# Graphs without large bicliques and well-quasi-orderability by the induced subgraph relation*

Aistis Atminas† Vadim V. Lozin‡ Igor Razgon§

**Abstract**

Recently, Daligault, Rao and Thomassé asked in [3] if every hereditary class which is well-quasi-ordered by the induced subgraph relation is of bounded clique-width. While the question has been shown to have a negative answer in general [9], in the present paper we show that the statement is true for a family of hereditary classes of graphs that exclude large bicliques as subgraphs. In particular, this implies (through the use of Courcelle theorem [2]) that any problem definable in Monadic Second Order Logic can be solved in a polynomial time for all well-quasi-ordered hereditary classes of graphs that exclude large bicliques.

*MSC codes:* 05C75 Structural characterization of families of graphs; 05C85 Graph algorithms.

## 1 Introduction

Well-quasi-ordering is a highly desirable property and a frequently discovered concept in mathematics and theoretical computer science [6, 8]. One of the most remarkable recent results in this area is the proof of Wagner's conjecture stating that the set of all finite graphs is well-quasi-ordered by the minor relation [12]. However, the subgraph or induced subgraph relation is not a well-quasi-order. On the other hand, each of these relations may become a well-quasi-order when restricted to graphs with some special properties.

A *graph property* (or a *class of graphs*) is a set of graphs closed under isomorphism. A property is *hereditary* if it is closed under taking induced subgraphs. It is well-known (and not difficult to see) that a graph property $X$ is hereditary if and only if $X$ can be described in terms of forbidden induced subgraphs. More formally, $X$ is hereditary if and only if there is a set $M$ of graphs such that no graph in $X$ contains any graph from $M$ as an induced subgraph. We call $M$ the set of *forbidden induced subgraphs* for $X$ and say that the graphs in $X$ are $M$-free.

Of our particular interest in this paper are graphs *without large bicliques*. We say that the graphs in a hereditary class $X$ are *without large bicliques* if there is a natural number $t$ such that no graph in $X$ contains $K_{t,t}$ as a (not necessarily induced) subgraph. Equivalently, there are $q$ and $r$ such $K_{q,q}$ and $K_r$ appear in the set of forbidden induced subgraphs for $X$. According to [11], these are precisely the graphs with a subquadratic number of edges. This

*This works was supported by EPSRC grant EP/L020408/1.

†Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK; Department of Computer Science and Information Systems, Birkbeck, University of London, E-mail: aistis.atminas@gmail.com

‡Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK. E-mail: V.Lozin@warwick.ac.uk.

§Department of Computer Science and Information Systems, Birkbeck, University of London, E-mail: igor@dc.bbk.ac.uk

family of properties includes many important classes, such as graphs of bounded vertex degree, of bounded tree-width, all proper minor closed graph classes. In all these examples, the number of edges is bounded by a linear function in the number of vertices and all of the listed properties are rather small (see e.g. [10] for the number of graphs in proper minor closed graph classes). In the terminology of [1], they all are at most factorial. In fact, the family of classes without large bicliques is much richer and contains classes with a superfactorial speed of growth, such as projective plane graphs (or more generally $C_4$-free bipartite graphs), in which case the number of edges is $\Theta(n^{\frac{3}{2}})$.

Recently, Daligault, Rao and Thomassé asked in [3] if every hereditary class which is well-quasi-ordered by the induced subgraph relation is of bounded clique-width. While the question has been shown to have a negative answer in general [9], the relationship holds true for some families of hereditary graph classes. Investigating such families is interesting because it connects two seemingly unrelated notions and leads to a strong algorithmic consequence. Indeed, it follows (through the use of Courcelle theorem [2]) that for such families any problem definable in Monadic Second Order Logic can be solved in a polynomial time on any class well-quasi-ordered by the induced subgraph relation.

