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Stability Analysis of Higher-Order Delta-Sigma Modulators for Sinusoidal Inputs

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Abstract- The aim of this paper is to determine the stability of higher-order Δ-Σ modulators for sinusoidal inputs. The nonlinear gains for the single bit quantizer for a dual sinusoidal input have been derived and the maximum stable input limits for a fifth-order Chebyshev Type II based Δ-Σ modulators are established. These results are useful for optimising the design of higher-order Δ-Σ modulators.

I. INTRODUCTION

The stable input amplitude limits for Δ-Σ modulators is complicated to predict due to the non-linearity introduced by the quantizer in the feedback loop. Various approaches have been employed to explain this nonlinear behaviour. Using quasilinear modeling, a new interpretation of the instability mechanism for Δ-Σ modulators based on the noise amplification curve is given in [1]. This is restricted for DC inputs and unity quantizer gains. The quasilinear method can be extended to more than one input with each input represented by a separate equivalent gain. This concept forms the basis for the Describing Function (DF) method [2]. In [3] the stability analysis for higher-order Δ-Σ modulators based on the noise amplification curve was done using the DF method for DC and (single-tone) sinusoidal inputs for non-unity quantizer gain values. In this paper the analysis is extended for multiple (dual) tone sinusoidal inputs.

II. QUASILINEAR STABILITY ANALYSIS OF Δ-Σ MODULATORS

A generic Δ-Σ modulator having its quantizer replaced by a gain factor K followed by additive quantization noise $q(k)$ [1] is shown in Figure 1.

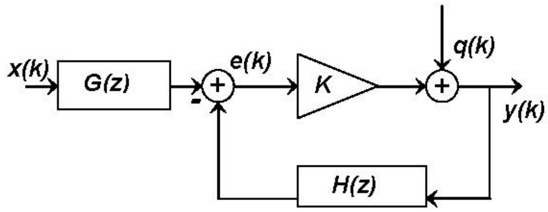

Figure 1. Quasilinear Δ-Σ modulator Quantizer Model

The output of the modulator in the z-domain is given by :

$$Y(z) = STF(z)X(z) + NTF(z)Q(z) \qquad (1)$$

where, $Y(z)$, $X(z)$ and $Q(z)$ are the z-transforms of the output, input and quantizer noise signals respectively. Also, $STF(z)$ and $NTF(z)$ are the Signal and Noise Transfer functions of the Δ-Σ modulator derived from Figure 1.

$$STF(z) = \frac{K.G(z)}{1 + K.H(z)} \qquad (2)$$

$$NTF(z) = \frac{1}{1 + K.H(z)} \qquad (3)$$

Since the poles of the denominator $(1+KH(z))$ determine the stability of the modulator, for a given $H(z)$, there will be a certain interval $[K_{min}, K_{max}]$ for which the modulator is stable [4]. Assuming $q(k)$ to be Gaussian white stochastic $G(0, \sigma_q^2)$ and the transfer function between $q(k)$ and $y(k)$ to be known, then the output noise variance is given by:

$$Var\{y(k)\} = \sigma_q^2 \int_0^1 \left|NTF(e^{j\pi f})\right|^2 df = \sigma_q^2 A(K) \qquad (4)$$

where, σ_q^2 is the variance of $q(k)$ and $A(K)$ is the total output noise power amplification factor. Using Parseval's relation, $A(K)$ can be found in the time domain as [1]:

$$A(K) = \sum_{k=0}^{\infty} \left|ntf(k)\right|^2 \triangleq \left\|ntf\right\|_2^2 \qquad (5)$$

where $ntf(k)$ is the impulse response corresponding to $NTF(z)$ and $A(K)$ is the squared two-norm of $NTF(z)$. The $A(K)$ curves of the loop-filter are crucial for the stability analysis of the Δ-Σ modulators. Typical curves for Type II Chebyshev 3rd and 4th order based modulators are shown in Figure 2. The A_{min} value is the global minimum of the curve. It has been shown in [1] that for stable operation $A(K)>A_{min}$. As the amplitude of the input signal increases the value of $A(K)$ decreases on the right side (monotonically decreasing) of the curve till it finally reaches A_{min} . This characterizes the onset of instability and the value of $A(K)$ then slips onto the left side of the curve which is monotonically increasing. In this region the quantizer gain values are such that the modulator is always unstable. This is an irreversible state.

