A novel spectral estimation method by using periodic nonuniform sampling.

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A Novel Spectral Estimation Method by Using Periodic Nonuniform Sampling

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Abstract-**In this paper we present a method of estimating power spectrum density of random ergodic signals. The method allows use of arbitrarily low sampling rates to achieve the goal. We compare our method with similar schemes reported in research literature and argue superiority of our approach in terms of its suitability for practical implementations. The most visible difference between our approach and the previously reported ones consists in replacing Poisson additive random sampling with deterministic sampling. Comparing with the approaches based on the Poisson additive random sampling, where theoretically infinitely large resources are needed to implement them accurately, our approach clearly relies on limited and well defined resources.**

I. INTRODUCTION

Spectrum estimation problems of the signal can be divided into two domains: estimation of Fourier transform of the deterministic signals and estimation of Power Spectral Density (PSD) of the stationary processes. Uniform sampling is a popular scheme used in both areas. In most cases it is required that the processed signal is sampled at a rate exceeding at least twice the signal bandwidth. If the spectrum estimation is carried out in wide frequency ranges such sampling rates may be economically or technically not feasible. For example, a leading manufacturer of Analog to Digital Converter (ADC) Maxim Dallas offers a range of converters [1] whose fastest sampling rates are up to about 100 times slower than the doubled bandwidths of converters. Therefore it is not surprising that significant research has been carried out to circumvent such limitations of uniform sampling and alternative solutions based on various forms of nonuniform sampling have been explored. The idea is to use sub-Nyquist average sampling rates that would still allow proper estimation of signal spectrum. How much the sampling rates can be lowered depends very much on which aspect of spectrum estimation is tackled.

In the case of Fourier transform, the average sampling rate can be as low as the Landau rate which is theoretically the lowest sampling rate that still allows perfect reconstruction of the sampled signal. It equals to the total bandwidth of the processed signal. The research reported so far indicates that such methods can be formulated even if the actual spectral support of the processed signal is not known. In [2] unbiased spectrum estimators have been proposed. In [3] we have extended these results [2] by incorporating a larger class of random sampling schemes and minimizing the errors of the estimators. Masry [4-5] have further developed this concept by investigating statistical properties of the estimates, including precise expressions and rate of convergence of the mean-square errors. In [6-7] it has been demonstrated that use of Periodic Nonuniform Sampling (PNS) can facilitate the estimation of Fourier transform of the bandpass/multiband signals.

In the case of PSD estimation, the average sampling rate can be arbitrarily slow. In order to elaborate on these issues more precisely we introduce two notions of sampling frequencies for nonuniform sampling: (a) the average sampling rate, defined by $f_a = \lim_{N\to\infty} \frac{N}{t_N - t_0}$ where N is the total number of collected samples and t_n are sampling instants; and (b) the instantaneous sampling rate defined by $f_n = \frac{1}{t_{n+1} - t_n}$. Note that f_n may change all the time. In [8] Shapiro and Silverman proposed an additive random sampling scheme that facilitated alias-free PSD estimation of ergodic signals while using arbitrarily low average sampling rates. Masry [9] furthered this concept. In [10-11] he investigated the quadratic-mean consistency of various estimates of PSD for Poisson sampling process, including the performance of consistent PSD estimates constructed from finite sets of signal samples. In [12] the authors presented a method for estimating PSD of a stochastic, stationary process. However, in order to implement data acquisition systems satisfying requirements described in [8-12] the sampling instants have to be generated using additive random scheme, $t_{n+1} = t_n + \varepsilon_n$ where the random variables ε_n have Poisson distribution. Although the average sampling rate could be arbitrarily low the method does not impose an upper limit either on the instantaneous sampling rate or on the duration of time over which the high instantaneous sampling rates are used. This means that these sampling schemes do not guarantee any minimum distance between two consecutive and even nonconsecutive sampling times. Therefore, those methods are not suitable for practical applications.

In this paper we revisit a method that allows estimating PSD of ergodic signals using arbitrarily low average sampling rates by use of PNS while the instantaneous sampling rates are kept

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under control to match the capabilities of used hardware. It has to be emphasized that PNS has been recommended for use in DSP in many publications [6-7, 13-15]. This sampling scheme is mainly used to process multiband signals at the rates that are just slightly above the Landau rate. In our paper we depart from these classical uses of PNS and propose a sampling scheme that utilizes two ADCs working in parallel. We demonstrate that if such hardware is properly deployed one can construct sampling sequences with arbitrarily slow average sampling rates and also acceptable instantaneous sampling rates, and still accomplish the estimation of PSD.

