On the Existence of Minimal Surfaces with Free Boundary

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to

my parents ,

&

Lucia & Jessica

Contents

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Summary

Let M be a surface with $\partial M \neq \emptyset$ and N a n-manifold, also consider S a embedded surface in N. The problem treated in this thesis is the existence of a smooth map $\varphi: M \to N$ satisfying the following conditions :

(i) $\Delta_M \varphi + {}^N\Gamma(\varphi)(d\varphi, d\varphi) = 0$, Δ_M = Laplace-Beltrami operator on (M,γ)
${}^N\Gamma$ = Christoffel symbols of (N,h)

(ii) There exist a strictly positive function $\lambda: M \to \mathbb{R}$ such that the pull-back metric φ^*h on each fiber of the pull-back vector bundle $\varphi^{-1}(TN)$ over M satisfies the relation $\varphi^*h = \lambda.\gamma$

(iii) $\varphi(\partial M) \subset S$

(iv) $\frac{\partial\varphi}{\partial n}(w) \perp T_{\varphi(w)}S$ for all $w \in \partial M$; $\frac{\partial\varphi}{\partial n} = d\varphi.n$ where n is the normal direction along ∂M induced by the orientation on M .

The technique used is based on Critical Point Theory applied to Variational Analysis . Instead of finding a solution φ , of the elliptic system in (i) , as a solution for the Euler-Lagrange equations of the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 dM$, the solution is found by considering φ as the limit when $\alpha \to 1$ of solutions for the Euler-Lagange equations associated with the α-energy functionals

$$E_\alpha(\varphi) = \frac{1}{2}\int_M [\,1+|d\varphi|^2\,]^\alpha dM \quad \text{for } \alpha > 1.$$

The condition in (ii) is proved assuming a condition known as Douglas Condition , which statement guarantees that the minimal surface in the homotopy class of φ has the same number of boundaries components and the same genus (as a topological space) as M .

CHAPTER 1.

Introduction

Let (M^2,g,k) be a C^∞ compact surface with genus g and connectivity k. Let (N^n,h) be a C^∞ n-manifold isometrically embedded in $\mathbb{R}^k$ and h a complete Riemannian metric defined on N. We assume that for all $x \in N$ and all $v \in T_xN$ there exist strictly positive constants k and K such that

$$(1.0) \qquad k.\sum_{i=1}^{n} v^i v^i \le h(x)(v,v) \le K.\sum_{i=1}^{n} v^i v^i \ .$$

In this thesis we are interested in proving the existence of minimal surfaces of genus g for certain homotopy classes of maps from M into N, such that the image of the boundary of M lies on a closed C^∞-surface S embedded in N. This is done by reducing to two problems ;

1st : To prove the existence of solutions for an elliptic operator in an appropriate space of maps .

2nd : To prove that the minimal surface is topologically equivalent to a surface of genus g .

Let $\alpha:\pi_1(M) \to \pi_1(N)$, $\beta:\pi_1(\partial M) \to \pi_1(S)$ be fixed homomorphisms and assume $\pi_2(N) = 0$. So , a homotopy class $\mathcal{F}_{\alpha\beta}$ of continuous maps from M into N is defined as the set of all maps $f: M \to N$, $f(\partial M) \subset S$ such that $f_* = \alpha$

and $(f|_{\partial M})_* = \beta$, where $f_*:\pi_1(M) \to \pi_1(N)$ and $(f|_{\partial M})_*:\pi_1(\partial M) \to \pi_1(S)$ are the homomorphisms induced by f . If $\pi_2(N) \neq o$ then the homomorphisms α and β do not define a class of homotopy . In fact , as proved in [19] , if $\pi_2(N) \neq 0$ the non-trivial elements of $\pi_2(N)$ are an obstruction to prove the existence of an harmonic map minimizing the energy in each class .

A curve c on S is called essential if the homotopy class of c in $\pi_1(S)$ is not the identity , otherwise it is not essential . Unless $S=S^2$, we will avoid the homotopy classes where β: $\pi_1(\partial M) \to \pi_1(S)$ takes some boundary component of M to a null homotopic curve on S . If the image of one components of the boundary of M is not essential on S we have no methods to avoid the decrease of k or even the triviality of the solution in certain classes of maps .

The results obtained in this thesis are restricted to the following homotopy classes $\mathcal{F}_{\alpha\beta}$ where S is fixed , β is a class of homomorphisms satisfying some condition and α is any possible homomorphism :

(1.1) Let $S=S^2$, so β is trivial .

(1.2) Let S be any closed surface and consider $c_1 , c_2 , \ldots , c_k$ the boundary components of M . Let β: $\pi_1(\partial M) \to \pi_1(S)$ be an homomorphism such that $\beta([c_i]) \neq e$ (e=identity in $\pi_1(S)$) for all boundary components c_i ($i=1, \ldots , k$) of M .

(1.3) Let S be any closed surface and β as in (1.2) . Define $\mathcal{F}_{\alpha\beta}$ as a class of maps associated to α , β and satisfying the following property H :

(H) There exists a continuous curve $p: I \to \mathcal{F}_{\alpha\bar{\beta}}$ inducing a map $\hat{p}: I\times\partial M \to S$ such that $[\hat{p}(I\times\partial M)] = \mu$ in $H_2(S,U;\mathbb{Z})$, where μ is the fundamental class and U is a subspace of S such that $H_2(S,U;\mathbb{Z})$ is isomorphic to $H_2(S;\mathbb{Z})$ (see examples in appendix 3).

The question which we pose in this thesis is the existence in each of the homotopy classes defined in (1.1),(1.2) and (1.3) of a map $\varphi: M \to N$ satisfying the following properties :

(1.5) $\Delta_M\varphi + {}^N\Gamma(\varphi)(d\varphi,d\varphi) = 0$, Δ_M = Laplace-Beltrami operator on (M,γ)
${}^N\Gamma$ = Christoffel symbols of (N,h)

(1.6) There exists a positive function $\lambda: M \to \mathbb{R}$ such that the pull-back metric φ^*h on each fiber of the pull-back vector bundle $\varphi^{-1}(TN)$ over M satisfies the relation $\varphi^*h = \lambda.\gamma$ for a metric γ defined on M , and such that φ is a critical point for the energy and area functionals defined on (M,γ).

(1.7) $\varphi(\partial M) \subset S$

(1.8) $\partial_n\varphi(w) \perp T_{\varphi(w)}S$ for all $w \in \partial M$; $\partial_n\varphi = d\varphi.n$ where n is the normal direction along ∂M induced by the orientation on M .

Define the set $S_h = \{ \varphi: S^2 \to (N,h) \mid \varphi$ is non-trivial and harmonic $\}$

The results which were obtained are the following :

1.9 – Theorem : Let γ be a Riemannian metric on M and let S be a compact surface without boundary, embedded in N and class C^{∞}. If $\pi_2(N) = 0$, then :

(i) For the homotopy classes described in (1.2) there exists an harmonic map $\varphi: M \to N$ satisfying the conditions (1.5), (1.7) and (1.8) and minimizing the energy in its class.

(ii) Assuming $S_h = \emptyset$ we have in the homotopy classes described by (1.1) and (1.3) that there exists an harmonic map $\varphi: M \to N$ satisfying the conditions (1.5), (1.7) and (1.8) but not minimizing the energy in its class.

1.10 – Remark: The set of classes described by (1.3) are subsets of classes described in (1.2); therefore the result in (1.9) claims that for the classes (1.3) both (i) and (ii) are true.

By assuming an extra hypothesis called the Douglas Condition, which is defined in (7.35), we are able to prove (1.6) for the map φ of (1.9).

1.11 – Theorem : Consider $S \subset N$ as in (1.9). Assume the Douglas Condition for the homotopy classes defined in (1.1), (1.2), (1.3) and $\pi_2(N) = 0$. Then in the homotopy classes defined in (1.2) there exists a map $\varphi: M \to N$ satisfying (1.5), (1.6), (1.7) and (1.8). If $S_h = \emptyset$ then such φ also exists in the homotopy classes defined in (1.1) and (1.3). It follows that

(i) In (1.2) it minimize the energy and the area,

(ii) In (1.1) and (1.3) it does not minimize the energy and the area, It follows that $\varphi: M \to N$ is a minimal surface.

The results which are known for (1.11) can be summarized as follows;

(i) In the cases of homotopy classes in (1.2);

– R. Courant gives a proof in [3] for the situation where $M = D^2$, $N = \mathbb{R}^3$ and $S = T^2$

– J. Jost proves in [20] the case when M is a surface of genus g, N is a three manifold whose boundary ∂N has non-negative mean curvature and S is a closed subset of N. His techniques relies on geometric measure theory and he assumes that in the homotopy class of maps considered there exists an embedding in order to obtain a embedded minimal surface of genus g.

(ii) In the cases of homotopy classes in (1.1);

– M. Struwe gives a proof in [7] for the case when $M = D^2$, $N = \mathbb{R}^3$ and $S = S^2$ (D^2 and $\mathbb{R}^3$ with the standards euclidean metrics).

The method we used relies essentially on the techniques and ideas developed by Sacks and Uhlenbeck in [4]. The result obtained by M.Struwe in [7] was an encouragement to extend his result to surfaces in general.

The basic tool is the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 dM$, from which it is natural to introduce the Sobolev space $L^{1,2}(M,N)_S = \{ f:M \to N \mid f(\partial M) \subset S , \int_M (|f|^2 + |df|^2)dM < \infty\}$ as the space of maps we use in the rest of the thesis.

However , $L^{1,2}(M,N)_S$ fails to be a differentiable Banach manifold and the Critical Point Theory cannot be used . One of the troubles is the impossibility to verify a condition like the Palais-Smale (PS)-Condition (defined in 2.14) for the energy functional .

In [4] Sacks-Uhlenbeck introduced a perturbed functional , which we call the α-energy functional , defined as $E_\alpha(\varphi)=\frac{1}{2}\int_M (1+|d\varphi|^2)^\alpha dM$. The interesting properties of this functional are for the situation when $\alpha>1$, then it is naturally defined on the space of maps $L^{1,2\alpha}(M,N)_S \subset C^0(M,N)_S$, which is a differentiable Banach manifold and on it the α-energy satisfies the Palais-Smale Condition.

So, to prove theorem (1.9) we first have to prove the existence of a critical point for the α-energy functional when $\alpha>1$ and then we take the limit $\alpha \to 1$. In this process we can guarantee a priori estimates over all M^2 except for a finite number of points where the limit blows up . If $\pi_2(N) = 0$ then we can manage to avoid this by using a result, first proved in [4] , that an harmonic map from the punctured disk with finite energy can be extended to a harmonic map from the disk.

The regularity along the boundary was first proved in [8] by a different approach to ours. The method we have used relies strongly on the fact that the critical points of α-energy satisfy the condition (1.8) and S is embedded in N. The interior regularity of critical points of α-energy is C^{∞} and it is proved in [4].

As far as the conformality condition is concerned, all we need is to ensure the convergence of a minimizing (energy) sequence in the moduli space associated with M^2, once a sequence in $C^1(M,N)_s$ with finite energy is equicontinuous by the Lebesgue-Courant Lemma. The ingredients to achieve such convergence are the Douglas Condition (7.35) and Munford's Compactness Theorem (7.28).

The Theorems (1.9) and (1.11) can be extended to the cases below, where $\dim(S) > 2$, and for all homotopy classes if it satisfies the condition (H) in the following form (H'):

(H') There exists a continuous curve $p: I \to \mathcal{F}_{\alpha\beta}$ inducing a map $\bar{p}: I\times\partial M \to S$ such that $[\bar{p}(I\times\partial M)] = \mu$ in $H_p(S,U;\mathbb{Z})$, where $p = \dim(S)$, μ is the fundamental class and U is a subspace of S such that $H_p(S,U;\mathbb{Z})$ is isomorphic to $H_p(S;\mathbb{Z})$.

- If S is a compact manifold embedded in N and for situations where $S \subseteq \partial M$.
- For the case where S has more than one connected component in N.

Possible results obtained from observation above are :

1.12 - Theorem : Let $S = S^p \subset N$ $(p<n)$ be a embedded submanifold

diffeomorphic to the p-sphere. Assuming that $\pi_2(N) = 0$, $S_h = \emptyset$ (as in 1.9) and the Douglas Condition, then there exists a map $\varphi: (M,\gamma,g,k) \to (N,h)$ satisfying (1.5), (1.6), (1.7) and (1.8), i.e. φ is a minimal surface.

1.13 - Theorem : Let $S = S_1 \cup S_2$ be a embedded submanifold of N such that $S_1 \cap S_2 = \emptyset$. Assuming $\pi_2(N) = 0$ and the Douglas Condition, then for each homotopy class of continuous maps in $C^0(M,N)_s$ there exists a minimal surface $\varphi: (M,\gamma,g,k) \to (N,h)$ satisfying (1.5), (1.6), (1.7) and (1.8) and minimizing the energy and the area among all maps in the same class. Furthermore, if $S_h = \emptyset$ and in the homotopy class there is a curve satisfying the H'-condition then there exists a map $\varphi: (M,\gamma,g,k) \to (N,h)$ satisfying (1.5), (1.6), (1.7) and (1.8), i.e. φ is a minimal surface which is a saddle point for the energy and the area functionals.

CHAPTER 2.

Preliminaries.

In this section the main point is to give a description of the main tools and the fundamental facts for later use.

Consider (M,γ) and (N,h), $N \subset \mathbb{R}^k$, as C^∞ Riemannian manifolds of dimension m and n, define the vector bundle $\eta = M \times \mathbb{R}^k$ and the fibre bundle $\zeta = M \times N$, $\zeta \subset \eta$.

As is well known from Riemannian Geometry, the Riemannian structure on the vector bundle TM over M induces a Riemannian structure on the bundles $\otimes^k TM$ (k^{th}-tensor product) and $\otimes^k T^*M$ ($*$ = dual) over M.

Each map $f : M \to N$ can be considered as a section $f : M \to \zeta$ in the bundle ζ. The $(k+1)^{th}$-order derivative of f associated with the Riemannian structures on M and N induce the sections $\nabla^k(df) : M \to \otimes^{k+1} T^*M \otimes f^{-1}(TN)$, $\nabla^k = \nabla \circ \ldots \circ \nabla$ (k times), because

$$\nabla^k(df)(x) \in \mathrm{Hom}(\otimes^{k+1} T_x M, T_{f(x)}N) = \otimes^k T_x^* M \otimes T_{f(x)}N .$$

The Riemannian metrics induced on these vector spaces let us define the norm

$$|\nabla^k(df)|(x) = \mathrm{trace}\,[(\nabla^k(df))^* h)](x) , \tag{2.0}$$

where the trace is taken relative to the metric on $\otimes^k T^* M$, induced by the metric on TM.

2.1. Definition: The L^p-spaces associated with the maps from M into N, considering the element of volume as the measure, are defined as $L^p(\zeta) = \{f: M \to \zeta \mid f$ is a section and $\int_M |f|^p dM < \infty\}$. Define the L^p-norm as $\|f\|_p = [\int_M |f|^p dM]^{1/p}$, where $|\cdot|$ = norm in $\mathbb{R}^k$.

As in the cases for domains in $\mathbb{R}^n$ these spaces are known as Sobolev spaces and they are complete with respect to their natural norm. In general they are not Banach spaces because of the non-linear character of N. Assuming the existence of a theory for weak-derivatives of sections in the bundle ζ (see [2]), we give the following definition:

2.2. Definition: The Sobolev spaces associated with maps from M into N, considering the element of volume as the measure, are defined as $L^{r,p}(\zeta) = \{f: M \to \zeta \mid f \in L^p(\zeta)$ and

$\sum_{i=0}^{r-1} \int_M |\nabla^i(df)|^p \, dM < \infty\}$. Define the norm $\|\ \|_{L^{r,p}}$ as

$$\|f\|_{(r,p)} = [\sum_{i=0}^{r-1} \int_M |\nabla^i(df)|^p \, dM]^{1/p} + \|f\|_{(0,p)}$$

The fundamental fact about this space is contained in the theorem below; in the reference ([10], pg.97) it is proved in general.

2.3. Theorem: (Sobolev's embedding theorem) Let (M,γ) and (N,h) be C^∞ Riemannian manifolds of dimension m and n. Let r be a non-negative integer and p satisfy $1 \leq p < \infty$. Then there exist the following embeddings:

Case A: Suppose $rp < m$, then

$$L^{r,p}(\zeta) \to L^{q}(\zeta) \text{ for all } p \le q \le \frac{mp}{m-rp}.$$

Case B: Suppose $rp = m$, then

$$L^{r,p}(\zeta) \to L^{q}(\zeta) \text{ for all } p \le q < \infty.$$

Case C: Suppose $rp > m$, then

$$L^{r,p}(\zeta) \to C^{j}(\zeta) \text{ for all } 0 \le j < r - \frac{m}{p}, \text{ where}$$

$C^{s}(\zeta) = \{f : M \to \zeta \mid \nabla^{s-1}(df)$ exists pointwise and is continuous$\}$.

Proof: See [10], pg.97. □

The case C is of particular interest for later developments because estimates for the norm in $L^{r,p}(\zeta)$ gives the class of pointwise differentiability.

Assuming N is isometrically embedded in $\mathbb{R}^k$ we have the following results which are vital for Critical Point Theory.

2.4. Theorem: Let $\eta = M \times \mathbb{R}^k$ be a vector bundle. If $rp > m$ then the Sobolev spaces $L^{r,p}(\eta)$ are Banach spaces.

Proof: See [2]. □

2.5. Theorem: If we assume $rp > m$ and N is isometrically embedded in $\mathbb{R}^k$, then the Sobolev spaces $L^{r,p}(\zeta)$ have a C^{∞} differentiable structure as a Banach submanifold of $L^{r,p}(\eta)$.

Proof: See [2], pg.49. □

In the situation in which this work is developed, M is a surface, i.e. $\dim M = 2$ and since the α-energy functional is the basic tool to be used, we are particularly interested in the Sobolev spaces $L^{1,2\alpha}(\zeta)$, $\alpha > 1$. As we are interested in situations where M has non-empty boundary, consider the Sobolev space $L^{1,2\alpha}(\zeta)_s = \{f \in L^{1,2\alpha}(\zeta) \mid f(\partial M) \subset S\}$, where S is a closed C^∞-embedded submanifold of N as in (1.8). Then, for $\alpha > 1$ $L^{1,2\alpha}(\zeta)_s$ has a C^∞ differentiable structure as a submanifold of $L^{1,2\alpha}(\eta)$ and the theorems (2.3) and (2.5) are also true for this class of spaces.

2.6. Definition: The α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$ is defined as

$$E_\alpha(f) = \tfrac{1}{2}\int_M (1 + |df|^2)^\alpha dM - \tfrac{1}{2}\int_M dM \ ; \quad |df|^2 = \operatorname{trace}(f^*h)$$

If $1 < \alpha < 2$ then $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$ is C^2-differentiable. Because the 2nd term is independent of f we consider on many occasions $E_\alpha(f) = \frac{1}{2}\int_M (1 + |df|^2)^\alpha dM$

It is useful to consider the α-energy above defined as a restriction of the functional $E_\alpha : L^{1,2\alpha}(\eta) \to \mathbb{R}$, given by the same expression in (2.6).

Considering that N is embedded in $\mathbb{R}^k$, we can define local projections $P(x) : \mathbb{R}^k \to T_xN$ which induce C^∞ sections $P : N \to \operatorname{Hom}(\mathbb{R}^k, TN)$. Taking the orthogonal complement in $\mathbb{R}^k$, we also have $Q(x) = I - P(x) : \mathbb{R}^k \to (T_xN)^\perp$ inducing a C^∞ section $Q : N \to \mathrm{Hom}(\mathbb{R}^k, (TN)^\perp)$ ($(TN)^\perp$ = normal tangent bundle induced by the Riemannian structure on $\mathbb{R}^k$).

Likewise, we can define the morphisms P_s and Q_s associated with the embedded surface S in N.

The tangent space of $L^{1,2\alpha}(\zeta)_s$ at the point f can be described as $T_f L^{1,2\alpha}(\zeta)_s = \{v: M \to f^*(TN) \mid v \in L^{1,2\alpha}(\zeta)_s,\ Q(v) = 0,\ Q_s(v|_{\partial M}) = 0\}$[1]. Therefore differentiating the C^2-functions $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$, for $\alpha > 1$, we define a section $dE_\alpha : L^{1,2\alpha}(\zeta)_s \to (TL^{1,2\alpha}(\zeta)_s)^*$.

The next steps are concerned with introducing concepts to formulate a sufficient condition for a general functional to ensure that the (PS)–Condition (defined in 2.14) is satisfied by the functional.

2.7. Theorem: Let $1 \leq p,\ q < \infty$ and let k and ℓ be real numbers with $k - (\frac{m}{p}) \geq \ell - (\frac{m}{q})$ and $k \geq \ell$. Then $L^{k,p}(\zeta)_s \subseteq L^{\ell,q}(\zeta)_s$ and the inclusion map is continuous. If $k - (\frac{m}{p}) > \ell - (\frac{m}{q})$ and $k > \ell$ then the inclusion map is completely continuous.

Proof: See A.P. Calderon, Vol.4. AMS, Symposia in Pure Mathematics, "Lebesgue spaces of differentiable functions and distributions". □

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1 - Its is important to note that if $\varphi \in L^{1,2\alpha}(\zeta)_s$ then the bundle $\varphi_*(TN)$ is of class $L^{1,2\alpha}$, i.e., it does make sense to talk about the space $L^{1,2\alpha}(\varphi_*(TN)) = \{s: M \to \varphi_* TN \mid s \text{ is } L^{1,2\alpha} \text{ integrable}\}$

2.8. Proposition: If $T \in \mathrm{Hom}(L^{r,p}(\eta), L^{r,p}(\eta))$ then for $rp > m$ T maps bounded sets in $L^{r,p}(\eta)$ into bounded sets in $L^{r,p}(\eta)$.

Proof: See [2], pg.112.

2.9. Proposition: Given any sequence $(\varphi_i)_{i=1}^{\infty}$ in $L^{r,p}(\zeta)_s$, for $rp > m$, which is bounded in $L^{r,p}(\eta)$, by taking a subsequence we can suppose that $\|Q(\varphi_i)(\varphi_i-\varphi_j)\|_{L^{r,p}(\eta)} \to 0$ as $i,j \to \infty$ ($\bar{Q} = Q$ by abuse of notation).

Proof: (See [2],19.15) The hypothesis that $rp > m$ implies (by (2.3)) that $L^{r,p}(\zeta) \subset C^0(\zeta)$ and therefore $C^\infty(\zeta)$ is dense in $L^{r,p}(\zeta)$. Choose a finite set of smooth vector fields on M, say $X_1,\ldots,X_m$, such that each $V(x) \in T_xM$ can be written as a linear combination of the $X_i(x)$, $i = 1,\ldots,m$.

(2.10) $\|Q(\varphi_i)(\varphi_i-\varphi_j)\|_{L^{r,p}(\eta)} = \|Q(\varphi_i)(\varphi_i-\varphi_j)\|_{L^p(\eta)} + \sum_{i=1}^{m} \|X_i(Q(\varphi_i)(\varphi_i-\varphi_j))\|_{L^{r-1,p}}$.

In the first term on the right hand side of e.q. (2.10) we know that $\|Q(\varphi_i(x))(\varphi_i(x)-\varphi_j(x))\|_{R^k} \le c\|\varphi_i(x)-\varphi_j(x)\|_{R^k}$, $c > 0$ a constant, because $Q(\varphi_i(x))$ is a projection, so integrating

$$\|Q(\varphi_i)(\varphi_i-\varphi_j)\|_{L^p(\eta)} \le c\|\varphi_i-\varphi_j\|_{L^p(\eta)} .$$

The embedding $L^{r,p}(\eta) \to C^0(\eta)$ is completely continuous, which implies that we can consider the subsequence $(\varphi_i)_{i=1}^{\infty}$ as a Cauchy sequence in $C^0(\eta)$.

Hence, $\|Q(\varphi_i)(\varphi_i-\varphi_j)\|_{L^p(\eta)} < \varepsilon/2$.

For the second term on the right hand side of (2.10), assume

$(\varphi_i)_{i=1}^{\infty} \subset C^{\infty}(\zeta)$, so

$$(2.11) \qquad X(Q(\varphi_i)(\varphi_i-\varphi_j)) = X(Q(\varphi_i))(\varphi_i-\varphi_j) + Q(\varphi_i)X(\varphi_i) - Q(\varphi_i)X(\varphi_j).$$

Using the fact that $Q = I-P$ and $P(\varphi_i)X(\varphi_i) = X(\varphi_i)$, we have

$$(2.12) \qquad X(Q(\varphi_i)(\varphi_i-\varphi_j)) = X(P(\varphi_i))(\varphi_i-\varphi_j) + (P(\varphi_i)-P(\varphi_j))X(\varphi_j),$$

but the bilinear map $\quad X \circ Q : C^{\infty}(\zeta) \times C^{\infty}(\eta) \to C^{\infty}$

$$(\varphi,s) \to X(Q(\varphi)s)$$

is C^{∞}, so by the fact that $C^{\infty}(\eta)$ is dense in $L^{r,p}(\eta)$, the bilinear map above can be extended to $X \circ Q : L^{r,p}(\zeta) \times L^{r,p}(\eta) \to L^{r-1,p}(\eta)$, thus (2.12) can be extended to $(\varphi_i)_{i=1}^{\infty} \subset L^{r,p}(\eta)$.

