_\times J\boldsymbol{\omega} - \kappa_1\boldsymbol{\zeta}_{d_1} - \kappa_2\boldsymbol{\omega}_{d_2}\right)$$
$$= -4\mathfrak{a}\dot{\eta} - 2\kappa_1\mathfrak{b}\boldsymbol{\omega}^T\boldsymbol{\zeta}_{d_1} - 2\kappa_2\mathfrak{b}\boldsymbol{\omega}^T\boldsymbol{\omega}_{d_2}$$
$$= 2\mathfrak{a}\boldsymbol{\zeta}^T\boldsymbol{\omega} - 2\kappa_1\mathfrak{b}\boldsymbol{\omega}^T\boldsymbol{\zeta}_{d_1} - 2\kappa_2\mathfrak{b}\boldsymbol{\omega}^T\boldsymbol{\omega}_{d_2}$$
$$= \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathfrak{a}\mathbf{I} & \mathbf{0} \\ * & \mathbf{0} & -\kappa_1\mathfrak{b}\mathbf{I} & \mathbf{0} \\ * & * & \mathbf{0} & -\kappa_2\mathfrak{b}\mathbf{I} \\ * & * & * & \mathbf{0}\end{bmatrix} \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}. \tag{14}$$

$$\dot{V}_{1b} = 2\mathfrak{c}\dot{\boldsymbol{\zeta}}^T J\boldsymbol{\omega} + 2\mathfrak{c}\boldsymbol{\zeta}^T J\dot{\boldsymbol{\omega}}$$
$$= \mathfrak{c}\boldsymbol{\omega}^T\left(\eta\mathbf{I} + [\boldsymbol{\zeta}]_\times\right)^T J\boldsymbol{\omega}$$
$$+ 2\mathfrak{c}\boldsymbol{\zeta}^T\left(-[\boldsymbol{\omega}]_\times J\boldsymbol{\omega} - \kappa_1\boldsymbol{\zeta}_{d_1} - \kappa_2\boldsymbol{\omega}_{d_2}\right)$$
$$\leq \mathfrak{c}\boldsymbol{\omega}^T J\boldsymbol{\omega} + \mathfrak{c}\boldsymbol{\zeta}^T[\boldsymbol{\omega}]_\times J\boldsymbol{\omega} - 2\mathfrak{c}\boldsymbol{\zeta}^T[\boldsymbol{\omega}]_\times J\boldsymbol{\omega}$$
$$- 2\kappa_1\mathfrak{c}\boldsymbol{\zeta}^T\boldsymbol{\zeta}_{d_1} - 2\kappa_2\mathfrak{c}\boldsymbol{\zeta}^T\boldsymbol{\omega}_{d_2}$$
$$\leq \mathfrak{c}\boldsymbol{\omega}^T J\boldsymbol{\omega} + \mathfrak{c}\boldsymbol{\zeta}^T[\boldsymbol{\omega}]_\times J\boldsymbol{\omega} - \mathfrak{c}\boldsymbol{\zeta}^T[\boldsymbol{\omega}]_\times J\boldsymbol{\omega}$$
$$+ M_J\mathfrak{c}\boldsymbol{\omega}^T\boldsymbol{\omega} - 2\kappa_1\mathfrak{c}\boldsymbol{\zeta}^T\boldsymbol{\zeta}_{d_1} - 2\kappa_2\mathfrak{c}\boldsymbol{\zeta}^T\boldsymbol{\omega}_{d_2}$$
$$\leq 2M_J\mathfrak{c}\boldsymbol{\omega}^T\boldsymbol{\omega} - 2\kappa_1\mathfrak{c}\boldsymbol{\zeta}^T\boldsymbol{\zeta}_{d_1} - 2\kappa_2\mathfrak{c}\boldsymbol{\zeta}^T\boldsymbol{\omega}_{d_2}$$
$$= \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}^T \begin{bmatrix} \mathbf{0} & -\kappa_1\mathfrak{c}\mathbf{I} & \mathbf{0} & -\kappa_2\mathfrak{c}\mathbf{I} \\ * & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & 2M_J\mathfrak{c}\mathbf{I} & \mathbf{0} \\ * & * & * & \mathbf{0}\end{bmatrix} \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}. \tag{15}$$

where the inequalities stem from unit-quaternion norm constraint (3), cross-product's cyclic property, J being positive definite, and $\mathfrak{c}$ being a positive real number:

$$\mathfrak{c}\boldsymbol{\omega}^T\left(\eta\boldsymbol{I} + [\boldsymbol{\zeta}]_\times\right)^T J\boldsymbol{\omega} = \mathfrak{c}\eta\boldsymbol{\omega}^T J\boldsymbol{\omega} - \mathfrak{c}\boldsymbol{\omega}^T[\boldsymbol{\zeta}]_\times J\boldsymbol{\omega}$$
$$\leq \mathfrak{c}\boldsymbol{\omega}^T J\boldsymbol{\omega} + \mathfrak{c}\boldsymbol{\omega}^T[J\boldsymbol{\omega}]_\times\boldsymbol{\zeta}$$
$$\leq M_J\mathfrak{c}\boldsymbol{\omega}^T\boldsymbol{\omega} + \mathfrak{c}\boldsymbol{\zeta}^T[\boldsymbol{\omega}]_\times J\boldsymbol{\omega}$$
$$-2\mathfrak{c}\boldsymbol{\zeta}^T\left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right) \leq -\mathfrak{c}\boldsymbol{\zeta}^T[\boldsymbol{\omega}]_\times J\boldsymbol{\omega} + \mathfrak{c}\|\boldsymbol{\zeta}\|\|J\|\|\boldsymbol{\omega}\|^2$$
$$\leq -\mathfrak{c}\boldsymbol{\zeta}^T[\boldsymbol{\omega}]_\times J\boldsymbol{\omega} + M_J\mathfrak{c}\boldsymbol{\omega}^T\boldsymbol{\omega}.$$

The combination of derivative terms $\dot{V}_{1a}$ (14) and $\dot{V}_{1b}$ (15) results in

$$\dot{V}_1 = \dot{V}_{1a} + \dot{V}_{1b}$$
$$\leq \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathfrak{a}\mathbf{I} & \mathbf{0} \\ * & \mathbf{0} & -\kappa_1\mathfrak{b}\mathbf{I} & \mathbf{0} \\ * & * & \mathbf{0} & -\kappa_2\mathfrak{b}\mathbf{I} \\ * & * & * & \mathbf{0}\end{bmatrix} \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}$$
$$+ \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}^T \begin{bmatrix} \mathbf{0} & -\kappa_1\mathfrak{c}\mathbf{I} & \mathbf{0} & -\kappa_2\mathfrak{c}\mathbf{I} \\ * & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & 2M_J\mathfrak{c}\mathbf{I} & \mathbf{0} \\ * & * & * & \mathbf{0}\end{bmatrix} \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}$$
$$= \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}^T \begin{bmatrix} \mathbf{0} & -\kappa_1\mathfrak{c}\mathbf{I} & \mathfrak{a}\mathbf{I} & -\kappa_2\mathfrak{c}\mathbf{I} \\ * & \mathbf{0} & -\kappa_1\mathfrak{b}\mathbf{I} & \mathbf{0} \\ * & * & 2M_J\mathfrak{c}\mathbf{I} & -\kappa_2\mathfrak{b}\mathbf{I} \\ * & * & * & \mathbf{0}\end{bmatrix} \begin{bmatrix}\boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2}\end{bmatrix}. \tag{16}$$

