Article:

Dong, Y.Y., Dong, C.X., Liu, W. orcid.org/0000-0003-2968-2888 et al. (2 more authors) (2018) Robust DOA Estimation for Sources with Known Waveforms Against Doppler Shifts via Oblique Projection. IEEE Sensors Journal, 18 (16). pp. 6735-6742. ISSN 1530-437X

https://doi.org/10.1109/JSEN.2018.2851099

Robust DOA Estimation for Sources with Known Waveforms Against Doppler Shifts via Oblique Projection

Yang-Yang Dong, Chun-Xi Dong, Wei Liu, *Senior Member, IEEE*, Ming-Ming Liu, and Zheng-Zhao Tang

***Abstract*—As known, utilization of the information about signal waveform can improve the direction of arrival (DOA) estimation results. However, with a fast moving platform, Doppler effect occurs, which distorts the known waveforms and may result in large DOA estimation bias and even error for conventional DOA estimation methods for sources with known waveforms. To deal with this problem, a robust DOA estimation method for sources with known waveforms against Doppler shifts is developed. The proposed method first transforms the nonlinear mixing of Doppler shifts in the model to an approximately linear one using discrete-time Fourier transform (DTFT) and finite Taylor series expansion. Then, multiple oblique projectors are constructed to separate each component corresponding to different order of derivatives. Finally, estimations of DOAs, complex amplitudes and Doppler shifts are obtained simultaneously. Simulation results show that the proposed method has a much more robust DOA estimation performance than existing methods for sources with known waveforms.**

***Index Terms*—Direction of arrival estimation, known waveform, Doppler shift, Taylor series expansion, oblique projection.**

I. Introduction

DIRECTION of arrival (DOA) estimation of multiple sources is a key problem in array signal processing, and it has been applied widely in wireless communications, radar, sonar, and electronic reconnaissance, etc [1]–[4]. Conventional DOA estimation methods, such as multiple signal classification (MUSIC) [5], estimation of signal parameters via rotational invariance technique (ESPRIT) [6], and the propagator method (PM) [7], can only utilize the statistical properties of the array received data. However, in many real applications, such as communications [8] and radar [9], prior information of the signal waveform can be available. It has been proved that the Cramer-Rao bound (CRB) of DOA estimation for signals with known waveforms is much lower than the case without [10], and there has been an increasing interest in studying the DOA estimation problem for known waveform sources [10]–[20]. They can be classified into two classes: the first one can only handle uncorrelated sources, such as decouple maximum likelihood (DEML) [11], subarray beamforming (SB) [12], and linear regression (LR) [15], while the other one can handle coherent sources in the presence of multipath, such as coherent decoupled maximum likelihood (CDEML) [17], white coherent decoupled maximum likelihood (WCDEML) [18], parallel decomposition (PADEC) [19], and linear propagator (LP) [20].

This work was supported in part by the China Postdoctoral Science Foundation under Grant 2017M623123 and in part by the Fundamental Research Funds for the Central Universities of China under Grant XJS18033.

Y.-Y. Dong, C.-X. Dong, M.-M. Liu, and Z.-Z. Tang are with the School of Electronic Engineering, Xidian University, Xi'an 710071, China (dongyangyang2104@126.com; chxdong@mail.xidian.edu.cn;18710981090@163.com;zztangzz@163.com).

W. Liu is with the Department of Electronic and Electrical Engineering, University of Sheffield, S1 4ET, U.K. (w.liu@sheffield.ac.uk).

In this work, the DOA estimation problem with known signal waveforms is further studied and the case with fast moving platforms is considered. When the array system is placed on such a platform, the Doppler effect cannot be neglected, which results in Doppler shifts from the known waveforms. If we apply the above mentioned methods directly, the estimation result may have a large bias and even some error.

Moreover, although the estimation problem for Doppler shifts and DOA angles can also be solved in the context of joint angle and frequency estimation [21]–[24], to our best knowledge, there has not been any method available which can exploit the known waveform information in the solution. To solve the problem, we first construct a new DOA estimation model incorporating the Doppler effect. Then, to avoid multidimensional spectrum peak search or nonlinear optimization, we transform the new model into an approximate linear model in digital frequency domain via discrete-time Fourier transform (DTFT) and finite order Taylor series expansion. To handle the large number of components of different order derivatives, an oblique projector is employed with the aid of known waveforms. Finally, the DOAs, complex amplitudes and Doppler shifts are obtained via the inherent relationship of these derivatives simultaneously. Simulation results show that the proposed method can achieve a much more robust estimation performance than the DEML method [11] in the presence of unknown Doppler shifts.

This remaining part of the paper is organised as follows. In Section 2, the studied signal model is introduced, while the proposed estimation method is derived in Section 3. Simulation results are provided in Section 4 and conclusions are drawn in Section 5.

Notations: matrices and vectors are denoted by boldfaced capital letters and lower-case letters, respectively. $(\cdot)^*$, $(\cdot)^T$ $(\cdot)^H$, $(\cdot)^{-1}$, and $(\cdot)^\dagger$ stand for conjugate, transpose, conjugate transpose, inverse, and Moore-Penrose inverse, respectively. $E\{\cdot\}$, $\circ$, $\mathrm{diag}\{\cdot\}$, $\mathrm{Re}\{\cdot\}$, and $\mathrm{Im}\{\cdot\}$ denote the statistical expectation, Hadamard product, diagonalization, real part and imaginary part of a complex number, respectively.

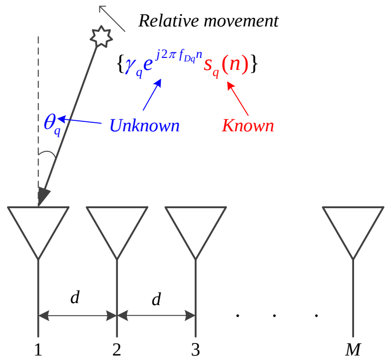


Fig. 1. Configuration for DOA estimation in the presence of Doppler effect.

II. Signal Model

As shown in Fig. 1, an M-element uniform linear array (ULA) with inter-sensor spacing d is placed on a high speed moving platform. Q narrowband far-field uncorrelated sources with known waveforms $\{s_q(n)\}_{q=1}^{Q}$ $(n = 0, \cdots, N-1,\ N$ is the number of snapshots) of wavelength λ from distinct directions $\{\theta_q\}_{q=1}^{Q}$ (unknown) impinge on the array. The signal received by the mth element $(m = 1, \cdots, M)$ can be expressed as

$$x_m(n) = \sum_{q=1}^{Q} a_m(\theta_q)\gamma_q e^{j2\pi f_{Dq}n} s_q(n) + w_m(n) \tag{1}$$

where $a_m(\theta_q) = \exp[-j2\pi(m-1)d\sin\theta_q/\lambda]$, γ_q denotes the complex amplitude of the received qth known waveform signal, f_{Dq} denotes the Doppler shift of the qth signal resulting from the relative movement of the source to the ULA, [1] and $w_m(n)$ represents the noise.