Now choose $0 < \varepsilon < 1$ so that $k - \varepsilon > \frac{m}{p}$. Consider $(\varphi_i)_{i=1}^{\infty} \subset L^{r,p}(\zeta)$, then by the Theorem (2.7) the inclusion $L^{r,p}(\zeta)_s \hookrightarrow L^{r-\varepsilon,p}(\zeta)_s$ is completely continuous, so taking a subsequence we can suppose that $\varphi_i \to \varphi_0$ in $L^{r-\varepsilon,p}(\zeta)$. Since the map $L^{r-\varepsilon,p}(\zeta)$ into $L^{r-\varepsilon,p}(\mathrm{Hom}(\eta,\eta))$ given by $(\varphi \to P(\varphi))$ is C^{∞} and hence continuous. So it follows that $\|P(\varphi_i)-P(\varphi_j)\| \to 0$ in $L^{r-\varepsilon,p}(\mathrm{Hom}(\eta,\eta))$, and also $\|\varphi_i-\varphi_j\|_{L^{r-\varepsilon,p}(\zeta)} \to 0$.

Thus, from the second term on the right hand side of the eq. (2.10)

(2.13) $\|X(Q(\varphi_i)(\varphi_i-\varphi_j))\|_{L^{r-1,p}(\eta)} = \|X(P(\varphi_i))(\varphi_i-\varphi_j) + (P(\varphi_i)-P(\varphi_j))X(\varphi_j)\| \leq$
$\leq \|X(P(\varphi_i))\|_{L^{r-1,p}(\eta)} \cdot \|\varphi_i-\varphi_j\|_{L^{r-\varepsilon,p}(\eta)} + \|P(\varphi_i)-P(\varphi_j)\|_{L^{r-\varepsilon,p}(\eta)} \|X(\varphi_j)\|_{L^{r-1,p}(\eta)} \leq$
$\leq K\|P(\varphi_i)\|_{L^{r-1,p}(\eta)} \|\varphi_i-\varphi_j\|_{L^{r-\varepsilon,p}(\eta)} + \|P(\varphi_i)-P(\varphi_j)\|_{L^{r-\varepsilon,p}(\eta)} \|\varphi_j\|_{L^{r,p}(\eta)}.$

The result follows by (2.8), the hypothesis, observations above and inequality (2.13). □

Although the Sobolev spaces $L^{r,p}(\zeta)_s$, for $rp > m$, have a C^∞ differentiable structure as a Banach manifold, they are manifolds with infinite dimension and are not locally compact. The (PS)-Condition is a sufficient condition to replace the lack of compactness.

2.14. Definition: A functional $F : L^{r,p}(\eta) \to \mathbb{R}$, $rp > m$, satisfies the (PS)-Condition if for any sequence $(\varphi_n)_{n=1}^{\infty}$, such that $(F(\varphi_n))_{n=1}^{\infty}$ is bounded and $\|dF(\varphi_n)\| \to 0$, there exists a subsequence (φ_{nk}) so that $\varphi_{nk} \to \varphi$ in $L^{r,p}(\eta)$ and $\|dF(\varphi)\| = 0$, i.e., φ is a critical point for the functional F.

2.15. Proposition: Consider the C^2-functionals $F : L^{r,p}(\zeta)_s \to \mathbb{R}$ and $\bar{F} : L^{r,p}(\eta) \to \mathbb{R}$ $(F = \bar{F}|_N)$, and a sequence $(\varphi_n)_{n=1}^{\infty} \subset L^{r,p}(\zeta)_s$ bounded in $L^{r,p}(\eta)$ such that $\|dF(\varphi_n)\| \to 0$ (in $L^{r,p}(\zeta)_s$) . Then passing to a subsequence we have that

$$\|d\bar{F}(\varphi_n)(\varphi_n-\varphi_m)\| \to 0 \text{ in } L^{r,p}(\eta) .$$

Proof: (See [2],19.17)

$$d\bar{F}(\varphi_n)(\varphi_n-\varphi_m) = d\bar{F}(\varphi_n)(P(\varphi_n)(\varphi_n-\varphi_m)) + d\bar{F}(\varphi_n)(Q(\varphi_n)(\varphi_n-\varphi_m)) =$$

$$= dF(\varphi_n).(\varphi_n-\varphi_m) + dF(\varphi_n).(Q(\varphi_n).(\varphi_n-\varphi_m)) \,, \text{ so}$$

$$|d\bar{F}(\varphi_n).(\varphi_n-\varphi_m)| < |dF(\varphi_n)|\, \|\varphi_n-\varphi_m\|_{L^{r,p}(\eta)} + |d\bar{F}(\varphi_n)|.\|Q(\varphi_n)(\varphi_n-\varphi_m)\|_{L^{r,p}(\eta)}$$

and it then follows by (2.8) and (2.9). $\square$

CHAPTER 3.

(PS)-Condition for the functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$, $\alpha > 1$.

The proof of the (PS)-condition (as in [2]) depends essentially on the properties of the "density of α-energy" (Lagrangian) . In this section we do not consider the general situation , for this see [2]-Chapter 19 .

3.0. Definition: Let $V_1, \ldots, V_s$ be orthogonal vector spaces and let $\zeta_i = M \times V_i$ be the corresponding Riemannian vector bundles over M . If A_i is a differential operator carrying sections of $\eta = M \times \mathbb{R}^k$ to sections of ζ_i , $i = 1,\ldots,s$ then we say that $\{A_i\}$ is an *ample family* of r^{th}-order linear operators for η , provided that there exist constants c_1 and c_2 such that for all $\varphi \in L^{r,p}(\eta)$

$$\|\varphi\|_{L^{r,p}(\eta)} \le c_1 \sum_{i=1}^{s} \|A_i\varphi\|_{L^p(\zeta_i)} + c_2\|\varphi\|_{L^p(\eta)}$$

and we shall say that $\{A_i\}$ is strongly ample if we can choose $c_2 = 0$.

3.1. Example: $A_i = X_{i_1}^{\alpha_1} \circ X_{i_2}^{\alpha_2} \circ \ldots \circ X_{i_m}^{\alpha_m}$, where $\alpha_1 + \ldots + \alpha_m \le r$

$X_i = \dfrac{\partial}{\partial x_i}$ and $X^\alpha = X \circ \ldots \circ X$ (α times).

It follows from the definition of $\|\cdot\|_{L^{r,p}(\eta)}$ that $\{A_i\}$ is ample.

We are really interested in the case $\dim M = 2$, because by (2.7) if $\alpha > 1$ we have

$L^{1,2\alpha}(\eta) \subset C^{o}(\eta)$.

Let (U,X) be a local chart for M, then the α-energy can be expressed as

$$(3.2)\quad \bar{E}_\alpha(\varphi) = \tfrac{1}{2}\int_U [1+\gamma^{\mu\upsilon}h_{ij}\partial_\mu\varphi^i\partial_\upsilon\varphi^j]^\alpha dU\,, \qquad dU = \sqrt{\gamma}\,dx^1\wedge\ldots\wedge dx^m$$

$$\gamma = \det(\gamma_{ij})$$

(summation in repeated index).

From (3.1) the family of 1^{st}-order differential operators $\{\partial_\mu\}$, $\mu = 1,2$ is ample.

The next definition are taken from [2] in a particular case which suit ours purposes.

3.3. Definition ([2],19.1): A 1^{st} order lagrangian on $\eta = M\times\mathbb{R}^k$ is an element of $D^1(\eta,\mathbb{R}_M) = \{$set of all differential operators of order 1 from η to the bundle $\mathbb{R}_M = M\times\mathbb{R}\}$. We denote the vector space of 1^{st} order lagrangian on η by $Lg_1(\eta)$.

3.4. Definition : Consider a functional $F: L^{1,p}(\eta) \to \mathbb{R}$ $\left(F(\varphi) < \infty \text{ if } \varphi \in L^{1,p}(\eta)\right)$ defined by $F(\varphi) = \int_M L(\varphi)dM$, where $L \in Lg_1(\eta)$. We say that L is the lagrangian associated with the functional F.

3.5. Definition: Let L be a lagrangian associated to a functional $F: L^{1,p}(\eta) \to \mathbb{R}$ as in 3.4. Then we say that L is (strongly) p-coersive if there exists a (strongly) ample family $\{A_i\}$ of 1st-order linear differential operators such that for all φ, $\Lambda \in C^{\infty}(\eta)$

$$d^2L_{\varphi}(\Lambda,\Lambda) \geq \sum_{i=1}^{s} |A_i\varphi|^{p-2}|A_i\Lambda|^2 .$$

3.6. **Example:** In this example we prove that the α-energy functional has a p-coersive Lagrangian according to (3.5). This example is fundamental to prove the (PS)-Condition for the α-energy, $\alpha > 1$. It is divided in 3 steps.

Step 1: Because γ is a Riemannian metric on a compact surface M, there exist constants $c > 0$ and $k > 0$ such that

(3.7) $$c|v|^2_{\mathbb{R}^2} < |v|^2_M < k|v|^2_{\mathbb{R}^2} ,$$

where $|\cdot|_{\mathbb{R}^2}$ = euclidean norm in $\mathbb{R}^2$

$|\cdot|_M$ = norm on M induced by the metric γ.

Let us consider $A_\mu = \partial_\mu$ and $E_\alpha(\varphi) = \int_M e_\alpha(\varphi)dM$, then by (3.2)

$$e_\alpha(\varphi) = \tfrac{1}{2}[1 + \gamma^{\mu\upsilon}h_{ij}A_\mu\varphi^i A_\upsilon\varphi^j]^\alpha \geq \tfrac{1}{2}[1 + c_1|A_1\varphi|^2 + c_2|A_2\varphi|^2]^\alpha .$$

We claim that there exists a constant $c > 0$ such that

(3.8) $$[1 + c_1|A_1\varphi|^2 + c_2|A_2\varphi|^2]^\alpha > c^\alpha(|A_1\varphi|^{2\alpha} + |A_2\varphi|^{2\alpha}) .$$

To prove (3.8) consider the function $f: \mathbb{R} \to \mathbb{R}$ defined as,

$$f(x) = (x+a)^{\alpha} - x^{\alpha} - a^{\alpha}\ , \quad \text{so } f(0) = 0$$

$$f'(x) = \alpha[(x+a)^{\alpha-1} - x^{\alpha-1}]\ , \text{ if } \alpha \geq 1 \text{ and } a > 0 \text{ then } f'(x) > 0\ .$$

Hence, if $x \geq 0 \Rightarrow f'(x) \geq 0$, that means f is an increasing function in $[0,\infty)$.

Take $c = \min(c_1,c_2)$.

Step 2: Define $\ell_{\alpha}(\varphi)(x) = \frac{1}{2}[(A_1\varphi,A_1\varphi)^{\alpha} + (A_2\varphi,A_2\varphi)^{\alpha}](x)$, $(.,.)$ = inner product in $\mathbb{R}^2$

$$[d\ell_{\alpha}(\varphi).\Lambda](x) = \frac{\alpha}{2}[(A_1\Lambda,A_1\varphi)|A_1\varphi|^{2(\alpha-1)} + (A_2\Lambda,A_2\varphi)|A_2\varphi|^{2(\alpha-1)}](x)$$

Step 3 ([2],19.31):

$$d^2\ell_{\alpha}(\varphi)(\Lambda,W)(x) = \frac{\alpha(\alpha-1)}{2}|A_1\varphi|^{2(\alpha-2)}(A_1\varphi,A_1W)(A_1\varphi,A_1\Lambda) +$$

$$+\frac{\alpha}{2}|\ |A_1\varphi)|^{2(\alpha-1)}(A_1W,A_1\Lambda) + \frac{\alpha(\alpha-1)}{2}|A_2\varphi|^{2(\alpha-2)}(A_2\varphi,A_2W)(A_2\varphi,A_2\Lambda) +$$

$$+\frac{\alpha}{2}|A_2\varphi|^{2(\alpha-1)}(A_2\Lambda,A_2W)\ .\ \text{So}\ ,$$

$$d^2\ell_{\alpha}(\varphi)(\Lambda,\Lambda)(x) = \frac{\alpha(\alpha-1)}{2}|A_1\varphi|^{2\alpha-4}((A_1\varphi,A_1\Lambda))^2 + \quad \frac{\alpha}{2}|A_1\varphi|^{2\alpha-2}|A_1\Lambda|^2 ++$$

$$\frac{\alpha(\alpha-1)}{2}|A_2\varphi|^{2\alpha-4}((A_2\varphi,A_2\Lambda))^2 + \frac{\alpha}{2}|A_2\varphi|^{2\alpha-2}|A_2\Lambda|^2 = \frac{\alpha}{2}|A_1\varphi|^{2\alpha-2}|A_1\Lambda|^2\Big[1$$

$$+ (\alpha-1)\frac{((A_1\varphi,A_1\Lambda))^2}{|A_1\varphi|^2|A_1\Lambda|^2}\Big] +$$

$$+\frac{\alpha}{2}|A_2\varphi|^{2\alpha-2}|A_2\Lambda|^2\Big[1 + (\alpha-1)\frac{((A_2\varphi,A_2\Lambda))^2}{|A_2\varphi|^2|A_2\Lambda|^2}\Big] \geq \frac{\alpha}{2}\Big[\sum|A_i\varphi|^{2\alpha-2}|A_i\Lambda|^2\Big](x)\ .$$

Thus by (3.8) e_α is 2α-coercive. ◻

3.9. Theorem ([2],19.29): Let $p \geq 2$ and $p > m$ and let L be a p-coercive Lagrangian . Let $\bar{F}: L^{1,p}(\eta) \to \mathbb{R}$ be a C^2-functional given by $\bar{F}(\varphi) = \int_M L(\varphi)dM$ such that $d\bar{F}: L^{1,p}(\eta) \to \left(TL^{1,p}(\eta)\right)^*$ maps bounded sets to bounded sets and let $F : L^{1,p}(\zeta)_s \to \mathbb{R}$ be the restriction . Then the functional F satisfies the (PS)-Condition.

Proof: It is divided into 3 steps.

Step 1: L is p-coercive, so there exist $c_1 > 0$ and $c_2 > 0$ such that for all $\varphi_1, \varphi_2 \in C^\infty(\eta)$ we have

$$d\bar{F}(\varphi_1).(\varphi_1-\varphi_2)-d\bar{F}(\varphi_2)(\varphi_1-\varphi_2) \geq c_1\|\varphi_1-\varphi_2\|_{L^{1,p}(\eta)}-c_2\|\varphi_1-\varphi_2\|_{L^p(\eta)}.$$

Proof: Take $a = \varphi_1(x)$, $b = \varphi_2(x)$, $c : I \to \mathbb{R}^k$ $c(t) = b + t(a-b)$

$$[dL(\varphi_1)-dL(\varphi_2)].(\varphi_1-\varphi_2) = \int_0^1 d^2L(c(t)).(\varphi_1-\varphi_2,\varphi_1-\varphi_2)dt \geq$$

$$\geq \int_0^1 \sum_{i=1}^2 |A_i(c(t))|^{p-2}.|A_i(\varphi_1-\varphi_2)|^2 dt \geq$$

$$\geq \sum_{i=1}^2 |A_i(\varphi_1-\varphi_2|^2.\int_0^1 |A_i(\varphi_2)+t.A_i(\varphi_1-\varphi_2)|^{p-2}dt \tag{3.10}$$

Using the fact that $\int_0^1 |x + ty|^m dt \geq c|y|^m$ (see [2], 19.27) in (3.10), it follows that the right hand side of 3.10 is lower bounded by

$$\geq c \sum_{i=1}^{2} |A_i(\varphi_1(x)-\varphi_2(x))|^p .$$

Hence, after integration on M and using the fact that A_i is an ample family, it follows that

$$[d\bar{F}(\varphi_1)-d\bar{F}(\varphi_2)]\,(\varphi_1-\varphi_2) \geq c_1\|\varphi_1-\varphi_2\|_{L^{1,p}(\eta)} - c_2\|\varphi_1-\varphi_2\|_{L^p(\eta)}$$

Step 2: The functional $\bar{F} : L^{1,p}(\eta) \to \mathbb{R}$ satisfies the (PS)-Condition.

Proof: By hypothesis $d\bar{F} : L^{1,p}(\eta) \to (TL^{1,p}(\eta))^*$ maps bounded sets to bounded sets ($(TL^{r,p}(\eta))^* = L^{r,q}(T^*\eta)$ where $\frac{1}{p}+\frac{1}{q}=1$)

Consider $c(t) = \varphi_2 + t(\varphi_1-\varphi_2)$ in $L^{r,p}(\eta)$. From step-1 we have

$$[d\bar{F}(c(t))-d\bar{F}(\varphi_1)].(t(\varphi_2-\varphi_1)) \geq tc_1\|\varphi_2-\varphi_1\|_{L^{1,p}(\eta)} - tc_2\|\varphi_2-\varphi_1\|_{L^p(\eta)}$$

$$\bar{F}(\varphi_2) = \bar{F}(\varphi_1) + \int_0^1 d\bar{F}(c(t)).(\varphi_2-\varphi_1)dt =$$

$$= \bar{F}(\varphi_1) + d\bar{F}(\varphi_1).(\varphi_1-\varphi_2) + \int_0^1 \frac{1}{t}[d\bar{F}(c(t))-d\bar{F}(\varphi_1)].t(\varphi_2-\varphi_1)dt \geq$$

$$\geq \bar{F}(\varphi_1) + d\bar{F}(\varphi_1).(\varphi_1-\varphi_2) + c_1\|\varphi_2-\varphi_1\|_{L^{1,p}(\eta)} - c_2\|\varphi_2-\varphi_1\|_{L^p(\eta)}. \qquad (3.11)$$

By taking a sequence $(\varphi_n)_{n=1}^{\infty}$ in $L^{1,p}(\zeta)$, bounded in $L^{1,p}(\eta)$ $\left(2.7 \Rightarrow (\varphi_n)_{n=1}^{\infty}\right.$ is Cauchy in $\left.L^p(\zeta)\right)$ and such that $|dF(\varphi_n)| \to 0$, it then follows by (3.11) that

$$(3.12)\quad \bar{F}(\varphi_n)-\bar{F}(\varphi_m)-d\bar{F}(\varphi_n)(\varphi_n-\varphi_m) \geq c_1\|\varphi_n-\varphi_m\|_{L^{1,p}(\eta)} - c_2\|\varphi_n-\varphi_m\|_{L^p(\eta)}$$

Now we have $|\bar{F}(\varphi_n) - \bar{F}(\varphi_m)| \to 0$, as $n,m \to \infty$, because $\bar{F} : L^{1,p}(\eta) \to \mathbb{R}$ is continuous. Also $|d\bar{F}(\varphi_n)(\varphi_n-\varphi_m)| \to 0$ because

$$d\bar{F}(\varphi_n)(\varphi_n-\varphi_m) = d\bar{F}(\varphi_n)(P(\varphi_n-\varphi_m)) + d\bar{F}(\varphi_n)(Q(\varphi_n)(\varphi_n-\varphi_m)) \ , \text{ so}$$

$$|d\bar{F}(\varphi_n)(\varphi_n-\varphi_m)| \leq |dF(\varphi_n)|\,\|\varphi_n-\varphi_m\|_{L^{1,p}(\eta)} + \|d\bar{F}(\varphi_n)\|\,\|Q(\varphi_n)(\varphi_n-\varphi_m)\|_{L^{1,p}(\eta)}$$

(by hypothesis $\|d\bar{F}(\varphi_n)\|$ is bounded and by (2.9) $\|Q(\varphi_n)(\varphi_n-\varphi_m)\|_{L^{1,p}(\eta)}$ goes to zero). So in the expression (3.12) it follows that $\|\varphi_n-\varphi_m\|_{L^{1,p}(\eta)} \to 0$ and by the completeness of $L^{1,p}(\eta)$ there is $\varphi \in L^{1,p}(\eta)$ such that $\varphi_n \to \varphi$ in $L^{1,p}(\eta)$.

Step 3: Let $\varphi = \lim \varphi_n$ in $L^{1,p}(\eta)$, then $\varphi \in L^{1,p}(\zeta)_s$ and $|dF(\varphi)| = 0$.

Proof: Since $(\varphi_n)_{n=1}^{\infty} \subset L^{1,p}(\zeta)_s$ and $L^{1,p}(\zeta)_s$ is a closed submanifold of $L^{1,p}(\eta)$, it follows that $\varphi \in L^{1,p}(\zeta)_s$ and by continuity $|dF(\varphi)| = 0$. □

3.13. Corollary. The α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta) \to \mathbb{R}$ satisfies the (PS)-Condition for $\alpha > 1$.

Proof: In example (3.6) we worked out the case for the α-energy's Langrangian and it was proved that it is 2α-coercive. In order of applying Thm 3.9 we need to prove that $dE_\alpha : L^{1,2\alpha}(\eta) \to (TL^{1,2\alpha}(\eta))^* = L^{1,\frac{2\alpha}{2\alpha-1}}(\eta^*)$ (2) takes bounded sets into bounded sets. Taking $\varphi \in L^{1,2\alpha}(\eta)$ then the we get a continuous linear operator $dE_\alpha(\varphi)$: $L^{1,2\alpha}(\varphi^* T\mathbb{R}^k) \to C^\infty(M)$ whose norm is defined (see [22]) by

$$\|dE_\alpha(\varphi)\|_{L^{1,\frac{2\alpha}{2\alpha-1}}} = \sup_{\|\Lambda\| \le 1} |dE_\alpha(\varphi).\Lambda|$$

The explicit formulae is given by equation (5.3),

$dE_\alpha(\varphi) = \alpha \int_M (1 + |d\varphi|^2)^{\alpha-1}.\langle d\varphi, \nabla\Lambda\rangle dM$. Applying the Holder inequality (see [2]) for $\frac{1}{2\alpha} + \frac{2\alpha-1}{2\alpha} = 1$ we get

$$|dE_\alpha(\varphi).\Lambda| \le \alpha\Big(\int_M (1 + |d\varphi|^2)^{\frac{2\alpha(\alpha-1)}{2\alpha-1}}.|d\varphi|^{\frac{2\alpha}{2\alpha-1}} dM\Big)^{\frac{2\alpha-1}{2\alpha}}.\Big(\int_M |\nabla\Lambda|^{2\alpha} dM\Big)^{1/2\alpha}$$

but $\int_M (1 + |d\varphi|^2)^{\frac{2\alpha(\alpha-1)}{2\alpha-1}}.|d\varphi|^{\frac{2\alpha}{2\alpha-1}} dM \le \int_M (1 + |d\varphi|^2)^{\frac{2\alpha(\alpha-1)}{2\alpha-1} + \frac{\alpha}{2\alpha-1}} dM = E_\alpha(\varphi)$

Then $|dE_\alpha(\varphi).\Lambda| \le \alpha.(E_\alpha(\varphi))^{\frac{2\alpha-1}{2\alpha}}$ since $\Lambda \in L^{1,2\alpha}(\varphi^* T\mathbb{R}^k) \Rightarrow$ $\|dE_\alpha(\varphi)\|_{L^{1,\frac{2\alpha}{2\alpha-1}}} \le \alpha.(E_\alpha(\varphi))^{\frac{2\alpha-1}{2\alpha}}$. So, if $(\varphi_n)_{n\in\mathbb{Z}}$ is a bounded sequence in

2- η^* = dual vector bundle associated with the vector bundle η

$L^{1,2\alpha}(\eta)$ it follows that $\|dE_\alpha(\varphi_n)\|_{L^{1,\frac{2\alpha}{2\alpha-1}}}$ is bounded for all $n \in \mathbb{Z}$. Thus by the Theorem above it follows that the α-energy $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$ satisfies the (PS)-Condition. $\square$

The geometric meaning of (PS)-Condition is that if there is no critical values in the interval $[a,b] \subset (-\infty,\infty)$ for a functional $F : L^{r,p}(\eta) \to \mathbb{R}$ satisfying the (PS)-Condition then the curves generated by the gradient flow (∇F) are transverse to the submanifolds $F^{-1}(a+\varepsilon)$ and $F^{-1}(b-\varepsilon)$ for all sufficiently small $\varepsilon > 0$. This is the main step to produce the deformations which leads to critical points for the functional, because the deformation along such curves cannot pass through a critical point.

3.14. Definition: Let $\mathcal{F}$ be a family of subsets of a Banach manifold. We shall say that $\mathcal{F}$ is isotopy invariant if there exists an isotopy $\{\varphi_t\}$ of M so that $\forall F \in \mathcal{F}$ we have $\varphi_t(F) \in \mathcal{F}$, $\forall t$.

3.15. Definition: Let M be a differentiable manifold and $f : M \to \mathbb{R}$. We define the minimax of f relative to $\mathcal{F}$ by

$$\operatorname{minimax}(f,\mathcal{F}) = \inf_{F\in\mathcal{F}} \sup_{x\in F} |f(x)| .$$

The next theorem is the main result to prove existence of critical points. It needs the definitions of Finsler structure to replace the Riemannian structure in case the dimension of M is ∞ (see [1]).

3.16. Minimax Theorem: Let M be a complete C^2-Finsler manifold without boundary and $f : M \to \mathbb{R}$ a C^2-function satisfying (PS)-Condition. Let $\mathcal{F}$ be an isotopy invariant family of subsets of M so that $-\infty < \operatorname{minimax}(f,\mathcal{F}) < \infty$. Then either minimax(f,$\mathcal{F}$) is a critical value or there is a sequence of distinct values (c_k), $k \in \mathbb{Z}$ with $c_k \to$ minimax(f,$\mathcal{F}$) and such that f assumes the constant value c_k on a component of M .