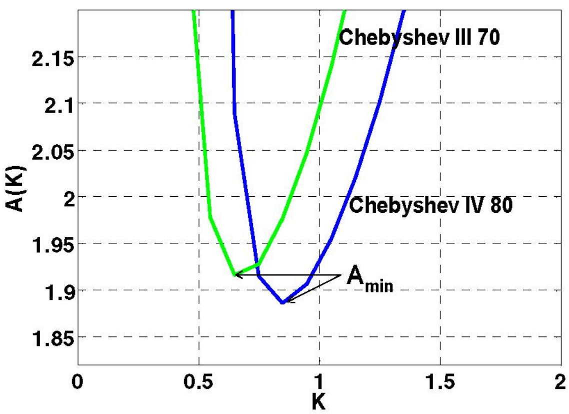

Figure 2. *A(K)* Curves for Type II Chebyshev NTF

III. NOISE AMPLIFICATION CURVES – DF METHOD

The quasilinear quantizer model in Figure 1 can be extended using separate gains K_x and K_n for the DF model as shown in Figures 3 and 4 [5].

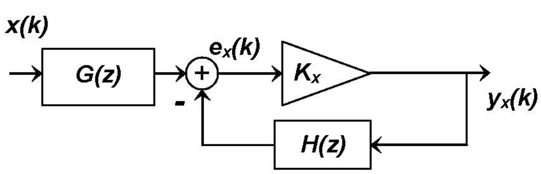

Figure 3. Δ-Σ modulator Quantizer Signal-Model

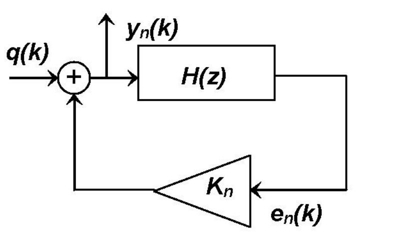

Figure 4. Δ-Σ modulator Quantizer Noise-Model

Figure 3 describes the model for the input signal with linear gain K_x. Figure 4 describes the noise signal model with linear gain K_n. The combined output signal is given by:

$$y(k) = y_x(k) + y_n(k) \tag{6}$$

The linearised gains for two sinusoidal input signals $x_a(t)=a\cos(\omega_1(t)+\phi_1)$, $x_b(t)=b\cos(\omega_2(t)+\phi_2)$ (where a, b are constants, ω_1, ω_2 the sinusoidal frequencies, ϕ_1 and ϕ_2 random phases) and a random Gaussian signal (feedback components) have been solved for the case of an one-bit quantizer with an output ±Δ in Appendix A where the final expressions are shown below:

$$K_a = \left(\frac{2}{\pi}\right)^{\frac{5}{2}}\left(\frac{\Delta}{\sigma}\right)\left(\frac{b}{a}\right)\left(\frac{1}{\frac{1}{2}-\rho_b^2}\right)\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_a^2\right)+\psi_a\right\} \tag{7}$$

$$K_b = \left(\frac{2}{\pi}\right)^{\frac{5}{2}}\left(\frac{\Delta}{\sigma}\right)\left(\frac{a}{b}\right)\left(\frac{1}{\frac{1}{2}-\rho_a^2}\right)\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_b^2\right)+\psi_b\right\} \tag{8}$$

$$K_n = \sqrt{\frac{2}{\pi}}\left(\frac{\Delta}{\sigma}\right)e^{-\rho_a^2}e^{-\rho_b^2}\zeta \tag{9}$$

where
$$\psi_a = \left\{\frac{4}{3}\rho_a^2 - \frac{16}{45}\rho_a^4 + \frac{16}{175}\rho_a^6 - \frac{128}{6615}\rho_a^8 + ..\right\} \tag{10}$$

$$\psi_b = \left\{\frac{4}{3}\rho_b^2 - \frac{16}{45}\rho_b^4 + \frac{16}{175}\rho_b^6 - \frac{128}{6615}\rho_b^8 + ..\right\} \tag{11}$$