Invoking Jensen's Inequality [7] and using delay bounds (5), it follows that $\dot{V}_2$ is also bounded:

$$\dot{V}_2 = \nu_1\mathfrak{p}_1\int_{-\nu_1}^{0}\left[\dot{\boldsymbol{\zeta}}(t)^T\dot{\boldsymbol{\zeta}}(t) - \dot{\boldsymbol{\zeta}}(t+l)^T\dot{\boldsymbol{\zeta}}(t+l)\right]dl$$
$$+ \nu_2\mathfrak{p}_2\int_{-\nu_2}^{0}\left[\dot{\boldsymbol{\omega}}(t)^T\dot{\boldsymbol{\omega}}(t) - \dot{\boldsymbol{\omega}}(t+l)^T\dot{\boldsymbol{\omega}}(t+l)\right]dl$$
$$= \nu_1^2\mathfrak{p}_1\dot{\boldsymbol{\zeta}}^T\dot{\boldsymbol{\zeta}} - \nu_1\mathfrak{p}_1\int_{t-\nu_1}^{t}\dot{\boldsymbol{\zeta}}(s)^T\dot{\boldsymbol{\zeta}}(s)\,ds$$
$$+ \nu_2^2\mathfrak{p}_2\dot{\boldsymbol{\omega}}^T\dot{\boldsymbol{\omega}} - \nu_2\mathfrak{p}_2\int_{t-\nu_2}^{t}\dot{\boldsymbol{\omega}}(s)^T\dot{\boldsymbol{\omega}}(s)\,ds$$

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$$
\begin{aligned}
&\leq \nu_1^2 \mathfrak{p}_1 \dot{\boldsymbol{\zeta}}^T \dot{\boldsymbol{\zeta}} - \nu_1 \mathfrak{p}_1 \int_{t-d_1(t)}^{t} \dot{\boldsymbol{\zeta}}(s)^T \dot{\boldsymbol{\zeta}}(s)\, ds \\
&+ \nu_2^2 \mathfrak{p}_2 \dot{\boldsymbol{\omega}}^T \dot{\boldsymbol{\omega}} - \nu_2 \mathfrak{p}_2 \int_{t-d_2(t)}^{t} \dot{\boldsymbol{\omega}}(s)^T \dot{\boldsymbol{\omega}}(s)\, ds \\
&\leq \nu_1^2 \mathfrak{p}_1 \dot{\boldsymbol{\zeta}}^T \dot{\boldsymbol{\zeta}} + \nu_2^2 \mathfrak{p}_2 \dot{\boldsymbol{\omega}}^T \dot{\boldsymbol{\omega}} \\
&- \nu_1 \frac{\mathfrak{p}_1}{d_1(t)} \left[\int_{t-d_1(t)}^{t} \dot{\boldsymbol{\zeta}}(s)\, ds \right]^T \left[\int_{t-d_1(t)}^{t} \dot{\boldsymbol{\zeta}}(s)\, ds \right] \\
&- \nu_2 \frac{\mathfrak{p}_2}{d_2(t)} \left[\int_{t-d_2(t)}^{t} \dot{\boldsymbol{\omega}}(s)\, ds \right]^T \left[\int_{t-d_2(t)}^{t} \dot{\boldsymbol{\omega}}(s)\, ds \right] \\
&\leq \nu_1^2 \mathfrak{p}_1 \dot{\boldsymbol{\zeta}}^T \dot{\boldsymbol{\zeta}} + \nu_2^2 \mathfrak{p}_2 \dot{\boldsymbol{\omega}}^T \dot{\boldsymbol{\omega}} - \mathfrak{p}_1 \left[\boldsymbol{\zeta} - \boldsymbol{\zeta}_{d_1}\right]^T \left[\boldsymbol{\zeta} - \boldsymbol{\zeta}_{d_1}\right] \\
&- \mathfrak{p}_2 \left[\boldsymbol{\omega} - \boldsymbol{\omega}_{d_2}\right]^T \left[\boldsymbol{\omega} - \boldsymbol{\omega}_{d_2}\right] \\
&\leq \nu_1^2 \mathfrak{p}_1 \dot{\boldsymbol{\zeta}}^T \dot{\boldsymbol{\zeta}} + \nu_2^2 \mathfrak{p}_2 \dot{\boldsymbol{\omega}}^T \dot{\boldsymbol{\omega}} \\
&+ \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2} \end{bmatrix}^T \begin{bmatrix} -\mathfrak{p}_1 \mathbf{I} & \mathfrak{p}_1 \mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & -\mathfrak{p}_1 \mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & -\mathfrak{p}_2 \mathbf{I} & \mathfrak{p}_2 \mathbf{I} \\ * & * & * & -\mathfrak{p}_2 \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2} \end{bmatrix}
\end{aligned}
$$

Now, let V_{2a}, V_{2b} be such that

$$
\begin{aligned}
&\dot{V}_2 = \dot{V}_{2a} + \dot{V}_{2b}, \\
&\dot{V}_{2a} = \nu_1^2 \mathfrak{p}_1 \dot{\boldsymbol{\zeta}}^T \dot{\boldsymbol{\zeta}} + \nu_2^2 \mathfrak{p}_2 \dot{\boldsymbol{\omega}}^T \dot{\boldsymbol{\omega}}, \\
&\dot{V}_{2b} \leq \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2} \end{bmatrix}^T \begin{bmatrix} -\mathfrak{p}_1 \mathbf{I} & \mathfrak{p}_1 \mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & -\mathfrak{p}_1 \mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & -\mathfrak{p}_2 \mathbf{I} & \mathfrak{p}_2 \mathbf{I} \\ * & * & * & -\mathfrak{p}_2 \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2} \end{bmatrix}.
\end{aligned} \tag{17}
$$

Using $\left\|[\boldsymbol{\zeta}]_\times\right\| \leq \|\boldsymbol{\zeta}\|$, $\left\|\dot{\boldsymbol{\zeta}}\right\|^2$ can be linearly bounded because

$$
\begin{aligned}
\dot{\boldsymbol{\zeta}}^T \dot{\boldsymbol{\zeta}} &= \frac{1}{4} \boldsymbol{\omega}^T \left(\eta \mathbf{I} - [\boldsymbol{\zeta}]_\times\right) \left(\eta \mathbf{I} + [\boldsymbol{\zeta}]_\times\right) \boldsymbol{\omega} \\
&= \frac{1}{4} \boldsymbol{\omega}^T \left(\eta^2 \mathbf{I} - [\boldsymbol{\zeta}]_\times^2\right) \boldsymbol{\omega} \\
&\leq \frac{\eta^2 + \|\boldsymbol{\zeta}\|^2}{4} \boldsymbol{\omega}^T \boldsymbol{\omega} = \frac{1}{4} \boldsymbol{\omega}^T \boldsymbol{\omega}.
\end{aligned}
$$