Proof: See [1] , Thm 5.18 □

3.17 - Theorem : Let M be a complete C^2 Finsler manifold without boundary and $f: M \to \mathbb{R}$ a C^1 function satisfying condition (PS) . If f is bounded below on a componente M_0 of M then $f|_{M_0}$ assumes its greatest lower bound . If f is bounded below then either f assumes its greatest lower bound or else there is a sequence $\{M_k\}$ of components of M , on each of which f is constant , and such that $f(M_k) \to \operatorname{Inf}\{f(x) \mid x \in M\}$

Proof : See [1] , Thm 5.7

CHAPTER 4.

Existence of critical points for the energy in $L^{1,2}(\zeta)_s$.

In this section we prove the existence of critical points for the energy functional defined in the homotopy classes (1.1), (1.2) and (1.3) in $L^{1,2}(\zeta)_s$. The strategy is to prove the existence of these critical points for the α-energy by using the result in (3.16) . Then we pass to the limit $\alpha \to 1$.

As enunciated in (1.9), the critical points in the classes (1.2) are minima for the energy whilst in (1.1) and in (1.3) the critical points are of saddle type .

In those homotopy classes in (1.2) the minimum can be degenerated into a point or a closed curve on S . The closed curve turns out to be a geodesic on S .

4.0. Proposition: If $\varphi : M \to N$, $\varphi(\partial M) \subset S$, is a critical point for the energy $E : L^{1,2}(\zeta)_s \to \mathbb{R}$ and $\varphi(M)$ is a curve on S , then $\varphi(M)$ is a geodesic on S with respect to the metric $\tilde{h}_{ij} = \int \gamma^{11} h_{ij} dx_2$ defined on S , where (h_{ij}) is the induced metric from (N,h).

Proof: Once we have such degeneration we can assume that $\partial_2 \varphi = 0$. Therefore, locally we get

$$(4.1)\quad E(\varphi) = \tfrac{1}{2}\iint \gamma^{11}\partial_1\varphi^i\partial_1\varphi^j h_{ij} dx_1 dx_2 = \tfrac{1}{2}\int \partial_1\varphi^i\partial_1\varphi^j \left(\int \gamma^{11} h_{ij} dx_2\right) dx_1 .$$

So by the hypothesis we can consider a variation $\varphi_t : M \to N$, $\varphi_t(\partial M) \subset S$ such that $\varphi_0 = \varphi$ and $\frac{d(E(\varphi_t))}{dt}\Big|_{t=0} = 0$.

Having done this in (4.1) we get the geodesic equation. □

4.2. Proposition: For all homotopy classes defined by (1.2), the infimum of the α-energy $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$ is achieved in the class and it is non-trivial.

Proof: Since the (PS)-Conditon is verified by the α-energy, Theorem 3.17 claims that the infimum of this functional is achieved for all homotopy classes of maps in $L^{1,2\alpha}(\zeta)_s$. Since in the homotopy classes $\mathcal{F}_{\alpha\beta}$, defined by (1.2), the images of the boundary components are essential on S none of them collapse to a point. Therefore the α-energy of the minima is positive. Thus it follows that there exists $\varphi_\alpha \subset L^{1,2\alpha}(\zeta)_s$ such that $E_\alpha(\varphi_\alpha) = \inf_{f \in \mathcal{F}_{\alpha\beta}} E_\alpha(f) > 0$,

□

4.3. Corollary: In all homotopy classes defined by (1.2), the infimum of the energy functional $E: L^{1,2}(\zeta)_s \to \mathbb{R}$ is achieved and is non-trivial.

Proof: As $0 < E_{\alpha'}(\varphi_{\alpha'}) < E_\alpha(\varphi_\alpha)$ if $\alpha' < \alpha$, the sequence $(\varphi_\alpha)_{\alpha>1}$ (given by (4.2)) is in $L^{1,2}(\zeta)_s$. It follows from the weak compactness of the ball in $L^{1,2}(\zeta)_s$ that there exists a subsequence of $(\varphi_\alpha)_{\alpha>1}$ such that $\varphi_\alpha \to \varphi$ in $L^{1,2}(\zeta)_s$ and therefore $E(\varphi) = \inf \lim E_\alpha(\varphi_\alpha)$ as $\alpha \to 1$

□

The next result proves the existence of a critical point of saddle type and the method used is quite illustrative for the general situation.

4.4. Proposition: There exists a critical point of saddle type for the α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$ in the homotopy class of maps defined by (1.1).

Proof: The idea is to use the minimax principle. In this homotopy class it is clear that the infimum is achieved by the trivial maps. As we are looking for critical points of saddle type consider the following construction (here the condition (H) defined in (1.3) becomes clear to the problem). Assume $k = 1$ (the same construction can be made for a general k) then a curve $p: I \to \mathcal{F}_{\alpha\beta}$ induces a map $\hat{p}: I\times S^1 \to S^2$.

Consider the identification given by the projection $\pi: I\times M \to \dfrac{I\times M}{(\{0\}\times S^1) \cup (\{1\}\times S^1)}$, the restriction to the boundary defines $\hat{\pi}: I\times S^1 \to \dfrac{I\times S^1}{(\{0\}\times S^1) \cup (\{1\}\times S^1)} \approx S^2$

Let $q: \dfrac{I\times M}{(\{0\}\times S^1) \cup (\{1\}\times S^1)} \to N$ be a continuous map inducing a map $\bar{q}: \dfrac{I\times S^1}{(\{0\}\times S^1) \cup (\{1\}\times S^1)} \to S^2$ and assume $\text{degree}(\bar{q}) = 1$. Define a curve $p: I \to \mathcal{F}_{\alpha\beta}$ by $p = q \circ \pi$. By $\text{degree}(\hat{p}) = 1$ we mean $\text{degree}(\bar{q}) = 1$

Claim: p is not contractible in $L^{1,2\alpha}(\zeta)_s$

If it were, then would exist a homotopy $H: I\times I \to L^{1,2\alpha}(\zeta)_s$ such that $H_0(t) = p(t)$ and $H_1(t) = *$, $*$ a trivial map. However, this would imply $\text{degree}(\bar{q} \circ \pi) = 0$ but this is a contradiction with the fact that $\text{degree}(\bar{q}) = 1$.

Thus the class of maps defined in (1.1) admit a family of subsets, namely $\mathcal{F} = \{p: I \to \mathcal{F}_{\alpha\beta} \mid p = q \circ \pi$ and degree$(\bar{p}) = 1\}$, homotopic invariant by the gradient flow of E_α and no subset of this family is homotopic to the subset of trivial maps in $L^{1,2\alpha}(\zeta)_s$. So, applying the Minimax Principle, there exists $\varphi_\alpha : M \to N$,

$$\varphi_\alpha(\partial M) \subset S^2 \text{ such that } E_\alpha(\varphi_\alpha) = \inf_{p \in \mathcal{F}} \sup_{t \in [0,1]} E_\alpha(p(t)). \qquad \square$$

4.5. Proposition: There exists a critical point of saddle type for the α-energy $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$ $(\alpha > 1)$, in the homotopy classes defined by (1.3).

Proof: The idea for proving this is the same as in (4.4). The condition (H) is a generalization for the existence of map $\bar{p}$ with degree$(\bar{p}) = 1$. The subspace $U \subset S$ is defined by a identification $(\sim)$ on $I \times \partial M$ to obtain a space homeomorphic to S. If $\pi: I \times M \to (I \times M)/\sim$ defines the projection on the identified space and $h: (I \times \partial M)/\sim \to S$ is a homeomorphism then $U = h \circ \pi([\{0\} \times \partial M] \cup [\{1\} \times \partial M])$, $[.]$ = class defined by the identification "$\sim$" (see appendix 3).

Fix a homotopy class $\mathcal{F}_{\alpha\beta}$ and consider $p : I \to \mathcal{F}_{\alpha\beta}$ a continuous curve satisfying condition H and such that $p(0) = p(1) = g$, where $g : M \to N$ achieves the minimum for the α-energy in its class. The curve p induces a map $\bar{p}: I \times \partial M \to S$.

Claim: p is non-contractible in $L^{1,2\alpha}(\zeta)_s$

Suppose it is contractible, then there exists a homotopy $H : I \times I \to L^{1,2\alpha}(\zeta)_s$ such that $H_0(t) = p(t)$ and $H_1(t) = g$, and there are two possibilities :

(i) Suppose $g(M)$ is a closed curve on S and is homotopically equivalent to

$\delta_1 v ... v \delta_\ell$ (v = wedge of curves) on S ; $\delta_i : S^1 \to S$ for $i = 1,...,l$. Since we have assumed condition H for p we have $[\bar{p}(I x \partial M)] = \mu$ in $H_2(S,\mathbb{Z})$. However the homotopy deforms $\bar{p}(I x \partial M)$ into $g(\partial M)$. But this is a contradiction because $H_2(S,\mathbb{Z}) = \mathbb{Z}$ and $H_2(\delta_1 v ... v \delta_\ell, \mathbb{Z}) = 0$.

(ii) If $g(M)$ is not a curve , then $\bar{p}$ is homotopic to $g(\partial M)$, a set of curves and therefore the same argument applied in (i) works here.

Consider $\mathcal{F} = \{p: I \to \mathcal{F}_{\alpha\bar{\beta}} \mid p$ is continuous and satisfies condition H$\}$ a family of subsets in the homotopy class $\mathcal{F}_{\alpha\beta}$. So, using the Minimax Principle we get the existence of a critical point of saddle type $\varphi_\alpha \in L^{1,2\alpha}(\zeta)$ satisfying $E_\alpha(\varphi_\alpha) = \inf_{p \in \mathcal{F}} \sup_{t \in [0,1]} E_\alpha(p(t))$. □

Thus by the propositions above ((4.4) and (4.5)), there is a sequence of maps $(\varphi_\alpha)_{\alpha>1}$, which are critical points of saddle type for the sequence of α-energy functionals $(E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R})_{\alpha>1}$. The next step is to pass the limit $\alpha \to 1$ of these sequences . Meanwhile for the homotopy classes in (1.1) it is interesting to prove that the limit for these critical points are non-trivial .

Observe that if $\alpha' < \alpha$ then $E_{\alpha'}(\varphi_{\alpha'}) < E_\alpha(\varphi_\alpha)$; therefore by (2.7) $L^{1,2\alpha'}(\zeta)_s \subset L^{1,2\alpha}(\zeta)_s$ continuously .

As S is an embedded compact surface in N there exists a tubular neighbourhood V_s and a number $\delta_s > 0$ such that if $\mathrm{diam}(V_s) = \sup \inf_{x,y \in V_s} \mathrm{dist}(x,y)$ is less than δ_s then there exists a well defined projection $\mathrm{proj}_s : V_s \to S$ induced by (N,h) .

4.6. Definition: Define $\operatorname{diam}(f,S) = \sup_{w \in M} \inf_{s \in S} \operatorname{dist}(f(w),s)$, for all $f \in C^0(\zeta)_s$.

4.7. Lemma: Let $(\varphi_\alpha)_{\alpha>1}$ be a sequence of critical points of saddle type for the sequence of functionals $(E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R})_{\alpha>1}$. Then the sequence $(\varphi_\alpha)_{\alpha>1}$ converges in $L^{1,2}(\zeta)_s$ to a map $\varphi : M \to N$, $\varphi(\partial M) \subset S$, which is a critical point for the energy functional $E : L^{1,2}(\zeta)_s \to \mathbb{R}$. If the convergence takes place in $C^1(\zeta)_s$ for the homotopy classes defined in (1.1) and (1.3), then there exists a constant $c > 0$ such that $E(\varphi) = \lim E_\alpha(\varphi_\alpha) > c$.

Proof: As $0 < E_{\alpha'}(\varphi_{\alpha'}) \leq E_\alpha(\varphi_\alpha)$ if $\alpha' \leq \alpha$, it follows from the weak compactness of the ball in $L^{1,2}(\zeta)_s$, (4.4) & (4.5) that passing to the limit $\alpha \to 1$ at a subsequence of $(\varphi_\alpha)_{\alpha>1}$ there exists a map $\varphi \in L^{1,2}(\zeta)_s$ with the property that $E(\varphi) = \inf \sup E_\alpha(\varphi_\alpha)$ and being a critical point of saddle type to the energy $E{:}L^{1,2}(\zeta)_s \to \mathbb{R}$. In the classes (1.3) the homomorphisms β takes the boundary components of S to essential curves on S, hence we have that $E(\varphi) > 0$. In the homotopy classes in (1.1) assume that there exists a constant $k > 0$, independent on α, such that $E_\alpha(\varphi_\alpha) > k > 0$; then passing to the limit we have $E(\varphi) > 0$.

Claim: There exists a constant $k > 0$ independent of α such that $E_\alpha(\varphi_\alpha) > k$ $\forall\, \alpha > 1$. From the regularity of φ_α (proved in (5.47)) we can consider a curve $p{:}\, I \to C^1(\zeta)_s$ with the following properties:

(i) The curve p satisfies condition H,

(ii) $p(\frac{1}{2}) = \varphi_\alpha$,

(iii) $E_\alpha(p(t)) \leq E_\alpha(\varphi_\alpha)$ $\forall\, t$.

Assume $\operatorname{diam}(p(t),S) \leq \delta_s$ $\forall\, t$. If for each t we have $[\operatorname{proj}_s(p(t)(M)] = \mu$ in $H_2(S,\mathbb{Z})$ then we can define a homotopy $H{:}I{\times}I \to S$ such that $H_0(t) = \operatorname{proj}_s(p(t)(M))$

and $H_1(t) = A$, where A is a subspace of S such that $H_2(A,\mathbb{Z}) \neq \mathbb{Z}$. By continuity we can extend this homotopy for all t, however, the fact that $H_2(A,\mathbb{Z}) \neq \mathbb{Z}$ leads to a contradiction because by the condition H we have that $[p(I\times\partial M)] = \mu$ in $H_2(S,\mathbb{Z})$. Therefore there exists t_0 such that $[proj_s(p(t_0))] = \mu$ or $diam(p(t_0),S) > \delta_s$.

1- If $[proj_s(p(t_0)(M)] = \mu$ then $E_\alpha(p(t_0)) > c > 0 \Rightarrow E_\alpha(\varphi_\alpha) > c$.

2- Let $diam(p(t_0),S) > \delta_s$. Consider $x_\alpha \in M$ such that $diam(p(t_0),S) = dist(p(t_0)(x_\alpha),S)$ and let $B_r(x_\alpha)$ be a ball such that $diam(p(t_0)|B_r(x_\alpha), S) > 0$. Therefore $E_\alpha(p(t_0)) > E_\alpha(p(t_0)|B_r(x_\alpha)) = c_\alpha > 0$ and by (iii) $E_\alpha(\varphi_\alpha) > c_\alpha$. However $c_\alpha = c_\alpha(\delta_s)$ must to be greater than a constant $c > 0$, independent on α, otherwise we go back to the situation in (1). If the convergence $\varphi_\alpha \to \varphi$ takes place in $C^1(\zeta)_s$ then follows that $E(\varphi) > c > 0$.

□

CHAPTER 5.

Regularity for the maps $\varphi_\alpha \in L^{1,2\alpha}(\zeta)_s$, $\alpha > 1$, which are critical points of the α-energy.

In the last chapter it was proved that for the homotopy classes of maps defined in (1.1), (1.2) and (1.3) there exists a non-trivial map $\varphi \in L^{1,2}(\zeta)_s$ which is a critical point for the energy functional $E : L^{1,2}(\zeta)_s \to \mathbb{R}$. Now we are interested in proving the C^1-differentiability of $\varphi = \lim \varphi_\alpha$. According to theorem (2.3) , to prove that φ is C^1 it is sufficient to estimate the norm $\|\varphi\|_{L^{2,p}(\zeta)_s}$ and this is done by estimating $\|\varphi_\alpha\|_{L^{2,p}}$ for the sequence $(\varphi_\alpha)_{\alpha>1}$ and passing to the limit $\alpha \to 1$. However , it is not possible to obtain apriori bounds to the norm $\|\varphi_\alpha\|_{L^{2,p}}$, so the convergence possibly does not take place in $C^1(\zeta)_s$. In this chapter we are concerned with estimating $\|\varphi_\alpha\|_{L^{2,p}}$ for the sequence of critical points $(\varphi_\alpha)_{\alpha>1} \subset L^{1,2}(\zeta)_s$. The differentiability of the limit will be handled in the next chapter .

5.0. Lemma: Let φ_α be a critical point of the α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$. Then φ_α is a weak solution for the system of P.D.E.

$$(5.1) \qquad -\Delta_M \varphi_\alpha + (\alpha-1)\frac{\operatorname{trace}\left[(\nabla d\varphi_\alpha, d\varphi_\alpha) d\varphi_\alpha\right]}{1 + |d\varphi_\alpha|^2} + {}^N\Gamma(\varphi_\alpha)(d\varphi_\alpha, d\varphi_\alpha) = 0$$

where Δ_M = Laplace-Beltrami operator on (M,γ) .

${}^N\Gamma$ = Christoffel symbol induced on the vector bundle $\varphi^{-1}(TN)$ by the Riemannian structure on (N,h) .

Proof: Let $V \subset \text{int}\, M$ be an open set and $\Lambda \in T_\varphi L^{1,2\alpha}(\zeta)_s$ such that $\text{supp}(\Lambda) \subset V$. Consider a family of maps $\varphi_t = \exp_\varphi t\Lambda$, then $\varphi_0 = \varphi$ and $\frac{\partial \varphi_t}{\partial t}\Big|_{t=0} = \Lambda$. (For more details about the proof see [14, pg.14]).

The index α will be deleted to simplify the notation

$$E_\alpha(\varphi_t) = \tfrac{1}{2}\int_M (1 + |d\varphi_t|^2)^\alpha dM, \qquad \text{so,}$$

$$\frac{dE_\alpha(\varphi_t)}{dt}\Big|_{t=0} = \tfrac{1}{2}\frac{d}{dt}\int_M (1 + |d\varphi_t|^2)^\alpha dM = \frac{\alpha}{2}\int_M (1 + |d\varphi_t|^2)^{\alpha-1}\frac{d}{dt}(\langle d\varphi_t, d\varphi_t\rangle)\Big|_{t=0} dM =$$

$$= \alpha \int_M (1 + |d\varphi_t|^2)^{\alpha-1} \langle \nabla_{\partial/\partial t} d\varphi_t, d\varphi_t\rangle\Big|_{t=0} dM, \tag{5.2}$$

where $\nabla_{\partial/\partial t}$ is the covariant derivative induced on the vector bundle $T^*(M\times\mathbb{R})\otimes\varphi^{-1}(TN)$ over $M\times\mathbb{R}$.

Now for $X \in TM$, we have

$$(\nabla_{\partial/\partial t} d\varphi_t)X = \nabla^{\varphi_t^{-1}(TN)}_{\partial/\partial t}(d\varphi_t X) - d\varphi_t \cdot \nabla^{T(M\times\mathbb{R})}_{\partial/\partial t} X =$$

$$= \nabla^{\varphi_t^{-1}(TN)}_X \Big(d\varphi_t \cdot \frac{\partial}{\partial t}\Big) + d\varphi_t\Big[\frac{\partial}{\partial t}, X\Big] = \nabla^{\varphi_t^{-1}(TN)}_X \Big(\frac{\partial \varphi_t}{\partial t}\Big).$$

Then (5.2) becomes

$$= \alpha \int_M (1 + |d\varphi_t|^2)^{\alpha-1} \, . <\nabla\left(\frac{\partial\varphi_t}{\partial t}\right), d\varphi_t>|_{t=0} \, dM =$$

(5.3) $$= \alpha \int_M (1 + |d\varphi|^2)^{\alpha-1} . <\nabla A, d\varphi> dM =$$

$$= \alpha \int_M <\nabla^*((1 + |d\varphi|^2)^{\alpha-1} d\varphi), A > dM = 0$$

where $\nabla^* : A^p(\varphi^{-1}(TN)) \to A^{p-1}(\varphi^{-1}(TN))$ is the codifferential operator $(A^p(\varphi^{-1}(TN)) = C(\wedge^p T^* M \otimes \varphi^{-1}(TN))$ $\therefore$ $C(V)$ = space of sections in the bundle V).

So , φ is a weak solution for the equation

(5.4) $$\nabla^*[(1 + |d\varphi|^2)^{\alpha-1} d\varphi] = 0 \ .$$

For the case when $\rho \in A^1(\varphi^{-1}(TN))$, there exists an easy representation for the codifferential operator as $\nabla^* \rho = -\operatorname{trace}(\nabla\rho)$. So , (5.4) becomes

(5.5) $$\nabla^*[(1 + |d\varphi|^2)^{\alpha-1} d\varphi] = -\operatorname{trace}\{\nabla[(1 + |d\varphi|^2)^{\alpha-1} d\varphi]\} = 0$$

By Sobolev's Embedding Theorem in (2.3), φ_α is of Holder class $C^{1-1/\alpha}(\zeta)_s \subset C^0(\zeta)_s$, thus we can take the covariant derivative of the term $(1+|d\varphi|^2)^{\alpha-1} d\varphi$. After this computation the expression obtained for the eq. (5.5) is

$$(5.6) \quad -\operatorname{trace}\left[(\alpha-1)(1+|d\varphi|^2)^{\alpha-2}(\nabla d\varphi,d\varphi)d\varphi + (1+|d\varphi|^2)^{\alpha-1}\nabla d\varphi\right] = 0$$

$$(\alpha-1)(1+|d\varphi|^2)^{\alpha-2}\operatorname{trace}[(\nabla d\varphi,d\varphi)d\varphi] + (1+|d\varphi|^2)^{\alpha-1}\operatorname{trace}(\nabla d\varphi) = 0$$

but $\quad \operatorname{trace}(\nabla d\varphi) = -\Delta_M\varphi + {}^N\Gamma(\varphi)(d\varphi,d\varphi)$, therefore

$$-\Delta_M\varphi + (\alpha-1)\frac{\operatorname{trace}[(\nabla d\varphi,d\varphi)d\varphi]}{1+|d\varphi|^2} + {}^N\Gamma(\varphi)(d\varphi,d\varphi) = 0 .$$

□

5.7. Remark: (a) The trace in eq. (5.1) is the trace defined by the metric induced on each fibre of the tangent bundle TM over M .

5.8. Corollary: The local expression for the eq. (5.1) is given by

$$(5.9) \qquad \sum_{i,j}\gamma^{ij}(\nabla d\varphi)^\alpha_{ij} + (\alpha-1)\cdot\frac{\displaystyle\sum_{\ell,k,i,j,\mu,\nu}\gamma^{\ell k}\gamma^{ij}h_{\mu\nu}\partial_i\varphi^\mu\partial_k\varphi^\alpha(\nabla d\varphi)^\nu_{\ell j}}{1+\displaystyle\sum_{i,j,\mu,\nu}\gamma^{ij}h_{\mu\nu}\partial_i\varphi^\mu\partial_j\varphi^\nu} = 0$$

where $(\nabla d\varphi)^\alpha_{ij} = \partial^2_{ij}\varphi^\alpha - \sum_k {}^M\Gamma^k_{ij}\partial_k\varphi^\alpha + \sum_{\gamma,\beta}{}^N\Gamma^\alpha_{\gamma\beta}\partial_i\varphi^\gamma\partial_j\varphi^\beta$, and $\ell,k,i,j = 1,2 \quad \mu,\nu = 1,\dots,n$.

Proof: (See [4],pg14) Consider $(A,(x^i))$ $(i = 1,2)$ a local chart of M and $(B,(u^\alpha))$ a local chart of N . We denote by γ_{ij} and ${}^M\Gamma^i_{jk}$ the components of the metric and the Christoffel symbols of the Levi-Civita connection on M, and use the notation $h_{\alpha\beta}$ and ${}^N\Gamma^{\alpha}_{\beta\gamma}$ on N. Let the Latin indices be for M and the Greek for N.

$$\nabla_{\partial/\partial x^i}(d\varphi) = \nabla_{\partial/\partial x^i}\Big(\sum_{\alpha j} \varphi^{\alpha}{}_j\, dx^j \frac{\partial}{\partial u^{\alpha}}\Big) = \qquad \therefore \Big(\varphi^{\alpha}{}_j = \frac{\partial \varphi^{\alpha}}{\partial x^j}\Big)$$

(5.10)

$$= \sum_{\alpha j} \Big(\frac{\partial \varphi^{\alpha}{}_j}{\partial x^i}\Big) dx^i \frac{\partial}{\partial u^{\alpha}} + \sum_{\alpha j} \varphi^{\alpha}{}_j \big(\nabla^{T^*M}_{\partial/\partial x^i} dx^i\big) \frac{\partial}{\partial u^{\alpha}} + \sum_{\alpha j} \varphi^{\alpha}{}_j \Big(\nabla^{\varphi^{-1}(TN)}_{\partial/\partial x^i} \frac{\partial}{\partial u^{\alpha}}\Big) =$$

but $\quad \nabla^{T^*M}_{\partial/\partial x^i} dx^j = -\sum_k {}^M\Gamma^j_{ik}\, dx^k$ and

$$\nabla^{\varphi^{-1}(TN)}_{\partial/\partial x^i} \frac{\partial}{\partial u^{\alpha}} = \overset{TN}{\nabla}\Big(\sum_{\beta} \varphi^{\beta}{}_{,j}\, \partial/\partial u^{\beta}\Big) \frac{\partial}{\partial u^{\alpha}} = \sum_{\beta} \varphi^{\beta}{}_{,i} \sum_{\gamma} {}^N\Gamma^{\gamma}_{\beta\alpha} \frac{\partial}{\partial u^{\gamma}}$$

so that (5.10) becomes

$$= \sum_{\alpha j} \varphi^{\alpha}{}_{,ij}\, dx^j \frac{\partial}{\partial u^{\alpha}} - \sum_{\alpha j} \varphi^{\alpha}{}_{,j} \sum_k {}^M\Gamma^j_{ik}\, dx^k \frac{\partial}{\partial u^{\alpha}} + \sum_{\alpha j} \varphi^{\gamma}{}_{,j}\, dx^j \sum_{\beta} \varphi^{\beta}{}_{,j} \sum_{\gamma} {}^N\Gamma^{\gamma}_{\beta\alpha} \frac{\partial}{\partial u^{\gamma}}$$

$$= \sum_{\alpha j} \Big[\varphi^{\alpha}{}_{,ij} - \sum_k {}^M\Gamma^k_{ij} \varphi^{\alpha}{}_{,k} + \sum_{\gamma\beta} {}^N\Gamma^{\alpha}_{\beta\gamma} \varphi^{\gamma}{}_{,j} \varphi^{\beta}{}_{,i}\Big] dx^j \frac{\partial}{\partial u^{\alpha}}.$$

then

$$(5.11) \qquad (\nabla d\varphi)^{\alpha}_{ij} = \frac{\partial^2 \varphi^{\alpha}}{\partial x^i \partial x^j} - \sum_k {}^M\Gamma^k_{ij} \frac{\partial \varphi^{\alpha}}{\partial x^k} + \sum_{\alpha,\beta} {}^N\Gamma^{\alpha}_{\gamma,\beta} \frac{\partial \varphi^{\gamma}}{\partial x^i} \frac{\partial \varphi^{\beta}}{\partial x^j}.$$

By applying the expression above in (5.1) we get the eq. (5.9) □

In (5.0) we have information about the differential behaviour of the critical point φ_α of the α-energy functional in int M. Meanwhile the boundary of M is non-empty and some behaviour is expected along the boundary ∂M.