The received signal vector of the ULA at the nth snapshot $\mathbf{x}(n)$ can be represented by

$$\mathbf{x}(n) = \mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\gamma})(\mathbf{s}_D(\mathbf{f}_D, n) \circ \mathbf{s}(n)) + \mathbf{w}(n) \tag{2}$$

where

$$\begin{aligned}
\mathbf{x}(n) &= [x_1(n), x_2(n), \cdots, x_M(n)]^T, \\
\mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\gamma}) &= \mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Gamma}(\boldsymbol{\gamma}), \\
\mathbf{s}_D(\mathbf{f}_D, n) &= [e^{j2\pi f_{D1}n}, e^{j2\pi f_{D2}n}, \cdots, e^{j2\pi f_{DQ}n}]^T, \\
\mathbf{s}(n) &= [s_1(n), s_2(n), \cdots, s_Q(n)]^T, \\
\mathbf{w}(n) &= [w_1(n), w_2(n), \cdots, w_M(n)]^T, \\
\mathbf{A}(\boldsymbol{\theta}) &= [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \cdots, \mathbf{a}(\theta_Q)], \\
\boldsymbol{\Gamma}(\boldsymbol{\gamma}) &= \mathrm{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_Q\}, \\
\mathbf{a}(\theta_q) &= [a_1(\theta_q), a_2(\theta_q), \cdots, a_M(\theta_q)]^T, \\
\boldsymbol{\theta} &= [\theta_1, \theta_2, \cdots, \theta_Q]^T. \\
\mathbf{f}_D &= [f_{D1}, f_{D2}, \cdots, f_{DQ}]^T,
\end{aligned}$$

Similar to [11], it is assumed that the additive noises are temporally and spatially white with zero-mean and variance σ_w^2, and are uncorrelated with the incident signals.

[1] The Doppler shift here is digital and can be easily transformed into analog via multiplication by the sampling rate.

III. Proposed Method

A. Proposed Method

Since the unknown Doppler shifts are nonlinearly mixed with the known waveforms, the existing methods, such as DEML [11], SB [12], and LR [15], cannot handle this problem effectively. According to the statistical parameter estimation theory [25], we can use the maximum likelihood (ML) method to solve this problem. However, a multidimensional search is needed, leading to extremely high computationsal complexity. To estimate DOAs in the presence of the unknown Doppler shifts with low computational complexity, we first transform the mth element received signal $x_m(n)$ into the digital frequency domain via discrete-time Fourier transform (DTFT) as follows,

$$\begin{aligned}
\tilde{x}_m(\omega) &= \sum_{n=-\infty}^{+\infty} x_m(n)e^{-j\omega n} \\
&= \sum_{q=1}^{Q} a_m(\theta_q)\gamma_q \sum_{n=0}^{N-1} e^{j2\pi f_{Dq}n} s_q(n)e^{-j\omega n} + \tilde{w}_m(\omega) \\
&= \sum_{q=1}^{Q} a_m(\theta_q)\gamma_q \sum_{n=0}^{N-1} s_q(n)e^{-j(\omega - 2\pi f_{Dq})n} + \tilde{w}_m(\omega) \\
&= \sum_{q=1}^{Q} a_m(\theta_q)\gamma_q \tilde{s}_q(\omega - 2\pi f_{Dq}) + \tilde{w}_m(\omega)
\end{aligned} \tag{3}$$

where $\omega \in [0, 2\pi)$, $\tilde{s}_q(\omega) = \sum_{n=0}^{N-1} s_q(n)e^{-j\omega n}$, $\tilde{w}_m(\omega) = \sum_{n=0}^{N-1} w_m(n)e^{-j\omega n}$.

From Eq. (3), $\tilde{s}_q(\omega - 2\pi f_{Dq})$ involves f_{Dq} and cannot be separated linearly. Since f_{Dq} is often much smaller than the frequency resolution $1/N$, we can approximate $\tilde{s}_q(\omega - 2\pi f_{Dq})$ with the Pth order Taylor series expansion around ω as follows, [2]

$$\tilde{s}_q(\omega - 2\pi f_{Dq}) \approx \sum_{p=0}^{P} \frac{\tilde{s}_q^{(p)}(\omega)}{p!}(-2\pi f_{Dq})^p \tag{4}$$

where $\tilde{s}_q^{(p)}(\omega)$ and ! denote the pth order derivative of $\tilde{s}_q(\omega)$ and the factorial operation, respectively. (See Appendix A for the calculation of $\tilde{s}_q^{(p)}(\omega)$.)

Obviously, Eq. (3) can be expressed approximately as

$$\tilde{x}_m(\omega) \approx \sum_{q=1}^{Q} a_m(\theta_q)\gamma_q \sum_{p=0}^{P} \frac{\tilde{s}_q^{(p)}(\omega)}{p!}(-2\pi f_{Dq})^p + \tilde{w}_m(\omega) \tag{5}$$

For numerical realization, we discretize ω in the manner of DFT, i.e., $\omega_k = 2\pi k/N$ with $k = 0, 1, \cdots, N-1$. Then, $\tilde{x}_m(\omega_k) = \tilde{x}_m(2\pi k/N)$, $\tilde{s}_q^{(p)}(\omega_k) = \tilde{s}_q^{(p)}(2\pi k/N)$, $\tilde{w}_m(\omega_k) = \tilde{w}_m(2\pi k/N)$. Without causing confusion and for simplicity, we use $\tilde{x}_m(k)$, $\tilde{s}_q^{(p)}(k)$, and $\tilde{w}_m(k)$ to denote them.

With Eqs. (2) and (5), we can express $\mathbf{x}(n)$ in the frequency domain with the Pth order Taylor series expansion as

$$\tilde{\mathbf{x}}(k) \approx \mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\gamma}) \sum_{p=0}^{P} \frac{(-2\pi)^p}{p!} \mathbf{F}_D^p \tilde{\mathbf{s}}^{(p)}(k) + \tilde{\mathbf{w}}(k) \tag{6}$$

[2] The choice of P depends on the application and is related to the relative value of Doppler shift to frequency resolution.

where $\tilde{\mathbf{x}}(k)$ is the simplification of $\tilde{\mathbf{x}}(\omega_k)$. $\tilde{\mathbf{s}}^{(p)}(k) = [\tilde{s}_1^{(p)}(k), \tilde{s}_2^{(p)}(k), \cdots, \tilde{s}_Q^{(p)}(k)]^T$, $\mathbf{F}_D = \mathrm{diag}\{f_{D1}, f_{D2}, \cdots, f_{DQ}\}$. The superscript p for $\mathbf{F}_D$ denotes its pth power.

Eq. (6) can be written in a matrix form compactly as follows,

$$\tilde{\mathbf{X}} = \mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\gamma})\mathbf{F}\tilde{\mathbf{S}} + \tilde{\mathbf{W}} \tag{7}$$

where

$$\begin{aligned}
\mathbf{F} &= [\mathbf{I}_Q, \mathbf{F}_D, \cdots, \mathbf{F}_D^P], \\
\tilde{\mathbf{S}} &= [\tilde{\mathbf{s}}(0), \tilde{\mathbf{s}}(1), \cdots, \tilde{\mathbf{s}}(N-1)], \\
\tilde{\mathbf{s}}(k) &= [(\tilde{\mathbf{s}}^{(0)}(k))^T, -2\pi(\tilde{\mathbf{s}}^{(1)}(k))^T, \cdots, \frac{(-2\pi)^p}{P!}(\tilde{\mathbf{s}}^{(P)}(k))^T]^T, \\
\tilde{\mathbf{W}} &= [\tilde{\mathbf{w}}(0), \tilde{\mathbf{w}}(1), \cdots, \tilde{\mathbf{w}}(N-1)].
\end{aligned}$$

Since $\tilde{\mathbf{s}}^{(p)}(k)$ is known, we can design an oblique projector $\mathbf{E}_p$ to separate the components corresponding to the pth order derivative of the expansion. According to [26], $\mathbf{E}_p$ can be designed as

$$\mathbf{E}_p = \tilde{\mathbf{S}}_p^T(\tilde{\mathbf{S}}_p^* \mathbf{P}_{\tilde{\mathbf{S}}_{rp}^T} \tilde{\mathbf{S}}_p^T)^{-1} \tilde{\mathbf{S}}_p^* \mathbf{P}_{\tilde{\mathbf{S}}_{rp}^T} \tag{8}$$

where $\tilde{\mathbf{S}}_p = \frac{(-2\pi)^p}{p!} \cdot [\tilde{\mathbf{s}}^{(p)}(0), \tilde{\mathbf{s}}^{(p)}(1), \cdots, \tilde{\mathbf{s}}^{(p)}(N-1)]$, $\mathbf{P}_{\tilde{\mathbf{S}}_{rp}^T} = \mathbf{I}_N - \tilde{\mathbf{S}}_{rp}^T(\tilde{\mathbf{S}}_{rp}^T)^\dagger$, and $\tilde{\mathbf{S}}_{rp} = [\tilde{\mathbf{S}}_0^T, \tilde{\mathbf{S}}_1^T, \cdots, \tilde{\mathbf{S}}_{p-1}^T, \tilde{\mathbf{S}}_{p+1}^T, \cdots, \tilde{\mathbf{S}}_P^T]^T$.