On the other hand, using cross-term bounds[2], and $\|J\|$ norm bounds (4)[3]

$$
\begin{aligned}
m_J^2 \dot{\boldsymbol{\omega}}^T \dot{\boldsymbol{\omega}} &\overset{(i)}{\leq} \dot{\boldsymbol{\omega}}^T J^T J \dot{\boldsymbol{\omega}} \\
&= \left(-[\boldsymbol{\omega}]_\times J\boldsymbol{\omega} - \kappa_1 \boldsymbol{\zeta}_{d_1} - \kappa_2 \boldsymbol{\omega}_{d_2}\right)^T \\
&\times \left(-[\boldsymbol{\omega}]_\times J\boldsymbol{\omega} - \kappa_1 \boldsymbol{\zeta}_{d_1} - \kappa_2 \boldsymbol{\omega}_{d_2}\right) \\
&= \left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right)^T \left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right) + 2\kappa_1 \left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right)^T \boldsymbol{\zeta}_{d_1} \\
&+ 2\kappa_2 \left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right)^T \boldsymbol{\omega}_{d_2} + \kappa_1^2 \boldsymbol{\zeta}_{d_1}^T \boldsymbol{\zeta}_{d_1} + 2\kappa_1\kappa_2 \boldsymbol{\zeta}_{d_1}^T \boldsymbol{\omega}_{d_2} \\
&+ \kappa_2^2 \boldsymbol{\omega}_{d_2}^T \boldsymbol{\omega}_{d_2} \\
&\leq 3 \underbrace{\left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right)^T \left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right)}_{=\|\boldsymbol{\omega}\|^2 \|J\boldsymbol{\omega}\|^2 - (\boldsymbol{\omega}^T J \boldsymbol{\omega})^2} + \kappa_1^2 \boldsymbol{\zeta}_{d_1}^T \boldsymbol{\zeta}_{d_1} + \kappa_2^2 \boldsymbol{\omega}_{d_2}^T \boldsymbol{\omega}_{d_2} \\
&+ \kappa_1^2 \boldsymbol{\zeta}_{d_1}^T \boldsymbol{\zeta}_{d_1} + 2\kappa_1\kappa_2 \boldsymbol{\zeta}_{d_1}^T \boldsymbol{\omega}_{d_2} + \kappa_2^2 \boldsymbol{\omega}_{d_2}^T \boldsymbol{\omega}_{d_2} \\
&\leq 3\left(M_J^2 - m_J^2\right) \|\boldsymbol{\omega}\|^4 + 2\kappa_1^2 \boldsymbol{\zeta}_{d_1}^T \boldsymbol{\zeta}_{d_1} + 2\kappa_1\kappa_2 \boldsymbol{\zeta}_{d_1}^T \boldsymbol{\omega}_{d_2} \\
&+ 2\kappa_2^2 \boldsymbol{\omega}_{d_2}^T \boldsymbol{\omega}_{d_2}.
\end{aligned} \tag{18}
$$

[2]The term $2\kappa_1 \left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right)^T \boldsymbol{\zeta}_{d_1}$ can also be checked using cross-term bound from Lemma 2.

[3]Since J is positive definite, it admits diagonal decomposition $Q^T D Q$, with Q orthonormal and D diagonal. Thus

$$
J^2 = \left(Q^T D Q\right)\left(Q^T D Q\right) = Q^T D^2 Q,
$$

which means $\lambda\left(J^2\right)$ and $\lambda^2(J)$ define the same set. This proves (i).

Imposing $m_J^{-1} \leq \mathfrak{b}$, the definition of V implies

$$
\boldsymbol{\omega}^T \boldsymbol{\omega} \leq \frac{m_J^{-1}}{\mathfrak{b}} \mathfrak{b} \boldsymbol{\omega}^T J \boldsymbol{\omega} \leq \mathfrak{b} \boldsymbol{\omega}^T J \boldsymbol{\omega} \leq V, \tag{19}
$$

which means that

$$
\dot{V}_{2a} \leq \boldsymbol{\vartheta}^T \Omega_{2a} \boldsymbol{\vartheta},
$$

$$
\Omega_{2a} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & (\Omega_{2a})_{22} & \mathbf{0} & (\Omega_{2a})_{24} \\ * & * & (\Omega_{2a})_{33} & \mathbf{0} \\ * & * & * & (\Omega_{2a})_{44} \end{bmatrix}, \quad \boldsymbol{\vartheta} = \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2} \end{bmatrix}, \tag{20}
$$

with

$$
\begin{aligned}
&(\Omega_{2a})_{22} = 2\frac{\nu_2^2}{m_J^2} \kappa_1^2 \mathfrak{p}_2 \mathbf{I}, \quad (\Omega_{2a})_{24} = \frac{\nu_2^2}{m_J^2} \kappa_1 \kappa_2 \mathfrak{p}_2 \mathbf{I}, \\
&(\Omega_{2a})_{44} = 2\frac{\nu_2^2}{m_J^2} \kappa_2^2 \mathfrak{p}_2 \mathbf{I}, \quad (\Omega_{2a})_{33} = \frac{\nu_1^2}{4} \mathfrak{p}_1 \mathbf{I} + 3\nu_2^2 \frac{M_J^2 - m_J^2}{m_J^2} \mathfrak{p}_2 V \mathbf{I}.
\end{aligned}
$$

Combining inequalities (16) and (20) with identity (17) results in

$$
\begin{aligned}
\dot{V} &= \dot{V}_1 + \dot{V}_2 \\
&\leq \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2} \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & -\kappa_1 \mathfrak{c} \mathbf{I} & \mathfrak{a} \mathbf{I} & -\kappa_2 \mathfrak{c} \mathbf{I} \\ * & \mathbf{0} & -\kappa_1 \mathfrak{b} \mathbf{I} & \mathbf{0} \\ * & * & 2M_J \mathfrak{c} \mathbf{I} & -\kappa_2 \mathfrak{b} \mathbf{I} \\ * & * & * & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2} \end{bmatrix} \\
&+ \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2} \end{bmatrix}^T \begin{bmatrix} -\mathfrak{p}_1 \mathbf{I} & \mathfrak{p}_1 \mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & -\mathfrak{p}_1 \mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & -\mathfrak{p}_2 \mathbf{I} & \mathfrak{p}_2 \mathbf{I} \\ * & * & * & -\mathfrak{p}_2 \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\zeta} \\ \boldsymbol{\zeta}_{d_1} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega}_{d_2} \end{bmatrix} \\
&+ \boldsymbol{\vartheta}^T \Omega_{2a} \boldsymbol{\vartheta} \\
&= \boldsymbol{\vartheta}^T \Omega \boldsymbol{\vartheta},
\end{aligned} \tag{21}
$$

where

$$
\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ * & * & \Omega_{33} & \Omega_{34} \\ * & * & * & \Omega_{44} \end{bmatrix},
$$

$$
\begin{aligned}
&\Omega_{11} = -\mathfrak{p}_1 \mathbf{I}, && \Omega_{22} = \left(2m_J^{-2} \kappa_1^2 \nu_2^2 \mathfrak{p}_2 - \mathfrak{p}_1\right) \mathbf{I}, \\
&\Omega_{12} = \left(\mathfrak{p}_1 - \kappa_1 \mathfrak{c}\right) \mathbf{I}, && \Omega_{33} = \frac{\nu_1^2}{4} \mathfrak{p}_1 \mathbf{I} + 2M_J \mathfrak{c} \mathbf{I} \\
&\Omega_{13} = \mathfrak{a} \mathbf{I}, && \quad + 3\nu_2^2 \frac{M_J^2 - m_J^2}{m_J^2} \mathfrak{p}_2 V \mathbf{I} \\
&\Omega_{14} = -\kappa_2 \mathfrak{c} \mathbf{I}, && \quad - \mathfrak{p}_2 \mathbf{I}, \\
&\Omega_{23} = -\kappa_1 \mathfrak{b} \mathbf{I}, && \Omega_{34} = \left(\mathfrak{p}_2 - \kappa_2 \mathfrak{b}\right) \mathbf{I}, \\
&\Omega_{24} = m_J^{-2} \nu_2^2 \kappa_1 \kappa_2 \mathfrak{p}_2 \mathbf{I}, && \Omega_{44} = \left(2m_J^{-2} \kappa_2^2 \nu_2^2 - 1\right) \mathfrak{p}_2 \mathbf{I}.
\end{aligned}
$$