5.12. Proposition: Let φ_α be a critical point of the α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$ and consider V, a C^∞ vector field tangent to S, then

$$h\left(\frac{\partial\varphi_\alpha}{\partial n}(w)\,,\, V(\varphi_\alpha(w))\right) = 0\,,\ \text{almost everywhere in } \partial M\,.$$

$\left(\frac{\partial\varphi_\alpha}{\partial n} = d\varphi_\alpha . n\right.$, n is the normal field along ∂M, induced by the orientation on M)

Proof: Consider $\bar{v}$ as the extension of V in $T_\varphi L^{1,2\alpha}(\zeta)_s$.

From eq. (5.3) we have

$$\int_{\partial M} (1 + |d\varphi_\alpha|^2)^{\alpha-1}\, h(v, d\varphi_\alpha . n)dw - \int_M \langle \nabla^*\left((1 + |d\varphi_\alpha|^2)^{\alpha-1} d\varphi_\alpha\right), \bar{v} \rangle dM = 0\,,$$

but by (5.0) the second expression is zero. Therefore

$$\int_{\partial M} (1 + |d\varphi_\alpha|^2)^{\alpha-1}\, h(v, d\varphi_\alpha . n)dw = 0$$

and $\qquad h(v, d\varphi_\alpha . n) = 0$ a.e. $\qquad \square$

Let $i : S \to N$ be the embedding of S in N and $\bar{i}^{-1}(TN^\perp)$ the pull back of the normal vector bundle over S induced by the Riemannian structure on N. This normal bundle has a natural inner product defined on each fibre coming from the inner product induced on each fibre of TN by the Riemannian structure on N. Because of the embedding of S in N, there exist a neighbourhood V of S in N and a well defined projection $proj_s : V \to S$. Assume $V = exp_s U$, where U is a neighbourhood of section zero in $\bar{i}^{-1}(TN^\perp)$, then we can define $proj_s$ through the Riemannian structure on $\bar{i}^{-1}(TN^\perp)$ and because $i : S \to N$ is C^∞ we can also define a C^∞ reflection $R : V \to V$ throughout S.

The same construction is used to extend M throughout its boundary. Let C be a C^∞ collar of ∂M in M, i.e. $C = \partial M \times [0,\delta)$ for some small $\delta > 0$, then define a C^∞ reflection $r : C \to r(C)$.

5.13. Definition. Using the notation above define the C^∞ surface

$\tilde{M} = M \underset{\partial M}{\cup} r(C)$, where the identification is done by $id : \partial M \to \partial M$. For the maps $f \in C^0(\zeta)_s$ define an extended continuous map $\tilde{f} : \tilde{M} \to N$ by

$$\tilde{f}(x) = \begin{cases} f(x), & x \in M \\ R \circ f \circ r^{-1}(x), & x \in r(C) \end{cases}$$

In this way we define a C^∞ metric on $\tilde{M}$ by

$$(5.14)\qquad \tilde{\gamma}(x) = \begin{cases} \gamma(x)\,, & x \in M \\ \hat{\gamma}(x) = (dr^{-1}(x))^{*}.\gamma(r^{-1}(x)).(dr^{-1}(x)), & x \in r(C) \end{cases}$$

We call the attention that the reflection $R: V \to R(V)$ is not an isometry in N. Lets define a C^{∞} metric on $V \bigcup R(V)$ by

$$(5.15)\qquad \tilde{h}(x) = \begin{cases} h(x)\,, & x \in V \\ \hat{h}(x) = (dR^{-1})^{*}(x).h(x).dR^{-1}(x) \end{cases}$$

If $f \in L^{1,2\alpha}(\zeta)_s$ the extended $\tilde{f}$ belongs to the extended space of maps $L^{1,2\alpha}(\tilde{\zeta})_s = \{\tilde{f} : \tilde{M} \to N \mid f \in L^{1,2\alpha}(\zeta)_s\}$, $\tilde{\zeta} = \tilde{M} \times N$.

Now the boundary of M is in int $\tilde{M}$ and the study of the regularity along the boundary becames simpler.

The extended map $\tilde{\phi}_\alpha : \tilde{M} \to N$ has the property $T_{\tilde{\phi}_\alpha(w)}\tilde{\phi}_\alpha(\partial M) \perp T_{\tilde{\phi}_\alpha(w)}S$ for almost all $w \in \partial M$.

5.16. Proposition: A critical point for the extended functional $E_\alpha : L^{1,2\alpha}(\tilde{\zeta})_s \to \mathbb{R}$ is a weak solution for the system of P.D.E. given by

$$(5.17)\qquad -\Delta_{\tilde{M}}\varphi + (\alpha-1)\,\frac{\mathrm{trace}[(\nabla d\varphi, d\varphi)d\varphi]}{1+|d\varphi|^2} + {}^{N}\tilde{\Gamma}(\varphi)(d\varphi,d\varphi) = 0$$

where $\Delta_{\tilde{M}} = \tilde{\gamma}^{ij}\partial_{ij} - \tilde{\gamma}^{ij}({}^{M}\tilde{\Gamma})^{k}_{ij}\partial_k$, ${}^{N}\tilde{\Gamma}$ and ${}^{M}\tilde{\Gamma}$ are the Levi-Civita connexions associated with $\tilde{\gamma}$ and $\tilde{h}$.

Proof: The local expression of $|d\varphi|^2$ is

(5.18) $$|d\tilde{\varphi}(x)|^2 = \begin{cases} \gamma^{ij} h_{\mu\nu} \partial_i\varphi^\mu \partial_j\varphi^\nu , & x \in M \\ \tilde{\gamma}^{ij}\hat{h}_{\mu\nu}\partial_i\varphi^\nu\partial_j\varphi^\nu , & x \in r(C) \end{cases}$$

Let $\tilde{\varphi}$ be a critical point and consider $\Lambda \in T_\varphi L^{1,2\alpha}(\zeta)_s$ with support in C; then extending Λ to $\tilde{\Lambda}$ by the same process as in (5.13), we get $\tilde{\Lambda} \in T_{\tilde{\varphi}} L^{1,2\alpha}(\tilde{\zeta})_s$ with support in $C \cup r(C)$. Assume $\Lambda|_{\partial M} = 0$.

Define as in (5.0) $\tilde{\varphi}_t = \exp_{\tilde{\varphi}} t\tilde{\Lambda}$. In order to prove (5.17) some care must be taken to assure that the process of extending φ does not introduce extra terms along ∂M, i.e. there is no "corner". The way to see this is considering the border terms of both integrals in the expression (5.3).

By using the calculations in (5.0) and (5.12) we have

$$\frac{dE_\alpha(\tilde{\varphi}_t)}{dt}\Big|_{t=0} = -\int_M \langle\nabla^*[(1+|d\varphi|^2)^{\alpha-1}d\varphi], \Lambda\rangle dM +$$

$$\int_{\partial M} (1+|d\varphi|^2)^{\alpha-1} h(\Lambda, d\varphi.n)dw + \int_{\partial M} (1+|d\varphi|^2)^{\alpha-1} h(\Lambda, dR.d\varphi.n)dw -$$

$$-\int_{r(C)} \langle\nabla^*[(1+|d(R\circ\varphi\circ r^{-1})|^2)^{\alpha-1} d(R\circ\varphi\circ r^{-1})], dR\circ\Lambda\circ dr^{-1}\rangle dM =$$

$$= -\int_{\tilde{M}} \langle\nabla^*[(1+|d\tilde{\varphi}|^2)^{\alpha-1}d\tilde{\varphi}], \tilde{\Lambda}\rangle d\tilde{M} = 0 , \quad \tilde{n} = -n .$$

The second and third integral are zero, so, as in (5.0), we get that $\tilde{\varphi}_\alpha$ is a weak solution for the system of P.D.E. defined in (5.17). □

Once the local behaviour of $\tilde{\varphi}_\alpha$ is equal to φ_α, the symbol "~" will be deleted and from now on we will always be dealing with the extended situation.

According to the Sobolev Embedding Theorem in (2.3), to prove that the critical points $\varphi_\alpha \in L^{1,2\alpha}(\zeta)_s$ of the α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$ are in $C^1(\zeta)_s$ $(\alpha > 1)$ we need to obtain an estimate for the norm $\|\varphi_\alpha\|_{L^{2,p}(\zeta)_s}$, $p > 2$.

In this way the first result obtained is called Morrey's Growth Condition for E_α, which tells us about the local growth of α-energy.

5.19. Lemma: Let φ_α be a critical point of the α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$, $\alpha > 1$, i_M = injectivity radius of M and consider $B_r(x_0)$ a small ball with center at $x_0 \in \operatorname{int} M$ and radius $r < i_M$. Then there exist strictly positive constants $C_0(\alpha)$ and $\gamma(\alpha) > 0$ such that

$$\int_{B_r(x_0)} (1 + |d\varphi_\alpha|^2)^\alpha \, dM < C_0(\alpha) r^{\gamma(\alpha)} .$$

Proof: Let τ be a C^∞ real valued function with support in $B_{2r}(x_0)$, identically 1 in $B_r(x_0)$ and $|\nabla\tau| < 1/r$ in $B_{2r}(x_0)$.

Define $\bar{\varphi} = \int_{B_{2r}/B_r} \varphi_\alpha dM$ (mean-value) and $\bar{A} : M \to \mathbb{R}^k$ as

$\bar{\Lambda} = (\varphi_\alpha - \phi).\tau^2$, then $d\bar{\Lambda}$ is a section in $T^*M \otimes \varphi^{-1}(T\mathbb{R}^k)$.

From eq. (5.3) we have

$$(5.20) \qquad \int_M (1+|d\varphi|^2)^{\alpha-1} \langle \nabla\Lambda, d\varphi \rangle dM = 0 \quad \text{(delete the index } \alpha\text{)},$$

where $\nabla\Lambda$ is a section in the vector bundle $T^*M \otimes \varphi^{-1}(TN)$. In local coordinates

$$(5.21). \qquad \langle \nabla\Lambda, d\varphi \rangle = \gamma^{\mu\nu} h_{ij} (\nabla\Lambda)^i_\mu \, \partial_\nu \varphi^j, \quad \therefore \partial_\nu = \frac{\partial}{\partial x^\nu}.$$

Now consider the section in $T^*M \otimes \varphi^{-1}(TN)$ given by $\nabla\Lambda(\varphi) = P(\varphi).d\bar{\Lambda}(\varphi)$, where P is defined as in Chapter 2. So, (5.21) becomes

$$\langle \nabla\Lambda, d\varphi \rangle = \gamma^{\mu\nu} h_{ij} P_{i\ell} \{ \tau^2 \partial_\mu \varphi^\ell + 2\tau(\varphi - \phi)^\ell \partial_\mu \tau \} \partial_\nu \varphi^j, \quad \text{i.e.}$$

$$\int_{B_{2r}(x_0)} (1 + |d\varphi|^2)^{\alpha-1} \, \gamma^{\mu\nu} h_{ij} P_{i\ell} \{ \tau^2 \partial_\mu \varphi^\ell + 2\tau(\varphi - \phi)^\ell \partial_\mu \tau \} \partial_\nu \varphi^j \, dM = 0.$$

This implies

$$(5.22) \qquad \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |d\varphi|^2 \tau^2 dM \le c_1 \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |\tau| \, |\varphi - \phi| \, |d\varphi| \, |d\tau| \, dM +$$

$$+ c_2 \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |d\varphi|^2 |\varphi-\phi|\tau^2 \, dM \,, \quad c_1, c_2 \text{ are constants depending}$$

only on γ,h and P .

Because all functions $\gamma^{\mu\nu}$, h_{ij} and $P_{i\ell}$, for all $\mu,\nu = 1,2$ and $i,j,\ell = 1,\ldots,k$, are C^∞ and bounded above and below we can consider the norms in eq. (5.22) as the norm in $\mathbb{R}^k$.

Since φ is uniformly continuous in $B_r(x_0)$ $(r = r(\alpha))$ we can take r in such a way that $|\varphi(x) - \phi(x)| < \frac{1}{2c_2}$ for all $x \in B_{2r}(x_0)$, therefore

$$\int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |d\varphi|^2 \tau^2 dM \le c_3 \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |\tau| \, |\varphi-\phi| \, |d\varphi| \, |d\tau| \, dM$$

by using the inequality $ab \le \frac{a^2}{\delta} + \delta b^2$, $\delta = 2c_3$

$$\int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |d\varphi|^2 \tau^2 dM \le \tfrac{1}{2} \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} \tau^2 |d\varphi|^2 dM +$$

$$+ c_4 \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |d\tau|^2 |\varphi-\phi|^2 dM$$

then

$$(5.23) \qquad \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |d\varphi|^2 \tau^2 dM \le c_4 \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |d\tau|^2 |\varphi-\phi|^2 dM \le$$

$$\leq c_4 \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} \frac{|\varphi-\bar{\varphi}|^2}{r^2} dM$$

by using Young's inequality $a.b \leq \frac{a^{\alpha/\alpha-1}}{\alpha/\alpha-1} + \frac{b^{\alpha}}{\alpha} \therefore \frac{1}{\alpha} + \frac{\alpha-1}{\alpha} = 1$.

The left-hand side of ineq. (5.23) becomes bounded by

$$(5.24) \qquad \leq c_4 \int_{B_{2r}/B_r} (1+|d\varphi|^2)^{\alpha} dM + c_5 \int_{B_{2r}/B_r} \frac{|\varphi-\bar{\varphi}|^{2\alpha}}{r^{2\alpha}} dM$$

Applying the Poincaré inequality: $\int_{B_{2r}/B_r} |\varphi|^{2\alpha} dx \leq c.r^{2\alpha} \int_{B_{2r}/B_r} |d\varphi|^{2\alpha} dx$, $\forall\ \varphi \in L^{1,2\alpha}(B_{2r})$.

the ineq. (5.24) becomes bounded by

$$(5.25) \qquad \leq c_6 \int_{B_{2r}/B_r} (1+|d\varphi|^2)^{\alpha}\, dM , \quad \text{i.e.}$$

$$(5.26) \qquad \int_{B_{2r}} (1+|d\varphi|^2)^{\alpha-1} |d\varphi|^2\, dM \leq c_6 \int_{B_{2r}/B_r} (1+|d\varphi|^2)^{\alpha}\, dM .$$

Define $\Phi(s) = \int_{B_s(x_o)} (1+|d\varphi|^2)^{\alpha}\, dM$.

$$(5.27)\qquad \int_{B_r}(1+|d\varphi|^2)^\alpha dM = \int_{B_r}(1+|d\varphi|^2)^{\alpha-1}|d\varphi|^2 dM + \int(1+|d\varphi|^2)^{\alpha-1}dM$$

by (5.26) and (5.27).

$$(5.28)\qquad \int_{B_r}(1+|d\varphi|^2)^\alpha dM \le c_0\int_{B_{2r}/B_r}(1+|d\varphi|^2)^\alpha dM + \int_{B_r}(1+|d\varphi|^2)^{\alpha-1}dM$$

but

$$\int_{B_r}(1+|d\varphi|^2)^{\alpha-1}dM \le \Big(\int_{B_r} dM\Big)^{\frac{1}{\alpha}}\Big(\int_{B_r}(1+|d\varphi|^2)^\alpha\Big)^{\frac{\alpha-1}{\alpha}} \le c.r^{\frac{2}{\alpha}}[\Phi(r)]^{\frac{1}{2}}\ ,\ \text{if } \alpha < 2\ .$$

The inequality above implies that

$$\Phi(r) \le c_6(\Phi(2r)-\Phi(r)) + c.r^{\frac{2}{\alpha}}[\Phi(r)]^{\frac{1}{2}} \le c_6(\Phi(2r)-\Phi(r)) + c_7\, r^{\frac{2}{\alpha}}$$

because $\Phi(r) < E_\alpha(\varphi) < \infty$ and $\alpha < 2$

$$\Phi(r) \le \frac{c_6}{1+c_6}\Phi(2r) + c_7\, r^{\frac{2}{\alpha}}\ ,\ \text{dividing by } r^\gamma$$

$$(5.29)\qquad \frac{\Phi(r)}{r^\gamma} \le \mu 2^\gamma\,\frac{\Phi(2r)}{(2r)^\gamma} + c_7\,\gamma^{\frac{2}{\alpha}-\gamma}\ ,\ \text{where } \mu = \frac{c_6}{1+c_6} < 1 \quad \text{and}$$

$$2^\gamma = \frac{1+\mu}{2\mu} > 1 \Rightarrow \gamma > 0\ ,\ \text{defining } \nu = \mu 2^\gamma \text{ and } \psi(s) = \frac{\Phi(s)}{r^\gamma}\ .$$

Then ineq. (5.29) implies that

$$\sup_{0<s<r} (\psi(s)) \le \nu \sup_{0<s<r} (\psi(s)) + \nu \sup_{r<s<2r} (\psi(s)) + c_7 r^{\frac{2}{\alpha}-\gamma}$$

but

$$\sup_{r<s<2r} (\psi(s)) = \sup_{r<s<2r} \int_{B_s(x_0)} \frac{(1+|d\varphi|^2)^{\alpha}}{s^{\alpha}} dM \le \frac{E_\alpha(\varphi)}{r^{\alpha}} < k\,,\ k \text{ a constant.}$$

Therefore $\sup_{0<s<r} (\psi(s)) \le \frac{C_0}{1-\nu}$, hence

$$\int_{B_r(x_0)} (1+|d\varphi|^2)^{\alpha}\, dM \le C_0 r^{\gamma}\,,\ C_0 = \frac{C_0}{1-\nu} \text{ and } \gamma > 0\,. \qquad \square$$

It is worth mentioning we are working with the extended situation , therefore the estimate which is interesting is that of the norm $\|\,.\,\|_{L_0^{2,p}(\zeta)_s}$ in

$$L_0^{2,p}(\zeta)_s = \{f \in L^{2,p}(\zeta)_s \mid \operatorname{supp}(f) \subset \operatorname{int} \tilde{M}\}\,.$$

The next proposition is a Sobolev type theorem where i_M is the injectivity radius of M .

5.30. Proposition: The Sobolev space $L_0^{1,2}(\zeta)_s$ is continuously embedded in $L_0^4(\zeta)_s$ and for all $f \in L_0^{1,2}(\zeta)_s$ with $\mathrm{supp}(f) \subset V$, V an open set in $\mathrm{int}\, M$ and $\mathrm{diam}(V) < i_M$, we get

$$\int_V |f|^4 \, dM \le \int_V |f|^2 \, dM \int_V |df|^2 \, dM \ . \tag{5.31}$$

Proof: The proof is carried in a simpler situation because M admits a local chart (U, x^1) with $x^1 \in C^\infty$ and $x(V) \subset U$.

Consider $Q = \{(x_1, x_2) \in u \mid a_i \le x_i \le b_i,\ i = 1,2\}$ and define $g(x_1,x_2) = |f(x_1,x_2)|$

$$g^4(x_1,x_2) = g^2(x_1,x_2) . g^2(x_1,x_2) = \int_{a_1}^{x_1} \frac{\partial g^2}{\partial t}(t,x_2)dt \int_{a_2}^{y_2} \frac{\partial g^2}{\partial s}(x_1,s)dt$$

$$\int_Q |g(x_1,x_2)|^4 dx_1 dx_2 = \int_Q | \int_{a_1}^{x_1} \frac{\partial g^2}{\partial t}(t,x_2)dt . \int_{a_2}^{y_2} \frac{\partial g^2}{\partial s}(x_1,s)ds \,|\, dx_1 dx_2 \le$$

$$\le \int_Q [\, | \int_{a_1}^{x_1} \frac{\partial g^2}{\partial t}(t,x_2)dt \,|\,|\int_{a_2}^{x_2} \frac{\partial g^2}{\partial s}(x_1,s)ds \,|\,] dx_1 dx_2 \le$$

$$\le \int_Q [\int_{-\infty}^{\infty} |\frac{\partial g^2}{\partial t}(t,x_2)|dt \int_{-\infty}^{\infty} |\frac{\partial g^2}{\partial s}(x_1,s)|ds] dx_1 dx_2 \le$$

$$\leq \int_{a_2}^{b_2} \int_{-\infty}^{\infty} \left|\frac{\partial g^2}{\partial t}(t,x_2)\right| dt\, dx_2 \int_{a_1}^{b_1} \int_{-\infty}^{\infty} \left|\frac{\partial g^2}{\partial s}(x_1,s)\right| ds\, dx_1 \leq$$

$$\leq \Big[\int_Q |dg^2| dM\Big]^2 \leq \int_Q |g|^2 dM \int_Q |dg|^2\, dM\,.$$

The result follows because $|g| = |f|$, $|dg| = |df|$ and U can be approximated by small squares. □

In what follows we will be searching for estimates of the norm $\|\varphi_\alpha\|_{L_0^{2,2}(\zeta)_s}$, where φ_α is the critical point of α-energy. Since $\varphi_\alpha \in L_0^{1,2}(\zeta)_s$ it will be sufficient to estimate $\|\nabla d\varphi\|_{L_0^2(\zeta)_s}$. In local coordinates the expression obtained in (5.11) for $\nabla d\varphi$ is

$$(5.32) \qquad (\nabla d\varphi)_{ij}^{\alpha} = \partial_{ij}\varphi^\alpha - \Gamma_{ij}^k \partial_k \varphi^\alpha + \Gamma_{\gamma\beta}^{\alpha}\partial_i\varphi^\gamma \partial_j\varphi^\beta\,.$$

Because $\nabla d\varphi$ is a section in the vector bundle $T^*M \otimes T^*M \otimes \varphi^{-1}(TN)$ over M, the norm $|\nabla d\varphi|^2$ can be written in local coordinates as (notation $\partial_i = \partial/\partial x_i$, $\partial_{ij} = \partial^2/\partial x_i \partial x_j$)

$$(5.33) \qquad |\nabla d\varphi|^2 = h_{\alpha\beta}\gamma^{i\ell}\gamma^{jk}\partial_{ij}\varphi^\alpha \partial_{\ell k}\varphi^\beta - 2h_{\alpha\beta}\gamma^{i\ell}\gamma^{jp}\Gamma_{\ell p}^{k}\partial_{ij}\varphi^\alpha\partial_k\varphi^\beta +$$
$$+ 2h_{\alpha\beta}\gamma^{i\ell}\gamma^{jp}\Gamma_{\gamma\delta}^{\beta}\partial_{ij}\varphi^\alpha\partial_\ell\varphi^\gamma\partial_p\varphi^\delta + h_{\alpha\beta}\gamma^{i\ell}\gamma^{jp}\Gamma_{ij}^{k}\Gamma_{\ell p}^{e}\partial_k\varphi^\alpha\partial_e\varphi^\beta -$$
$$- 2h_{\alpha\beta}\gamma^{i\ell}\gamma^{jp}\Gamma_{ij}^{k}\Gamma_{\gamma\delta}^{\beta}\partial_k\varphi^\alpha\partial_\ell\varphi^\gamma\partial_p\varphi^\delta + h_{\alpha\beta}\gamma^{i\ell}\gamma^{jp}\Gamma_{\gamma\delta}^{\alpha}\Gamma_{\theta\eta}^{\beta}\partial_i\varphi^\gamma\partial_j\varphi^\delta\partial_\ell\varphi^\theta\partial_p\varphi^\eta\,.$$

By using the inequalities

(5.34) $\quad a^3 \le a^2 + a^4$

(5.35) $\quad ab \le \varepsilon^2 a^2 + \frac{1}{\varepsilon^2} b^2$

in (5.33), and the fact that M and N are compact Riemannian manifolds we have

(5.36) $\quad |\nabla d\varphi|^2 \le c_1 |d^2\varphi|^2 + c_2 |d\varphi|^2 + c_3 |d\varphi|^4$, where $d^2\varphi = (\partial_{ij}\varphi^\alpha)$

and $d\varphi = (\partial_i \varphi^\alpha)$, i.e. assuming $\varphi \in L_0^{1,2\alpha}(\zeta)_s$ and $\mathrm{supp}(\varphi) \subset V$ we have

(5.37) $\quad \|\nabla d\varphi\|^2_{L_0^2(\zeta)_s} \le c_1 \|d^2\varphi\|^2_{L_0^2(\zeta)_s} + c_2 \|d\varphi\|^2_{L_0^2(\zeta)_s} + c_3 \|d\varphi\|^4_{L_0(\zeta)_s}$,

by (5.31) $\quad \|d\varphi\|^4_{L_0(\zeta)_s} \le \int_V |d\varphi|^2 dM . \int_V |d^2\varphi|^2 dM$.