$$\zeta = \left\{1+\rho_a^2\rho_b^2 + \frac{\rho_a^4\rho_b^4}{4} + \frac{\rho_a^6\rho_b^6}{36} + \frac{\rho_a^8\rho_b^8}{576} + ...\right\} \tag{12}$$

and $\rho_a^2=(1/2)(a^2/\sigma^2)$, $\rho_b^2=(1/2)(b^2/\sigma^2)$. $F(.)$ is the Confluent Hypergoemetric Function [6]. The output noise variance is given by:

$$Var\{y(k)\} = \sigma_{e_n}^2 K_n^2 + \sigma_{q_{ab}}^2 \tag{13}$$

where σ_{qab}^2 is the quantization noise power for the two uncorrelated sinusoidal inputs $x_a(t)$ and $x_b(t)$. Therefore from (4), (9) and (13) the noise amplification factor is given by:

$$A_{ab}(K) = \frac{\left(\frac{2}{\pi}\right)\left\{e^{-\rho_a^2}e^{-\rho_b^2}\right\}^2\zeta^2 + \sigma_{q_{ab}}^2}{\sigma_{q_{ab}}^2} \tag{14}$$

Since $x_a(t)$ and $x_b(t)$are uncorrelated, the power of the output signal is given by:

$$E\{y^2(k)\} = \sigma_{e_n}^2 K_n^2 + \sigma_{q_{ab}}^2 + \sigma_{e_b}^2 K_b^2 + \sigma_{e_a}^2 K_a^2 \tag{15}$$

where σ_{eb}^2 and σ_{ea}^2 are the powers of the sinusoidal inputs at the quantizer input which are given by:

$$\sigma_{e_b}^2 = \frac{1}{K_b^2}\sigma_b^2 \quad \text{and} \quad \sigma_{e_a}^2 = \frac{1}{K_a^2}\sigma_a^2 \tag{16}$$

From (9), (15) and (16) we get:

$$\Delta^2 = \frac{2}{\pi}\Delta^2\{e^{-\rho_a^2}e^{-\rho_b^2}\}^2\zeta^2 + \sigma_{q_{ab}}^2 + \frac{b^2}{2} + \frac{a^2}{2} \tag{17}$$

Rearranging (17), the quantization noise is given by:

$$\sigma_{q_{ab}}^2 = \Delta^2\left[1 - \frac{a^2}{2\Delta^2} - \frac{b^2}{2\Delta^2} - \frac{2}{\pi}\{e^{-\rho_a^2}e^{-\rho_b^2}\}^2\zeta^2\right] \tag{18}$$

From (8) and (16) we get:

$$\left(\frac{2}{\pi}\right)^5\left(\frac{a^2}{b^2}\right)\frac{\rho_b^2}{\left[\frac{1}{2}-\rho_a^2\right]^2}\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_b^2\right)+\psi_b\right\}^2 = \frac{b^2}{2} \tag{19}$$

Similarly from (7) and (16) for the sinusoid $x_a(t)$ we have:

$$\left(\frac{2}{\pi}\right)^5\left(\frac{b^2}{a^2}\right)\frac{\rho_a^2}{\left[\frac{1}{2}-\rho_b^2\right]^2}\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_a^2\right)+\psi_a\right\}^2 = \frac{a^2}{2} \tag{20}$$

(19) and (20) are two simultaneous equations that were solved using MATLAB in order to get the values of ρ_a and ρ_b for various values a and b.

IV. RESULTS & SIMULATIONS

From (19) and (20), values obtained of ρ_b have been plotted in Figure 5. The value of ρ_b is observed to get bigger as the sinusoidal amplitude b increases. However, the increase in ρ_b gets attenuated as the signal amplitude a increases from 0.2 to 0.8.