Now, suppose $V(\boldsymbol{\vartheta}(0))$ is less than $\mathfrak{m}$, with $\mathfrak{m}$ such that it makes Ω negative definite if V is replaced by $\mathfrak{m}$ in Ω_{33}; call it $\Omega|_{\mathfrak{m}}$. Then, $V(\boldsymbol{\vartheta}(t))$ is less than $\mathfrak{m}$ for all t greater than zero. Indeed, suppose, by contradiction, there is t_c greater than zero such that $V(\boldsymbol{\vartheta}(t_c))$ equals $\mathfrak{m}$—note that, by continuity of V, if there's V greater than $\mathfrak{m}$, there must also exist such t_c. This implies there exists some t_p in $[0, t_c]$ that makes $\dot{V}(\boldsymbol{\vartheta}(t_p))$ positive. Without loss of generality, assume t_p is the smallest instant of time in $[0, t_c]$ with this property. For t in $[0, t_c)$, $\dot{V}(\boldsymbol{\vartheta}(t))$ is nonpositive, meaning $V(\boldsymbol{\vartheta}(t))$ is less than $\mathfrak{m}$.

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Since V is continuous, $V(\boldsymbol{\vartheta}(t_p))$ must be less than or equal to $\mathfrak{m}$. Hence, inequality (21) implies

$$\dot{V}(\boldsymbol{\vartheta}(t_p)) \leq \boldsymbol{\vartheta}(t_p)^T \Omega|_{V(t_p)} \boldsymbol{\vartheta}(t_p) \leq \boldsymbol{\vartheta}(t_p)^T \Omega|_{\mathfrak{m}} \boldsymbol{\vartheta}(t_p) < 0$$

because $\Omega|_{\mathfrak{m}}$ is negative definite, which contradicts the hypothesis that $\dot{V}(\boldsymbol{\vartheta}(t_p))$ is positive. On the other hand, $V(\boldsymbol{\vartheta}(t))$ is positive for every t. This implies $\boldsymbol{\omega}(t)$ is bounded for every nonnegative t—see (19)—and from (1)-(2), also gives that $\dot{\boldsymbol{q}}(t)$ and $\dot{\boldsymbol{\omega}}(t)$ are both bounded. Thus, one concludes via mean-value theorem that $\boldsymbol{q}(t)$ and $\boldsymbol{\omega}(t)$ are both uniformly continuous.

Because Ω is negative definite for all nonnegative t, then

$$\dot{V}(\boldsymbol{\vartheta}(t)) \leq \boldsymbol{\vartheta}(t) \Omega|_{V(t)} \boldsymbol{\vartheta}(t) < 0$$

holds for every nonnegative t. Integrating previous inequality from 0 to t gives

$$V(\boldsymbol{\vartheta}(t)) - V(\boldsymbol{\vartheta}(0)) \leq \int_0^t \boldsymbol{\vartheta}(s)^T \Omega|_{V(s)} \boldsymbol{\vartheta}(s)\, ds < 0. \quad (22)$$

Let $\lambda_{\max}\left(\Omega|_{V(t)}\right)$ be the largest eigenvalue of $\Omega|_{V(t)}$, which is negative because $\Omega|_{V(t)}$ is negative definite. For $V(\boldsymbol{\vartheta}(t))$ is nonnegative and less than $\mathfrak{m}$, it can be concluded via last inequality, (22), that

$$\begin{aligned} -\lambda_{\max}(\Omega) \int_0^t \boldsymbol{\vartheta}(s)^T \boldsymbol{\vartheta}(s)\, ds &\leq -\int_0^t \boldsymbol{\vartheta}(s) \Omega|_{V(s)} \boldsymbol{\vartheta}(s)\, ds \\ &\leq V(\boldsymbol{\vartheta}(0)) < +\infty, \end{aligned}$$

i.e., $\int_0^t \boldsymbol{\vartheta}(s)^T \boldsymbol{\vartheta}(s)\, ds$ is finite. Since $\boldsymbol{\vartheta}(t)$ is uniformly continuous, from Barbalat's Lemma, one concludes that $\boldsymbol{\vartheta}(t)$ converges to zero as t increases, i.e., $\boldsymbol{q}(t)$ and $\boldsymbol{\omega}(t)$ both converge to zero as $t \to \infty$. Therefore, the system is asymptotically stable.

Now, it remains to obtain the conditions $\mathfrak{m}$ must satisfy in order to bound $V(\boldsymbol{\vartheta}(0))$. Before that, however, note term Ω_{33} is nonlinear with respect to the decision variables because $\mathfrak{m}$ multiplies $\mathfrak{p}_2$, both variables. Imposing an extra constraint

$$\mathfrak{p}_2 < M_{\mathfrak{p}_2},$$

with $M_{\mathfrak{p}_2}$ a positive real number considered a given parameter, Ω becomes linear with regard to the decision variables. It could be argued that, instead of imposing an extra constraint, a new decision variable $\mathfrak{p}_{\mathfrak{m}}$ accounting for the product $\mathfrak{p}_2\mathfrak{m}$ could have been defined. Nevertheless, the constraint necessary to ensure $\mathfrak{m}$ is, in fact, greater than $V(\boldsymbol{\vartheta}(0))$ would make constraints nonlinear again. Thus, Ω_{33} is considered

$$\Omega_{33} = \left(\frac{\nu_1^2}{4}\mathfrak{p}_1 + 2M_J\mathfrak{c} + 3\nu_2^2 \frac{M_J^2 - m_J^2}{m_J^2} M_{\mathfrak{p}_2}\mathfrak{m} - \mathfrak{p}_2\right)\mathbf{I}.$$