Then

(5.38) $\quad \|\nabla d\varphi\|^2_{L_0^2(\zeta)_s} \le (c_1 + E(\varphi|_V)) \|d^2\varphi\|^2_{L_0^2(\zeta)_s} + E(\varphi|_V)$.

5.39. – **Proposition.** If $\|\Delta_M \varphi\|^2_{L_0^2(\zeta)_s} < k(\varphi)$, where $k(\varphi) > 0$ depends on φ, then there exists constants $k_1(\varphi) > 0$ and $d > 0$ such that

$$\|d^2\varphi\|^2_{L^2_0(\zeta)_S} \leq k_1(\varphi) + d.\|d\varphi\|^2_{L^2_0(\zeta)_S} \quad (\Delta_M = \text{Laplace-Beltrami operator}).$$

Proof: Let $V \subset M$ be a open set contained in a local chart , then $(\Delta_M\varphi)^\alpha_{ij} = \gamma^{ij}\partial_{ij}\varphi^\alpha - \gamma^{ij}\Gamma^k_{ij}\partial_k\varphi^\alpha$ locally . As $\Delta_M\varphi$ is a section in $\varphi^{-1}(TN)$ we have

$$|\Delta_M\varphi|^2 = h_{\alpha\beta}\gamma^{ij}\gamma^{\ell p}\partial_{ij}\varphi^\alpha\partial_{\ell p}\varphi^\beta - 2\gamma^{ij}\gamma^{\ell p}h_{\alpha\beta}\Gamma^k_{\ell p}\partial_{ij}\varphi^\alpha\partial_k\varphi^\beta +$$

$$+ \gamma^{ij}\gamma^{\ell p}h_{\alpha\beta}\Gamma^k_{ij}\Gamma^e_{\ell p}\partial_k\varphi^\alpha\partial_e\varphi^\beta .$$

Therefore, by applying (5.35) to the inequality $\|\Delta_M\varphi\|^2_{L^2_0(\zeta)_S} < k(\varphi)$ we get

$$\|h_{\alpha\beta}\gamma^{ij}\gamma^{\ell p}\partial_{ij}\varphi^\alpha\partial_{\ell p}\varphi^\beta|^2 \leq k(\varphi) + \varepsilon^2\|d^2\varphi\|^2 + (1/\varepsilon)^2\|d\varphi\|^2 + c.\|d\varphi\|^2$$

However , the left hand side of inequality above has a lower bound given by the condition imposed on the metric h (see 1.0) . So , taking $v = (v^\alpha) = (\gamma^{ij}\partial_{ij}\varphi^\alpha)$ it follows that there exists a constant $k > 0$ such that

$$k.\|\gamma^{ij}\partial_{ij}\varphi^\alpha\|^2 \leq k(\varphi) + \varepsilon^2\|d^2\varphi\|^2 + (1/\varepsilon)^2\|d\varphi\|^2 + c.\|d\varphi\|^2$$

The metric γ has a lower bound $c_1 > 0$ because it is a Riemannian metric and M is compact . Thus , taking $\varepsilon > 0$ sufficiently small it follows that

$$|d^2\varphi|^2_{L^2_0(\zeta)_S} \le k_1(\varphi) + d.\|d\varphi\|^2_{L^2_0(\zeta)_S} \qquad \square$$

(5.40) **Proposition:** Let $\varphi_\alpha \in L^{1,2\alpha}(\zeta)_s$, $\alpha > 1$, be a critical point for the α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$, then there exists a constant k such that in a sufficiently small ball $B_r(x_0) \subset M$ ($r = r(\alpha)$)

$$\int_{B_r(x_0)} |d^2\varphi_\alpha|^2 \, dM < E(\varphi_\alpha|_{B_r(x_0)}).$$

Proof: (See [4],3.2) From (5.17) we know that φ_α is a weak solution for the equation

$$-\Delta_M\varphi_\alpha + (\alpha-1)\frac{\text{trace}[(\nabla d\varphi_\alpha, d\varphi_\alpha)d\varphi_\alpha]}{1+|d\varphi_\alpha|} + N_{\Gamma(\varphi_\alpha)}(d\varphi_\alpha, d\varphi_\alpha) = 0 .$$

Now, consider $\tau : B_{2r}(x_0) \to \mathbb{R}$ a C^∞ cut-off function such that $\tau = 1$ in $B_r(x_0)$, then (omit α)

$$\Delta_M(\tau\varphi) = \tau\Delta_M\varphi + 2d\varphi.d\tau + \Delta_M\tau.\varphi \text{, and (delete the index } \alpha)$$

$$\Delta_M(\tau\varphi) - 2d\varphi.d\tau - \Delta_M\tau.\varphi + (\alpha-1)\tau.\frac{\text{trace}[(\nabla d\varphi, d\varphi)d\varphi]}{1+|d\varphi|^2} + \tau\Gamma(\varphi)(d\varphi, d\varphi) = 0 .$$

Therefore

(5.41) $$|\Delta(\tau\varphi)| < c_1|d\varphi| + c_2|\varphi| + c_3(\alpha-1)\tau|\nabla d\varphi| + c_4\tau|d\varphi|^2 \ , \quad \text{and}$$

(5.42) $$|\Delta(\tau\varphi)|^2 < c_1|d\varphi|^2+c_2|\varphi|^2+c_3(\alpha-1)\tau^2|\nabla d\varphi|^2+c_4\tau^2|d\varphi|^4+$$
$$k_1|d\varphi|.|\varphi|+k_2\tau|d\varphi|.|\nabla d\varphi|+k_3\tau|d\varphi|^3+k_4(\alpha-1)\tau|d\varphi|.|\nabla d\varphi|+k_5\tau|d\varphi|.|d\varphi|^2+$$
$$+ k_6\tau^2|\nabla d\varphi|.|d\varphi|^2 .$$

By using the inequalities (5.34) and (5.35), the right-hand side of ineq (5.42) becomes:

(5.43) $$|\Delta(\tau\varphi)|^2 < c_2|\varphi|^2 + c_1|d\varphi|^2 + c_3(\alpha-1)\tau^2|\nabla d\varphi|^2 + c_4|d\varphi|^4 .$$

So, integrating on M, applying (5.38), (5.31) and the Poincaré inequality we get

$$\|\Delta(\tau\varphi)\|_{L^2(\zeta)_s} \leq k_1 E(\varphi|_{B_r(x_0)}) + k_2(\alpha-1)\|d^2\varphi\|^2_{L^2_0(\zeta)_s}$$
$$+ k_3\, E(\varphi|_{B_r(x_0)}).\|d^2\varphi\|^2_{L^2_0(\zeta)_s} + k_4\|\varphi\|_{L^{1,2}(\zeta)_s}$$

From (5.39) if we consider $k(\varphi) = k_1 E(\varphi|_{B_r(x_0)}) + k_2(\alpha-1)\|d^2\varphi\|^2_{L^2_0(\zeta)_s} +$
$k_3\, E(\varphi|_{B_r(x_0)}).\|d^2\varphi\|^2_{L^2_0(\zeta)_s} + k_4\|\varphi\|^2_{L^{1,2}(\zeta)_s}$

(also using $d^2(\tau\varphi) = \tau d^2\varphi + 2d\tau.d\varphi + d^2\tau.\varphi$), we get

$$[1 - (\alpha-1)k_2 - k_3E(\varphi|_{B_r(x_0)})]\, \|d^2\varphi\|^2_{L^2_0(\zeta)_s} \leq k.E(\varphi|_{B_r(x_0)})$$

Given $\varepsilon > 0$, assuming $C_0(\alpha).r^{\gamma(\alpha)} < \varepsilon$ in (5.19) and considering $\alpha\sim 1$ we get

(5.44) $$\|d^2\varphi_\alpha\|_{L^2_0(\zeta)_s} \leq k.E(\varphi_\alpha|_{B_r(x_0)}) .$$ □

5.45. – Remark: It will be a major problem to obtain a uniform bound to the energy of φ_α in (5.44) because it is not clear that the sequences $(C_0(\alpha))_{\alpha>1}$ and $(\gamma(\alpha))_{\alpha>1}$ are bounded above .

5.46. Corollary: If $\alpha > 1$ the critical point φ_α of the α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$ has a bounded norm $\|\varphi_\alpha\|_{L^{1,p}}$ for all $p > 2$, i.e. $\varphi_\alpha \in L^{1,p}(\zeta)_s$ for all $p > 2$.

Proof: Let (U,τ) and (V,λ) be C^∞ local charts , where $U \subset M$, $V \subset N$ and $\varphi:\tau(U) \to \lambda(V)$ is the local map . Define the curve $\sigma:I \to \tau(U)$ by $\sigma(t) = (y - x)t + x$,$x,y \in \tau(U) \subset \mathbb{R}^2$, and let $v:I \to \tau(U)$ be a C^∞-vector field such that $\nabla_{\sigma'}v = 0$. Finally define the curve $c:I \to \varphi^{-1}(T\mathbb{R}^n)$ by $c(t) = d(\varphi\circ\sigma)(t).v(t)$, then we have

$$c'(t) = [\nabla_{\sigma'}d(\varphi\circ\sigma)](t).v(t) = [\,(\nabla_{\sigma'}d\varphi)(t).(y - x)\,].v(t)$$

From $c(1) = c(0) + \int_0^1 c'(t)dt$ and the fact that $\varphi \in L_0^{2,2}(\zeta)_s$ we have

$$d\varphi(y).v(1) = \int_0^1 [(\nabla_{\sigma'}d\varphi)(t).(y - x)].v(t)\,dt\ , \quad \text{and}$$

$$|d\varphi(y)| \le K.[\int_0^1 |(\nabla_{\sigma'}d\varphi)(t)|^2 dt\,]^{1/2}.[\int_0^1 (y - x)^2 dt\,]^{1/2}$$

hence $\quad |d\varphi(y)|^p \le K\,\|\nabla d\varphi\|^p_{L^2_0(\zeta)_s}.\mathrm{diam}(M)^p$

and finally integrating on y we have

$$\|\varphi\|_{L_0^{1,p}(\zeta)_s} \leq K.\mathrm{diam}(M)^p.\mathrm{vol}(M).\|\varphi\|^p_{L_0^{2,2}(\zeta)_s} \qquad \square$$

5.47. Theorem: Let φ_α be a critical point of the α-energy functional $E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R}$, then φ_α is a C^1-differentiable map.

Proof: (See [4],2.3)Consider $B_r(x_0)$ a small ball and $\tau : B_r(x_0) \to \mathbb{R}$ a cut-off function as in (5.40). From inequality (5.41) and Minkowski's inequality

$$\|u + v\|_{L^p} < \|u\|_{L^p} + \|v\|_{L^p} \,. \tag{5.48}$$

It follows that there exist constants $c_i > 0$ $(i = 1,\dots,3)$ such that

$$\|\Delta(\tau\varphi_\alpha\|_{L^p} < c_1\|\varphi_\alpha\|_{L^{1,p}} + c_2(\alpha-1)\|\varphi_\alpha\|_{L^{2,p}} + c_3\|\varphi_\alpha\|_{L^{1,2p}} \,. \tag{5.49}$$

Let $c(p)$ be the norm of Δ^{-1} (see [4],2.3) as a map from $L_0^p \to (L^{2,p} \cap L_0^{1,2})$ on the disk.

Then from (5.43) we get

$$c(p)^{-1}\|\tau\varphi_\alpha\|_{L^{2,p}} < c_1\|\varphi_\alpha\|_{L^{1,p}} + c_2(\alpha-1)\|\varphi_\alpha\|_{L^{2,p}} + c_3\|\varphi_\alpha\|_{L^{1,2p}}$$

$$[c(p)^{-1} - c_2(\alpha-1)]\|\tau\varphi_\alpha\|_{L^{2,p}} < c_1\|\varphi_\alpha\|_{L^{1,p}} + c_3\|\varphi_\alpha\|_{L^{1,2p}} \,.$$

So, for $\alpha \sim 1$ and by (5.44) & (5.46) we have

$\|\varphi_\alpha\|_{L_0^{2,p}} < \infty$, i.e. $\varphi_\alpha \in L_0^{2,p}(\zeta)_s$. Therefore by Sobolev's Embedding Theorem $L_0^{2,p}(\zeta)_s \hookrightarrow C_0^1(\zeta)_s$, hence φ_α is C^1-differentiable. $\square$

5.50. Corollary: If $S \subset N$ is a C^∞ surface embedded in N then the curve $\varphi_\alpha(\partial M) \subset S$ is a C^1-differentiable curve.

Proof: This is a trivial consequence of (5.-47) once the boundary of M lies in $\mathrm{int}(\bar{M})$. The class of differentiability of embedding $i:S \to N$ appears in the method of extension through the reflection $R:V \to R(V)$ defined in (5.13) and decides which class of differentiability we have along the boundary.

$\square$

CHAPTER 6.

Existence of a harmonic map, proof of (1.9).

In the last section the main result was the C^1-differentiability of the critical points $(\varphi_\alpha)_{\alpha>1} \subset L^{1,2}(\zeta)_s$ of functionals $(E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R})_{\alpha>1}$. The bounds for the norm $|\varphi_\alpha|_{L^{2,p}}$ are dependent on α because the bounds to the norm $|\varphi_\alpha|_{L^{2,2}}$ were obtained by using sufficiently small balls with radius $r = r(\alpha)$ depending on the constants from lemma (5.19), therefore they are not apriori bounds. So, when we pass to the limit $(\alpha \to 1)$ may be the bound to the norm $|\varphi_\alpha|_{L^{2,p}}$ is tending to infinity

.6.0. **Proposition:** If for a given $\varepsilon > 0$ there exists a finite cover of M by balls $B_{r_i}(x_i)$ (with centre in x_i and radius r_i, $i = 1,\dots,l$) such that $\sup\limits_{1\le i\le l} \int_{B_{r_i}(x_i)} |d\varphi_\alpha|^2 < \varepsilon$, then as $\alpha \to 1$ there exists a subsequence of $(\varphi_\alpha)_{\alpha>1}$ such that $\varphi_{\alpha_k} \to \varphi$ in $C^1(\zeta)_s$.

Proof: With this apriori bound on the energy, we can see from the main estimates in (5.44) that there exists a constant $k > 0$, independent of α, such that $|\varphi_\alpha|_{L^{2,2}} < k$, hence (according to (5.46) and as in (5.47)) $|\varphi_\alpha|_{L^{1,p}}$ also have a bound independent of α for all $p > 0$ and we get that there exists a constant $c > 0$, independent of α, such that $|\varphi_\alpha|_{L^{2,p}} < c$. Hence it follows after passing to the limit that we have $|\varphi|_{L^{2,p}} < c$ and thus $\varphi \in C^1(\zeta)_s$.

□

The assumption in (6.0) is too strong because it is known from [21] (Eells–Wood) that does not exist any harmonic maps in $C^1(T^2, S^2)$ of degree 1. However, the assumption in (6.0) turns out to be true outside of a finite number of small balls in M.

In this section it is worth considering the extended situation as in (5.13). Here the symbol "~" will still be omitted unless it is necessary.

6.1. Proposition: Given a fixed $\varepsilon > 0$, there exists α_0 such that for $1 < \alpha < \alpha_0$ there are only a finite number of small balls on M where

$$\int_{Ball} (1 + |d\varphi_\alpha|^2)^\alpha \, dM > \varepsilon \quad \text{if} \quad \alpha < \alpha_0 .$$

Proof: (See [4], 4.3) Consider $(B_i)_{i=1}^{\ell}$ a finite cover of M with the property that anypoint $x \in M$ belongs to at most n balls. Because the energy of maps in $L^{1,2}(\zeta)_s$ is finite it follows that for all $1 < \alpha < \alpha_0$

$$\sum_i \int_{B_i} (1 + |d\varphi_\alpha|^2)^\alpha \, dM < n.K , \quad E_{\alpha_0}(\varphi_{\alpha_0}) < K .$$

Therefore the number of balls B_i such that $E(\varphi_\alpha|_{B_i}) > \varepsilon$ $(\alpha < \alpha_0)$ is bounded by $n.K/\varepsilon$.

□

6.2. Proposition: Given a sequence of critical points $(\varphi_\alpha)_{\alpha>1}$, there exists a subsequence of $(\varphi_\alpha)_{\alpha>1}$ and a finite subset of points $\{x_1,\dots,x_\ell\} \subset M$ such that the subsequence converges in $C^1(M\setminus\{x_1,\dots,x_\ell\},N)_s$ to a harmonic map $\varphi: M\setminus\{x_1,\dots,x_\ell\} \to N$ as $\alpha \to 1$.

Proof: (See [4],4.4)Consider a finite cover of M given by the balls $B(x_i,r_i)$ $(1\le i\le p)$ such that $B(x_i,r_i) \cap B(x_j,r_j) = \emptyset$ if $i,j = 1,\dots,\ell$. If we take for each $i = 1,\dots,\ell$ a small ball $B(x_i,\delta(\alpha)) \subset B(x_i,r_i)$ such that $\lim \delta(\alpha) = 0$ as $\alpha \to 1$, then from (6.1) given $\varepsilon > 0$ we can assume that $\sup E(\varphi_\alpha|_{M\setminus\bigcup B(x_i,\delta)}) < \varepsilon$. Therefore passing to the limit $\alpha\to 1$ we get a harmonic map in $C^1(M\setminus\{x_1,\dots,x_\ell\},N)_s$ as proved in (6.0).

□

The next lemma (first proved in [4]) is fundamental for the purpose of extending the harmonic map obtained in (6.2) to all M.

6.3 Lemma: Let $\varphi : D^2\setminus\{p\} \to N$ be a harmonic map with finite energy, then φ extends to a harmonic map $\Phi : D^2 \to N$.

Proof: (See [4],3.6) In Appendix 1.

6.4. Theorem ([4],4.6): If a sequence of critical points $(\varphi_\alpha)_{\alpha>1}$ admits a subsequence converging in $C^1(M\setminus\{x_1,\dots,x_\ell\}, N)_s$ but does not admit any subsequence converging in $C^1(M\setminus\{x_2,\dots,x_\ell\}, N)_s$, as $\alpha \to 1$, then there exists a non-trivial branched minimal immersion $\phi: S^2 \to N$.

Proof: Define $\mu_\alpha = \max_{x\in M} |d\varphi_\alpha(x)|$. From the compactness of M there exists

$x_\alpha \in M$ such that $\mu_\alpha = |d\varphi_\alpha(x_\alpha)|$ and $x_\alpha \to x_1$ as $\alpha \to 1$.

Let $V(x_\alpha, r_\alpha) = \exp_{x_\alpha} B(0, r_\alpha)$ be a neighbourhood of x_α , where $B(0, r_\alpha)$ is a ball in $T_{x_\alpha}M$ with centre at the origin and radius $r_\alpha < i_M$. then, we can define $T_\alpha : B(0, \mu_\alpha r_\alpha) \to B(0, r_\alpha)$ by

$$T_\alpha(x) = \mu_\alpha^{-1} x$$

and the sequence of maps $\phi_\alpha : B(0, \mu_\alpha r_\alpha) \to N$ by

$$\phi_\alpha(x) = \varphi_\alpha \circ \exp_{x_\alpha} \circ T_\alpha(x) = \varphi_\alpha(\exp_{x_\alpha}(\mu_\alpha^{-1} x)) .$$

This new sequence has the following properties:

(6.5) $\quad |d\phi_\alpha(x)| \leq 1$ for all $x \in B(0, \mu_\alpha r_\alpha)$.

Applying the chain rule we have $d\phi_\alpha(x) = (\mu_\alpha)^{-1} d\varphi_\alpha \circ d\exp_{x_\alpha}(x)$, hence

(6.6.) $\quad |d\phi_\alpha(0)| = 1$.

(6.7) $\quad \phi_\alpha$ is a weak solution of the PDE given by

$$\Delta_M \phi_\alpha + (\alpha - 1) \frac{\operatorname{trace}((\nabla d\phi_\alpha, d\phi_\alpha) d\phi_\alpha)}{\mu_\alpha^2 + |d\varphi_\alpha|^2} + \Gamma(\phi_\alpha)(d\phi_\alpha, d\phi_\alpha) = 0 . \tag{6.8}$$

once it is observed that $|d\phi_\alpha| = \mu_\alpha^{-2} |d\varphi_\alpha|$ (see proof of Thm 7.15)

$$\int_{B(0,\mu_\alpha r_\alpha)} (1+|d\phi_\alpha|^2)^\alpha dM = \mu_\alpha^{2(1-\alpha)} \int_{B(0,r_\alpha)} (\mu_\alpha^2 + |d\varphi_\alpha|^2)^\alpha dM \,. \tag{6.9}$$

We can assume as $\alpha \to 1$ that the sequence (r_α) tends to 0 and the sequence $(\mu_\alpha r_\alpha)$ tends to infinity. Then it follows from $0 < E(\varphi_\alpha) < E(\varphi_{\alpha'})$ for $\alpha < \alpha'$ and (6.5), that the sequence (ϕ_α) admits a subsequence converging to $\phi \in L^{1,2}(\mathbb{R}^2,N)$, since $B(0,\mu_\alpha r_\alpha) \to \mathbb{R}^2$ as $\alpha \to 1$. The metric γ_α in $T_{x_\alpha}M$ is induced by the metric γ on M, in fact $\gamma_\alpha(x) = \gamma(x_\alpha)$ for all $x \in T_{x_\alpha}M$. Hence the pair $\big(B(0,\mu_\alpha r_\alpha), \gamma_\alpha = \gamma(x_\alpha)\big)$ converges to $\big(\mathbb{R}^2, \bar\gamma = \gamma(x_1)\big)$, where $\bar\gamma$ is equivalent (as a quadratic form) to the Euclidean metric on $\mathbb{R}^2$.

So we can claim the following about $\phi : \mathbb{R}^2 \to N$:

(6.10) $\phi : \mathbb{R}^2 \to N$ is non-trivial by (6.6).

(6.11) ϕ is harmonic by (6.8).

(6.12) $E(\phi)$ is finite because

$$E(\phi) + E(\varphi|M-\{x_1\}) \le \overline{\lim_{\alpha\to 1}} \{E(\phi_\alpha|_{B(0,\mu_\alpha r_\alpha)}) + E(\varphi_\alpha|M-B(x_\alpha,r_\alpha))\} \le$$

$$\le \overline{\lim_{\alpha\to 1}} \{\mu_\alpha^{2(1-\alpha)} . E(\varphi_\alpha|_{B(x_\alpha,r_\alpha)}) + E(\varphi_\alpha|M-B(x_\alpha,r_\alpha)\} \le \overline{\lim_{\alpha\to 1}} E(\varphi_\alpha) \le E_\alpha(\varphi_\alpha)\,.$$

$(\, 2 > \alpha > 1 \Rightarrow \mu_\alpha^{2(1-\alpha)} \to 0$ as $\alpha \to 1\,)$

However, $\mathbb{R}^2$ with the Euclidean metric is conformally equivalent to the standard $S^2\setminus\{p\}$ (p is the north pole) through the stereographic projection. Applying (6.11), (6.12) and (6.3) we get a non-trivial harmonic map $\varphi : S^2 \to N$, which turns out to be a branched minimal immersion from the standard S^2 into N.

□

6.13. Theorem: Let $(\varphi_\alpha)_{\alpha>1}$ be a sequence of critical points of the functionals $(E_\alpha : L^{1,2\alpha}(\zeta)_s \to \mathbb{R})_{\alpha>1}$. Then either $\varphi_\alpha \to \varphi$ in $C^1(\zeta)_s$, or there exists a non-trivial harmonic map $\varphi : S^2 \to N$.

6.14. Remark: Describing the latest result with respect to φ instead of $\bar{\varphi}$ (the extended map according to (5.13)), the sphere which blew up at a point $x_1 \in \partial M$ turns out to be a disk when restricted to M only.

6.15. Theorem: If $\pi_2(N) = 0$ then the sequence $(\varphi_\alpha)_{\alpha>1}$ of critical points given by (4.2) admits a subsequence which converges in $C^1(\zeta)_s$ to a non-trivial harmonic map $\varphi : M \to N$ satisfying (1.5), (1.7) and (1.8) and minimizing the energy for each homotopy class defined in (1.2).