In the next section we review the concept of PSD estimation of a stationary process. In section III, a double-sampler PNS scheme for PSD estimation of a stationary process is introduced. In section IV numerical examples are presented to illustrate the proposed method.

II. The Estimation of PSD Function

Let $X(t)$ be a real-valued, ergodic, zero-mean random process. The autocorrelation function (ACF) of the process is defined as

$$R_x(\tau) = E\{X(t)X(t+\tau)\} \tag{1}$$

and its Power Spectral Density is

$$P_{x,\infty}(f) = \int_{-\infty}^{\infty} R_x(\tau)\exp(-j2\pi f\tau)d\tau \tag{2}$$

Consider a single realization of $X(t)$. Following [16], if $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|R_x(\tau)|^2 d\tau = 0$ then $\hat{R}_x(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} X(t)X(t+\tau)dt$ is an unbiased estimator of $R_x(\tau)$ when $0 \le \tau \le T$. Here we estimate ACF using discrete-time observations of $X(t)$:

$$\hat{R}_x(\tau) = \frac{1}{N}\sum_{n=1}^{N} X(t_{k_n}+\tau)X(t_{k_n}) \tag{3}$$

where $t_{k_n} \in [0, T-\tau]$ are time instants such that signal $X(t)$ is sampled at both t_{k_n} and $t_{k_n}+\tau$, and N is the number of such pairs. As shown in [17], (3) is an unbiased estimator of ACF. In practical cases $R_x(\tau)$ has a finite bandwidth $f_{\max}$. Hence PSD (2) can be calculated from the samples of $R_x(\tau)$:

$$P_{x,\infty}(f) = L\sum_{m=-\infty}^{\infty} R_x(mL)\exp(-j2\pi fmL) \quad \text{if} \quad L \le 0.5/f_{\max} .$$

Further simplifications are obtained by exploiting the fact that $R_x(\tau)$ is an even function and that $R_x(\tau)$ (or its estimate) is most often known only inside a window of a finite-length. Therefore we replace the "target" PSD $P_{x,\infty}(f)$ with its windowed-version counterpart

$$P_x(f) = 2L\sum_{m=1}^{M} R_x(mL)\cos(2\pi fml) + LR_x(0) \tag{4}$$

The estimate of $P_x(f)$, $\hat{P}_x(f)$, is obtained by replacing $R_x(mL)$ with $\hat{R}_x(mL)$

$$\hat{P}_x(f) = 2L\sum_{m=1}^{M} \hat{R}_x(mL)\cos(2\pi fml) + L\hat{R}_x(0) \tag{5}$$

It is not difficult to note that $\hat{P}_x(f)$ is an unbiased estimator. In fact $E\left[\hat{P}_x(f)\right] = 2L\sum_{m=1}^{M} E\left[\hat{R}_x(mL)\right]\cos(2\pi fml) + LE\left[\hat{R}_x(0)\right] = 2L\sum_{m=1}^{M} R_x(mL)\cos(2\pi fml) + LR_x(0) = P_x(f)$

In this paper we use (5) to estimate signal's PSD.

III. Periodic Nonuniform Sampling

Most of the data acquisition systems rely on uniform sampling signals using sampling rates at least twice as large as the bandwidth of the processed signals. While this approach works well for relatively low frequency ranges it may be economically or technically not viable when processing signals in the frequency ranges starting around 1GHz. This is confirmed by the fact that the processing bandwidths of the ADCs are much wider than the maximum sampling rates at which the ADCs can be triggered [1]. This particular deficiency does not have to prevent us from using them in PSD estimation systems and still exploit full bandwidth of the converters. When designing a sampling scheme we request that the following conditions are met:

(c1) Each sampling instant is a multiple of time interval L such that $L \le 0.5/f_{\max}$.

(c2) The time distance between any two samples collected by the same ADC is not shorter than $H = rL$, where r is a whole number.

The first constraint guarantees that by using (3) we can estimate $R_x(\tau)$ at the right points while the second one allows accommodating ADCs with unduly long processing intervals.

Periodic nonuniform sampling sequence takes samples at $t_{N_1k+n} = kT_s + \tau_n$, where $T_s = pL$ is the period of PNS, N_1 is the number of sampling points during each period, $1 \le n \le N_1$ and $0 \le \tau_1 < ... < \tau_{N_1} < T_s$ are the sampling points in the first period. When selecting a particular sampling sequence we ought to make sure that it provides sufficiently large numbers of pairs of sampling instants that are $L, 2L, ..ML$ apart so that one can estimate all ACF values required in (5). As we demonstrate in the following subsections, design of such sequences would not be possible if a single ADC was used. However, having two, even slow converters, makes this task achievable.