In the present paper, we establish the relationship between well-quasi-ordering and boundedness of clique-width for graphs without large bicliques. More precisely, we prove that if a class $X$ without large bicliques is well-quasi-ordered by the induced subgraph relation, then the graphs in $X$ have bounded path-width, i.e. there is a constant $c$ such that the path-width of any graph in $X$ is at most $c$. Since bounded path-width implies bounded clique-width, the result affirmatively answers the question in [3] for graphs without large bicliques. Thus the above algorithmic consequence is confirmed e.g. for classes of graphs of bounded degree.

Section 2 contains all preliminary information related to the topic. In this section we define an infinite family of graphs pairwise incomparable by the induced subgraph relation, which we call *canonical graphs*. In Section 3 we prove our main combinatorial result, Theorem 1, stating that a graph without large bicliques and having a large path-width has a large induced canonical graph. A consequence of this result is that if a class $X$ without large bicliques has unbounded path-width, then $X$ contains an infinite subset of canonical graphs, i.e. an infinite antichain. This implies that classes of graphs without large bicliques that are well quasi-ordered by the induced subgraph relation must have bounded path-width.

## 2 Notation and definitions

In this work we will be using standard graph theory terminology and notation consistent with the book of Diestel [4]. In particular, $K_n$ and $P_n$ denote the complete graph and the chordless path with $n$ vertices, respectively, and $K_{n,m}$ stands for a complete bipartite graph with parts of size $n$ and $m$.

Throughout the text, whenever we say that $G$ contains $H$, we mean that $H$ is a subgraph of $G$, unless we explicitly say that $H$ is an *induced* subgraph of $G$ (or $G$ contains $H$ as an *induced* subgraph). If $H$ is not an induced subgraph of $G$, we say that $G$ is $H$-free. By $R = R(k, r, m)$, we denote the Ramsey number, i.e. the minimum $R$ such that in every colouring of $k$-subsets of an $R$-set with $r$ colours there is a monochromatic $m$-set, i.e. a set of $m$ elements all of whose $k$-subsets have the same colour.

According to the celebrated Graph Minor Theorem of Robertson and Seymour, the set of all graphs is well-quasi-ordered by the graph minor relation [12]. This, however, is not the case for the more restrictive relations such as subgraph or induced subgraph. Indeed, a sequence of graphs $H_1, H_2, ...$, creates an infinite antichain with respect to both relations, where $H_i$ is the graph represented in Figure 1.

By connecting two vertices of degree one having a common neighbour in $H_i$, we obtain a graph represented on the left of Figure 2. Let us denote this graph by $H_i'$. By further connecting the other pair of vertices of degree one we obtain the graph $H_i''$ represented on the right of Figure 2.

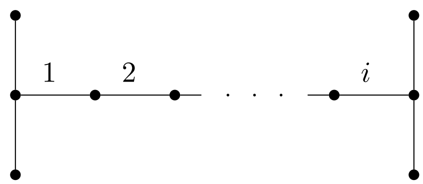


Figure 1: The graph $H_i$

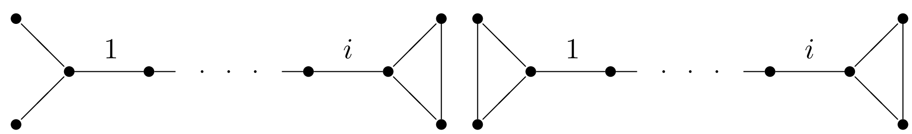


Figure 2: Graphs $H_i'$ and $H_i''$

We call any graph of the form $H_i$, $H_i'$ or $H_i''$ an $H$-graph. Furthermore, we will refer to $H_i''$ a *tight* $H$-graph and to $H_i'$ a *semi-tight* $H$-graph. In an $H$-graph, the path connecting two vertices of degree 3 will be called the *body* of the graph, and the vertices which are not in the body the *wings*.

Following standard graph theory terminology, we call a chordless cycle of length at least four a *hole*. Let us denote by

$\mathcal{C}$ the set of all holes and all $H$-graphs.