Then, we have [3]

$$\begin{aligned}
\tilde{\mathbf{X}}_p^{save} &= \mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\gamma})\mathbf{F}\tilde{\mathbf{S}}\mathbf{E}_p + \tilde{\mathbf{W}}\mathbf{E}_p \\
&= \mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\gamma})\mathbf{F}_D^p\tilde{\mathbf{S}}_p + \tilde{\mathbf{W}}\mathbf{E}_p
\end{aligned} \tag{9}$$

We can use the known $\tilde{\mathbf{S}}_p$ to calculate the following,

$$\hat{\mathbf{B}}_p = \tilde{\mathbf{X}}_p^{save}\tilde{\mathbf{S}}_p^\dagger \tag{10}$$

where $\mathbf{B}_p = \mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\gamma})\mathbf{F}_D^p$.

Repeat the process of Eq. (9)-(10) $(P+1)$ times, we can obtain $\hat{\mathbf{B}}_0$, $\hat{\mathbf{B}}_1$, $\cdots$, $\hat{\mathbf{B}}_P$.

Therefore, the DOAs and Doppler shifts can be estimated as

$$\hat{\theta}_q = \arcsin\{\frac{-\lambda}{2\pi d} \cdot \mathrm{angle}[\frac{1}{(P+1)(M-1)} \cdot \sum_{p=0}^{P}\sum_{m=1}^{M-1} \frac{\hat{\mathbf{B}}_p(m+1, q)}{\hat{\mathbf{B}}_p(m, q)}]\} \tag{11}$$

$$\hat{f}_{Dq} = \frac{1}{PM}\sum_{p=0}^{P-1}\sum_{m=1}^{M} \frac{\hat{\mathbf{B}}_{p+1}(m, q)}{\hat{\mathbf{B}}_p(m, q)} \tag{12}$$

where $\hat{\mathbf{B}}_p(m, q)$ denotes the (p, q)th element of $\hat{\mathbf{B}}_p$. $\arcsin\{\cdot\}$ and $\mathrm{angle}[\cdot]$ represents the arcsine value of a real number and the phase angle of a complex number.

With $\{\hat{\theta}_q\}_{q=1}^Q$, we can obtain $\{\mathbf{a}(\hat{\theta}_q)\}_{q=1}^Q$ and $\hat{\mathbf{A}} = [\mathbf{a}(\hat{\theta}_1), \mathbf{a}(\hat{\theta}_2), \cdots, \mathbf{a}(\hat{\theta}_Q)]$. Then, the estimations of complex amplitude γ_q $(q = 1, 2, \cdots, Q)$ can be calculated as follows,

$$\hat{\gamma}_q = \frac{1}{M}\sum_{m=1}^{M} \frac{\hat{\mathbf{B}}_0(m, q)}{\hat{\mathbf{A}}(m, q)} \tag{13}$$

[3] Since it can be proved that $\tilde{\mathbf{s}}^{(p)}(k)$ and $\tilde{\mathbf{s}}^{(q)}(k)$ $(p \neq q)$ are correlated, i.e., $E\{\tilde{\mathbf{s}}^{(p)}(k)\tilde{\mathbf{s}}^{(q)}(k)\} \neq 0$, the oblique projector instead of the orthogonal projector is applied in Eq.(9).

B. Computational Complexity Analysis

In this subsection, we analyse the computational complexity of the proposed method compared with the DEML method [11] in terms of the number of complex-valued multiplications.

For the proposed method, it consists of

(i) DFT using (6): $O\{MN\log_2 N\}$,

(ii) oblique projector construction via (8): $O\{(P+1)[N^3 + (P+2)QN^2 + (P^2+2)Q^2N + Q^3]\}$,

(iii) $\tilde{\mathbf{X}}_p^{save}$ and $\mathbf{B}_p$ estimation with (9)-(10): $O\{(P+1)(MN^2 + QMN + Q^2N)\}$,

(iv) DOA, complex amplitude, and Doppler shift estimation using (11)-(13): $O\{2PQM + QM\}$.

Then, with $N \gg M > Q$ and P being small for conventional applications (see Section IV for details), the overall computational complexity of the proposed method can be approximately expressed as $O\{MN\log_2 N + (P+1)N^3 + (P+1)(P+2)QN^2 + (P+1)QMN\}$.

Similarly, for the DEML method, it includes

(i) $\hat{\mathbf{B}}$ and $\hat{\mathbf{Q}}$ estimation using (17)-(18) in [11]: $O\{M^2N + QMN + Q^2(N+3M) + 2Q^3\}$,

(ii) DOA and complex amplitude estimation using Eq.(24)-(25) in [11]: $O\{2N_\theta M^2 + M^3\}$, where N_θ denotes the number of angle searches.

For conventional case, i.e., $N_\theta \approx N \gg M > Q$, its overall computational complexity is about $O\{2N_\theta M^2 + M^2N + QMN + M^3\}$.

According to the above analysis, the proposed method has a larger computational complexity than the DEML method owing to the oblique projector construction step.

Remark 1: N_θ and N are usually the same in terms of magnitude, since some simple search strategy such as in [27], can be applied to reduce the number of searches.

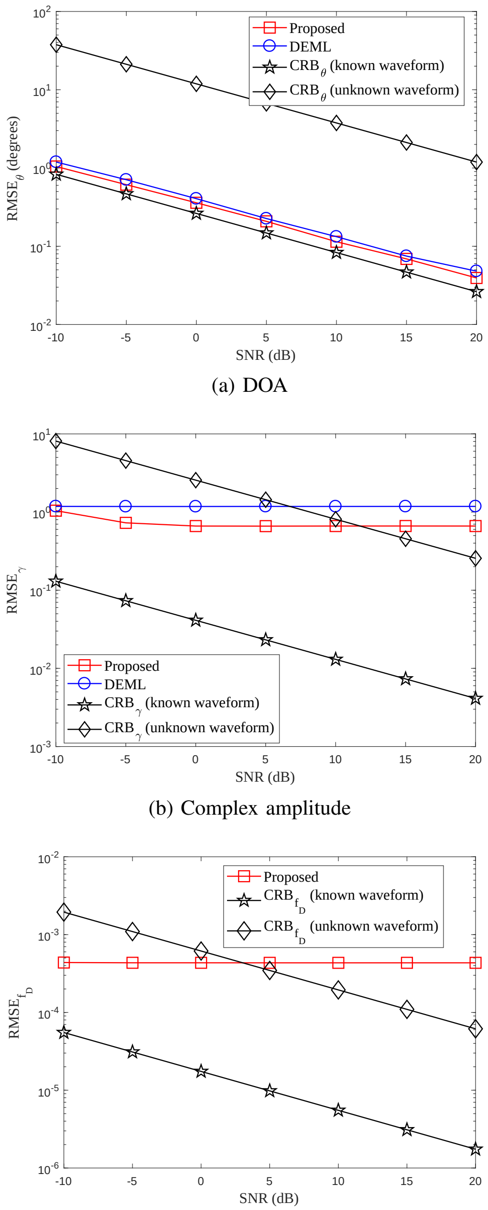
IV. Simulation Results

In this section, the performance of the proposed method is investigated in comparison with that of DEML [11] , and the Cramer-Rao bound (CRB) for known waveforms (see Appendix B for the derivation) and unknown waveforms [25], respectively. It is assumed that $d = \lambda/2$, and the waveforms of all sources are known with unit power. The angle search range for the DEML method is fixed as $[-90°, 90°]$ with an interval of $0.01°$.