Proof: (See [4],5.1) As in (6.4) consider the sequence $(x_\alpha) \subset M$ such that

$$|d\varphi_\alpha(x_\alpha)| = \sup_{x\in M} |d\varphi_\alpha(x)|, \quad \lim_{\alpha\to 1} |d\varphi_\alpha(x_\alpha)| = \infty \text{ and } x_\alpha \to x_1 \text{ in } M .$$

Let $B(x_1,2r)$ be a small disk with center in x_1, such that x_1 is the only point where the continuity of the derivative of the map $\varphi = \lim \varphi_\alpha$ fails in $B(x_1,2r)$. Let η be a C^∞ function with $\operatorname{supp}(\eta) \subset B(x_1,2r)$ and $\eta = 1$ in $B(x_1,r)$. In (6.2) it was proved that there exists a harmonic map $\varphi : B(x_1,2r)\setminus\{x_1\} \to N$ with finite energy. Then by (6.3) there exists an extension $\Phi : B(x_1,2r) \to N$ and it is harmonic. Define a sequence of maps $\psi_\alpha : B(x_1,2r) \to N$ as

$$\Psi_\alpha(x) = \exp_{\varphi_\alpha(x)}\left[\eta(|x|).\exp^{-1}_{\varphi_\alpha(x)}\Phi(x)\right]. \tag{6.16}$$

That means

$$\psi_\alpha(x) = \begin{cases} \Phi(x)\,, & x \in B(x_1,r) \\ \varphi_\alpha(x)\,, & x \notin B(x_1,2r) \end{cases} \tag{6.17}$$

Therefore as $\alpha \to 1$ we have $\lim \psi_\alpha = \Phi$. By (6.17) ψ_α can be extended to $\psi_\alpha \in C^1(\zeta)_s$. The assumption that $\pi_2(N) = 0$ implies that φ_α and ψ_α are in the same homotopy class because the induced homomorphism between the fundamental groups are the same. However, φ_α is a map which minimizes energy in its class, so

$$E_\alpha(\varphi_\alpha|_{B(x_1,2r)}) \le E_\alpha(\psi_\alpha|_{B(x_1,2r)}) \quad \forall \alpha > 1\,, \text{ also} \tag{6.18}$$

$$\lim_{\alpha\to 1} E_\alpha(\psi_\alpha|_{B(x_1,2r)}) = E(\Phi|_{B(x_1,2r)})\,. \tag{6.19}$$

Therefore,

$$\overline{\lim}\, E_\alpha(\varphi_\alpha|_{B(x_1,2r)}) \le E(\Phi|_{B(x_1,2r)}) \le (2r)^2\pi\, |\Phi|^2_{1,\infty}\,. \tag{6.20}$$

So, giving $\varepsilon > 0$ we choose r such that $(2r)^2\pi\,|\Phi|^2_{1,\infty} < \varepsilon$ and we obtain an apriori estimate for the α-energy restricted to $B(x_1,2r)$, i.e.

$$\sup \int_{B(x_1,2r)} (1+|d\varphi_\alpha|^2)^\alpha dM < \varepsilon$$

By (6.0) $\varphi_\alpha \to \varphi$ in $C^1(B(x_1,2r),N)_s$, hence $\varphi = \Phi$. By the same process , we can prove the same result for the others points where the C^1 convergence fails . Therefore $\varphi_\alpha \to \varphi$ $C^1(\zeta)_s$.

□

For the homotopy classes defined in (1.1) and (1.3) φ_α is a saddle point for the α-energy , so it is no longer a minimum in its homotopy class and therefore more care is necessary in using the inequality (6.18). In this case we need the assumption that the set $S_h = \{\varphi:S^2 \to (N,h) \mid \varphi$ is harmonic and non-trivial map $\} = \emptyset$

As in the last theorem, consider the sequence of maps in $C^1(\zeta)_s$ given by

$$(6.21) \qquad \psi_\alpha(x) = \begin{cases} \varphi_\alpha(x)\,, & x \in M \backslash B_{2r}(x_1) \\ \exp_{\varphi_\alpha(x)}\left[\eta(|x|)\exp^{-1}_{\varphi_\alpha(x)}\Phi(x)\right], & x \in B_{2r}(x_1) \end{cases}$$

6.22. Lemma: Consider $\varphi_\alpha \in C^1(\zeta)_s$ a critical point of saddle type for the α-energy functional and $\psi_\alpha \in C^1(\zeta)_s$ as in (6.21) . Assuming $S_h = \emptyset$, there exists $\delta > 0$ sufficiently small such that if $r < \delta$ in (6.21) then

$$(6.23) \qquad E_\alpha(\varphi_\alpha \mid_{B(x_1,r)}) \le E_\alpha(\psi_\alpha \mid_{B(x_1,r)}) .$$

Proof: Let $x_1 \in M$ be a point of discontinuity for the derivative of map

$\varphi = \lim \varphi_\alpha$, $\alpha \to 1$, and $B(x_1,\delta)$ be a small ball in M with center at x_1 and radius $\delta > 0$. From the fact that $\varphi_\alpha \in C^1(\zeta)_s$ there exists $\varepsilon = \varepsilon(\alpha,\delta) > 0$ such that

$$(6.24) \qquad C(\alpha,\delta) = E_\alpha(\varphi_\alpha|_{B(x_1,\delta)}) < \varepsilon(\alpha,\delta) .$$

Define $d_\alpha = \sup\limits_{\substack{x \in \varphi_\alpha(B_\delta) \\ y \in \Phi(B_\delta)}} \operatorname{dist}(\varphi_\alpha(x), \Phi(y))$. Then there exists a constant $k(\alpha) > 0$ depending also on the metric of N , such that $k(\alpha).d(\alpha) > 0$ and assuming $\delta > 0$ sufficiently small we get

$$(6.25) \qquad k(\alpha).d_\alpha < E_\alpha(\psi_\alpha|_{B(x_1,\delta)}) \ .$$

From theorem 6.4 if x_1 is a point where the C^1 convergence fails for $\varphi = \lim \varphi_\alpha$ then

$$(6.26) \qquad \lim_{\substack{\alpha \to 1 \\ \delta \to 0}} C(\alpha,\delta) = \text{Area}(\phi,N) \ , \text{ where } \phi: S^2 \to N \text{ is harmonic}$$

(on S^2 φ is harmonic $\Leftrightarrow$ φ is a minimal branched immersion).

However, the hypothesis that $S_h = \emptyset$ implies

$$(6.27) \qquad \text{Area}(\phi,N) = 0 .$$

Therefore $\varepsilon(\alpha,\delta) \to 0$ as $\alpha \to 1$ and $\delta \to 0$.

Then for $\alpha \sim 1$ and $\delta \sim 0$ we have $\varepsilon(\alpha,\delta) \leq k(\alpha).d_\alpha$ and therefore

$$E_\alpha(\varphi_\alpha|_{B(x_1,\delta)}) \leq E_\alpha(\psi_\alpha|_{B(x_1,\delta)}) . \qquad \square$$

6.28. Theorem: If $S_h = \emptyset$ then the sequence of maps $(\varphi_\alpha)_{\alpha>1}$ given by (4.5) admits a subsequence which converges in $C^1(\zeta)_s$ to a non-trivial map $\varphi : M \to N$ in each homotopy classes defined in (1.1) and (1.3). The limit is a critical point of saddle type for the energy functional $E: L^{1,2}(\zeta)_s \to \mathbb{R}$, therefore it does satisfy (1.5), (1.7) and (1.8).

Proof: As in the proof of (6.15) define the map $\psi_\alpha \in C^1(\zeta)_s$.

Then $\psi_\alpha \to \Phi$ in $C^1(\zeta)_s$ and by (6.24) it follows that

$$E_\alpha(\varphi_\alpha|_{B(x_1,\delta)}) < E_\alpha(\psi_\alpha|_{B(x_1,\delta)})$$

Because $\psi_\alpha \to \Phi$ in $C^1(\zeta)_s$ we have $\lim\limits_{\alpha \to 1} E_\alpha(\psi_\alpha|_{B(x_1,\delta)}) = E(\Phi|_{B(x_1,\delta)})$.

Therefore

$$\overline{\lim_{\alpha\to 1}}\, E_\alpha(\varphi_\alpha|_{B(x_i,\delta)}) \le E(\Phi|_{B(x_1,\delta)}) \le \pi\delta^2 \|\Phi\|^2_{1,\infty} \quad .$$

By choosing $\varepsilon > 0$ sufficiently small and $\delta = \dfrac{1}{\|\Phi\|_{1,\infty}} \sqrt{\dfrac{\varepsilon}{\pi}}$ we get

$$\sup \int_{B(x_i,\delta)} (1 + |d\varphi_\alpha|^2)^\alpha dM < \varepsilon \quad .$$

Then, by (6.0) it follows that $\varphi_\alpha \to \Phi$ in $C^1(\zeta)_s$. □

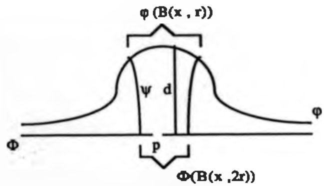


Fig 6.1 : $\pi_2(N) = 0$

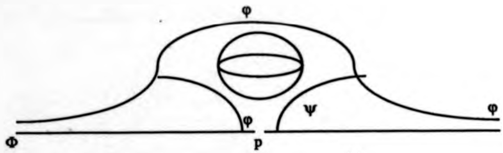


Fig 6.2 : $\pi_2(N) = 0$

CHAPTER 7.

Existence of a minimal surface, proof of (1.11).

The classical origin of the subject of minimal surface comes from the study of surfaces in a Riemannian manifold whose area is a minimum between all surfaces in a same homotopy class . It is natural to extend this notion defining minimal surfaces as critical points to the functional area.

7.0. **Definition:** A minimal surface (M,γ) in (N,h) is a map $\varphi \in C^r(\zeta)_s$ such that the following conditions are satisfied:

(7.1a) $\Delta_M \varphi + \Gamma(\varphi)(d\varphi, d\varphi) = 0$

(7.1b) There exist a Riemannian metric γ on M and a strictly positive function $\lambda : M \to \mathbb{R}$ such that
$\varphi^* h = \lambda\gamma$.

The class C^r is determined by the class of differentiability of the boundary conditions , since solutions to (7.1a) are C^∞ in int(M).

In order to define the area of a map in the general context of Riemannian Geometry it is convenient for our purpose to follow the ideas in [15] for the case in dimension 2.

Starting with the surface M and any smooth symmetric 2-covariant tensor field α on M , we fix a point $\varphi \in M$ and consider the eigenvalues of α relative to the metric tensor γ of M ; i.e. the n real roots of the equation

(7.2) $$\det[\lambda.\gamma(p) - \alpha(p)] = 0 .$$

In the case $\dim M = 2$, which we are interested in, the equation (7.2) can be written as:

(7.3) $$\lambda^2 + \mathrm{trace}(\alpha).\lambda + \frac{\det(\alpha)}{\det(\gamma)} = 0 ,\ \mathrm{trace}(\alpha) = \Sigma\, \gamma^{\mu\nu}\alpha_{\mu\nu} ,\ \gamma^{\mu\nu} = (\gamma^{-1})_{\mu\nu} .$$

Because $\alpha(\gamma)$ is a symmetric matrix there exist two real solutions, counted with multiplicity, for the equation (7.2) and therefore

(7.4) $$\left[\frac{\det(\alpha)}{\det(\gamma)}\right]^{\frac{1}{2}} \le \tfrac{1}{2}.\mathrm{trace}(\alpha) .$$

Now, for a map $f : (M,\gamma) \to (N,h)$ of class $L^{1,2}$ consider the symmetric non-degenerate 2-covariant tensor field f^*h .

7.5. Definition: Given a map $f :(M,\gamma) \to (N,h)$ of class $L^{1,2}$, the area of f is defined as:

(7.6) $$A(f) = \int_M \left[\frac{\det(f^*h)}{\det(\gamma)}\right]^{\frac{1}{2}} dM .$$

7.7 – Remark: For faster computation an equivalent definition for the area is given by:

$$A(f) = \int_M \sqrt{EG-F^2}\, dM \text{, where } E = f^*h(e_1,e_1),\ G = f^*h(e_2,e_2),$$

$F = f^*h(e_1,e_2)$ and e_1,e_2 are vector fields on M such that for all $p \in M$ the tangent vector space T_pM is spanned by $\{e_1(p), e_2(p)\}$.

7.8 – Definition: A map $\varphi : (M,\gamma) \to (N,h)$ is said to be conformal if there exists a strictly positive function $f : M \to \mathbb{R}$ such that

$$\varphi^*h = \lambda\gamma \text{, i.e. } h(\varphi(x))(d\varphi_x.,\, d\varphi_x.) = \lambda(x)\gamma(x)(.,.) \text{ for all } x \in M.$$

7.9 – Remark: The map φ is said to be branched conformal if there exists a point $p \in M$ such that $\lambda(p) = 0$; then p is a critical point of φ and is called a branched point.

The relation between the energy and the area is given by the following result from [15]:

7.10 – Proposition: Let M be a surface, then for any $f \in L^{1,2}(\zeta)_s$ we have $A(f) \leq E(f)$ and the equality holds if and only if f is a conformal map.

Proof: Consider $\alpha = f^*h$ in (7.3), then the inequality

(7.11) $\qquad A(f) \leq E(f)$

comes out from (7.4).

Assume $A(f) = E(f)$. This is equivalent to:

$$(7.12) \qquad \left[\frac{\det(f^*h)}{\det(\gamma)}\right]^{\frac{1}{2}} = \tfrac{1}{2}\,\mathrm{trace}(f^*h)\,.$$

By fixing a point $p \in M$ and looking at the equation (7.3) at p together with equality (7.12) we have that:

$$(7.13) \qquad \left[\lambda(p) - \frac{\mathrm{trace}(f^*h(p))}{2}\right]^2 = 0\,. \text{ i.e.}$$

$$(7.14) \qquad \lambda(p) = \tfrac{1}{2}\,\mathrm{trace}(f^*h(p))\,,\ \lambda(p) \geq 0 \quad \forall\, p \in M\,.$$

From the diagonalization process described by (7.2), we have that $f^*h(p) = \lambda(p).\gamma(p)$. From the C^∞ character of γ and $\lambda : M \to \mathbb{R}$ defined by (7.14), it follows that $f^*h = \lambda\gamma$ on M and therefore f is conformal.

The converse follows by assuming f is conformal, i.e. $f^*h = \lambda\gamma$ then

$$\det(f^*h) = \lambda^2 \det(\gamma) \text{ and } \mathrm{trace}(f^*h) = 2\lambda\,.$$

Hence the equality (7.12) is satisfied. □

As a consequence of proposition (7.10), the fundamental idea in this section is to replace the functional area by the functional energy in order to obtain minimal surfaces. However, that will succeed only if we can obtain a conformal map as a critical point for the energy.

The next proposition from [15] is one of the fundamental facts about the energy for our purpose of proving existence of minimal surfaces .

7.15 – Proposition: Let φ be any map in $L^{1,2}(\zeta)_s$. Consider M a surface and $\tau : (M,\gamma_1) \to (M,\gamma_2)$ a conformal diffeomorphism where γ_1 and γ_2 are Riemannian metrics on M . Then $E(\varphi\circ\tau) = E(\varphi)$.

Proof: According to the definition of energy, we get

$$E(\varphi\circ\tau) = \tfrac{1}{2}\int e_1(\varphi\circ\tau)(x)dM_1 \qquad\qquad N$$

$$E(\varphi) = \tfrac{1}{2}\int e_2(\varphi)(x')dM_2 \qquad\qquad (M,\gamma_1) \qquad\qquad (M,\gamma_2)$$

where $\quad e_2(\varphi)(x') = \gamma_2^{\mu\nu}(x')h_{ij}(\varphi(x'))\partial_\mu\varphi^i(x')\partial_\nu\varphi^j(x')$ and

$$e_1(\varphi\circ\tau)(x) = \gamma_1^{\mu\nu}(x)h_{ij}(\varphi\circ\tau(x))\partial_\mu(\varphi^i\circ\tau)(x)\partial_\nu(\varphi^j\circ\tau)(x)$$

but $\quad e_1(\varphi\circ\tau)(x) = \gamma_1^{\mu\nu}(x).\partial_\mu\tau^\rho(x).\partial_\nu\tau^\eta(x).h_{ij}(\varphi\circ\tau(x)).\partial_\rho\varphi^i(\tau(x)).\partial_\eta\varphi^j(\tau(x))$.

Note that $(d\tau.\gamma_1^{-1}.(d\tau)^t)_{\rho\eta} = \gamma_1^{\mu\nu}\partial_\mu\tau^\rho.\partial_\nu\tau^\eta$.

The hypothesis that τ is conformal implies that $\lambda\gamma_1 = \tau^*\gamma_2$, i.e.

$$\lambda(x).\gamma_1(x) = (d\tau)^t(x).\gamma_2(\tau(x))(d\tau)(x) \Rightarrow \gamma_2^{-1}(\tau(x)) = \lambda^{-1}(x).(d\tau)(x)\gamma_1^{-1}(x).(d\tau)^t(x).$$

Therefore $\quad e_1(\varphi\circ\tau)(x) = \lambda(\tau(x)).\gamma_2^{\mu\nu}(\tau(x)).h_{ij}(\varphi\circ\tau(x)).\partial_\rho\varphi^i(\tau(x))\,\partial_\eta\varphi^j(\tau(x)) =$

$$= \lambda(x').\gamma_2^{\mu\nu}(x').h_{ij}(\varphi(x')).\partial_\rho\varphi^i(x').\partial_\eta\varphi^j(x')\,.$$

Hence $e_1(\varphi\circ\tau)(x) = \lambda(x')e_2(\varphi)(x')$. The elements of area dM_1 and dM_2 are related by $dM_1 = \sqrt{\det[d\tau^{-1}]}\,dM_2$, but by the conformality $\det[(d\tau)^{-1}] = \lambda^{-2}$. Thus

$$E(\varphi\circ\tau) = E(\varphi)\,. \qquad \square$$

At this stage the concept of critical point for the energy will be extended by considering variations of the critical map in the target and variations of the conformal structure on M , i.e. variations on different Riemannian metrics such that there is no conformal diffeomorphism between them . The right description of this discussion is made by introducing the moduli space of conformal structures associated to a surface M .

In order to give an accurate definition of the moduli space of conformal structures we need some basic definitions. From the classification of surfaces we known that the genus g classifies topologically a compact oriented surface without boundary . From now on g means the genus of M .

7.16 – Definition: Consider M a closed surface with genus g .

(i) Consider the C^∞ vector bundle $\pi:(T^*M*T^*M)\otimes\mathbb{R} \to M$ ("$*$"=symetric product) . Define the space of C^∞ symmetric and positive definite bilinear forms on M as the space

$\mathcal{B}(g) = \{s : M \to (T^*M \otimes T^*M) \otimes \mathbb{R} \mid \pi \circ s = \text{identity},\ s \text{ is } C^\infty \text{ and } s(p)(v,v) > 0 \text{ for all } p \in M \text{ and } v \in T_pM\}$.

(ii) Define the following equivalence relation in $\mathcal{B}(g)$:

Given γ_1 and γ_2 in $\mathcal{B}(g)$ we say that $\gamma_1 \sim \gamma_2$ <=> there exist a diffeomorphism $f : M \to M$ and a strictly positive function $\lambda : M \to \mathbb{R}$ such $\lambda\gamma_1 = f^*\gamma_2$.

7.17 – Definition: The moduli space of conformal structures $\mathcal{R}(g)$ on a surface of genus g is defined as $\mathcal{R}(g) = \mathcal{B}(g)/\sim$. "$\sim$" = the equivalent relation defined in (7.16–(ii)) .

7.18 – Remark:

(i) Another definition could be given by defining an action $\alpha : (C_+(M) \otimes \mathrm{Diff}(M)) \times \mathcal{B}(g) \to \mathcal{B}(g)$ given by $(\lambda \otimes f)(\gamma) = \lambda f^*\gamma$; $C_+(M) = \{\lambda : M \to \mathbb{R} \mid \lambda > 0 \text{ and } \lambda \text{ is } C^\infty\}$ and $\mathrm{Diff}(M) = \{\tau : M \to M \mid \tau \text{ is a diffeomorphism}\})$. Then $\mathcal{R}(g)$ is the orbit space of action α .

(ii) $\mathcal{R}(g)$ turns out to be a real differentiable non-compact manifold with dimension $6g-6$ if $g \geq 2$ (see [16]), dimension 2 if $g = 1$ and dimension 0 if $g = 0$.

(iii) Each point of $\mathcal{R}(g)$ is a conformal structure on M .

7.19 – Proposition: If φ is a critical map of energy related to variations of φ and the conformal structure on (M,γ) , then φ is a minimal surface.

Proof:

(i) By considering a variation of φ as in (5.0) with $\alpha = 1$ and the regularity in (5.49), φ satisfies the equation

$$\Delta_M \varphi + \Gamma(\varphi)(d\varphi, d\varphi) = 0\,,\ \text{i.e. } \varphi \text{ is a harmonic map.}$$

(ii) A variation of conformal structure $\gamma \in \mathfrak{R}(g)$ is given by a curve $\sigma : (-\varepsilon,\varepsilon) \to \mathfrak{R}(g)$ with $\sigma(0) = \gamma$. A map $\varphi \in L^{1,2}(\zeta)_g$ is a critical point with respect to the conformal structure γ if

$$\frac{d}{dt} E_t(\varphi)|_{t=0} = \tfrac{1}{2}\frac{d}{dt}\Big[\int (\sigma(t))^{\mu\nu}(x) h_{ij}(\varphi(x))\partial_\mu\varphi^i(x)\partial_\nu\varphi^j(x)\sqrt{\det(\sigma(t))}\,dx\Big] = 0\,.$$

Consider $\frac{d\gamma}{dt}(0) = \Lambda$ and $\sigma(t) = \sigma_t$.

$$E_t(\varphi) = \tfrac{1}{2}\int \sigma_t^{\mu\nu} h_{ij}\,\partial_\mu\varphi^i\,\partial_\nu\varphi^j \frac{1}{\sqrt{\sigma_t^{11}\sigma_t^{22} - (\sigma_t^{12})^2}}\,dx$$

$$\frac{dE_t}{dt}(\varphi)|_{t=0} = \tfrac{1}{2}\int\Big[\Lambda^{\mu\nu}\frac{h_{ij}\partial_\mu\varphi^i\partial_\nu\varphi^j}{\sqrt{\det(\gamma^{-1})}} +$$

$$+\,\gamma^{\mu\nu}\frac{h_{ij}\partial_\mu\varphi^i\partial_\nu\varphi^j}{2(\det(\gamma^{-1}))^{3/2}}(\Lambda^{11}\gamma^{22} + \Lambda^{22}\gamma^{11} - 2\Lambda^{12}\gamma^{12})\Big]\,dx =$$

$$= \frac{1}{2}\int\Big\{\Lambda^{11}\Big[\frac{h_{ij}\partial_1\varphi^i\partial_1\varphi^j}{\sqrt{\det(\gamma^{-1})}} - \frac{(\gamma^{\mu\upsilon}h_{ij}\partial_\mu\varphi^i\partial_\upsilon\varphi^j)\gamma^{22}}{2[\det(\gamma^{-1})]^{3/2}}\Big] +$$

$$+ \Lambda^{22}\Big[\frac{h_{ij}\partial_2\varphi^i\partial_2\varphi^j}{\sqrt{\det(\gamma^{-1})}} + \frac{(\gamma^{\mu\nu}h_{ij}\partial_\mu\varphi^i\partial_\nu\varphi^j)\gamma^{11}}{2(\det(\gamma^{-1}))^{3/2}}\Big] +$$

$$+ 2\Lambda^{12}\Big[\frac{h_{ij}\partial_1\varphi^i\partial_2\varphi^j}{\sqrt{\det(\gamma^{-1})}} - \frac{\gamma^{\mu\nu}h_{ij}\partial_\mu\varphi^i\partial_\nu\varphi^j\gamma^{12}}{2(\det(\gamma^{-1}))^{3/2}}\Big]\Big\}\, dx = 0\,.$$

However, Λ is arbitrary, hence $\gamma^{11} = -\dfrac{\gamma_{22}}{(\det(\gamma))^2}$, $\quad \gamma^{22} = -\dfrac{\gamma_{11}}{(\det(\gamma))^2}$

and $\gamma^{12} = \dfrac{\gamma_{12}}{(\det(\gamma))^2}$, therefore

$$\gamma_{11} = \frac{2\varphi^* h(e_1,e_2)}{\mathrm{trace}(\varphi^* h)},\quad \gamma_{22} = \frac{2\varphi^* h(e_2,e_2)}{\mathrm{trace}(\varphi^* h)},\quad \gamma_{12} = \frac{2\varphi^* h(e_1,e_2)}{\mathrm{trace}(\varphi^* h)}.$$

So $\lambda\gamma = \varphi^* h$ and $\lambda(x) = \dfrac{\mathrm{trace}(\varphi^* h(x))}{2}$. Then according to (7.8) φ is conformal and by (i) and (ii) φ is a minimal surface.

□

7.20 – Remark: The proposition (7.19) justifies the definition of minimal surface given in (7.1). From (7.10) the Euler–Lagrange equations for the area functional is the same as the Euler Lagrange equation of the energy functional if φ is conformal.

According to the result obtained in (7.19), we shall be searching for a pair (γ,φ), in each homotopy class (1.1), (1.2) and (1.3), such that

$$(7.21) \qquad E(\varphi) = \inf_{\gamma\in R(g)} \ \inf_{F\in\mathcal{F}} \ \max_{f\in F} E_\gamma(f)$$

where E_γ is the energy defined on the Sobolev space $L^{1,2}(\zeta)_s$ which is defined by the metrics γ on M and h on N and take $\mathcal{F}$ as in (3.16). If we consider a sequence $(\gamma_n)\subset\mathcal{R}(g)$, then Theorem (1.9) gives us a sequence of smooth harmonic maps $\varphi_n:(M,\gamma_n)\to(N,h)$, $\varphi_n(\partial M)\subset S$ such that (7.21) becomes

$$(7.22) \qquad E(\varphi) = \inf_{\gamma_n\in\mathcal{R}(g)} E_{\gamma_n}(\varphi_n)\ .$$

We assume that the sequence (γ_n,φ_n) minimizes the energy.