It is not difficult to see that any two distinct (i.e. non-isomorphic) graphs in $\mathcal{C}$ are incomparable with respect to the induced subgraph relation. In other words,

**Claim 1.** *$\mathcal{C}$ is an antichain with respect to the induced subgraph relation.*

Moreover, from the poof of Theorem 1 we will see that for classes of graphs without large bicliques which are of unbounded path-width this antichain is unavoidable, or *canonical*, in the terminology of [5]. Suggested by this observation, we introduce the following definition.

**Definition 1.** *The graphs in the set $\mathcal{C}$ will be called* CANONICAL.

The *order* of a canonical graph $G$ is either the number of its vertices, if $G$ is a hole, or the the number of vertices in its body, if $G$ is an $H$-graph.

## 3 Main result

In this section we prove the following theorem which is the main result of the paper.

**Theorem 1.** *If $X$ is a hereditary subclass of $(K_t, K_{q,q})$-free graphs which is well-quasi-ordered by the induced subgraph relation, then graphs in $X$ have a bounded path-width.*

To prove the theorem, we will show that a large path-width combined with the absence of large bicliques implies the existence of a large induced canonical graph, which is a much richer structural consequence than just the existence of a long induced path. An important part of showing the existence of a large canonical graph is verifying that its body (see Section 2 for the terminology) is induced. This will be done by application of the following theorem proved in [7].

**Theorem 2.** *For every $s$, $t$, and $q$, there is a number $Z = Z(s,t,q)$ such that every graph with a path of length at least $Z$ contains either $P_s$ or $K_t$ or $K_{q,q}$ as an induced subgraph.*

A plan of the proof of Theorem 1 is outlined in Section 3.1. Sections 3.2, 3.3, 3.4, 3.5 contain various parts of the proof.

### 3.1 Plan of the proof

To prove Theorem 1 we will show that graphs of arbitrarily large path-width contain either arbitrarily large bicliques as subgraphs or arbitrarily large canonical graphs as induced subgraphs. The main notion in our proof is that of a *rake-graph*.

A *rake-graph* (or simply a *rake*) consists of a chordless path, the *base* of the rake, and a number of pendant vertices, called *teeth*, each having a private neighbour on the base. The only neighbour of a tooth on the base will be called the *root* of the tooth, and a rake with $k$ teeth will be called a $k$-rake. We will say that a rake is *$\ell$-dense* if any $\ell$ consecutive vertices of the base contain at least one root vertex. An example of a 1-dense 9-rake is given in Figure 3.

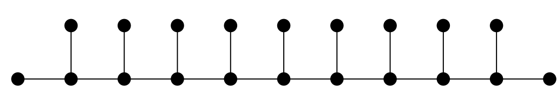

Figure 3: 1-dense 9-rake

We will prove Theorem 1 through a number of intermediate steps as follows.

1. In Section 3.2, we observe that any graph of large path-width contains a rake with many teeth as a subgraph.
2. In Section 3.3 we show that any graph containing a rake with many teeth as a subgraph contains either
   - a *dense* rake with many teeth as a subgraph or
   - a large canonical graph as an *induced* subgraph.
3. In Section 3.4 we prove that dense rake subgraphs necessarily imply either

– a large canonical graph as an *induced* subgraph or

– a large biclique as a subgraph.

4. In Section 3.5, we use the results of sections 3.2-3.4 to deduce Theorem 1.

### 3.2 Rake subgraphs in graphs of large path-width

**Lemma 1.** *For any natural $k$, there is a number $f(k)$ such that every graph of path-width at least $f(k)$ contains a $k$-rake as a subgraph.*

*Proof.* In [13], Robertson and Seymour has shown that for any tree $T$ there is a constant $c_T$ such that any graph of path-width is at least $c_T$ contains $T$ as a minor. Taking $T$ to be some fixed $k$-rake, we obtain that there exist a constant $f(k)$ such that any graph of path-width at most $f(k)$ contains a $k$-rake as a minor. Finally, it is not hard to see that if a graph contains a $k$-rake as a minor, then it also contains a $k$-rake as a subgraph. This observation completes the proof. □