Example 1: In the first example, we focus on the selection of the optimum order P of Taylor series expansion under different f_D and different number of snapshots. DOAs and complex amplitudes of two sources are set to $10°$, $12°$, $e^{j0.3\pi}$, and $e^{-j0.4\pi}$, respectively. For convenience, the two sources have the same Doppler shift f_D. With $M = 4$ and SNR = 10 dB, for each fixed f_D, N and P, 500 Monte Carlo trials are performed.

The optimum values of P are shown in Table I.

It can be seen that the smaller the value of f_D and N, the smaller the optimum P. The reason may be that with a larger P, the power of $\frac{(-2\pi)^P}{P!}\mathbf{F}_D^P\tilde{\mathbf{s}}^{(P)}(k)$ becomes smaller, which may be lower than that of noise, leading to a larger P used for the proposed method, and therefore a larger error occurs. Besides, if the optimum $P = 0$, the Doppler shifts cannot be

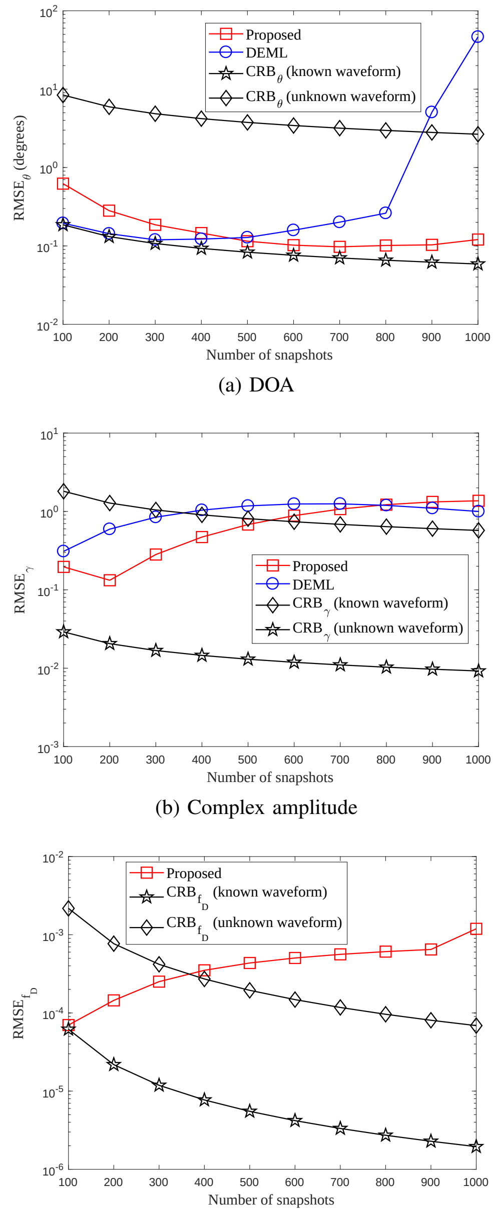


(a) DOA

(b) Complex amplitude

(c) Doppler shift

Fig. 2. RMSE versus SNR, K = 2, M = 4, N = 500.



(a) DOA

(b) Complex amplitude

(c) Doppler shift

Fig. 3. RMSE versus number of snapshots, K = 2, M = 4, SNR = 10 dB.

estimated. To avoid this problem, we can set $P = 1$ to balance the performance of DOA and Doppler shift estimation.

TABLE I
OPTIMUM P FOR DOA ESTIMATION UNDER DIFFERENT f_D AND N

	$f_D = 10^{-4}$	$f_D = 10^{-3}$	$f_D = 10^{-2}$	$f_D = 10^{-1}$
$N = 100$	0	0	2	> 50
$N = 500$	0	1	> 50	> 50
$N = 1000$	0	2	> 50	> 50

Example 2: In this example, the performance of the proposed method with respect to SNR is investigated. The DOAs, complex amplitudes and Doppler shifts of two sources are set to $10°$, $12°$, $e^{j0.3\pi}$, $e^{-j0.4\pi}$, 10^{-3} and 10^{-3}, respectively. With $M = 4$, $N = 500$, and $P = 1$, the input SNR varies from -15 dB to 30 dB with an interval of 5 dB. The root mean square error (RMSE) results are shown in Fig. 2.

Example 3: In this example, we examine the performance of the proposed method against the number of snapshots. The settings are the same as Example 2 except that SNR = 10 dB and N ranges from 100 to 1000 with an interval of 100. The estimation results are provided in Fig. 3.

For DOA estimation, as shown in Fig. 2a and 3a, the proposed method can work for values of N from 100 to 1000 effectively, while the DEML method can only work for $N \leq 800$ under the simulation conditions here, which shows that the proposed method is more robust to Doppler shifts for DOA estimation. Furthermore, the DOA estimation performance of the DEML method becomes worse as N increases, which is completely in contrast to the performance of conventional DOA estimation methods for sources with unknown waveforms. The reason is that for a fixed f_D, with the increase of N, the difference between true waveforms and known waveforms becomes larger, which results in worse DOA estimation performance. Besides, the DOA estimation RMSEs of the proposed method are always lower than the CRB with unknown waveforms, which proves the superiority of proposed method in comparison with conventional joint DOA, complex amplitude and Doppler shift estimation methods without prior information of the signal waveform.

In terms of complex amplitude estimation, from Fig. 2b and 3b, we can see that the proposed method and the DEML method have a similar estimation performance. Especially, for the proposed method, when N is small, such as $N \leq 500$, it outperforms the DEML method. That is to say, the proposed method has a robust complex amplitude estimation performance against Doppler shifts. In addition, compared with the CRB for unknown waveforms, the proposed method has a better performance for lower SNR and a smaller number of snapshots. When the SNR and the number of snapshots increase, the error resulting from the finite order Taylor series expansion approximation dominates, which degrades the estimation performance.

For the estimation of Doppler shifts, according to Figs. 2c and 3c, the performance mainly depends on the ratio of Doppler shift to frequency resolution (i.e., $f_D/(1/N) = f_D N$), showing an approximately negative relationship, i.e., when $f_D N$ is very small, the proposed method has a good Doppler shift estimation performance, which may be helpful for velocity measurement and is a great advantage of the proposed method. Moreover, due to the same reason, the estimation performance for Doppler shifts using the proposed method still has a similar problem as that of complex amplitude estimation in comparison with the CRB of sources with unknown waveforms.

V. Conclusions

A robust DOA estimation method for sources with known waveforms in the presence of unknown Doppler shifts has been introduced. It first transforms the nonlinear model including Doppler shifts into an approximately linear one using DTFT and Taylor series expansion; then, with the known waveforms and their derivatives, components corresponding to derivatives of different order are separated via a series of oblique projectors; finally, DOAs, complex amplitudes and Doppler shifts of the impinging signals are estimated simultaneously. As demonstrated by simulation results, the proposed method has a robust DOA estimation performance against Doppler shifts in comparison with algorithms available for known waveforms which do not take the Doppler effect into consideration. However, further research is needed in the future regarding its performance in Doppler shift estimation.