At this point, we obtain an expression that bounds $V(\boldsymbol{\vartheta}(0))$, so that $\mathfrak{m}$ can be greater than this expression, satisfying the assumption required to prove the theorem. Suppose $\|\boldsymbol{\omega}(t)\|$ is less than M_ω for all t in $[-2\nu, 0]$, with ν given by $\max\{\nu_1, \nu_2\}$. This implies

$$\begin{aligned} \boldsymbol{u}(t)^T \boldsymbol{u}(t) = &\kappa_1^2 \boldsymbol{\zeta}(t - d_1(t))^T \boldsymbol{\zeta}(t - d_1(t)) \\ &+ 2\kappa_1\kappa_2 \boldsymbol{\zeta}(t - d_1(t))^T \boldsymbol{\omega}(t - d_2(t)) \\ &+ \kappa_2^2 \boldsymbol{\omega}(t - d_2(t))^T \boldsymbol{\omega}(t - d_2(t)) \\ \leq &\kappa_1^2 + 2\kappa_1\kappa_2 \|\boldsymbol{\omega}(t - d_2(t))\| \\ &+ \kappa_2^2 \|\boldsymbol{\omega}(t - d_2(t))\|^2 \\ \leq &M_u^2, \end{aligned}$$

for all t in $[-\nu, 0]$—note $t - d_2(t)$ belongs to $[-2\nu, 0]$. Then, substituting (1) for $\dot{\boldsymbol{\zeta}}$ and (2) for $\dot{\boldsymbol{\omega}}$, and using initial conditions upper bound M_ω, it can be concluded that, for t equal to zero, the inequalities

$$\begin{aligned} \int_{-\nu_1}^{0}\int_{t+l}^{t} \dot{\boldsymbol{\zeta}}(s)^T \dot{\boldsymbol{\zeta}}(s)\, ds dl &= \int_{-\nu_1}^{0}\int_{l}^{0} \left\| \frac{1}{2}\left(\eta(s)\mathbf{I} + [\boldsymbol{\zeta}(s)]_\times\right)^T \boldsymbol{\omega}(s) \right\|^2 ds dl \\ &\leq \frac{1}{4}\int_{-\nu_1}^{0}\int_{l}^{0} \|\boldsymbol{\omega}(s)\|^2\, ds dl \\ &\leq \frac{1}{4}\int_{-\nu_1}^{0}\int_{l}^{0} M_\omega^2\, ds dl \\ &= \frac{\nu_1^2}{8} M_\omega^2, \end{aligned} \quad (23)$$

and

$$\begin{aligned} \int_{-\nu_2}^{0}\int_{t+l}^{t} \dot{\boldsymbol{\omega}}(s)^T \dot{\boldsymbol{\omega}}(s)\, ds dl &\leq m_J^{-2} \int_{-\nu_2}^{0}\int_{l}^{0} \dot{\boldsymbol{\omega}}(s)^T JJ\dot{\boldsymbol{\omega}}(s)\, ds dl \\ &= m_J^{-2} \int_{-\nu_2}^{0}\int_{l}^{0} \left\| -[\boldsymbol{\omega}(s)]_\times J\boldsymbol{\omega}(s) + \boldsymbol{u}(s) \right\|^2 ds dl \\ &\leq m_J^{-2} \int_{-\nu_2}^{0}\int_{l}^{0} \left(M_J M_\omega^2 + M_u\right)^2 ds dl \\ &= m_J^{-2} \frac{\nu_2^2}{2}\left(M_J M_\omega^2 + M_u\right)^2, \end{aligned} \quad (24)$$

must hold. Note that, because t equals zero and l belongs to $[-\nu_1, 0]$, the limits of integral $\int_{t+l}^{t} \|\boldsymbol{\omega}(s)\|^2 ds$ belong to $[-\nu, 0]$, which means $\|\boldsymbol{\omega}(s)\|$ is less than or equal to M_ω, by hypothesis. Similar rationale allows one to conclude $\|\boldsymbol{\omega}(s)\|$ and $\|\boldsymbol{u}(s)\|$ are bounded by M_ω and M_u in the second integral.

Combining inequalities (23) and (24) with initial condition hypothesis yields

$$\begin{aligned} V_1(\boldsymbol{\vartheta}(0)) &= 2\left[\boldsymbol{\zeta}^T\boldsymbol{\zeta} + (1-\eta)^2\right]\mathfrak{a} + \boldsymbol{\omega}^T J\boldsymbol{\omega}\mathfrak{b} + 2\boldsymbol{\zeta}^T J\boldsymbol{\omega}\mathfrak{c} \\ &\leq 4(1-\eta)\mathfrak{a} + M_J M_\omega^2\mathfrak{b} + 2\|\boldsymbol{\zeta}\|\|J\boldsymbol{\omega}\|\mathfrak{c} \\ &\leq 8\mathfrak{a} + M_J M_\omega^2\mathfrak{b} + 2M_J M_\omega\mathfrak{c}, \\ V_2(\boldsymbol{\vartheta}(0)) &= \nu_1\mathfrak{p}_1 \int_{-\nu_1}^{0}\int_{l}^{0} \dot{\boldsymbol{\zeta}}(s)^T \dot{\boldsymbol{\zeta}}(s)\, ds dl \\ &\quad + \nu_2\mathfrak{p}_2 \int_{-\nu_2}^{0}\int_{l}^{0} \dot{\boldsymbol{\omega}}(s)^T \dot{\boldsymbol{\omega}}(s)\, ds dl \\ &\leq \nu_1\mathfrak{p}_1 \frac{\nu_1^2}{8} M_\omega^2 + \nu_2\mathfrak{p}_2 \frac{\nu_2^2}{2} m_J^{-2}\left(M_J M_\omega^2 + M_u\right)^2 \\ &= \frac{\nu_1^3}{8} M_\omega^2\mathfrak{p}_1 + \frac{\nu_2^3}{2} m_J^{-2}\left(M_J M_\omega^2 + M_u\right)^2 \mathfrak{p}_2. \end{aligned}$$

Summing the two inequalities results in

$$\begin{aligned} V(\boldsymbol{\vartheta}(0)) \leq &8\mathfrak{a} + M_J M_\omega^2\mathfrak{b} + 2M_J M_\omega\mathfrak{c} + \frac{\nu_1^3}{8} M_\omega^2\mathfrak{p}_1 \\ &+ \frac{\nu_2^3}{2} m_J^{-2}\left(M_J M_\omega^2 + M_u\right)^2 \mathfrak{p}_2 \\ = &M_V. \end{aligned}$$

Therefore, if $\mathfrak{m}$ is greater than M_V, validating the hypothesis on which the conditional proof is based. ■

In the proof of Theorem 4, $\left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right)^T \left([\boldsymbol{\omega}]_\times J\boldsymbol{\omega}\right)$ imposes a challenge in finding a linear upper bound to $\dot{V}_2$ since only

$\boldsymbol{\omega}\left(t-d(t)\right)$ is available for feedback. The proposed approach is to manipulate this term as in (18) and upper bound it by $\left(M_J^2-m_J^2\right)\|\boldsymbol{\omega}\|^4$. Still, since our goal is to obtain conditions in form of LMIs, a bound for $\|\boldsymbol{\omega}\|^2$ is needed, which is why condition (19) is convenient. From Theorem 4, V is bounded by its initial condition and monotonically decreasing, but $\boldsymbol{\omega}$ is not—in fact it is often physically required that $\boldsymbol{\omega}$ increases to stabilize attitude (see Figure 3, Section V).

IV. Controller Design

By imposing relaxations on the variables from stability conditions of Theorem 4, it is possible to obtain a controller designing procedure based on LMIs. To do that, we must rewrite such conditions using decision variables that are consistent with respect to the power of controller gains being multiplied by the decision variables of that theorem. In other words, all powers of κ_1 and κ_2 must have the same degree throughout. To this end, we impose additional constraints.