### 3.3 From rake subgraphs to dense rake subgraphs

**Lemma 2.** *Let $k$ and $s$ be natural numbers. Every graph containing a $k+2$-rake as a subgraph contains either*

- *an $s+5$-dense $k$-rake as a subgraph or*
- *a canonical graph of order at least $s$ as an induced subgraph.*

*Proof.* Consider a graph that contains a $k+2$-rake as a subgraph and choose such a $k+2$-rake with the minimal number of vertices. We denote the base of the rake by $P$. Let $\{u_1, u_2, \ldots, u_{k+2}\}$ denote the roots of the rake that are indexed respecting the linear order of the path $P$, i.e. so that $u_1$ and $u_{k+2}$ are the endpoints of $P$ and the subpaths of $P$ from $u_i$ to $u_{i+1}$, which we denote by $P_i$, are all mutually disjoint apart from the endpoints. Note that by minimality of the rake it follows that each endpoint of the path $P$ is indeed a root vertex of the rake and that each $P_i$ is an induced path. If each $P_i$ for $i = 2, 3, \ldots, k$ has at most $s+5$ vertices, then we have an $s+5$-dense $k$-rake as required. So assume now that $P_i$ for some $i = 2, 3, \ldots k$ has size more than $s+5$. To complete the proof we will show that this $P_i$ gives rise to a canonical graph of order at least $s$ as an induced subgraph. We proceed with some notation.

Let $P_i = w_1 w_2 \ldots w_r$ with $w_1 = u_i$ and $w_r = u_{i+1}$. Extend $P_i$ by adding the vertex $w_0$ of $P_{i-1}$ that is adjacent to $w_1$ and the vertex $w_{r+1}$ of $P_{i+1}$ that is adjacent to $w_r$ (unique choice as $P_{i-1}$ and $P_{i+1}$ are induced paths). Note that $w_0 w_1 w_2 \ldots, w_r w_{r+1}$ is a subpath of $P$, the tooth $v_i$ is adjacent to $w_1$ and the tooth $v_{i+1}$ is adjacent to $w_r$. Let $G$ be a graph induced by vertices $\{w_0, w_1, \ldots, w_{r+1}\} \cup \{v_i, v_{i+1}\}$ and note that $G$ contains an H-graph formed by edges $\{w_0 w_1, w_1 w_2, \ldots, w_r w_{r+1}\} \cup \{v_i w_1, v_{i+1} w_r\}$ as a subgraph but not necessarily as an induced subgraph. Note that the body of the $H$-graph, spanned by vertices $\{w_1, w_2 \ldots, w_r\}$, is a chordless path $P_i$. For the rest of the proof we will be arguing on the adjacencies of the wings of the $H$-graph in $G$, i.e. adjacencies of vertices $w_0, w_r, v_i$ and $v_{i+1}$ in $G$. It will follow $G$ contains a canonical subgraph of order at least $s$ as an induced subgraph.

We first claim that $w_0$ is not adjacent to $w_l$ for any $l = 2, 3, \ldots, r-1$. Indeed, suppose for contradiction that $w_0$ is adjacent to some $w_l$ for $l = 2, 3, \ldots, r-1$. Let a path $P'$ be obtained from path $P$ by replacing subpath $w_0 w_1 \ldots w_r$ of $P$ by path $w_0 w_l w_{l+1} \ldots w_r$. The path $P'$ has smaller number of vertices than path $P$, and note that the missing root vertex $w_1$ can be replaced by $w_l$ with the new tooth being $w_{l-1}$. This gives us a $k+2$-rake that has smaller number of vertices than the original, which contradicts our minimality assumption.