Appendix A
Calculation of the pth order derivative of $\tilde{s}_q(\omega)$

For $p = 1$, we have

$$\begin{aligned}\tilde{s}_q^{(1)}(\omega) &= \frac{\partial}{\partial\omega}\tilde{s}_q(\omega) = \frac{\partial}{\partial\omega}\sum_{n=0}^{N-1} s_q(n)e^{-j\omega n} \\ &= \sum_{n=0}^{N-1}[-jn \cdot s_q(n)]e^{-j\omega n}\end{aligned} \tag{14}$$

With Eq.(14), for $p = 2$, it can be derived that

$$\begin{aligned}\tilde{s}_q^{(2)}(\omega) &= \frac{\partial^2}{\partial\omega^2}\tilde{s}_q(\omega) = \frac{\partial}{\partial\omega}\tilde{s}_q^{(1)}(\omega) \\ &= \sum_{n=0}^{N-1}[(-jn)^2 \cdot s_q(n)]e^{-j\omega n}\end{aligned} \tag{15}$$

Hence, we can conclude that for any integer p,

$$\tilde{s}_q^{(p)}(\omega) = \sum_{n=0}^{N-1}[(-jn)^p \cdot s_q(n)]e^{-j\omega n} \tag{16}$$

It is noticed that for $p = 0$, $\tilde{s}_q^{(0)}(\omega)$ is the DTFT of $s_q(n)$.

Appendix B
Derivation of the Cramer-Rao bound

To obtain the CRB, we collect all real-valued unknown variables of the model in Eq. (2) into a vector as

$$\boldsymbol{\mu} = [\boldsymbol{\theta}^T, \boldsymbol{\xi}^T, \boldsymbol{\eta}^T, \mathbf{f}_D^T]^T \tag{17}$$

where $\boldsymbol{\theta} = [\theta_1, \cdots, \theta_Q]^T$, $\boldsymbol{\xi} = [\xi_1, \cdots, \xi_Q]^T = [\mathrm{Re}(\gamma_1), \cdots, \mathrm{Re}(\gamma_Q)]^T$, $\boldsymbol{\eta} = [\xi_1, \cdots, \xi_Q]^T = [\mathrm{Im}(\gamma_1), \cdots, \mathrm{Im}(\gamma_Q)]^T$, $\mathbf{f}_D = [f_{D1}, f_{D2}, \cdots, f_{DQ}]^T$.

For simplicity, $\mathbf{A}(\boldsymbol{\theta})$ and $\boldsymbol{\Gamma}(\boldsymbol{\gamma})$ are denoted as $\mathbf{A}$ and $\boldsymbol{\Gamma}$. Besides, $\widehat{\mathbf{s}}(n) = \mathbf{s}_D(\mathbf{f}_D, n) \circ \mathbf{s}(n)$, and $\mathbf{x}_0(n) = \mathbf{A}\boldsymbol{\Gamma}\widehat{\mathbf{s}}(n)$.

According to [25], when the noise is white guassian, the Fisher information matrix can be calculated with the gradient of $\mathbf{x}_0(n)$ with respect to $\boldsymbol{\mu}$,

$$\mathbf{I}(\boldsymbol{\mu}) = \frac{2}{\sigma_w^2}\mathrm{Re}\left(\sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\mu})\mathbf{D}_n(\boldsymbol{\mu})\right) \tag{18}$$

where

$$\begin{aligned}
\mathbf{D}_n(\boldsymbol{\mu}) &= [\mathbf{D}_n(\boldsymbol{\theta}), \mathbf{D}_n(\boldsymbol{\xi}), \mathbf{D}_n(\boldsymbol{\eta}), \mathbf{D}_n(\mathbf{f}_D)], \\
\mathbf{D}_n(\boldsymbol{\theta}) &= [\frac{\partial \mathbf{x}_0(n)}{\partial\theta_1}, \cdots, \frac{\partial \mathbf{x}_0(n)}{\partial\theta_Q}] = [\frac{\partial \mathbf{A}}{\partial\theta_1}\boldsymbol{\Gamma}\widehat{\mathbf{s}}(n), \cdots, \frac{\partial \mathbf{A}}{\partial\theta_Q}\boldsymbol{\Gamma}\widehat{\mathbf{s}}(n)], \\
\mathbf{D}_n(\boldsymbol{\xi}) &= [\frac{\partial \mathbf{x}_0(n)}{\partial\xi_1}, \cdots, \frac{\partial \mathbf{x}_0(n)}{\partial\xi_Q}] = [\mathbf{A}\frac{\partial \boldsymbol{\Gamma}}{\partial\xi_1}\widehat{\mathbf{s}}(n), \cdots, \mathbf{A}\frac{\partial \boldsymbol{\Gamma}}{\partial\xi_Q}\widehat{\mathbf{s}}(n)], \\
\mathbf{D}_n(\boldsymbol{\eta}) &= [\frac{\partial \mathbf{x}_0(n)}{\partial\eta_1}, \cdots, \frac{\partial \mathbf{x}_0(n)}{\partial\eta_Q}] = [\mathbf{A}\frac{\partial \boldsymbol{\Gamma}}{\partial\eta_1}\widehat{\mathbf{s}}(n), \cdots, \mathbf{A}\frac{\partial \boldsymbol{\Gamma}}{\partial\eta_Q}\widehat{\mathbf{s}}(n)], \\
\mathbf{D}_n(\mathbf{f}_D) &= [\frac{\partial \mathbf{x}_0(n)}{\partial f_{D1}}, \cdots, \frac{\partial \mathbf{x}_0(n)}{\partial f_{DQ}}] = [\mathbf{A}\boldsymbol{\Gamma}\frac{\partial \widehat{\mathbf{s}}(n)}{\partial f_{D1}}, \cdots, \mathbf{A}\boldsymbol{\Gamma}\frac{\partial \widehat{\mathbf{s}}(n)}{\partial f_{DQ}}].
\end{aligned}$$

$\mathbf{I}(\boldsymbol{\mu})$ in (18) can be expressed compactly in matrix form as follows,

$$\mathbf{I}(\boldsymbol{\mu}) = \frac{2}{\sigma_w^2}\mathrm{Re}\left(\begin{bmatrix} \mathbf{I}_{\boldsymbol{\theta}\boldsymbol{\theta}} & \mathbf{I}_{\boldsymbol{\theta}\boldsymbol{\xi}} & \mathbf{I}_{\boldsymbol{\theta}\boldsymbol{\eta}} & \mathbf{I}_{\boldsymbol{\theta}\mathbf{f}_D} \\ \mathbf{I}_{\boldsymbol{\xi}\boldsymbol{\theta}} & \mathbf{I}_{\boldsymbol{\xi}\boldsymbol{\xi}} & \mathbf{I}_{\boldsymbol{\xi}\boldsymbol{\eta}} & \mathbf{I}_{\boldsymbol{\xi}\mathbf{f}_D} \\ \mathbf{I}_{\boldsymbol{\eta}\boldsymbol{\theta}} & \mathbf{I}_{\boldsymbol{\eta}\boldsymbol{\xi}} & \mathbf{I}_{\boldsymbol{\eta}\boldsymbol{\eta}} & \mathbf{I}_{\boldsymbol{\eta}\mathbf{f}_D} \\ \mathbf{I}_{\mathbf{f}_D\boldsymbol{\theta}} & \mathbf{I}_{\mathbf{f}_D\boldsymbol{\xi}} & \mathbf{I}_{\mathbf{f}_D\boldsymbol{\eta}} & \mathbf{I}_{\mathbf{f}_D\mathbf{f}_D} \end{bmatrix}\right) \tag{19}$$