7.23 - Lemma (Courant-Lebesgue): Let $f\in C^1(\zeta)_s$ and consider the set $C_r(x)=\{y\in M \mid \mathrm{dist}_M(x,y)=r\}$. Let δ be a constant, $0<\delta<1$. Then there is an r_0 with $\delta<r_0<\sqrt{\delta}$ and a function $\mathcal{E}:(0,1)\to\mathbb{R}$, depending only on K and M, such that $\mathcal{E}(t)\to 0$ as $t\to 0$ and such that

$$\mathrm{length}(f(C_{r_0}))\le\mathcal{E}(\delta)\ .$$

Proof: Consider $B_r(x)$ a small ball with center at x. If we consider $B_r(x)$ contained in a local chart (U,ψ) parametrized by polar coordinates then the energy of f is given by

(7.24) $$E(f) = \tfrac{1}{2}\int_{B_r(x)} \Big[a(r,\theta)h\Big(\frac{\partial f}{\partial r},\frac{\partial f}{\partial r}\Big) + b(r,\theta)h\Big(\frac{\partial f}{\partial r},\frac{\partial f}{\partial \theta}\Big) + \frac{c(r,\theta)}{r^2}h\Big(\frac{\partial f}{\partial \theta},\frac{\partial f}{\partial \theta}\Big)\Big]\sqrt{\det(\gamma)}\, r dr d\theta$$

where $a, b, c : u \to \mathbb{R}$ are C^∞-functions and $c(r,\theta) = \gamma^{11}\sin^2\theta + \gamma^{22}\cos^2\theta - r\gamma^{12}\sin 2\theta$. So, considering r small enough there exists $k' > 0$ such that $c(r,\theta) > k'$.

Therefore, there exists a constant $k > 0$ such that

(7.25) $$k\int_{B_r(x)} \frac{1}{r^2} h\Big(\frac{\partial f}{\partial \theta},\frac{\partial f}{\partial \theta}\Big)\, r\, dr\, d\theta < K .$$

The length of $f(c_r)$ is given by

(7.26) $$L(f(c_r)) = \int_0^{2\pi} \sqrt{h\Big(\frac{\partial f}{\partial \theta},\frac{\partial f}{\partial \theta}\Big)}\, d\theta < \sqrt{2\pi}.\Big[\int_0^{2\pi} h\Big(\frac{\partial f}{\partial \theta},\frac{\partial f}{\partial \theta}\Big)\, d\theta\Big]^{1/2}.$$

Let δ be a constant, $0 < \delta < 1$ such that $\delta < r < \sqrt{\delta}$. Because of the differentiability of f, the function $H(r) = \int_0^{2\pi} h(\partial_\theta f, \partial_\theta f)(r,\theta)d\theta$ achieves its minimum in the interval $[\delta, \sqrt{\delta}]$ at $r = r_0$. So, integrating in (7.25) we obtain

$$k.\ell n\left(\frac{1}{\sqrt{\delta}}\right)\left[\int_0^{2\pi} h\left(\frac{\partial f}{\partial \theta},\frac{\partial f}{\partial \theta}\right)d\theta\right](r_0) < K \Rightarrow \int_0^{2\pi} h\left(\frac{\partial f}{\partial \theta},\frac{\partial f}{\partial \theta}\right)d\theta < \frac{K}{k}.\left[\ell n\left(\frac{1}{\sqrt{\delta}}\right)\right]^{-1}$$

Then by (7.26) we have

$$L(f(c_{r_0})) < \sqrt{\frac{2\pi K}{\ell n(\frac{1}{\sqrt{\delta}})}}\,.$$

Therefore setting $\mathcal{E}(\delta) = \sqrt{\frac{2\pi K}{\ell n(\frac{1}{\sqrt{\delta}})}}$ the result follows. □

7.27 – Corollary: Let $(\gamma_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{R}(g)$. Consider $(\varphi_n)_{n=1}^{\infty}$ a sequence of harmonic maps obtained in (1.9), such that φ_n is a critical point to the energy with respect to the conformal structure γ_n defined on M. Then $(\varphi_n)_{n=1}^{\infty}$ is equicontinuous, whichever the conformal structure on M.

Proof: The maps φ_n are smooth. Consider $\bigcup B(x_\lambda, r_\lambda)$ a covering of M by balls of radius r_λ, where for all $\lambda \in \Lambda$ we have that r_λ satisfies lemma (7.23) for a constant δ ($0 < \delta < 1$). Therefore if $x_1, x_2 \in M$ and $dist_M(x_1, x_2) < r_\lambda$ for some $\lambda \in \Lambda$ then length of the geodesic of minimal length joining $\varphi_n(x_1)$ to $\varphi_n(x_2)$ is less than $L(\varphi_n(C_{r_\lambda}))$ $\forall n > n_0$. Since

$L(\varphi_n(C_r)) < \mathcal{E}(\delta)$ (by (7.23)) it follows that

$\text{dist}_N(\varphi_n(x_1), \varphi_n(x_2)) < \mathcal{E}(\delta) \Rightarrow (\varphi_n)_{n=1}^{\infty}$ is equicontinuous.

□

At this point it looks as if a sequence of harmonic maps which minimize energy must admit a convergent subsequence. Meanwhile, it is not clear if the limit is a conformal branched immersion. The reason for this doubt lies in the fact that we do not have any information about the sequence of conformal structures $(\gamma_n)_{n=1}^{\infty} \subset \mathcal{R}(g)$. For instance , it is easy to realise geometrically situations where the length of a curve c_n , with respect to the metric γ_n on M , can be approaching 0 as $n \to \infty$ and at the limit we get a surface topologically different . In this case the limit can not be a conformal branched immersion .

From now on the problem is to look for conditions to achieve the convergence of $(\gamma_n)_{n=1}^{\infty}$ in $\mathcal{R}(g)$.

A characterization of compact subsets in $\mathcal{R}(g)$ is described by the next result from [13].

7.28 - Theorem (Mumford's compactness): Let $\mathcal{R}(g)$ be the moduli space of conformal structures on a surface of genus g . For all $\varepsilon > 0$ the subset:

$\{\gamma \in \mathcal{R}(g) \mid$ all closed geodesics on (M,γ) have length $\geq \varepsilon\}$

is compact.

Proof: See Appendix 2.

The Sobolev Space $L^{1,2}(\zeta)_s$ depends on the Riemannian metrics defined on M and N. If (γ_n) is a sequence in $\mathcal{B}(g)$ $\left(7.16(i)\right)$ and h is a fixed metric on N, we use the notation $L^{1,2}(\zeta_n)_s$ when refering to the Sobolev Space associated to the metrics γ_n and h $\therefore$ $L^{1,2}(\zeta_n)_s = L^{1,2}((M,\gamma_n),(N,h))_s$.

It is important for our purposes to note that the definition of Sobolev Spaces $L^{1,2}(\zeta)_s$ is invariant by conformal transformation, i.e. if $f \in L^{1,2}(\zeta_1)_s$ and $f \in L^{1,2}(\zeta_2)_s$, then supposing $[\gamma_1] = [\gamma_2]$ it follows that $\|f\|_{L^{1,2}(\zeta_1)_s} = \|f\|_{L^{1,2}(\zeta_2)_s}$ (Thm 7.15). It is also easy to note that if $\gamma_n \to \gamma$ in $\mathcal{R}(g)$ then $\|f\|_{L^{1,2}(\zeta_n)_s} \to \|f\|_{L^{1,2}(\zeta)_s}$ $\left(\text{just look the local expression of } \|f\|_{L^{1,2}(\zeta_n)_s}\right)$, where $L^{1,2}(\zeta)_s = L^{1,2}((M,\gamma),(N,h))$

7.29 – Definition: Let $(\gamma_n)_{n\in\mathbb{Z}} \subset \mathcal{R}(g)$ be a sequence. If $\gamma_n \to \gamma$ in $\mathcal{R}(g)$, then we say that the sequence of Sobolev Spaces $L^{1,2}(\zeta_n)_s$ converge to $L^{1,2}(\zeta)_s$.

7.30 – Theorem: Let M be a surface with genus g and $F_{\alpha\beta}$ a homotopy class of maps defined in (1.1), (1.2) or (1.3). Assume $\pi_1(N) \neq 0$ and $g \geq 1$. If the homomorphisms $\alpha: \pi_1(M) \to \pi_1(N)$ and $\beta: \pi_1(\partial M) \to \pi_1(S)$ are monomorphisms, then for each such class there exists a minimal surface of genus g.

Proof: Let $(\gamma_n,\varphi_n)_{n=1}^{\infty}$ be a pair of sequence where $(\gamma_n)_{n=1}^{\infty} \subset \mathcal{R}(g)$, φ_n is a harmonic map given by (1.9) with respect to the conformal structure γ_n on M and $(\gamma_n,\varphi_n)_{n=1}^{\infty}$ minimizes the energy.

Because α is a monomorphism, all classes in $\pi_1(N)$ can be represented by the image of geodesics representing the generators of $\pi_1(M)$ and therefore the length of geodesics on (M,γ_n) must be greater than $\varepsilon > 0$ for all $n \in \mathbb{Z}$. In this way, it is impossible for the length of a geodesic representing a class in $\pi_1(M)$ to approach 0 for a sequence of maps according to the hypothesis. If we look the situation on the double of M, we can make the same analysis using the monomorphism β and conclude that no boundary component of (M,γ_n) has length less than $\varepsilon > 0$. Therefore by (7.26) the sequence (γ_n) converge to $\gamma \in \mathfrak{R}(g)$.

Then the sequence of spaces $L^{1,2}(\zeta_n)_s$ (associated with γ_n) converge to $L^{1,2}(\zeta)_s$ (associated with γ) and by the lower semicontinuity of the energy functional on $L^{1,2}(\zeta)_s$ it follows that $\varphi_n \to \varphi$ in $L^{1,2}(\zeta)_s$ (weakly). By the equicontinuity of sequence $(\varphi_n)_{n=1}^{\infty}$ it follows that $\varphi_n \to \varphi$ in $C^1(\zeta)_s = C^1((M,\gamma),(N,h))$ and that φ is a minimal surface. □

The assumption $\alpha : \pi_1(M) \to \pi_1(N)$ is a monomorphism is very particular in the general context. However, a better sufficient condition to prove the convergence in $\mathfrak{R}(g)$ was described by J. Douglas in [11]. Before defining the Douglas condition it is necessary to describe the notions involved.

In our context the surface M has k boundary components. The moduli space $\mathfrak{R}(g)$ has been defined for closed surfaces, so we define the double $\overline{M}$ of M by

$$\overline{M} = M \bigcup_{\partial} M, \text{ identified by } id : \partial \to \partial .$$

Then $\overline{M}$ is a closed surface with genus $g = 2g + k-1$ and admits an isometric involution $\sigma : \overline{M} \to \overline{M}$ ($\sigma^2 = id$) . The locus of points fixed under σ consist of k closed curves, called curves of transition ; $\sigma(C_i) = C_i$, $i = 1,...,k$ where $\{C_i \mid i = 1, ..., k\} = \partial M$.

A curve $C : I \to M$ will be called a proper curve on M when it is not contractible and either

(7.31)

(i) C is a simple closed curve on M , or

(ii) C is a simple arc with boundary on ∂M .

Therefore a proper curve can either have 1 component or 2 components on $\overline{M}$.

7.32 - Definition: Let M be a surface with genus g and k boundary components . Let $\overline{M}$ be its double , i.e. , a surface of genus $g = 2g+k-1$. A surface M' is a primary reduction of M if it results from M by a process divided in two stages :

- a proper curve $\tilde{c}$ is identified into a point $p \in M$ by a continuous map $\sigma: M \to M/\sim$, $(c \sim \{p\})$
- There exists a map $\tau: M/\sim \to M'$ such that it has only one discontinuity and it is at p .

So M' is a surface in one of the following cases:

(i) $M' = M_1 \cup M_2$ (2 components) , $g' = g_1 + g_2$, $k' = k_1 + k_2$, $g_i \geq 1$, $k_i \geq 1$, $i = 1,2$.

(ii) M' has genus $g' = g-1$ and $k' = k$.

(iii) M' has genus $g' = g$ and $k' = k-1$.

(iv) M' has genus $g' = g - 1$ and $k' = k + 1$

(v) M' has genus $g' = g - 1$ and $k' = k - 1$

A general reduction is a transformation of M such that it decreases the genus of M or the number of boundary components of M . The special case is in (iv) , but there the genus is decreased . All general reductions can be described by a sequence of primary reductions.

A map $\varphi : M \to N$, $\varphi(\partial M) \subset S$ can be considered as a map $\phi : \overline{M} \to N$ such that $\varphi = \phi \circ \sigma$ and $\phi(c_i) \subset S$, $i = 1 ,\dots, k$ where c_i are the transition curves.

As mentioned before a homotopy class $\mathcal{F}_{\alpha\beta}$ of maps $f \in C^0(\zeta)_s$ (assuming $\pi_2(N) = 0$) is fixed by the induced homomorphisms $\alpha : \pi_1(M) \to \pi_1(N)$ and $\beta : \pi_1(\partial M) \to \pi_1(S)$, where $f_* = \alpha$ and $(f|_{\partial M})_* = \beta$. In this way we define

(7.33) $$d(S, g, k, \alpha, \beta) = \inf_{f \in \mathcal{F}_{\alpha\beta}} E(f)$$

and for all possible general reductions M' of M we define

(7.34) $$d^*(S, g, k, \alpha, \beta) = \inf_{M'} d(S, g', k, \alpha, \beta) .$$

If $g = 0$ and $k = 1$ define $d^*(S, 0, 1, \alpha , \beta) = \infty$.

7.35 – Douglas Condition: Let $\mathcal{F}_{\alpha\beta}$ be a homotopy class of continuous maps from M into N with boundary lying on S and induced by α and β, then

$$d(S, g, k, \alpha, \beta) < d^*(S, g, k, \alpha, \beta).$$

The description of a surface according to the Uniformization Theorem for surfaces (see [17]) will be useful for the next steps in order to prove (1.10).

7.36 – Theorem (Uniformization):

(A) Let (M,γ) be a simply connected surface and γ a metric on M. Then M is conformally equivalent to one and only one of the following surfaces:

(i) $\mathbb{C}$

(ii) $\mathbb{C} \cup \{\infty\}$

(iii) $\mathbb{H} = \{z \in \mathbb{C} \mid |z| < 1\}$.

(B) Let (M,γ) be a closed surface of genus g and metric γ on M. Then it is conformally equivalent to one of the following cases:

(i) $g = 0 \Rightarrow M \sim \mathbb{C} \cup \{\infty\}$

(ii) $g = 1 \Rightarrow M \sim \mathbb{C}/\Gamma$

(iii) $g \geq 2 \Rightarrow M \sim \mathbb{H}/\Gamma$,

where Γ in (ii) and (iii) is a freely acting group of Möbius transformations on $\mathbb{C}$ (in (ii)) and $\mathbb{H}$ (in (iii)). Furthermore in both cases $\Gamma \cong \pi_1(M)$.

Proof: See [17].

7.37 - Remark:

(i) $\mathbb{C} \cup \{\infty\}$ is topologically equivalent to S^2.

(ii) In (7.36-(iii)) when $g \geq 2$, the group Γ is usually called a Fuchsian Group.

7.38 - Theorem: Let M be a surface with genus g and k boundary components. If we assume the Douglas Condition in (7.34) for the homotopy classes of maps defined by (1.1), (1.2) and (1.3), then there exists a minimal surface with genus g in each of these homotopy classes.

Proof: The proof will be given for the case of surfaces of genus ≥ 2 (see [12]) . For the cases $g = 0$ and $g = 1$ the proof is exactly the same . Because the surfaces with which we are working have boundaries, we will always be working with the double $\overline{M}$, so in order to avoid confusion we mind ourselves that the bar will be deleted and M and φ means $\overline{M}$ and $\overline{\varphi}$.

By the Uniformization Theorem a sequence of conformal structures can be considered as a sequence of Fuchsian Groups $(\Gamma_n)_{n=1}^{\infty}$. Let $(\Gamma_n,\varphi_n)_{n=1}^{\infty}$ be a pair of sequences where φ_n is a harmonic map given by (1.9) with respect to the structure Γ_n on M and $(\Gamma_n,\varphi_n)_{n=1}^{\infty}$ minimize the energy , i.e. , if $n_1 > n_2 \Rightarrow E_{n_1}(\varphi_{n_1}) < E_{n_2}(\varphi_{n_2})$.

(7.39) Suppose that the length of some closed geodesic of $\overline{M}$ approaches zero as $n \to \infty$, then one of the following situations occurs (Fig 7.1):

(i) The length of an interior geodesic in $\overline{M}$ approaches 0.

(ii) The length of a transition curve C_i in $\overline{M}$ approaches 0.

(iii) The length of a geodesic in $\overline{M}$ intersecting two different transition curves approaches 0.

(iv) The length of a geodesic in $\overline{M}$ intersecting one transition curve twice approaches 0.

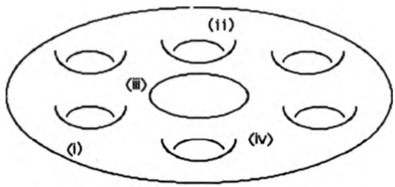


Fig 7.1 : $\overline{M} = M \bigcup_{\partial M} M$

We need two results in order to prove that the Douglas Condition is sufficient in achieving theorem (7.38).

Assume $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$.

7.40 - Lemma: Let c be a simple closed geodesic of length ℓ in $\mathbb{H}/\Gamma$. Then there is a collar of area $\frac{\ell}{\sinh \ell/2}$ around γ, i.e. $\mathbb{H}/\Gamma$ contains an isometric copy of the region

$$\{re^{i\varphi} \in \mathbb{H} \mid 1 \leq r < e^{\ell} \mid \arctan \sinh(\ell/2) < \varphi < \pi - \arctan \sinh(\ell\backslash 2)\}$$

where c corresponds to $\{re^{i\pi/2}, 1 \leq r \leq e^{\ell}\}$; $\{r = 1\}$ and $\{r = e^{\ell}\}$ are identified via $z \to e^{\ell}z$.

Proof: See [18].

7.41 - Lemma: Suppose $c\colon I \to M$ is a curve which is absolutely continuous on $\partial B(x_0,r)$ and

$$\int_0^{2\pi} |c'(\theta)|\, d\theta \leq \frac{\lambda}{\pi}.$$

Then there exists $f \in L^{1,2} \cap C^0(B(x_0,r),M)$ with

$$f|_{\partial B(x_0,r)} = c \quad \text{and} \quad E(f,B(x_0,r)) \leq \frac{\mu}{\lambda}\int_0^{2\pi} |c'(\theta)|^2\, d\theta.$$

Proof: See ([19], Lemma 9.4.8b).

Now we prove that the cases described in (7.39) (i),(ii),(iii) and (iv) cannot occur once we have assumed the Douglas Condition.

7.39–(i): Assume that the length of an interior geodesic c_n with respect to the structure induced by Γ_n approaches 0, i.e.

$$\ell(c_n) = \varepsilon_n \to 0 \quad \text{as } n \to \infty .$$

By the Lemma 7.40, (M,Γ_n) contains a collar isometric to

$$\{re^{i\psi} \in \mathbb{H} \mid 1 \le r < e^{\varepsilon_n},\ \arctan\left(\sinh \frac{\varepsilon_n}{2}\right) < \psi < \pi - \arctan(\sinh\frac{\varepsilon_n}{2})\},$$

where c_n corresponds to $\{re^{i\pi/2}\}$ and the boundaries are identified via $z \to e^{\varepsilon_n} z$.

Consider the parametrization for the unit disk given by

$$\text{(7.42)} \qquad t : \{e^{i\theta} \mid 0 \le \theta \le 2\pi\} \longrightarrow \{re^{i\psi} \mid 1 \le r \le \exp(\varepsilon_n)\}$$

$$(1,\theta) \longrightarrow \exp(i\psi).\exp\left(\frac{\varepsilon_n}{2\pi}\theta\right)$$

where ψ will be fixed.

Now consider curves $d_n^{\,i} : I \to N$, $i = 1,2$, defined as

$d_n^{\,i}(1,\theta) = \varphi_n\left(\exp\left(\frac{\varepsilon_n}{2\pi}\theta\right),\psi\right)$, and let $[-\varepsilon,\varepsilon] \times c_n$ be a neighbourhood of c_n in M, where $\{-\varepsilon\} \times c_n = d_n^{\,1}$ and $\{\varepsilon\} \times c_n = d_n^{\,2}$.

$$\text{(7.43)} \qquad \int_0^{2\pi} \left| \frac{\partial}{\partial\theta} d_n(1,\theta) \right|^2 d\theta = \int_1^{\exp(\varepsilon_n)} \left| \frac{\partial}{\partial r} \varphi_n(r,\psi) \right|^2 \left(\frac{\varepsilon_n}{2\pi}\right) r \, dr.$$

(7.44) Since

$$\int_{\pi/4}^{\pi/2}\int_{1}^{\exp(\varepsilon_n)} \left|\frac{\partial}{\partial r}\varphi_n(r,\psi)\right|^2 \frac{r\,dr\,d\psi}{r^2\sin^2\psi} \le 2\,E(\varphi_n)\ ,$$

there exists $\psi_n^{\,i} \in (\frac{\pi}{4},\frac{\pi}{2})$, $i=1,2$, such that choosing $\psi=\psi_n^{\,i}$ in (7.41) we get

(7.45) $$\int_0^{2\pi} \left|\frac{\partial}{\partial\theta}\varphi_n(1,\theta)\right|^2 d\theta \le k_1\varepsilon_n\ , \quad (E(\varphi_n) \text{ is uniformly bounded}).$$

If ε_n is small enough then by (7.41) $d_n : I \to N$ can be extended to a continuous function $d_n : D^2 = \{z \in \mathbb{C} \mid |z| \le 1\} \to N$ satisfying

(7.46) $E(d_n) \le k_2\,\varepsilon_n$ (k_2 independent of n).

Now we make a surgery on (M,Γ_n) by cutting off the collar $[-\varepsilon,\varepsilon] \times c_n$ on M and pasting two disks D_1 and D_2 by identifying ∂D_1 to $\{-\varepsilon\} \times c_n$ and ∂D_2 to $\{\varepsilon\} \times c_n$. In this way we can define a new map

$\phi: (M - \{[-\varepsilon,\varepsilon] \times c_n\}) \bigcup D_1 \bigcup D_2 \to N$ by

$$\phi(x) = \begin{cases} \varphi(x)\ , & x \in (M - \{[-\varepsilon,\varepsilon] \times c_n\}) \\ d_n^{\,i}(x)\ , & x \in D_i \end{cases}$$

Throughout the surgery process we obtain a primary reduction of M where the genus of M was decreased , also it follows that

(7.47) $$E(\phi) = E\left(\varphi_n|M - \{[-\varepsilon,\varepsilon] \times c_n\}\right) + E(d_n^1|D_1) + E(d_n^2|D_2) \le E(\varphi_n) + 2\,k_2\,\varepsilon_n\,.$$

Since $\varepsilon_n \to 0$ and $E(\varphi_n) \to d(S, g, \alpha, \beta)$ we conclude that

$d^*(s, g, \alpha, \beta) \le d(s, g, \alpha, \beta)$, a contradiction. □

7.39-(ii): In this case we can use the same details used in case 7.39-(i) by supposing the length of a transition curve is approaching 0 . After the surgery we obtain a primary reduction where the genus is maintained but the number of boundary components is decreased . Then we get a inequality like (7.47) and a contradiction for the assumption of Douglas Condition .

7.39-(iii): Assume that the length of a geodesic c_n on (M, Γ_n) intersecting two transaction curves t_1 and t_2 approaches 0 .

By (7.38) c_n admits a collar isometric to

$$\{re^{i\psi} \in \mathbb{H} \mid 1 \le r \le \exp\left(\frac{\varepsilon_n}{2}\right), \frac{\pi}{4} \le \psi \le \frac{3\pi}{4}\}, \text{ where}$$

t_1 corresponds to the image of $\{e^{i\psi} \mid \frac{\pi}{4} \le \psi \le \frac{3\pi}{4}\}$,

t_2 corresponds to $\{\exp\left(\frac{\varepsilon_n}{2}\right).e^{i\psi} \mid \frac{\pi}{4} \le \psi \le \frac{3\pi}{4}\}$ and c_n corresponds to $\{re^{i\psi_n} \mid 1 \le r \le \exp(\varepsilon_n)\}$.

As in case (7.39-(ii)) consider $\ell(c_n) = \varepsilon_n$ and define the curve $d_n : I \to N$ by $d_n(1,\theta) = \varphi_n\left(\exp\left(\frac{\varepsilon_n}{2\pi}\theta\right), \psi_n\right)$.

By (7.45)

$$\int |d_n'(\theta)|^2 \, d\theta < k_1 \, \varepsilon_n .$$

Then cutting M along c_n we can paste 2 disks $d_n^1 : D_1 \to N$, $d_n^2 : D_2 \to N$ along each curve resulting from c_n , where each disk satisfies $E(d_n^1) < k_1 \, \varepsilon_n$. Therefore

$$E\big(\varphi_n | M-(D_1 \cup D_2)\big) + E(d_n^1 | D_1) + E(d_n^2 | D_2) \le E(\varphi_n) + 2\, k_1 \, \varepsilon_n .$$

Since $\varepsilon_n \to 0$ and $E(\varphi_n) \to d(s, g, \alpha, \beta)$ it follows that

$$d^*(S, g, \alpha, \beta) \le d(S, g, \alpha, \beta) .$$

However , from the surgery it results a primary reduction of M by decreasing the number of boundary components by 1 and keeping the same genus . Thus we get a contradiction with the assumption of Douglas's Condition.