Next, we show that $v_i$ is not adjacent to $w_4, w_5, \ldots, w_r$. Again, suppose for contradiction that $v_i$ is adjacent to $w_l$ for some $l = 4, 5, \ldots, r$. Let the path $P'$ be obtained from path $P$ by replacing the subpath $w_1 w_2 \ldots w_r$ of $P$ by path $w_1 v_i w_l w_{l+1} \ldots w_r$. Again, the path $P'$ has fewer vertices than path $P$, all the root vertices of $P$ remain in path $P'$, but as $v_i$ is now in the path $P'$, we assign a new tooth $w_2$ to correspond to the root $w_1$. Again, we obtain a $k+2$-rake that has smaller number of vertices than the original, a contradiction.

By symmetry, we can show that $w_{r+1}$ is not adjacent to $w_l$ for any $l = 2, 3, \ldots, r-1$ and $v_{i+1}$ is not adjacent to any of $w_1, w_2, \ldots, w_{r-3}$. We conclude that none of the wings of the $H$-graph are adjacent to any of $w_4, w_5, \ldots, w_{r-3}$. In other words, vertices $w_4, w_5, \ldots, w_{r-3}$ are of degree 2 in $G$. If $w_4 w_5$ is a cut-edge of $G$, we have that no vertex of $\{w_0, w_1, w_2, w_3, v_i\}$ is adjacent to any of the vertex of $\{w_{r-2}, w_{r-1}, w_r, w_{r+1}, v_{i+1}\}$. Let $l \leq 3$ be the largest possible such that $w_l$ has degree at least 3 in $G$, $p \geq r-2$ the smallest possible such that $w_p$ has degree at least 3 in $G$. Taking the path $w_l w_{l+1} \ldots w_p$ together with another two neighbours of $w_l$ and $w_p$ provides us with an induced $H$-graph whose base $w_l w_{l+1} \ldots w_p$ has at least $s+1$ vertices. On the other hand, if $w_4 w_5$ is not a cut-edge in $G$, then there is a chordless cycle in $G$ containing the edge $w_4 w_5$ and hence this cycle must contain $w_3 w_4 w_5 \ldots w_{r-2}$ (because of vertices of degree 2). Therefore, we obtain an induced cycle of $G$ with at least $r-4 \geq s+1$ vertices. Hence in both cases we obtain a canonical graph of order at least $s$ as an induced subgraph. This finishes the proof.

□

### 3.4 Dense rake subgraphs

**Lemma 3.** *For every $s, q$ and $\ell$, there is a number $D = D(s, q, \ell)$ such that every graph containing an $\ell$-dense $D$-rake as a subgraph contains either*

- *a canonical graph of order at least $s$ as an induced subgraph or*
- *a biclique of order $q$ as a subgraph.*

*Proof.* To define the number $D = D(s, q, \ell)$, we introduce intermediate notations as follows: $b := 2(q-1)s^q + 2sq + 4$ and $c := R(2, 2, \max(b, 2q))$, where $R$ is the Ramsey number. With these notations the number $D$ is defined as follows: $D = D(s, q, \ell) := Z(\ell c^2, 2q, q)$, where $Z$ is the number defined in Theorem 2.

Consider a graph $G$ containing an $\ell$-dense $D$-rake $R^0$ as a subgraph. The base of this rake is a path $P^0$ of length at least $D$ and hence, by Theorem 2, the subgraph of $G$ induced by the base contains either a biclique of order at least $q$ as a subgraph (in which case we are done) or an *induced* path $P$ of length at least $\ell c^2$. Let us call any (inclusionwise) maximal sequence of consecutive vertices of $P^0$ that belong to $P$ a *block*. Assume the number of blocks is more than $c$. Let $P'$ be the subpath of $P$ induced by the first $c$ blocks. Let $w_1, \ldots, w_c$ be the rightmost vertices of the blocks. Let $v_1, \ldots, v_c$ be the vertices such that each $v_i$ is the vertex of $P_0$ immediately following $w_i$. Then $P'$ together with $v_1, \ldots, v_c$ create a $c$-rake with $P'$ being the induced base, $v_1, \ldots, v_c$ being the teeth and $w_1, \ldots, w_c$ being the respective roots. If the number of blocks is at most $c$, then $P^0$ must contain a block of size at least $\ell c$, in which case this block also forms an induced base of a $c$-rake (since $R^0$ is $\ell$-dense). We see that in either case $G$ has a $c$-rake with an induced base. According to the definition of $c$, the $c$ teeth of this rake induce a graph which has either a clique of size $2q$ (and hence a biclique of order $q$ in which case we are done), or an independent set of size $b$. By ignoring the teeth outside this set we obtain a $b$-rake $R$ with an induced base and with teeth forming an independent set.