where $\mathbf{I}_{\boldsymbol{\theta}\boldsymbol{\theta}} = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\theta})\mathbf{D}_n(\boldsymbol{\theta})$, $\mathbf{I}_{\boldsymbol{\theta}\boldsymbol{\xi}} = \mathbf{I}_{\boldsymbol{\xi}\boldsymbol{\theta}}^H = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\theta})\mathbf{D}_n(\boldsymbol{\xi})$, $\mathbf{I}_{\boldsymbol{\theta}\boldsymbol{\eta}} = \mathbf{I}_{\boldsymbol{\eta}\boldsymbol{\theta}}^H = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\theta})\mathbf{D}_n(\boldsymbol{\eta})$, $\mathbf{I}_{\boldsymbol{\theta}\mathbf{f}_D} = \mathbf{I}_{\mathbf{f}_D\boldsymbol{\theta}}^H = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\theta})\mathbf{D}_n(\mathbf{f}_D)$, $\mathbf{I}_{\boldsymbol{\xi}\boldsymbol{\xi}} = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\xi})\mathbf{D}_n(\boldsymbol{\xi})$, $\mathbf{I}_{\boldsymbol{\xi}\boldsymbol{\eta}} = \mathbf{I}_{\boldsymbol{\eta}\boldsymbol{\xi}}^H = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\xi})\mathbf{D}_n(\boldsymbol{\eta})$, $\mathbf{I}_{\boldsymbol{\xi}\mathbf{f}_D} = \mathbf{I}_{\mathbf{f}_D\boldsymbol{\xi}}^H = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\xi})\mathbf{D}_n(\mathbf{f}_D)$, $\mathbf{I}_{\boldsymbol{\eta}\boldsymbol{\eta}} = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\eta})\mathbf{D}_n(\boldsymbol{\eta})$, $\mathbf{I}_{\boldsymbol{\eta}\mathbf{f}_D} = \mathbf{I}_{\mathbf{f}_D\boldsymbol{\eta}}^H = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\boldsymbol{\eta})\mathbf{D}_n(\mathbf{f}_D)$, and $\mathbf{I}_{\mathbf{f}_D\mathbf{f}_D} = \sum_{n=0}^{N-1}\mathbf{D}_n^H(\mathbf{f}_D)\mathbf{D}_n(\mathbf{f}_D)$.

To derive the analytical expressions of $\mathbf{I}_{\boldsymbol{\theta\theta}}$, $\mathbf{I}_{\boldsymbol{\theta\xi}}$,$\mathbf{I}_{\boldsymbol{\theta\eta}}$, $\mathbf{I}_{\boldsymbol{\theta}\mathbf{f}_D}$, $\mathbf{I}_{\boldsymbol{\xi\xi}}$, $\mathbf{I}_{\boldsymbol{\xi\eta}}$, $\mathbf{I}_{\boldsymbol{\xi}\mathbf{f}_D}$, $\mathbf{I}_{\boldsymbol{\eta\eta}}$, $\mathbf{I}_{\boldsymbol{\eta}\mathbf{f}_D}$ and $\mathbf{I}_{\mathbf{f}_D\mathbf{f}_D}$, respectively, we calculate the (p,q)th element of them firstly, as follows

$$\begin{aligned}\mathbf{I}_{\theta_p\theta_q} &= \sum\nolimits_{n=0}^{N-1}\widehat{\mathbf{s}}^H(n)\boldsymbol{\Gamma}^H\dot{\bar{\mathbf{A}}}_{\theta_p}^H\dot{\bar{\mathbf{A}}}_{\theta_q}\boldsymbol{\Gamma}\widehat{\mathbf{s}}(n)\\ &= N\cdot\mathrm{tr}\{\boldsymbol{\Gamma}^H\dot{\bar{\mathbf{A}}}_{\theta_p}^H\dot{\bar{\mathbf{A}}}_{\theta_q}\boldsymbol{\Gamma}\mathbf{R}_{\widehat{s}\widehat{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\boldsymbol{\Gamma}^H\dot{\mathbf{A}}^H\dot{\mathbf{A}}\boldsymbol{\Gamma}\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\widehat{s}\widehat{s}}^T\mathbf{e}_q) \qquad (20)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\theta_p\xi_q} &= \sum\nolimits_{n=0}^{N-1}\widehat{\mathbf{s}}^H(n)\boldsymbol{\Gamma}^H\dot{\bar{\mathbf{A}}}_{\theta_p}^H\mathbf{A}\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_q}\widehat{\mathbf{s}}(n)\\ &= N\cdot\mathrm{tr}\{\boldsymbol{\Gamma}^H\dot{\bar{\mathbf{A}}}_{\theta_p}^H\mathbf{A}\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_q}\mathbf{R}_{\widehat{s}\widehat{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\boldsymbol{\Gamma}^H\dot{\mathbf{A}}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_\xi\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\widehat{s}\widehat{s}}^T\mathbf{e}_q) \qquad (21)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\theta_p\eta_q} &= \sum\nolimits_{n=0}^{N-1}\widehat{\mathbf{s}}^H(n)\boldsymbol{\Gamma}^H\dot{\bar{\mathbf{A}}}_{\theta_p}^H\mathbf{A}\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_q}\widehat{\mathbf{s}}(n)\\ &= N\cdot\mathrm{tr}\{\boldsymbol{\Gamma}^H\dot{\bar{\mathbf{A}}}_{\theta_p}^H\mathbf{A}\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_q}\mathbf{R}_{\widehat{s}\widehat{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\boldsymbol{\Gamma}^H\dot{\mathbf{A}}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_\eta\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\widehat{s}\widehat{s}}^T\mathbf{e}_q) \qquad (22)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\theta_p f_{Dq}} &= \sum\nolimits_{n=0}^{N-1}\widehat{\mathbf{s}}^H(n)\boldsymbol{\Gamma}^H\dot{\bar{\mathbf{A}}}_{\theta_p}^H\mathbf{A}\boldsymbol{\Gamma}\frac{\partial\widehat{\mathbf{s}}(n)}{\partial f_{Dq}}\\ &= N\cdot\mathrm{tr}\{\boldsymbol{\Gamma}^H\dot{\bar{\mathbf{A}}}_{\theta_p}^H\mathbf{A}\boldsymbol{\Gamma}\mathbf{E}_q\mathbf{R}_{\breve{s}\widehat{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\boldsymbol{\Gamma}^H\dot{\mathbf{A}}^H\mathbf{A}\boldsymbol{\Gamma}\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\breve{s}\widehat{s}}^T\mathbf{e}_q) \qquad (23)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\xi_p\xi_q} &= \sum\nolimits_{n=0}^{N-1}\widehat{\mathbf{s}}^H(n)\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_p}^H\mathbf{A}^H\mathbf{A}\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_q}\widehat{\mathbf{s}}(n)\\ &= N\cdot\mathrm{tr}\{\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_p}^H\mathbf{A}^H\mathbf{A}\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_q}\mathbf{R}_{\widehat{s}\widehat{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\dot{\boldsymbol{\Gamma}}_\xi^H\mathbf{A}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_\xi\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\widehat{s}\widehat{s}}^T\mathbf{e}_q) \qquad (24)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\xi_p\eta_q} &= \sum\nolimits_{n=0}^{N-1}\widehat{\mathbf{s}}^H(n)\dot{\boldsymbol{\Gamma}}_{\xi_p}^H\mathbf{A}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_{\eta_q}\widehat{\mathbf{s}}(n)\\ &= N\cdot\mathrm{tr}\{\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_p}^H\mathbf{A}^H\mathbf{A}\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_q}\mathbf{R}_{\widehat{s}\widehat{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\dot{\boldsymbol{\Gamma}}_\xi^H\mathbf{A}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_\eta\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\widehat{s}\widehat{s}}^T\mathbf{e}_q) \qquad (25)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\xi_p f_{Dq}} &= \sum\nolimits_{n=0}^{N-1}\widehat{\mathbf{s}}^H(n)\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_p}^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma}\frac{\partial\widehat{\mathbf{s}}(n)}{\partial f_{Dq}}\\ &= N\cdot\mathrm{tr}\{\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_p}^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma}\mathbf{E}_q\mathbf{R}_{\breve{s}\widehat{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\dot{\boldsymbol{\Gamma}}_\xi^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma}\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\breve{s}\widehat{s}}^T\mathbf{e}_q) \qquad (26)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\eta_p\eta_q} &= \sum\nolimits_{n=0}^{N-1}\widehat{\mathbf{s}}^H(n)\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_p}^H\mathbf{A}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_{\eta_q}\widehat{\mathbf{s}}(n)\\ &= N\cdot\mathrm{tr}\{\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_p}^H\mathbf{A}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_{\eta_q}\mathbf{R}_{\widehat{s}\widehat{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_p}^H\mathbf{A}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_{\eta_q}\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\widehat{s}\widehat{s}}^T\mathbf{e}_q) \qquad (27)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{\eta_p f_{Dq}} &= \sum\nolimits_{n=0}^{N-1}\widehat{\mathbf{s}}^H(n)\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_p}^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma}\frac{\partial\widehat{\mathbf{s}}(n)}{\partial f_{Dq}}\\ &= N\cdot\mathrm{tr}\{\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_p}^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma}\mathbf{E}_q\mathbf{R}_{\breve{s}\widehat{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\dot{\boldsymbol{\Gamma}}_\eta^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma}\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\breve{s}\widehat{s}}^T\mathbf{e}_q) \qquad (28)\end{aligned}$$