First, assume the ratio between κ_1 and κ_2 is given by R_κ, a design parameter, such that

$$\frac{\kappa_2}{\kappa_1}=R_\kappa, \tag{25}$$

and suppose κ_1 is confined in $[m_\kappa, M_\kappa]$—an interval that is also defined by the controller designer. This implies

$$m_\kappa\leq\kappa_1\leq M_\kappa, \quad R_\kappa m_\kappa\leq\kappa_2\leq R_\kappa M_\kappa. \tag{26}$$

Now, the goal is to have decision variables incorporate controller gains, implicitly making them decision variables as well. This means all instances of a variable that are multiplied by κ_i must be done so consistently with respect to the power of κ_i. For example, either $\mathfrak{c}\kappa_1$ or $\mathfrak{c}\kappa_1^2$ can be present in design criteria, but not both. This applies to all decision variables but $\mathfrak{p}_2$, which will be used as an extra degree of freedom to determine the gains.

Theorem 5. *Let ν_1 and ν_2 be nonnegative delay bounds (5), let M_ω be a positive real number such that $\|\boldsymbol{\omega}(t)\|$ is less than or equal to M_ω for all t in $[-2\nu,0]$, where ν is given by $\max\{\nu_1,\nu_2\}$, and consider parameters m_κ, M_κ, R_κ, and $M_{\mathfrak{p}_2}$ given positive real numbers. If there exist positive real numbers $\mathfrak{a}$, $\mathfrak{b}_\kappa$, $\mathfrak{c}_\kappa$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_{2,\kappa}$ and $\mathfrak{m}$ such that*

$$\Omega=\begin{bmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14}\\ * & \Omega_{22} & \Omega_{23} & \Omega_{24}\\ * & * & \Omega_{33} & \Omega_{34}\\ * & * & * & \Omega_{44}\end{bmatrix}<0, \tag{27}$$

and

$$M_\kappa<m_J\mathfrak{b}_\kappa,\ \mathfrak{p}_2<M_{\mathfrak{p}_2},\ M_V^d<\mathfrak{m},\ m_\kappa^2\mathfrak{p}_2\leq\mathfrak{p}_{2,\kappa}\leq M_\kappa^2\mathfrak{p}_2, \tag{28}$$

as well as

$$\begin{array}{llll}\mathfrak{a}>0, & 2\mathfrak{a}>M_J m_\kappa^{-1}\mathfrak{c}_\kappa, & \mathfrak{b}_\kappa>0, & \mathfrak{b}_\kappa>\mathfrak{c}_\kappa,\\ \mathfrak{c}_\kappa>0, & \mathfrak{p}_1>0, & \mathfrak{p}_2>0, & \mathfrak{p}_{2,\kappa}>0,\end{array} \tag{29}$$

hold, where

$$\begin{aligned}
&\Omega_{11}=-\mathfrak{p}_1\mathbf{I}, && \Omega_{22}=\left(2m_J^{-2}\nu_2^2\mathfrak{p}_{2,\kappa}-\mathfrak{p}_1\right)\mathbf{I},\\
&\Omega_{12}=\left(\mathfrak{p}_1-\mathfrak{c}_\kappa\right)\mathbf{I}, && \Omega_{33}=\frac{\nu_1^2}{4}\mathfrak{p}_1\mathbf{I}+2m_\kappa^{-1}M_J\mathfrak{c}_\kappa\mathbf{I},\\
&\Omega_{13}=\mathfrak{a}\mathbf{I}, && \qquad +3\nu_2^2\frac{M_J^2-m_J^2}{m_J^2}M_{\mathfrak{p}_2}\mathfrak{m}\mathbf{I}\\
&\Omega_{14}=-R_\kappa\mathfrak{c}_\kappa\mathbf{I}, && \qquad -\mathfrak{p}_2\mathbf{I},\\
&\Omega_{23}=-\mathfrak{b}_\kappa\mathbf{I}, && \Omega_{34}=\left(\mathfrak{p}_2-R_\kappa\mathfrak{b}_\kappa\right)\mathbf{I},\\
&\Omega_{24}=m_J^{-2}\nu_2^2R_\kappa\mathfrak{p}_{2,\kappa}\mathbf{I}, && \Omega_{44}=2m_J^{-2}\nu_2^2R_\kappa^2\mathfrak{p}_{2,\kappa}\mathbf{I}-\mathfrak{p}_2\mathbf{I},
\end{aligned}$$

$$\begin{aligned}M_V^d=&8\mathfrak{a}+m_\kappa^{-1}M_JM_\omega^2\mathfrak{b}_\kappa+2m_\kappa^{-1}M_JM_\omega\mathfrak{c}_\kappa+\frac{\nu_1^3}{8}M_\omega^2\mathfrak{p}_1\\ &+\frac{\nu_2^3}{2}m_J^{-2}M_J^2M_\omega^4\mathfrak{p}_2+\frac{\nu_2^3}{2}\frac{2m_\kappa^{-1}M_JM_\omega^2+M_R}{m_J^2}M_R\mathfrak{p}_{2,\kappa}\end{aligned}$$

with M_R given by $1+R_\kappa M_\omega$, then closed-loop system (1)-(2)-(6) is asymptotically stabilized by controller gains

$$\kappa_1=\sqrt{\frac{\mathfrak{p}_{2,\kappa}}{\mathfrak{p}_2}}, \qquad \kappa_2=R_\kappa\kappa_1. \tag{30}$$

Proof: Considering the stability conditions from Theorem 4, and assumption (25), define variables

$$\mathfrak{b}_\kappa=\mathfrak{b}\kappa_1,\ \ \mathfrak{c}_\kappa=\mathfrak{c}\kappa_1,\ \ \mathfrak{p}_{2,\kappa}=\mathfrak{p}_2\kappa_1^2, \tag{31}$$

and also let

$$M_R=1+R_\kappa M_\omega. \tag{32}$$

Since κ_1 and κ_2 are both positive, $\mathfrak{b}_\kappa>0$, $\mathfrak{c}_\kappa>0$, and $\mathfrak{b}_\kappa>\mathfrak{c}_\kappa$ imply $\mathfrak{b}>0$, $\mathfrak{c}>0$, $\mathfrak{b}>\mathfrak{c}$. In addition, by imposing (26), and since $2\mathfrak{a}>M_Jm_\kappa^{-1}\mathfrak{c}_\kappa$, it follows that

$$2\mathfrak{a}>M_Jm_\kappa^{-1}\mathfrak{c}_\kappa=M_Jm_\kappa^{-1}\mathfrak{c}\kappa_1\geq M_J\mathfrak{c},$$

satisfying all positivity conditions of Theorem 4.