Let us denote the base of $R$ by $U$, its vertices by $u_1, \ldots, u_m$ (in the order of their appearances in the path), and the teeth of $R$ by $t_1, \ldots, t_b$ (following the order of their root vertices).

Denote $r := (q-1)s^q + 2$ and consider two sets of teeth $T_1 = \{t_2, t_3, \ldots, t_r\}$ and $T_2 = \{t_{b-1}, t_{b-2}, \ldots, t_{b-r+1}\}$. By definition of $r$ and $b$, there are $2sq$ other teeth between $t_r$ and $t_{b-r+1}$, and hence there is a set $M$ of $2sq$ consecutive vertices of $U$ between the root of $t_r$ and the root of $t_{b-r+1}$. We partition $M$ into $2q$ subsets (of consecutive vertices of $U$) of size $s$ each and for $i = 1, \ldots, 2q$ denote the $i$-th subset by $M_i$.

If each vertex of $T_1$ has a neighbour in each of the first $q$ sets $M_i$, then by the Pigeonhole Principle there is a biclique of order $q$ with $q$ vertices in $T_1$ and $q$ vertices in $M$. Similarly, a biclique of order $q$ arises if each vertex of $T_2$ has a neighbour in each of the last $q$ sets $M_i$. Therefore, we assume that there are two vertices $t_a \in T_1$ and $t_b \in T_2$ and two sets $M_x$ and $M_y$ with $x < y$ such that $t_a$ has no neighbours in $M_x$, while $t_b$ has no neighbours in $M_y$.

By definition, $t_a$ has a neighbour in $U$ (its root) on the left of $M_x$. If additionally $t_a$ has a neighbour to the right of $M_x$, then a chordless cycle of length at least $s$ arises (since $|M_x| = s$ and $t_a$ has no neighbours in $M_x$), in which case the lemma is true. This restricts us to the case, when all neighbours of $t_a$ in $U$ are located to the left of $M_x$. By analogy, we assume that all neighbours of $t_b$ in $U$ are located to the right of $M_y$. Let $u_i$ be the rightmost neighbour of $t_a$ in $U$ and $u_j$ be the leftmost neighbour of $t_b$ in $U$. According to the above discussion, $i < j$ and $j - j > 2s$. But then the vertices $t_a, t_b, u_{i-1}, u_i, \ldots, u_j, u_{j+1}$ induce an $H$-graph (possibly tight or semi-tight) of order more than $s$ (the existence of vertices $u_{i-1}$ and $u_{j+1}$ follows from the fact that $T_1$ does not include $t_1$, while $T_2$ does not include $t_b$). □

### 3.5 Proof of Theorem 1

Combining the results of Lemma 1, Lemma 2 and Lemma 3, we conclude that for every $s, q$, there is a number $X = X(s,q)$ such that every graph of path-width at least $X$ contains either

- a canonical graph of order at least $s$ as an induced subgraph or
- a biclique of order $q$ as a subgraph.

From this it is not hard to conclude that a class of graphs with unbounded path-width that excludes a biclique of order $q$ must contain an infinite family of distinct canonical graphs, hence the class must be not well-quasi-ordered. Therefore, well-quasi-ordered classes that exclude a biclique of order $q$ for some $q$, must be of bounded path-width, as required.

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