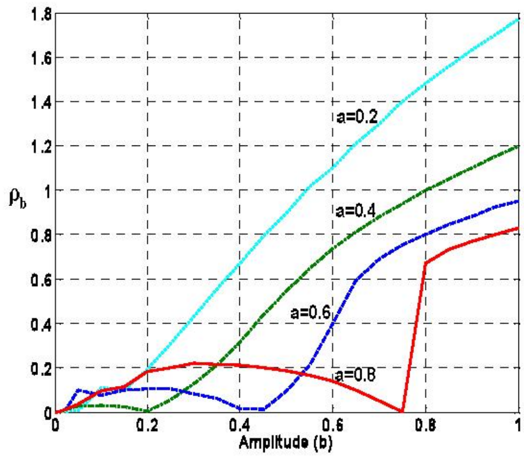

Figure 5. Variation of ρ_b versus b for different a amplitudes

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Using (18) the quantization noise σ^2_{qab} is plotted in Figure 6. The σ^2_{qab} in the regions $b < 0.2$, $b < 0.4$ and $b < 0.6$ for the curves I (a=0.2), II (a=0.4) and III (a=0.6) respectively increases mainly due to ρ_a. As ρ_a becomes bigger when the amplitude a increases from 0.2 to 0.6 in so does σ^2_{qab}. The increase in σ^2_{qab} in the regions $b > 0.2$, $b > 0.4$ and $b > 0.6$ for the curves I, II, and III respectively is mainly attributed to increase in ρ_b. As ρ_b increases with a reduction in the amplitude a from 0.6 to 0.2 in Figure 5 so does σ^2_{qab}.

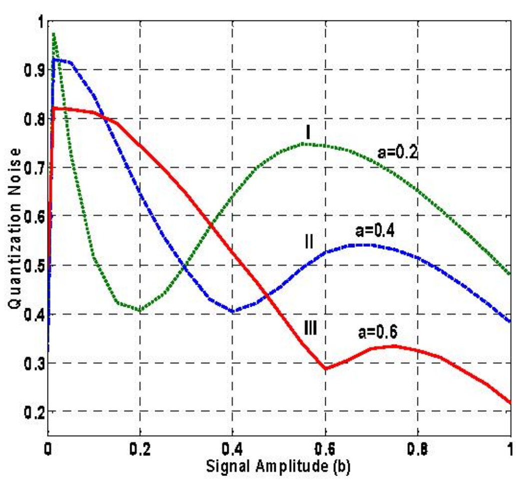

Figure 6. Variation of quantization noise versus the two sine amplitudes

Figure 7 shows the $A(K)$ curves obtained from (40) for a = 0.2, 0.4 and 0.6.

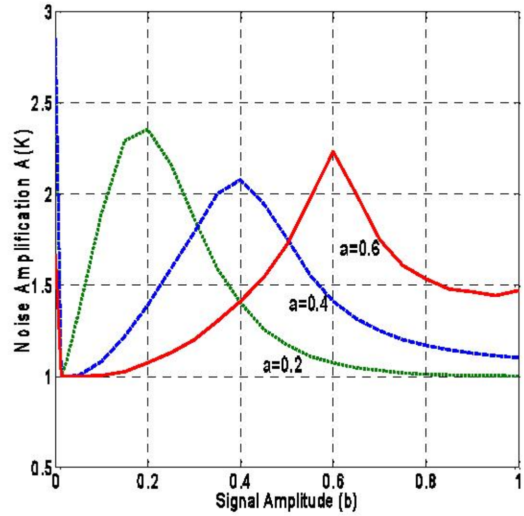

Figure 7. A(K) variation versus the two sine amplitudes

From $A_{ab}(K)$, the stable amplitude limits b have been plotted for the 5[th]- Chebyshev Type II NTF for a = 0.2 in Figure 8. The stable amplitude b is seen to vary with the quantizer gain K and the stop-band attenuation. Maximum stable limit of b is achieved when the quantizer gain value is close to one for a given stop-band attenuation. This stable amplitude limit decreases as K increases to 1.5 and beyond. It is also observed that for given value of the quantizer gain K, the stable amplitude limit decreases as the stop-band attenuation increases.