A. PNS with a Single ADC

The problem of designing PNS that uses single ADC satisfying constraints (c1) and (c2) has been tackled in [6-7]. In those cases however, the objective of processing the signal was to estimate the Fourier transform of deterministic signals. It is easy to note that because of (c2) single ADC PNS cannot generate pairs of sampling points that are closer to each other than $H = rL$. Therefore, (4) cannot be used directly to estimate PSD. To circumvent this problem we propose using two ADCs working in parallel.

B. PNS Designed for Two ADCs

When two ADCs are used to alternately take samples it is possible to create pairs of sampling instants that are $L, 2L, ..ML$ apart. We review this by an example when $T_s = 2000L$, $H = 20L$ and $T_o = 200000L$. Here $[0, T_o]$ is the window within which we observe the signal. Fig. 1 shows the resultant numbers of pairs, characterised by distances from 1 to 99 times L, as fractions of the total number of samples N. It has to be emphasised that these numbers have been achieved without violating (c1) or (c2). Moreover, the resources needed to get them are well defined – two ADC. These features contrast favourably with the requirements imposed by Poisson additive random sampling where no finite number of ADCs can guarantee that the resources are plentiful enough to perfectly implement the sampling scheme. It is also worth comparing the proposed PNS scheme with uniform sampling. In order to avoid aliasing the uniform sampling time ought to be at most L. Since for a single ADC the minimum distance between samples has to be $20L$ we would need 20+ multiplexed converters to implement such scheme.

IV. Numerical Examples

In this section we present numerical analysis of the proposed method of estimating PSD. The results are compared with those where the data was collected using uniform and additive random, Poisson sampling. The scenario we deal here with is as follows. A random signal of unknown spectral support is placed somewhere between 0 and 2GHz. In other words the signal could be located anywhere between 0 and 2GHz. The test signal was actually band-limited to $[300, 500]$MHz. We use three sampling schemes: uniform, PNS and additive random. In each case we maintain identical average density of sampling instants (240MHz) and collect the samples inside the observation window of length 100µs. For PNS analysis we use the two-ADC sampling scheme designed in the previous section. The maximum sampling rate of the ADC we use is 200MHz. We chose $L = H/20 = 0.25$ ns and $T_s = 2000L$. The rectangular window applied to estimate $R(\tau)$ was $(-25 \quad 25)$ ns for double-sided Fourier transform or equivalently $(0 \quad 25)$ ns for the single-sided cosine transform. For uniform and PNS schemes we use (3) and (5) to estimate the PSD. In the case of Poisson additive random scheme we follow [11, 18] and use the following estimate of PSD:

$$P_{xN}(f) = \frac{1}{\pi f_a N} \sum_{n=1}^{N-1} \sum_{k=1}^{N-n} X(t_{k+n}) X(t_k) \cos 2\pi f (t_{k+n} - t_k) \quad (6)$$

where f_a is the average sampling rate and N is the total number of samples.

The average and maximum instantaneous sampling rates are listed in table 1 for uniform, Poisson additive and PNS sampling. Note that the average sampling rate f_a = 240MHz was the same for all three sampling schemes. The results are shown in Fig. 2-7. Fig. 2 shows the target spectrum of the signal. Note that although the signal was strictly limited to frequency range $[300, 500]$MHz our target spectrum has been derived using windowed ACF. Therefore the ripples and power leakage to neighbouring frequencies are visible. Fig. 3 shows the estimate obtained from uniform sampling. Obviously the sampling rate is much below the required 4GHz and the aliasing badly affects the results. Fig. 4 shows the outcome of spectrum estimation using Poisson additive sampling. Ten results for different realisations of the random process and random sampling sequences are superimposed. Finally Fig. 5 shows the results when PNS is used. Once again ten different realisations of the same random process were tested. The figures confirmed that low rate uniform sampling is not suitable for such tasks. Suitably chosen nonuniform sampling delivers acceptable results.

Fig. 6–7 presents the standard deviation of the estimated spectra. These plots have been created by processing results shown in Fig. 2, 4–5. Among them, PNS is characterised by slightly larger errors than the estimation based on Poisson additive random sampling. At first these differences may look strange if taking into account that the same number of samples was used for processing the signals and similar methods of spectrum estimation were deployed. However by examining (3) and (6) we realise that this is not the number of samples which directly affects the quality of spectrum estimation. More importantly we should be looking at the number of pairs of samples that are within the distance of the window which was used to truncate $R_x(\tau)$. We analysed two examples from this point of view. PNS generates 167739 such pairs while Poisson additive random sampling generates 194577. Clearly, Poisson additive random sampling had generally larger number of closely placed pairs than PNS. This is further illustrated by Fig. 8 where we show the histogram of the numbers of pairs of samples as percentage of the total number of collected samples at distances shorter than $H = 20L = 5$ ns. Note that the number of sample pairs apart from each other less than L is 1455. In fact this overwhelming presence of closely placed samples is one of the reasons why such sampling scheme is difficult to implement in practice. The quality of spectrum estimation of both nonuniform sampling schemes is very similar. However, PNS used here has the advantage of controlling the instantaneous sampling frequencies for ADC used in data acquisition.