Now, it remains to rewrite the negativity constraints of Theorem 4 using only the new variables. Assuming (26), since m_κ and κ_1^{-1} are positive, then

$$\begin{aligned}2M_J\mathfrak{c}&\leq 2M_J\left(m_\kappa^{-1}\kappa_1\right)\mathfrak{c}=2m_\kappa^{-1}M_J\mathfrak{c}_\kappa,\\ M_JM_\omega^2\mathfrak{b}&\leq M_JM_\omega^2\left(m_\kappa^{-1}\kappa_1\right)\mathfrak{b}=m_\kappa^{-1}M_JM_\omega^2\mathfrak{b}_\kappa,\\ 2M_JM_\omega\mathfrak{c}&\leq 2M_JM_\omega\left(m_\kappa^{-1}\kappa_1\right)\mathfrak{c}=2m_\kappa^{-1}M_JM_\omega\mathfrak{c}_\kappa,\end{aligned}$$

and because

$$\begin{aligned}\left(M_JM_\omega^2+M_u\right)^2&=\left(M_JM_\omega^2+M_R\kappa_1\right)^2\\ &=M_J^2M_\omega^4+2M_JM_\omega^2M_R\kappa_1+M_R^2\kappa_1^2\\ &\leq M_J^2M_\omega^4+2M_JM_\omega^2\frac{\kappa_1}{m_\kappa}M_R\kappa_1+M_R^2\kappa_1^2\\ &=M_J^2M_\omega^4+\left(2\frac{M_JM_\omega^2}{m_\kappa}+M_R\right)M_R\kappa_1^2,\end{aligned}$$

where M_u equals $\kappa_1+\kappa_2M_\omega$ and M_R is given by (32), it also follows that

$$\begin{aligned}\frac{\nu_2^3}{2}\left(\frac{M_JM_\omega^2+M_u}{m_J}\right)^2\mathfrak{p}_2\leq&\frac{\nu_2^3}{2}m_J^{-2}M_J^2M_\omega^4\mathfrak{p}_2\\ &+\frac{\nu_2^3}{2m_J^2}\left(\frac{2M_JM_\omega^2}{m_\kappa}+M_R\right)M_R\mathfrak{p}_{2,\kappa}.\end{aligned}$$

Thus, M_V, defined in (13), is upper bounded by M_V^d, given by

$$\begin{aligned}M_V^d=&8\mathfrak{a}+m_\kappa^{-1}M_JM_\omega^2\mathfrak{b}_\kappa+2m_\kappa^{-1}M_JM_\omega\mathfrak{c}_\kappa+\frac{\nu_1^3}{8}M_\omega^2\mathfrak{p}_1\\ &+\frac{\nu_2^3}{2}m_J^{-2}M_J^2M_\omega^4\mathfrak{p}_2+\frac{\nu_2^3}{2}\frac{2m_\kappa^{-1}M_JM_\omega^2+M_R}{m_J^2}M_R\mathfrak{p}_{2,\kappa},\end{aligned}$$

which means (28) implies $M_V\leq M_V^d<\mathfrak{m}$. In addition, since $M_\kappa<m_J\mathfrak{b}_\kappa$, then

$$m_J^{-1}<M_\kappa^{-1}\mathfrak{b}_\kappa=\frac{\kappa_1}{M_\kappa}\mathfrak{b}\leq\mathfrak{b},$$

that is, all inequalities from (12) in Theorem 4 are satisfied.

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Since $\mathfrak{b}$ and $\mathfrak{c}$ give a degree of freedom to variables $\mathfrak{b}_\kappa$ and $\mathfrak{c}_\kappa$, and because κ_1, m_κ, M_κ and $\mathfrak{p}_2$ are positive,

$$m_\kappa^2 \mathfrak{p}_2 \leq \mathfrak{p}_{2,\kappa} \leq M_\kappa^2 \mathfrak{p}_2 \tag{33}$$

implies (26).

Therefore, both stability and design constraints are fulfilled, meaning that if they are all valid, the resulting controller $\{\kappa_1, \kappa_2\}$ extracted from the variables as in

$$\kappa_1 = \sqrt{\frac{\mathfrak{p}_{2,\kappa}}{\mathfrak{p}_2}}, \qquad \kappa_2 = R_\kappa \kappa_1.$$

stabilizes the closed-loop system given by (1)-(6). ∎

V. Quantitative analysis

This section explores quantitative aspects of the proposed stability and stabilization criteria presented in Theorems 4 and 5 under different settings. All scenarios hereafter assume both attitude and angular velocity subject to time-delays as described in (1)-(6) and an unknown rigid body's inertia matrix J bounded by m_J and M_J equal to 0.046 and 0.051.[4]

To illustrate the influence of the initial rigid body's angular velocity on stability analysis, we obtain the feasible control gain region according to Theorem 4 under different values of M_ω—up to a precision of $1e{-}3$ rad/s.[5] The system delays are assumed time-varying with identical upper bounds, ν_1 and ν_2, equal to 100 ms. The resulting feasible gain region shown in Figure 1 suggests more aggressive initial conditions cause the valid region to shrink faster; the higher the value of M_ω, the smaller the control gain region is. Under the described delay conditions and body's inertia uncertainties, the maximum allowable value for $||\omega(t)||$ at time $[-2\nu, 0]$ from Theorem 4 is 11.65 rad/s with $\{\kappa_1, \kappa_2\}$ equal to $\{0.001, 0.045\}$.

From the stabilizing controller region observed in Figure 1, we note higher κ_1 gains are admissible, in general, compared to κ_2—in contrast with trends observed in [16]—which is corroborated by simulated results in Table I. To allow further comparison, the initial rigid body's angular velocity upper bound M_ω is set to 0.03 rad/s; sufficiently small to yield a feasible control region from [16]. As illustrated in Figure 2, the strategy adopted in Theorem 4 allows roughly twenty four times higher κ_1 gains, and approximately κ_2 gain eighty percent higher, resulting in a substantially larger area of feasible gain pairs that contains the one from [16]. This means there exist faster controller than the ones presented in that work that can stabilize closed-loop system (1)-(6). In fact, Theorem 4 guarantees these faster controllers are also stabilizing in the more general case of time-varying delays, as opposed to [16], which concerns only constant delays.

The discrepancy between convergence velocities of feasible controllers according to [16] and Theorem 4 is outlined in Figure 3, which superimposes the attitude quaternion vector part and angular velocity norms assuming $\boldsymbol{q}(0)$ equals $\frac{1}{4}\left[-2\sqrt{2}\,\sqrt{3}\,2\,1\right]$, and $\boldsymbol{\omega}(0)$ equals $3e^{-2}\mathbf{1}$. Considering a 2% settling time criterion, controller $\{0.1, 0.076\}$, which is feasible according to Theorem 4 but not [16], reaches steady state before 8 seconds, whereas the stable controller from [16], $\{0.01, 0.024\}$, takes up to 25 seconds to reach steady state—a threefold increase. Yet, the proposed solution still provides conservative delay bounds compared to simulated results. Table I illustrates this point by pairing theoretical—according to Theorem 4—and simulated maximum allowable delays, where $\{\kappa_1, \kappa_2\}$ is the same controller as before, i.e., $\{0.1, 0.076\}$. Table I also shows that, in general, larger controller gains result in smaller maximum allowable delays.

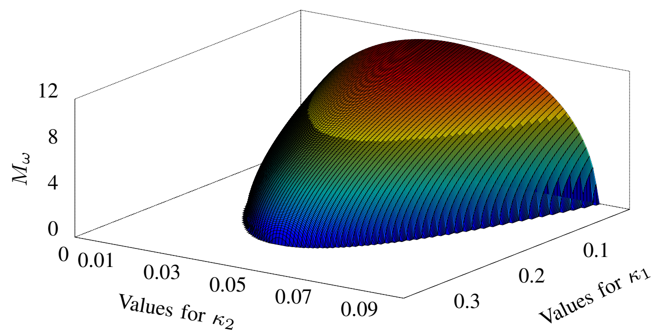

Figure 1. Feasible κ_1, κ_2 regions for different values of M_ω and ν_1, ν_2 equal to 100 ms.