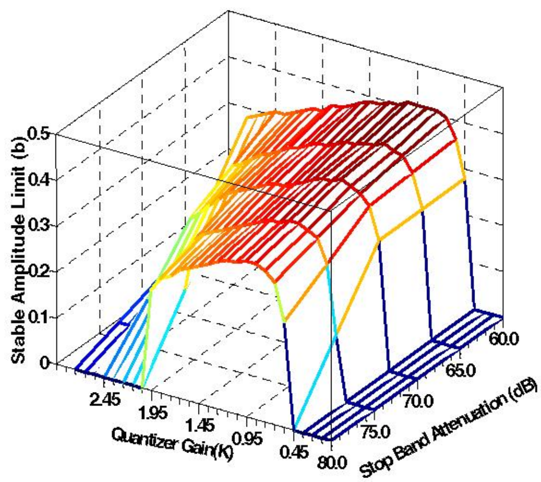

Figure 8. Stable limits of amplitude b of 5[th]-order for a = 0.2

Simulations for the 5[th]-order Chebyshev Type II Δ-Σ modulator shown in Figure 9 were performed for 1638400 samples. The input amplitude was increased in steps of 0.1. The maximum stable amplitude limits were obtained and compared with simulations as shown in Figure 9. Results obtained in [3] were used for the DC and single sinusoidal inputs.

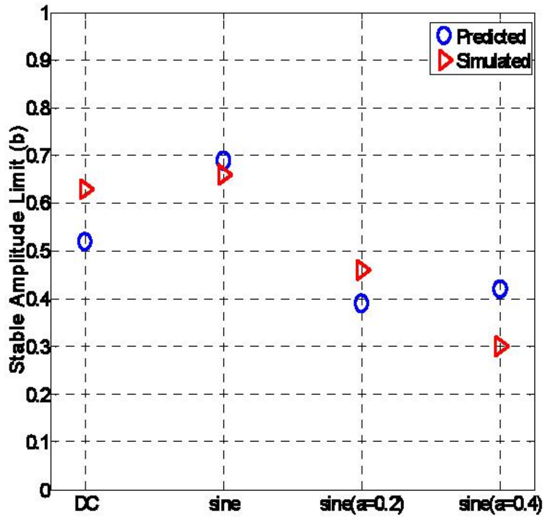

Figure 9. Simulation results for DC, sine & two sinusoidal inputs

The reason for variation can be attributed to the fact that the derivation of the three gains (i.e. 2 sinusoids and one Gaussian) is based on the modified non-linearity concept. In order to compute the gain for any of the 3 inputs, it is assumed that the non-linear function has been modified in turn by each of the 2 remaining inputs. However, in real-time this may not be the case as all the 3 inputs coexist simultaneously.

V. CONCLUSION

The stability of higher-order Δ-Σ modulators for dual tone sinusoidal inputs using the Describing Function Method has been predicted. The nonlinear gains for the single bit quantizer for a dual sinusoidal input have been derived and the maximum stable input limits for 5th-order Chebyshev Type II based Δ-Σ modulator have been established. Accurate results for the stable amplitude curves can be obtained for a range of values of quantizer gain K in which the Δ-Σ modulators are likely to operate.

APPENDIX A

In this Appendix, the derivation of the gains for two inputs (a dual-tone sinusoidal one Gaussian) for a single-bit quantizer is made. If the inputs to the nonlinearity are of different (Probability Density Functions) PDFs or of different magnitudes of similar waveforms, the output component from one of these inputs depends not only on the magnitude of this particular input but also on the magnitudes of all the other inputs. The concept used here is the modified linearity concept [7], whereby to determine the response to a particular input, the nonlinear characteristic is modified in turn by each of the input signals present to obtain a modified nonlinearity to which the input is applied.

Sinusoidal Gains

The two sinusoidal inputs considered here are $x_a(t)=a\cos(\omega_1(t)+\phi_1)$ and $x_b(t)=b\cos(\omega_2(t)+\phi_2)$ where a, b are constants, ω_1, ω_2 the sinusoidal frequencies, assumed to be incommensurate, ϕ_1 and ϕ_2 are RVs each having a uniform PDF in the interval $[0, 2\pi]$. The second input is the quantization noise assumed to be Gaussian $G(0, \sigma)$ i.e. with zero mean and variance σ^2.The modified nonlinearity of single-bit quantizer with a random input is given by [8]:

$$n_1(\gamma) = 2\Delta \int_0^{\gamma} q(y)dy \quad \text{(A1)}$$

where $\pm\Delta$ is the output of the quantizer and $q(y)$ is the PDF of the random input. Therefore for a Gaussian input:

$$n_1(\gamma) = 2\Delta \int_0^{\gamma} \left(\frac{1}{\sigma\sqrt{2\pi}}\right) e^{\frac{-y^2}{2\sigma^2}} dy \quad \text{(A2)}$$