V. Conclusions

In this paper we have proposed a method that allows estimating Power Spectrum Density of ergodic signals using arbitrarily low sampling rates. Use of low sampling rates removes the burden and possibly large cost of deploying very fast ADCs that could otherwise be needed if traditional approaches based on uniform sampling were used. As we have mentioned earlier there are reports in research literature on possibility of using low sampling rates to alias-free estimation of PSD. However in these approaches the guarantees of low sampling rates apply to the average values. At the same time, the instantaneous sampling rates could be arbitrary high and could last for arbitrary long periods. In the approach we propose here we suggest use of two ADCs. When designing sampling schemes we are able to accommodate upper limits on the instantaneous sampling frequencies for each of them.

The proposed method has been compared with two other approaches that put much more demanding requirements on the data acquisition hardware than what we propose here. Without doubts our method imposes the least stringent requirements on the speed and quantity of hardware used. The quality of the results can be further improved by using either of two methods proposed below. First one can use longer observation windows to collect more data and obtain better estimate. The other possibility is to gradually enhance hardware either by increasing the number of ADCs or by increasing their sampling rates. In either case the number of data collected per second and, more importantly, the number of sufficiently closely placed pairs of samples generated every second will increase. Further research will be devoted to search for methodologies of designing such sampling schemes.

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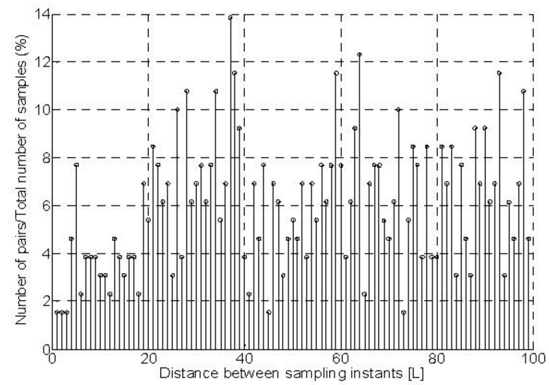

Fig. 1. The number of pairs distant from each other by mL , $m = 1,2,...,99$ for double-sampler PNS.

TABLE I
COMPARISON OF AVERAGE AND INSTANTANEOUS SAMPLING RATES FOR DIFFERENT SAMPLING SCHEMES

Sampling Scheme	Avarage sampling rate [MHz]	Maximum instantaneous sampling rate [MHz]
Uniform	240	240
Poisson additive	240	Infinity
PNS for sampling scheme	240	4000
PNS for each ADC	120	200

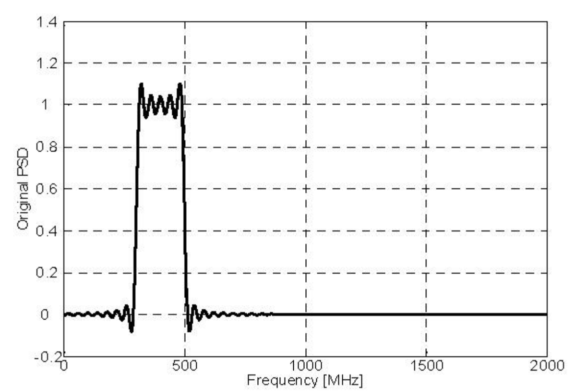

Fig. 2. The original PSD of the signal.

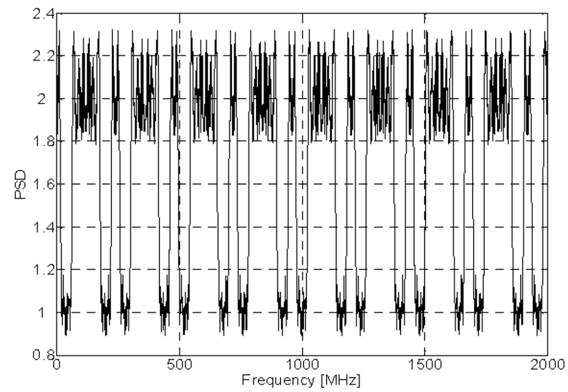

Fig. 3. The PSD estimation obtained from uniform sampling method (T_o =100μs and f_a =240MHz).