7.39–(iv): In this case the process is exactly the same. The contradiction comes from the fact that the surgery leads to a primary reduction of M by decreasing the genus of M by 1 and increasing the number of boundary components by 1 .

We conclude that the Douglas Condition implies that no closed geodesic on (M,Γ_n) has length approaching 0 , therefore by Mumford's Theorem the sequence of groups $(\Gamma_n)_{n=1}^{\infty}$ converges to Γ , which defines a conformal structure on $\mathfrak{R}(g)$.

Then the sequence of Sobolev Spaces $L^{1,2}(\zeta_n)_s$ defined on (M,Γ_n) , converges to the Sobolev Space $L^{1,2}(\zeta)_s$ defined on (M,Γ) , hence the lower semicontinuity of the energy plus the equicontinuity of sequence $(\varphi_n)_{n\in\mathbb{Z}}$ implies that $\varphi_n \to \varphi$ in $C^1(\zeta)_s$. Therefore $\varphi: (M,\gamma,g,k) \to (N,h)$ is a minimal surface , completing the proof of (7.38).

□

8. Appendix 1 : Proof of Lemma 6.3.

Consider (N,h) a Riemannian manifold with metric h, isometrically embedded in $\mathbb{R}^k$, and $D^2 = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ with the Euclidean metric.

Lemma 6.3: Let $\varphi : D^2-\{0\} \to N$ be a harmonic map with finite energy, then φ extends to a harmonic map $\Phi : D^2 \to N$.

The proof given here follows the ideas as in ([4], Theorem 3.6). The main idea is to prove that if $\varphi : D^2-\{0\} \to N$ is a harmonic map with $E(\varphi) < \infty$ then $\varphi \in L^{1,2\alpha}(D^2,N) \subset C^0(D^2,N)$ for $\alpha > 1$. Therefore φ can be continuously extended and is a weak solution for the Euler-Lagrange equation associated to the energy $E : L^{1,2}(D^2,N) \to \mathbb{R}$. By the regularity of such weak solutions, it follows that $\varphi : D^2 \to N$ is a harmonic map in the classical sense, i.e., φ is smooth.

Let us fix $x_0 \in D^2-\{0\}$ and remember that for all points $x_0 \in D^2-\{0\}$ and $\varepsilon > 0$ there exists a ball $B_r(x_0) = \{x \in D^2-\{0\} \mid |x-x_0| < r\}$

s.t. $E\left(\varphi|_{B_r(x_0)}\right) < \varepsilon$ (φ as in (6.3)).

The proof is divided into 3 steps.

8.1. - Step 1: There exists a constant $c > 0$ such that

(8.1) $$|d\varphi(x_0)|\,|x_0| < c.E\left(\varphi|_{B_{|x_0|}(x_0)}\right) \text{ for all } x_0 \in D^2-\{0\}.$$

Proof: By a conformal transformation we identify $B_r(x)$ with $B_2(0)$, then $E\left(\varphi|_{B_2(0)}\right) < \varepsilon$. Define the transformation $T : B_1(0) \to B_{|x_0|}(x_0)$, $x_0 \in B_2(0)$, and the map $\tilde{\varphi} : B_1(0) \to N$ by $\tilde{\varphi}(x) = \varphi(x_0 + |x_0|x)$, so, $\tilde{\varphi} : B_1(0) \to N$ is also harmonic and $E(\tilde{\varphi}) < \varepsilon$.

By (5.30), (5.41) and (2.3) there exists a constant $k > 0$ such that

$$\max_{x \in B_1(0)} |\, d\tilde{\varphi}(x)\,| \le k.E(\tilde{\varphi}),$$

therefore

$$|d\varphi(x_0)|.|x_0| = |d\tilde{\varphi}(0)| \le k.E\left(\varphi|_{B_{|x_0|}(x_0)}\right).$$

□

8.2. – **Step 2:** Let $\varphi : D^2 - \{0\} \to N$ be a smooth harmonic map such that $E(\varphi) < \infty$, then

$$\text{(8.3)} \qquad \int_0^{2\pi} |\partial_\theta \varphi(z)|^2 \, d\theta = r^2 \int_0^{2\pi} |\partial_r \varphi(z)|^2 \, d\theta .$$

Proof: Let $\eta(z) = w(z)dz^2$ be the holomorphic quadratic differential form where $w(z) = (|\partial_x \varphi|^2 - |\partial_y \varphi|^2) + 2i(\partial_x \varphi, \partial_y \varphi)$.

From (8.1) we get:

(8.4) $$|w(z)| \le 2|d\varphi(z)|^2 \le c^1|z|^{-2}\,.$$

By (8.4) and the fact that $\int |w(z)|d\mu \le 2\int |d\varphi|^2\, d\mu < \infty$, ($\mu$ the Lebesgue measure in D^2), the order of the pole in $z = 0$ is at most one.

From Cauchy's theorem

$$\int_{|z|=r}^{2\pi} \mathrm{Re}(w(z)z^2)d\theta = \int_0^{2\pi} \left[|\partial_\theta\varphi(r,\theta)|^2 - r^2|\partial_r\varphi(r,\theta)|^2\right] d\theta = 0\,.$$

So (8.3) follows from above.

□

8.5. – Step 3: Let $\varphi : D^2 - \{0\} \to N$ be a harmonic map with $E(\varphi) < \infty$, then $\varphi \in L^{1,2\alpha}(D^2,N)$ for $\alpha > 1$.

Proof: Assume $\int_{B_2(0)} |d\varphi|^2 < \varepsilon$. We approximate φ by a function g which is piecewise linear in $\log r$, depending only on the radial coordinate and $f(2^{-m}) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(2^{-m},\theta)d\theta$. Then f is harmonic for $2^{-m} < r < 2^{-m+1}$, $m \ge 1$.

Now for $2^{-m} \le r \le 2^{-m+1}$

$$|q(r) - \varphi(r,\theta)| \le |q(2^{-m}) - q(2^{-m+1})| + |\varphi(r,\theta) - q(2^{-m+1})|\,.$$

Since

$$\max\{|\varphi(x)-\varphi(y)| \mid 2^{-m} \le |x|,|y| \le 2^{-m+1}\} \le 2^{-m+3} \max\{|d\varphi(x)|,\ 2^{-m} \le |x| \le 2^{-m+1}\}$$

$$\le C\, 2^3\left(\int |d\varphi|^2\, d\mu\right)^{\frac{1}{2}},$$

we can assume

$$|q(r) - \varphi(r,\theta)| \le 2^4 C\left(\int_{|x|\le 2^{-m+1}} |d\varphi|^2\, d\mu\right)^{\frac{1}{2}} \le 2^4 . \varepsilon^{\frac{1}{2}} .$$

An estimate to the $L^{1,2}$ norm of the difference between q and φ is made as follows.

$$(8.6) \quad \int |dq-d\varphi|^2\, d\mu = \sum_{m=1}^{\infty} r \int_0 \left(q(r)-\varphi(r,\theta),\ \partial_r\varphi(r,\theta) - q'(r)\right) d\theta \,\Big|_{r=2^{-m}}^{r=2^{m+1}}$$

$$- \int \left(q-\varphi,\ \Delta(q-\varphi)\right) d\mu .$$

On the right side of (8.6) the boundary integral vanishes because the terms containing $q'(r)$ disappears ; this is because q is the average of φ at 2^{-m}, and the terms with $\partial_r\varphi(r,\theta)$ cancel with succeeding and preceeding terms because $\partial_r\varphi(r,\theta)$ is continuously defined.

The estimate of (8.6) is then :

$$(8.7)\quad \int |dq-d\varphi|^2 d\mu = -\int\big((q-\varphi),\ \Delta(q-\varphi)\big)d\mu = \int \big(q-\varphi,\ \Gamma(\varphi)(d\varphi,d\varphi)\big)d\mu \le$$
$$\le \|\Gamma\|_\infty\ \|q-\varphi\|_\infty\ \|d\varphi\|_{L^{0,2}} \le \|\Gamma\|_{0,\infty}\ 2^4.c.\sqrt{\varepsilon}\ \|d\varphi\|_{L^{0,2}}.$$

Let $\|\Gamma\|_{0,\infty}\ 2^4\ c.\sqrt{\varepsilon} < \delta$. Then we get

$$(8.8)\quad \int_{B_1(0)} |d(\varphi-q)|^2\ d\mu \le \Big[\int_{r=1}^{2\pi} |\varphi-q|^2 d\theta\Big]^{\frac12}\ \Big[\int_{r=1}^{2\pi} |\partial_r\varphi|^2 d\theta\Big]^{\frac12} + \delta \int_{B_1(0)} |d\varphi|^2\ d\mu\ .$$

From (8.3) we get:

$$(8.9)\quad \tfrac12 \int_{B_1(0)} |d\varphi|^2 d\mu = \int_{B_1(0)} \frac{|\partial_\theta\varphi|^2}{r^2} d\mu\ .$$

Applying (8.9) to the right-hand side in (8.8) we get:

$$(8.10)\qquad \int_{B_1(0)} |d(\varphi-q)|^2\ d\mu = \int_{B_1(0)} \frac{|\partial_\theta\varphi|^2}{r^2} d\mu\ = \tfrac12 \int_{B_1(0)} |d\varphi|^2\ d\mu\ .$$

Because q is the average value of φ:

$$(8.11)\qquad \Big[\int_{r=1} |\varphi-q|^2\ d\theta\Big]^{\frac12} \le \Big[\int_{r=1} |\partial_\theta\varphi|^2\ d\theta\Big]^{\frac12} = \Big(\tfrac12 \int_{r=1} |d\varphi|^2\ d\theta\Big)^{\frac12}\ .$$

So, inserting (8.9), (8.10) and (8.11) in (8.8) we get:

$$(1-2\delta)\int_{B_1(0)} |d\varphi|^2\,d\mu \le \int_{r=1} |d\varphi|^2\,d\theta\,;$$

By translating the expression above (by expansion and contraction) into a disk of any radius, we get for $r \le 1$:

$$(8.12) \qquad (1-2\delta)\int_{B_r(0)} |d\varphi|^2\,d\mu \le r\int_{r=1} |d\varphi|^2\,d\theta\,.$$

Now integrating (8.12) as in ([4], Theorem 3.6) and applying (8.1) for $0 < |x_0| < \frac{1}{2}$ we get

$$(8.13) \qquad |d\varphi(x_0)|.|x_0| \le c.|2x_0|^{\frac{1-2\delta}{2}}\Big(\int_{B^2} |d\varphi|^2 d\mu\Big)^{\frac{1}{2}} \text{ for all } x_0 \in D^2-\{0\}\,.$$

Then

$$\int |d\varphi(x)|^{2\alpha} d\mu \le k\,.\int |x|^{-(1+2\delta).\alpha} d\mu \;<\; \infty \;\text{ for } 2 > \alpha > 1$$

and $\varphi \in L^{1,2\alpha}(D^2,N) \subset C^0(D^2,N) \Rightarrow \varphi$ is a harmonic map from D^2 into N .

□

9. Appendix 2 : Proof of Theorem 7.26.

For a complete proof of Mumford's compactness theorem ([13]), it would be necessary to go into the Lie Group Theory, which is too far from our aim. Therefore the proof is based on strong results (without proof) adapted for our concern. The ideas are the same as used in [13].

By the Uniformization Theorem (7.33), the only classes of simply connected spaces up to conformal equivalence are $\mathbb{C}$, $\mathbb{C} \cup \{\infty\}$ and $\mathbb{H}$, and all closed surface are conformaly equivalent to $\tilde{M}/\Gamma$, where $\tilde{M}$ is one of those simply connected space (according to the genus (M)) and Γ is a discrete subgroup of the group of Conformal Automorphisms of $\tilde{M}$ acting freely and discontinuously on $\tilde{M}$.

The group of Conformal Automorphisms of $\tilde{M}$ is a Lie Group, in fact:

(9.0) (i) $\mathrm{Aut}(\mathbb{C} \cup \{\infty\}) \cong PL(2, \mathbb{C}) \cong SL(2, \mathbb{C})/\pm I$

(ii) $\mathrm{Aut}(\mathbb{C}) \cong P\Delta(2, \mathbb{C}) = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in SL(2, \mathbb{C}) \mid a.c \neq 0\}/\pm I$.

(iii) $\mathrm{Aut}(\mathbb{H}) \cong PL(2, \mathbb{R}) \cong SL(2, \mathbb{R})/\pm I$.

So, the Uniformization Theorem reduces the problem of studying conformal structures on M to studying discrete subgroups of groups of Conformal Automorphisms of $\tilde{M}$ $\left(\tilde{M} = \text{universal cover of } M \text{ according to (7.33)}\right)$ acting freely and discontinuously on $\tilde{M}$.

Because all surfaces of genus $g = 0$ are conformally equivalent to $\mathbb{C} \cup \{\infty\}$, we are particularly interested in describing the situation for surfaces with genus $g \geq 1$.

From now on $G = \mathrm{Aut}(\tilde{M})$, Γ = discrete subgroup of G acting freely and discontinuously on $\tilde{M}$. Define the action $\alpha : G \times \tilde{M} \to \tilde{M}$ by $\alpha(g,\tilde{m}) = g(\tilde{m})$. Let ρ be the standard metric on $\tilde{M}$, i.e. if $\tilde{M} = \mathbb{C}$, $\rho(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\forall\, z = x+iy \in \mathbb{C}$, if $\tilde{M} = \mathbb{H}$, $\rho(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1/y^2 \end{bmatrix}$ $\forall\, (x,y) \in \mathbb{H}$. Define a function d on G by

$$d(g) = \inf_{z \in M} \rho(z,g(z)). \tag{9.1}$$

For all $\varepsilon > 0$, define an open subset of G by

$$U_\varepsilon = \{g \in \mid d(g) < \varepsilon\}.$$

A subgroup Γ acts freely and discontinuously on $\tilde{M}$ if and only if it is discrete and $\Gamma \cap U_\varepsilon = \{e\}$ for some $\varepsilon > 0$.

For a general Lie Group G, define the subsets

$$\mathcal{M}_G = \{\Gamma \subset G \mid \Gamma \text{ is a discrete subgroup}\}$$

$$\mathcal{M}_G^c = \{\Gamma \subset G \mid \Gamma \in \mathcal{M}_G,\ G/\Gamma \text{ is compact}\}.$$

The following are general results for Lie Groups which are not too pathological; see [13] for further references.

9.2. – Theorem: Let U be an open neighbourhood of e, C a positive number. Then $\{\Gamma \in \mathcal{M}_G \mid \Gamma \cap U = \{e\}$ and measure $(G/\Gamma) \le C\}$ is compact.

9.3. – Theorem: Assume that G is a Lie Group (connected). Suppose a sequence $(\Gamma_i)_{i=1}^{\infty} \subset \mathcal{M}_G^C$ converges to $\Gamma \subset \mathcal{M}_G^C$. Then for i sufficiently large there exist isomorphisms of groups $\varphi_i : \Gamma \tilde{\to} \Gamma_i$, such that for all $\gamma \in \Gamma$, $\varphi_i(\gamma) \in G$ and converges to γ. Moreover, there is a compact set $K \subset G$ and an open neighbourhood $U \subset G$ of e (identity in G) such that $K.\Gamma = G$, $K.\Gamma_i = G$, $U \cap \Gamma = \{e\}$ and $U \cap \Gamma_i = \{e\}$ if i is sufficiently large.

□

9.4. – Lemma: Let $M = \tilde{M}/\Gamma$ be a closed surface with genus $g \ge 1$. Then there is a constant k such that

(9.5) $$\mathrm{diam}(M).\ell \le k\ \mathrm{area}(M)$$

where ℓ = length of smallest closed geodesic on M.

Proof: From the uniformization theorem $\tilde{M} = \mathbb{C}$ or $\mathbb{H}$. As a real surface $\tilde{M} = \mathbb{C}$ it has curvature $K(x,y) = 0$ $\forall$ $z = x+iy \in \mathbb{C}$ and $\tilde{M} = \mathbb{H}$ has curvature $K(x,y) = -1$ $\forall (x,y) \in \mathbb{H}$. Therefore M has sectional curvature $K \le 0$.

Let $d = \mathrm{diam}(X)$, $x,y \in X$ such that $\mathrm{dist}(x,y) = d$ and $\sigma : I \to M$ a geodesic from x to y of length d. Consider a tubular neighbourhood of σ defined as

$$T = \{\exp_{\sigma(t)} s.v(t) \mid v(t) \in T_{\sigma(t)}M \,,\ |v(t)| = 1 \,,\ 0 \le s \le \frac{\ell}{4}\}$$

There are two possibilities :

(i) No two geodesics δ_1 , δ_2 perpendicular to σ of length $\frac{\ell}{4}$ meet themselves,

or

(ii) some pair δ_1 , δ_2 do meet.

In the first case the exponential map from the normal bundle N to σ in M maps an $\frac{\ell}{4}$–tube T_0 around the 0–section in N injectively to M .Then

(9.6) $$\text{area}(M) \ge \text{area}(T) \ge \text{area}(T_0) \ge \frac{\ell}{2} . d \ .$$

In the second case suppose two geodesics δ_1 and δ_2 meet themselves , where $\delta_1(s) = \exp_{z_1}(s.v(t_1))$ and $\delta_2(s) = \exp_{z_2}(s.v(t_2))$, $z_1 = \sigma(t_1)$ and $z_2 = \sigma(t_2)$. Let e be the distance from z_1 to z_2 along σ . Then we can go from x to y by going from x to z_1 on σ_1 following δ_1 , then following δ_2 and going from z_2 to y on σ . This path has length $\le d - e + \frac{\ell}{2}$, and since σ is the shortest path from x to y , $d \le d - e + \frac{\ell}{2}$, i.e. $e \le \frac{\ell}{2}$. But then δ_1 , δ_2 and part of σ , between z_1 and z_2 , is a closed path $\tau : I \to M$ of length at most ℓ . Because $g \ge 1$, τ is certainly not homotopic to 0 since on the universal covering space $\tilde{M}$ of M , the exponential from the cover of N into $\tilde{M}$ is injective. Moreover, τ has corners and so is not a geodesic itself. Therefore there is a closed geodesic freely homotopic to τ of length $< \ell$. This contradicts the definition of ℓ and so the possibility in (i) is the only correct one and (9.4) follows from (9.5).

□

9.7. – **Theorem:** Let $\Gamma \in \mathcal{M}_G$, $\varepsilon > 0$ and $\Gamma \cup U_\varepsilon = \{e\}$. Then there is a constant k and a compact set K_c, with radius $C = \dfrac{k.\text{measure}(G/\Gamma)}{\varepsilon}$, such that $\Gamma.K_c = G$. Hence for all positive D, the subset $S = \{\Gamma \in \mathcal{M}_G^{\varepsilon} \mid \Gamma \cup U_\varepsilon = \{e\}$, measure $(G/\Gamma) \le D\}$ is compact $\left(G = \text{Aut}(\tilde{M})\right)$.

Proof: Consider the metric on $M = \tilde{M}/\Gamma$ induced by the metric ρ on $\tilde{M}$. The closed geodesics of X/Γ are all images of geodesics in $\tilde{M}$ joining two points, x and $g(x)$, where $x \in \tilde{M}$ and $g \in \Gamma$. Since $\Gamma \cap U_\varepsilon = \{e\}$, these all have length at least ε. It follows from lemma (9.3) that

$$\text{diam}(M) \le \frac{k.\text{area}(M)}{\varepsilon} = \frac{k.\text{measure}(G/\Gamma)}{\varepsilon} = c\,.$$

Hence, the ball $B_c(x)$ with centre at x $(\forall\, x \in \tilde{M})$ and radius c is mapped onto M by the covering map $\pi : \tilde{M} \to M$. Thus the action $\alpha : \Gamma \times B_c(x) \to \tilde{M}$ is onto, i.e. $\Gamma(B_c) = \tilde{M}$.

(i) $\Gamma(B_c) = \tilde{M} \Rightarrow$ there exists a compact set K_c such that $\Gamma.K_c = G$.

To prove (i) consider the compact set $K_c(x) = \{g \in G \mid \text{dist}(x, g(x)) \le c\}$. Let $x, y \in \tilde{M}$ and $g \in G$ such that $g(x) = y$. By hypothesis there is $\gamma \in \Gamma$ and $z \in B_c(x)$ such that $\gamma(z) = y$. Let $h \in K_c(x)$ and $h(x) = z$. Then

$g = \gamma h \Rightarrow \Gamma.K_c = G$.

(ii) $\Gamma.K_c = G \Rightarrow G/\Gamma$ is compact.

Otherwise $G/\Gamma \subset K_c$ would be open, but this cannot happen because G is path-connected.

So, to prove S is compact it is sufficient to prove that if a sequence $(\Gamma_i)_{i=1}^{\infty} \subset S$ converges to $\Gamma \in \mathcal{M}_G$ (this is assured by (9.2)) then G/Γ is compact, i.e. $\Gamma \in S$. From (i) we have $\Gamma_i(B_c) = \tilde{M}$. Therefore by (ii) we have $\Gamma_i.K_c = G$ and passing to the limit we get $\Gamma.K_c = G$. Then by the same argument in (ii), it follows that G/Γ is compact. By (9.3), $\Gamma \cong \Gamma_i$ for i sufficiently large.

□

It must be checked that measure $(G/\Gamma) < \infty$ for all $M = \tilde{M}/\Gamma$, but this comes from the following arguments:

(9.8) If genus $(M) \geq 2$ then $\tilde{M} = \mathbb{H}^2$, $G = PSL(2, \mathbb{R})$ and $\Gamma \cong \pi_1(M)$. From Gauss-Bonnet Theorem:

$$\text{measure}(G/\Gamma) = \text{area}(\mathbb{H}/\Gamma) = 4\pi(g-1) \quad (g = \text{genus}).$$

(9.9) If genus $(M) = 1$ then $\tilde{M} = \mathbb{C}$, $G = P\Delta(2, \mathbb{C})$ and $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$. In this case the generators of Γ can be represented by the translations $z \to z + z_1$, $z \to z + z_2$, where $0 < Re(z_1) < \infty$ and $0 < Re(z_2) < \infty$. In this case we have

$$\text{measure}(G/\Gamma) = \text{area}(\mathbb{C}/\Gamma) = Re(z_1).Im(z_2).$$

9.10. Corollary: Let genus $g > 1$ and let $\mathfrak{R}(g)$ be the moduli space of compact surfaces of genus g. For all $\varepsilon > 0$, the subset

$$\{\gamma \in \mathfrak{R}(g) \mid \text{all geodesics in } (M,\gamma) \text{ have length} \geq \varepsilon\}$$

is compact.

Proof: It follows from (9.7) ; from (9.3) we have that the limit has genus g.

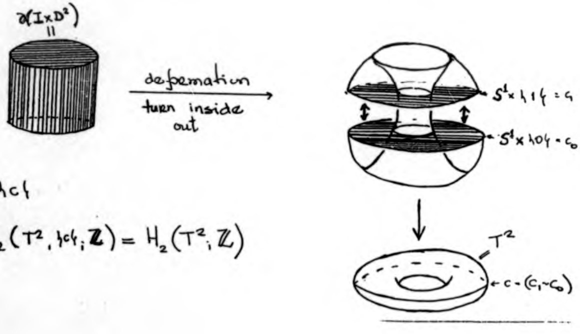
□

10. Appendix 3 : Examples satisfying condition (H)

In this chapter we sketch some pictures where the condition (H) is verified. It is important to note that given M we take $\partial(M\mathrm{x}I)$ and the identification is made on $(\partial M\mathrm{x}\{0\})\bigcup(\partial M\mathrm{x}\{1\})$.

Because the condition (H) depend just on the topology of M and S we do not need to specify the metrics.

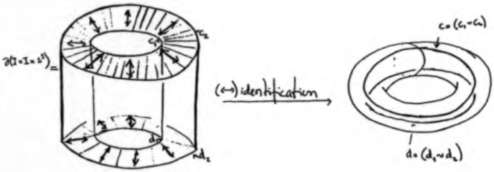
Example 1 : Let $S = T^2$ and let $M = D^2$.



$\mathcal{U} = \{c\}$

$$H_2(T^2,\{c\};\mathbb{Z}) = H_2(T^2;\mathbb{Z})$$

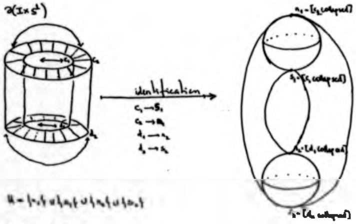
Example 2 : $S = T^2$ **and** $M = I{\times}S^1$



$c_1 = \{1\} \times \{0\} \times S^1$ | $d_1 = \{0\} \times \{0\} \times S^1$ | $U = \{c\} \cup \{d\}$

$c_2 = \{1\} \times \{1\} \times S^1$ | $d_2 = \{0\} \times \{1\} \times S^1$ | $H_2(T^2, \{c\} \cup \{d\}; \mathbb{Z}) = H_2(T^2; \mathbb{Z})$

Example 3 : $S = S^2 \cup S^2$ **and** $M = I{\times}S^1$



$U = \{s_1\} \cup \{n_1\} \cup \{n_2\} \cup \{s_2\}$

$H_2(S, U; \mathbb{Z}) = H_2(S; \mathbb{Z})$

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