On integration (A2) simplifies to:

$$n_1(\gamma) = \Delta erf\left(\frac{\gamma}{\sigma\sqrt{2}}\right) \quad \text{(A3)}$$

The non-linearity $n_1(\gamma)$ further modified to $n_2(\gamma)$ by one of the sinusoidal signals say $x_a(t)$ which is given by [7]:

$$n_2(\gamma) = \int_{-a}^{a} p(x) n_1(x+\gamma)dx \quad \text{(A4)}$$

where $p(x)$ is the PDF of $x_a(t)$. Therefore:

$$n_2(\gamma) = \int_{-a}^{a} \frac{1}{\pi} \frac{1}{\sqrt{a^2 - x^2}} \Delta erf\left(\frac{x+\gamma}{\sigma\sqrt{2}}\right) dx \quad \text{(A5)}$$

$n_2(\gamma)$ is now the nonlinearity of the quantizer which has been modified by the sinusoidal input $x_a(t)$ and the quantization noise $G(0,\sigma)$. The next step is to evaluate the gain for $x_b(t)$ to this modified nonlinearity. The gain K_b of the sinusoidal input $x_b(t)$ to this non-linearity $n_2(\gamma)$ is given by [8]:

$$K_b = \frac{1}{\sigma_b^2} \int_{-b}^{b} x n_2(x) r(x) dx \quad \text{(A9)}$$

where $\sigma_b^2 = b^2/2$, is the variance and $r(x)$ the PDF of $x_b(t)$. On integrating (A9) we get the gain for b K_b as:

$$K_b = \left(\frac{2}{\pi}\right)^{\frac{5}{2}} \left(\frac{\Delta}{\sigma}\right)\left(\frac{a}{b}\right)\left(\frac{1}{\frac{1}{2}-\rho_a^2}\right)\left\{ {}_1F_1\left(1, \frac{3}{2}, -\rho_b^2\right) + \psi_b \right\} \quad \text{(A10)}$$

where,

$$\psi_b = \left\{ \frac{4}{3}\rho_b^2 - \frac{16}{45}\rho_b^4 + \frac{16}{175}\rho_b^6 - \frac{128}{6615}\rho_b^8 + .. \right\} \quad \text{(A11)}$$

In order to obtain the gain for $x_a(t)$, we proceed as in above to get:

$$K_a = \left(\frac{2}{\pi}\right)^{\frac{5}{2}} \left(\frac{\Delta}{\sigma}\right)\left(\frac{b}{a}\right)\left(\frac{1}{\frac{1}{2}-\rho_b^2}\right)\left\{ {}_1F_1\left(1, \frac{3}{2}, -\rho_a^2\right) + \psi_a \right\} \quad \text{(A12)}$$

Noise Gain

The modified nonlinearity of order 1 for a Gaussian input to an single bit quantizer is given by [8]:

$$n(\sigma,\gamma)_1 = \int_{-\infty}^{\infty} n(y+\gamma) H_1\left(\frac{y}{\sigma}\right) q(y)\,dy \qquad \text{(A13)}$$

where H_1 is the Hermite Polynomial of the first order. Substituting for $q(y)$ and $n(y+\gamma)$ in (A20):

$$n(\sigma,\gamma)_1 = \frac{\Delta}{\sigma^2\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \sqrt{\frac{2}{\pi}} \Delta e^{-\frac{\gamma^2}{2\sigma^2}} \qquad \text{(A14)}$$

The noise gain K_n in the presence of another random input with PDF $p(r)$ is given by [8]

$$K_n = \frac{1}{\sigma} \int_{-\infty}^{\infty} n(\sigma,r)_1\, p(r)\,dr \qquad \text{(A15)}$$