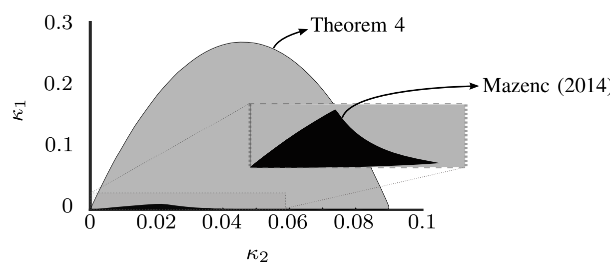

Figure 2. Comparison of feasible gains for ν_1 and ν_2 equal to 100 ms, and M_ω equal to 0.03 rad/s.

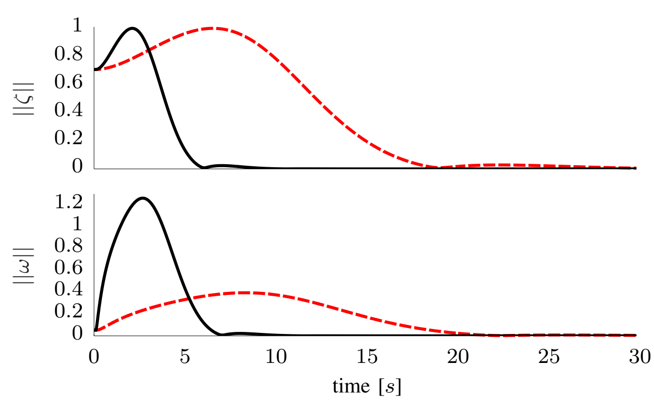

Figure 3. Quaternion vector and angular velocity norms using controller $\{0.01, 0.024\}$ from [16] (dotted line) and $\{0.1, 0.076\}$ from Theorem 4 (solid line).

To conclude this section, we show how to design controllers using Theorem 5, and investigate how they are affected by parameters R_κ, m_κ, M_κ and $M_{\mathfrak{p}_2}$. Letting time-delay and initial condition settings remain the same—d_1 and d_2 both taking values in $[0, 0.1]$, and M_ω equal to 0.03 rad/s—we cast controller designing as minimizing $\mathfrak{p}_2 - \mathfrak{p}_{2,\kappa}$ subjected to

[4]All simulations have been performed setting $M_{\mathfrak{p}_2}$ to 1 and using Sedumi [22]. The rigid body's inertia and its bounds stemmed from the cube-satellite system described in [16].

[5]The feasible control gain region from Theorem 4 was obtained using a binary-search-like algorithm with a precision of $1e{-}4$ for κ_1 and κ_2.

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Table I
MAXIMUM ν ACCORDING TO SIMULATION, ν_{sim}, AND THEOREM 4, ν_{thm} FOR $\kappa_1 = 0.1$ AND $\kappa_2 = 0.076$.

Controller	$\frac{1}{2}\{\kappa_1, \kappa_2\}$	$\{\kappa_2, \kappa_1\}$	$\{\kappa_1, \kappa_2\}$	$2\{\kappa_1, \kappa_2\}$
$\{\nu_{sim}, \nu_{thm}\}$ [s]	$\{1.0, 0.28\}$	$\{0.64, \text{NF}^6\}$	$\{0.7, 0.12\}$	$\{0.42, \text{NF}^6\}$

[6]NF: Not Feasible.

Table II
MAXIMUM $\{\kappa_1, \kappa_2\}$ ACCORDING TO THEOREMS 4 AND 5.

$\{R_\kappa, m_\kappa, M_\kappa\}$	$\{\frac{1}{6}, 0.267, 0.268\}$	$\{1, 0.082, 0.083\}$	$\{2, 0.042, 0.043\}$
Theorem 4	$\{0.268, 0.045\}$	$\{0.083, 0.083\}$	$\{0.044, 0.087\}$
Theorem 5	$\{0.268, 0.045\}$	$\{0.083, 0.083\}$	$\{0.043, 0.086\}$

constraints from Theorem 5, thereby obtaining the stabilizing controller with maximum gains $\{\kappa_1, \kappa_2\}$. Thus, contrary to Theorem 4, a stability test, this procedure actually returns a controller, if existence conditions from Theorem 5 are feasible.

Table II shows the controllers returned by the designing procedure using Theorem 5 described in the previous paragraph assuming tight intervals $[m_\kappa, M_\kappa]$ (and $M_{\mathfrak{p}_2}$ equal to 1) are virtually the same as the ones obtained using a search algorithm (e.g., the binary-search-like algorithm that was used to determine feasible region in Figure 1) together with Theorem 4. This is not surprising since the fundamental difference between the two theorems is that the latter assumes controllers bounded on $[m_\kappa, M_\kappa]$. When larger intervals come into play, however, discrepancies become pronounced. Table III shows designing controllers with less information about the stabilizing controller gain region (according to Theorem 4), i.e., using larger $[m_\kappa, M_\kappa]$ intervals, decreases the maximum gains that can be achieved. The same table also highlights the sensitivity of the procedure with respect to parameter $M_{\mathfrak{p}_2}$, suggesting certain values of $M_{\mathfrak{p}_2}$ allow for larger $[m_\kappa, M_\kappa]$ intervals than others, and that even larger regions of feasible controller gains according to Theorem 4 are possible compared to the one on Figure 2 if $M_{\mathfrak{p}_2}$ is properly tunned.

VI. FINAL REMARKS

Seeking for design conditions in form of LMIs feasibility tests represents a fundamentally distinct approach from most results in time-delayed attitude control literature. This approach allowed us to preserve cross-terms and rely on decision variables to derive substantially less conservative stability conditions compared to previous results from the literature. These conditions concern the more general case when attitude and angular velocity measurements are subjected to independent (but possibly equal) time-varying delays, and when no model information is available, besides bounds on the matrix of inertia. Building upon this result, to the best of authors knowledge, we obtained the first controller design conditions, also in form of an LMI feasibility problem, to the dynamic attitude control problem. This allows automatic controller design, avoiding the use of algorithms with no guarantee of convergence to obtain controller parameters. In numerical experiments, it was observed the relaxations that enabled the transition from analysis to design have not imposed noticeable conservatism to the range of feasible controller gains when appropriate parameters are chosen.

Table III
MAXIMUM$\{\kappa_1, \kappa_2\}$ ACCORDING TO THEOREM 5 FOR R_κ EQUAL TO 2.

$[m_\kappa, M_\kappa] \backslash M_{\mathfrak{p}_2}$	1	30	50
$[0.042, 0.043]$	$\{0.043, 0.086\}$	$\{0.043, 0.086\}$	$\{0.043, 0.086\}$
$[0.010, 0.800]$	NF^7	$\{0.036, 0.072\}$	$\{0.018, 0.037\}$
$[0.010, 0.830]$	NF^7	$\{0.017, 0.034\}$	NF^7

[7]NF: Not Feasible.

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