$$\begin{aligned}\mathbf{I}_{f_{Dp}f_{Dq}} &= \sum\nolimits_{n=0}^{N-1}\frac{\partial\widehat{\mathbf{s}}^H(n)}{\partial f_{Dp}}\boldsymbol{\Gamma}^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma}\frac{\partial\widehat{\mathbf{s}}(n)}{\partial f_{Dq}}\\ &= N\cdot\mathrm{tr}\{\mathbf{E}_p\boldsymbol{\Gamma}^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma}\mathbf{E}_q\mathbf{R}_{\breve{s}\breve{s}}\}\\ &= N\cdot(\mathbf{e}_p^T\boldsymbol{\Gamma}^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma}\mathbf{e}_q)\cdot(\mathbf{e}_p^T\mathbf{R}_{\breve{s}\breve{s}}^T\mathbf{e}_q) \qquad (29)\end{aligned}$$

where $\mathbf{R}_{\widehat{s}\widehat{s}} = 1/N\sum_{n=0}^{N-1}\widehat{\mathbf{s}}(n)\widehat{\mathbf{s}}^H(n)$, $\dot{\mathbf{A}} = [\frac{\partial\mathbf{a}(\theta_1)}{\partial\theta_1},\cdots,\frac{\partial\mathbf{a}(\theta_Q)}{\partial\theta_Q}]$, $\dot{\bar{\mathbf{A}}} = [\dot{\bar{\mathbf{A}}}_{\theta_1},\cdots,\dot{\bar{\mathbf{A}}}_{\theta_Q}] = [\frac{\partial\mathbf{A}}{\partial\theta_1},\cdots,\frac{\partial\mathbf{A}}{\partial\theta_Q}]$, $\dot{\bar{\mathbf{A}}}_{\theta_q} = \dot{\mathbf{A}}\mathbf{e}_q\mathbf{e}_q^T$,$\dot{\bar{\boldsymbol{\Gamma}}}_\xi = [\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_1},\cdots,\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_Q}] = [\frac{\partial\boldsymbol{\Gamma}}{\partial\xi_1},\cdots,\frac{\partial\boldsymbol{\Gamma}}{\partial\xi_Q}]$, $\dot{\bar{\boldsymbol{\Gamma}}}_\eta = [\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_1},\cdots,\dot{\bar{\boldsymbol{\Gamma}}}_{\eta_Q}] = [\frac{\partial\boldsymbol{\Gamma}}{\partial\eta_1},\cdots,\frac{\partial\boldsymbol{\Gamma}}{\partial\eta_Q}]$, $\dot{\boldsymbol{\Gamma}}_\xi = [\frac{\partial\gamma_1}{\partial\xi_1},\cdots,\frac{\partial\gamma_Q}{\partial\xi_q}]$, $\dot{\boldsymbol{\Gamma}}_\eta = [\frac{\partial\gamma_1}{\partial\xi_1},\cdots,\frac{\partial\gamma_Q}{\partial\xi_q}]$, $\dot{\bar{\boldsymbol{\Gamma}}}_{\xi_q} = \dot{\boldsymbol{\Gamma}}_\xi\mathbf{e}_q\mathbf{e}_q^T$, $\boldsymbol{\Gamma} = [\gamma_1,\cdots,\gamma_Q]$, $\frac{\partial\widehat{\mathbf{s}}(n)}{\partial f_{Dq}} = \mathbf{E}_q\breve{\mathbf{s}}(n)$, $\breve{\mathbf{s}}(n) = j2\pi n\widehat{\mathbf{s}}(n)$, $\mathbf{R}_{\breve{s}\breve{s}} = 1/N\sum_{n=0}^{N-1}\breve{\mathbf{s}}(n)\breve{\mathbf{s}}^H(n)$, $\mathbf{E}_q = \mathbf{e}_q\mathbf{e}_q^T$, $\mathbf{e}_q$ is the qth column of an identity matrix.

Hence,

$$\begin{aligned}
\mathbf{I}_{\boldsymbol{\theta\theta}} &= N\cdot(\boldsymbol{\Gamma}^H\dot{\mathbf{A}}^H\dot{\mathbf{A}}\boldsymbol{\Gamma})\circ\mathbf{R}_{\widehat{s}\widehat{s}}^T && (30)\\
\mathbf{I}_{\boldsymbol{\theta\xi}} &= N\cdot(\boldsymbol{\Gamma}^H\dot{\mathbf{A}}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_\xi)\circ\mathbf{R}_{\widehat{s}\widehat{s}}^T && (31)\\
\mathbf{I}_{\boldsymbol{\theta\eta}} &= N\cdot(\boldsymbol{\Gamma}^H\dot{\mathbf{A}}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_\eta)\circ\mathbf{R}_{\widehat{s}\widehat{s}}^T && (32)\\
\mathbf{I}_{\boldsymbol{\theta}\mathbf{f}_D} &= N\cdot(\boldsymbol{\Gamma}^H\dot{\mathbf{A}}^H\mathbf{A}\boldsymbol{\Gamma})\circ\mathbf{R}_{\breve{s}\widehat{s}}^T && (33)\\
\mathbf{I}_{\boldsymbol{\xi\xi}} &= N\cdot(\dot{\boldsymbol{\Gamma}}_\xi^H\mathbf{A}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_\xi)\circ\mathbf{R}_{\widehat{s}\widehat{s}}^T && (34)\\
\mathbf{I}_{\boldsymbol{\xi\eta}} &= N\cdot(\dot{\boldsymbol{\Gamma}}_\xi^H\mathbf{A}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_\eta)\circ\mathbf{R}_{\widehat{s}\widehat{s}}^T && (35)\\
\mathbf{I}_{\boldsymbol{\xi}\mathbf{f}_D} &= N\cdot(\dot{\boldsymbol{\Gamma}}_\xi^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma})\circ\mathbf{R}_{\breve{s}\widehat{s}}^T && (36)\\
\mathbf{I}_{\boldsymbol{\eta\eta}} &= N\cdot(\dot{\boldsymbol{\Gamma}}_\eta^H\mathbf{A}^H\mathbf{A}\dot{\boldsymbol{\Gamma}}_\eta)\circ\mathbf{R}_{\widehat{s}\widehat{s}}^T && (37)\\
\mathbf{I}_{\boldsymbol{\eta}\mathbf{f}_D} &= N\cdot(\dot{\boldsymbol{\Gamma}}_\eta^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma})\circ\mathbf{R}_{\breve{s}\widehat{s}}^T && (38)\\
\mathbf{I}_{\mathbf{f}_D\mathbf{f}_D} &= N\cdot(\boldsymbol{\Gamma}^H\mathbf{A}^H\mathbf{A}\boldsymbol{\Gamma})\circ\mathbf{R}_{\breve{s}\breve{s}}^T && (39)
\end{aligned}$$