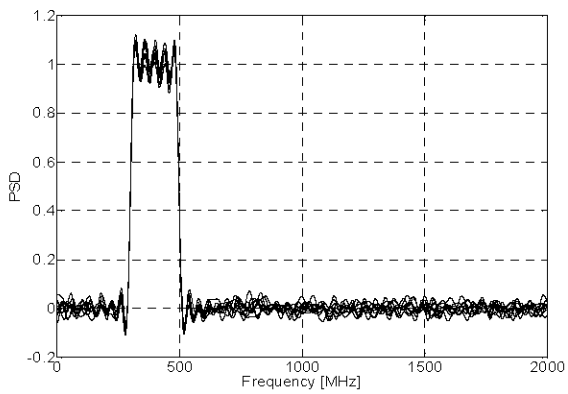

Fig. 4. The PSD estimation obtained from Poisson additive random sampling method (T_o =100μs and f_a =240MHz).

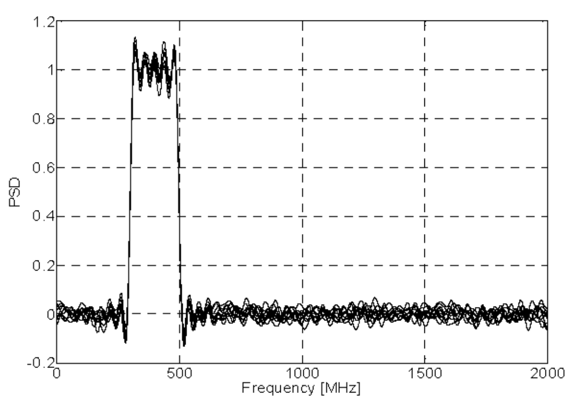

Fig. 5. The PSD estimation obtained from double-sampler PNS method (T_o =100μs and f_a =240MHz).

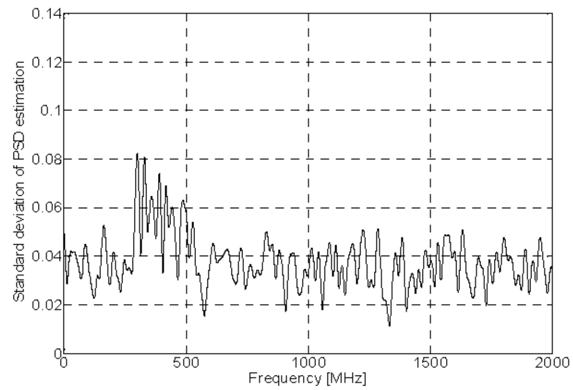

Fig. 6. Standard deviation of the PSD estimation for Poisson additive random sampling case.

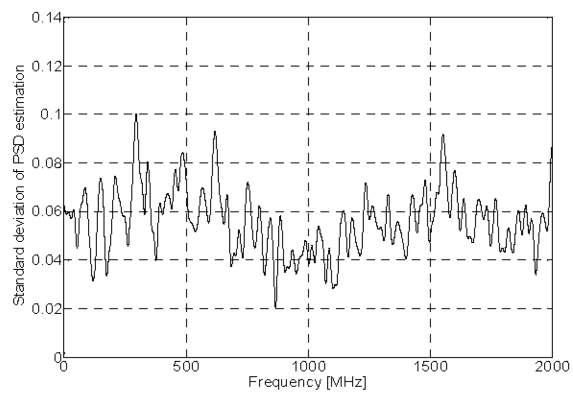

Fig. 7. Standard deviation of the PSD estimation for PNS case.

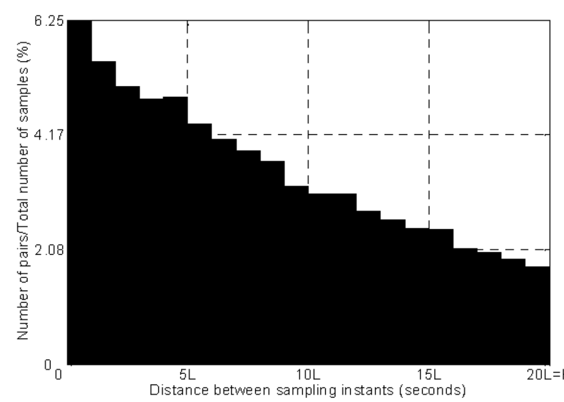

Fig. 8. The number of pairs distant from each other less than H=20L for Poisson additive random sampling.