Here we consider the additional random input as a combination of two uncorrelated sinusoidal inputs. The joint PDF $p(r)$ of the two sinusoidal signals having amplitudes a and b, with incommensurate frequencies is: $p(r)=(r/\pi ab)(1/\sin\theta)$, where $\theta=\cos^{-1}\{[a^2+b^2-r^2]/2ab\}$. Putting the value of $p(r)$ in (22) we get:

$$K_n = \sqrt{\frac{2}{\pi}}\left(\frac{\Delta}{\sigma}\right) \int_{a-b}^{a+b} e^{-\frac{r^2}{2\sigma^2}} \left(\frac{r}{\pi ab}\right)\left(\frac{1}{\sin\theta}\right) dr \qquad \text{(A16)}$$

Changing the variable from $r \rightarrow \theta$,

$$K_n = \sqrt{\frac{2}{\pi}}\left(\frac{\Delta}{\sigma\pi}\right) e^{-\frac{a^2}{2\sigma^2}} e^{-\frac{b^2}{2\sigma^2}} \int_0^{\pi} e^{k\cos\theta} d\theta \qquad \text{(A17)}$$

where $k = ab/\sigma^2$. Solving the integral above we get the noise gain as:

$$K_n = \sqrt{\frac{2}{\pi}}\left(\frac{\Delta}{\sigma}\right) e^{-\rho_a^2} e^{-\rho_b^2} \zeta \qquad \text{(A18)}$$

where:

$$\zeta = \left\{1 + \rho_a^2\rho_b^2 + \frac{\rho_a^4\rho_b^4}{4} + \frac{\rho_a^6\rho_b^6}{36} + \frac{\rho_a^8\rho_b^8}{576} + \ldots\right\} \qquad \text{(A19)}$$

Solving the integral above we get the noise gain as:

$$K_n = \sqrt{\frac{2}{\pi}}\left(\frac{\Delta}{\sigma}\right) e^{-\rho_a^2} e^{-\rho_b^2} \zeta \qquad \text{(A20)}$$

where, $$\zeta = \left\{1 + \rho_a^2\rho_b^2 + \frac{\rho_a^4\rho_b^4}{4} + \frac{\rho_a^6\rho_b^6}{36} + \frac{\rho_a^8\rho_b^8}{576} + \ldots\right\} \qquad \text{(A21)}$$

REFERENCES

[1] Risbo, L., "Stability Predictions for Higher-Order Sigma-Delta Modulators Based on Quasilinear Modeling", *IEEE International Symposium on Circuits & Systems,* Volume 5, page 361 – 364, 1994.

[2] Gelb, A and Vander Velde, W., E., *Multiple-Input Describing Functions and Nonlinear System Design,* New York McGraw-Hill, 1968.

[3] Lota, Jaswinder., Al Janabi, Mohammed and Kale, Izzet., "Stability Analysis of Higher-Order Sigma-Delta Modulators using the Describing Function Method", *IEEE International Symposium on Circuits & Systems,* Volume 5, page 361 – 364, 2006.

[4] Stikvoort, E., F., "Some Remarks on the Stability and Performance of the Noise Shaper or Sigma-Delta Modulator" *IEEE Trans. On Communications*, Volume 36, no.10, page 1157-1162, Oct 1988.

[5] Ardalan, S., H., Paulos, J., "An Analysis of Nonlinear Behavior in Delta-Sigma Modulators", *IEEE Transactions on Circuits & Systems,* Volume CAS-34, No.6, June 1987.

[6] Haddad, A., H., *Nonlinear Systems,* Benchmark Papers in Electrical Engineering and Computer Science, vol. 10, Dowden, Hutchinson & Ross, Inc and Halsted Press, page 197, 1975.

[7] Atherton, D.P. and G.F. Turnbull, "Response of Nonlinear Characteristics to Several Inputs and the use of the Modified Linearity Concept in Control Systems", *Proc IEE,* vol.111, No.1, pages 157-164, January 1964.

[8] Atherton, D., P., *Nonlinear Control Engineering-Describing Function Analysis & Design,* Van Notsrand Reinhold London, page 383-388, 1982.

[9] Mahalanabis, A.K and A.K. Nath, "On Dual-Input Describing Functions of a Nonlinear Element", *IEEE Trans. Automatic Control,* vol.10, issue 2, pages 203-204, 1965.