Therefore, given the relationship between CRB and the Fisher information matrix, define $\boldsymbol{\Delta} = \mathbf{I}^{-1}(\boldsymbol{\mu})$, and consequently we have

$$\mathrm{CRB}_\theta = \sqrt{\frac{1}{Q}\sum_{q=1}^{Q}\boldsymbol{\Delta}_{q,q}} \qquad (40)$$

$$\mathrm{CRB}_\gamma = \sqrt{\frac{1}{Q}\sum_{q=1}^{Q}(\boldsymbol{\Delta}_{Q+q,Q+q}+\boldsymbol{\Delta}_{2Q+q,2Q+q})} \qquad (41)$$

$$\mathrm{CRB}_{f_D} = \sqrt{\frac{1}{Q}\sum_{q=1}^{Q}\boldsymbol{\Delta}_{3Q+q,3Q+q}} \qquad (42)$$

where $\mathrm{CRB}_{\mathrm{DOA}}$, CRB_γ, and CRB_{f_D} represent the absolute Cramer-Rao bounds for DOAs, complex amplitudes, and Doppler shifts, respectively. $\boldsymbol{\Delta}_{p,q}$ denotes the (p,q)th element of $\boldsymbol{\Delta}$.

References

[1] H. L. Van Trees, *Detection, estimation, and modulation theory, optimum array processing*. John Wiley & Sons, 2004.

[2] G.-Q. Zhao, *Principle of Radar Countermeasure*. Xidian University Publisher House, 2nd ed., 2012.

[3] Y. Luo, X. Xin, F. Du, X. Tang, and Y. Li, "Comparison of DOA algorithms applied to ultrasonic arrays for PD location in oil," *IEEE Sensors J.*, vol. 15, no. 4, pp. 2316–2323, 2015.

[4] J. Shi, G. Hu, B. Zong, and M. Chen, "DOA estimation using multipath echo power for MIMO radar in low-grazing angle," *IEEE Sensors J.*, vol. 16, no. 15, pp. 6087–6094, 2016.

[5] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propag.*, vol. 34, pp. 276–280, Mar 1986.

[6] R. Roy and T. Kailath, "ESPRIT-estimation of signal parameters via rotational invariance techniques," *IEEE Trans. Acoust., Speech, Signal Process.,*, vol. 37, no. 7, pp. 984–995, 1989.

[7] S. Marcos, A. Marsal, and M. Benidir, "The propagator method for source bearing estimation," *Signal Process.*, vol. 42, no. 2, pp. 121–138, 1995.

[8] J. Ward and J. R. T. Compton, "Improving the performance of a slotted ALOHA packet radio network with an adaptive array," *IEEE Trans. Commun.*, vol. 40, no. 2, pp. 292–300, 1992.

[9] A. Jakobsson, A. L. Swindlehurst, and P. Stoica, "Subspace-based estimation of time delays and doppler shifts," *IEEE Trans. Signal Proces.*, vol. 46, no. 9, pp. 2472–2483, 1998.

[10] J. Li and R. Compton, "Maximum likelihood angle estimation for signals with known waveforms," *IEEE Trans. Signal Process.*, vol. 41, no. 9, pp. 2850–2862, 1993.

[11] J. Li, B. Halder, P. Stoica, and M. Viberg, "Computationally efficient angle estimation for signals with known waveforms," *IEEE Trans. Signal Process.*, vol. 43, no. 9, pp. 2154–2163, 1995.

[12] N. Wang, P. Agathoklis, and A. Antoniou, "A new DOA estimation technique based on subarray beamforming," *IEEE Trans. Signal Process.*, vol. 54, no. 9, pp. 3279–3290, 2006.

[13] J.-F. Gu, S. C. Chan, W.-P. Zhu, and M. Swamy, "DOA estimation and tracking for signals with known waveform via symmetric sparse subarrays," in *2012 IEEE 55th Int. Midwest Symp. Circuits Syst. (MWSCAS)*, pp. 952–955, IEEE, 2012.

[14] J.-F. Gu, W.-P. Zhu, and M. Swamy, "Sparse linear arrays for estimating and tracking DOAs of signals with known waveforms," in *2013 IEEE Int. Symp. Circuits Syst. (ISCAS)*, pp. 2187–2190, IEEE, 2013.

[15] J.-F. Gu, W.-P. Zhu, and M. Swamy, "Fast and efficient DOA estimation method for signals with known waveforms using nonuniform linear arrays," *Signal Process.*, vol. 114, pp. 265–276, 2015.

[16] J.-F. Gu, W.-P. Zhu, and M. Swamy, "Direction of arrival tracking for signals with known waveforms based on block least squares techniques," *J. Frankl. Inst.*, vol. 354, no. 11, pp. 4573–4594, 2017.

[17] M. Cedervall and R. L. Moses, "Efficient maximum likelihood DOA estimation for signals with known waveforms in the presence of multipath," *IEEE Trans. Signal Process.*, vol. 45, no. 3, pp. 808–811, 1997.

[18] H. Li, G. Liu, and J. Li, "Angle estimator for signals with known waveforms," *Electron. Lett.*, vol. 35, no. 23, pp. 1992–1994, 1999.

[19] L. N. Atallah and S. Marcos, "DOA estimation and association of coherent multipaths by using reference signals," *Signal Process.*, vol. 84, no. 6, pp. 981–996, 2004.

[20] J.-F. Gu, P. Wei, and H.-M. Tai, "Fast direction-of-arrival estimation with known waveforms and linear operators," *IET Signal Process.*, vol. 2, no. 1, pp. 27–36, 2008.

[21] A. N. Lemma, A.-J. Van Der Veen, and E. F. Deprettere, "Analysis of joint angle-frequency estimation using ESPRIT," *IEEE Trans. Signal Process.*, vol. 51, no. 5, pp. 1264–1283, 2003.

[22] J.-D. Lin, W.-H. Fang, Y.-Y. Wang, and J.-T. Chen, "FSF MUSIC for joint DOA and frequency estimation and its performance analysis," *IEEE Trans. Signal Process.*, vol. 54, no. 12, pp. 4529–4542, 2006.

[23] H. Wang and S. Kay, "Maximum likelihood angle-Doppler estimator using importance sampling," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 46, no. 2, pp. 610–622, 2010.

[24] C. Cui, W. Wu, and W.-Q. Wang, "Carrier frequency and DOA estimation of sub-Nyquist sampling multi-band sensor signals," *IEEE Sensors J.*, vol. 17, no. 22, pp. 7470–7478, 2017.

[25] S. M. Kay, *Fundamentals of statistical signal processing*. Prentice Hall PTR, 1993.

[26] R. T. Behrens and L. L. Scharf, "Signal processing applications of oblique projection operators," *IEEE Trans. Signal Process.*, vol. 42, no. 6, pp. 1413–1424, 1994.

[27] D. Malioutov, M. Cetin, and A. S. Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays.," *IEEE Trans. Signal Process.*, vol. 53, no. 8, pp. 3010–3022, 2005.