FIR filters for systems with input clock jitter.

Andrzej Tarczynski

Cavendish School of Computer Science

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A. Tarczyński

Department of Electronic Systems, University of Westminster
115 New Cavendish Street, London, W1W 6UW, UK.
tarczya@wmin.ac.uk

ABSTRACT

A method of designing fixed–coefficient FIR filters whose input signals are sampled irregularly due to clock jitter is presented. The approach does not require direct measuring of the jitter. Instead it is assumed that the jitter is a strictly stationary stochastic process for which some statistical information is available. Preliminary analysis of degradation of filter performance due to presence of jitter is also presented. Some numerical analyses illustrate the main assertions of the paper.

1. INTRODUCTION

Virtually all classical approaches towards designing digital filters require that the sampling instants are uniformly distributed in time domain [1], [2]. Usually this assumption is satisfied with sufficient accuracy and the filters constructed with use of these methods maintain quality that was predicted at the designing stage. However, there are situations that the sampling instants depart from the uniform grid so much that the quality of filtering is adversely affected. For example, in extremely fast DSP systems even relatively small jitter may prove to be significant in comparison to a very short sample time. In such cases the designer may try to re-design the system (typically using more expensive components) so that the sampling period is kept constant with required accuracy or to modify the filter so that provisions against sampling rate variations are taken. In this paper we discuss the problem of designing FIR filters in such a way that the effect of input clock jitter is diminished. It is intuitively obvious that the best compensation would be achieved if the jitter were measured. In such case however, one would need to modify the coefficients of the filter every sample period - ending up with a time varying filter [3]- [5]. We propose a different approach. The amount of jitter is not measured but instead the stochastic properties of the jitter are taken into account and used for tuning the coefficients of the designed filter. We assume that the jitter is a stationary process. Therefore the filter's coefficients are designed once and kept constant all the time. Similar approach was discussed in [5] for the purpose of signal reconstruction. In [5], however, the signals considered were polynomials and the fixed coefficient filter was created as a result of approximating time-varying solution with time-invariant one. Here we consider a class of band-limited signals and use direct optimization of the filter coefficients to get a solution that is optimal in the sense of a selected criterion. Of course, the approach proposed here is limited in terms of how good quality of filtering can be achieved and how much the sampling jitter can be compensated for. The filters designed with the method proposed in this paper do better than traditional filters, for which the effect of jitter is ignored at the designing stage. However, their performance is still limited regardless of how big the filter is and how accurate the information about the properties of the jitter is available. Some preliminary analysis of those limitations is presented in this paper as well.

2. OPTIMIZATION METHOD FOR DESIGNING FIR FILTER WITH JITTERED INPUT CLOCK

Consider a prototype of a digital FIR filter

$$y(t)=\sum_{n=-N}^{N} a_n\, u(t-n) \tag{1}$$

where $u(t)$ and $y(t)$ are the filter's input and output signals respectively, $t=0,\pm1,\pm2,\cdots$ is normalized time and $[a_{-N},\cdots,a_N]^T=\mathbf{a}$ is a vector of coefficients whose values have to be selected in such a way that the filter's frequency response is shaped according to given specification. Note that filter (1) is in a non-causal form. Therefore it can be implemented only after its input-output delay is increased by at least N samples. In the reminder of the paper we will assume that such operation is always performed before practical realization of the filter and therefore in all our discussions we will freely use non-causal representations similar to (1).

Assume that the time instants at which input signal is sampled are affected by random not measurable clock jitter. Consequently (1) is replaced with

$$y(t)=\sum_{n=-N}^{N} a_n\, u(t-n+\gamma_{t-n})\,. \tag{2}$$

In (2) jitter γ_t is a sequence of random, stochastically independent variables having all the same probability density function (p.d.f.). We assume that the p.d.f. is known.

Let ν be normalized frequency such that the sample frequency is represented by $\nu=1$. Denote the spectrum of the continuous-time input signal as $U(\nu)$. For obvious reason we demand that $U(\nu)=0$ when $|\nu|\geq 0.5$. Let $H(\nu)$ be a complex-valued function representing desired frequency response of the filter. Hence, the spectrum of the perfectly filtered signal is $\hat{Y}(\nu)=H(\nu)U(\nu)$. Let $\hat{y}(t)$ be the time-domain signal whose spectrum is $\hat{Y}(\nu)$. We will aim at selecting the filter's coefficients a_n in such a way that the output of filter (2) closely follows $\hat{y}(t)$. This can be expressed by demanding that

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$$\hat{J} \doteq E\left\{\left[\hat{y}(t) - y(t)\right]^2\right\}. \tag{3}$$

is minimized. In (3) $E\{\cdot\}$ denotes expectation operator.

A popular class of methods for designing good quality filters is based on using optimization techniques. These, however, can be deployed only when the designer can formulate an optimization index (cost) that properly reflects the objectives of the design and that can be calculated any time when needed during optimization process. When we substitute relation (2) in (3) and use inverse Fourier transforms to express $\hat{y}(t)$ and $u(t)$ we get

$$\hat{J} = \mathrm{E}\left\{\left[\int_{-0.5}^{0.5} \mathrm{U}(\nu)\exp(\mathrm{j}2\pi\nu t) \times \left(\mathrm{H}(\nu) - \sum_{n=-N}^{N} a_n \exp(\mathrm{j}2\pi\nu(-n+\gamma_{t-n})\right) \mathrm{d}\nu\right]^2\right\}. \tag{4}$$

Now it is obvious that although $\hat{J}$ looks attractive as an optimization criterion for designing our filter, we cannot use it directly in computations since in order to calculate it we should know exact spectrum of the input signal. This kind of information is practically never available at the stage of designing a filter. Therefore we need to modify (3) - (4) to create a more tractable cost function. It follows from Schwartz inequality that for any integrable function $f(x)$: $\left|\int_a^b f(x)\,\mathrm{d}x\right|^2 \le (b-a)\int_a^b |f(x)|^2\,\mathrm{d}x$.

Therefore cost (4) is never greater than $\bar{J}$ defined as

$$\bar{J} \doteq \mathrm{E}\left\{\int_{-0.5}^{0.5} |\mathrm{U}(\nu)|^2 \times \left|\mathrm{H}(\nu) - \sum_{n=-N}^{N} a_n \exp(\mathrm{j}2\pi\nu(-n+\gamma_{t-n})\right|^2 \mathrm{d}\nu\right\}. \tag{5}$$

Note that (5) is better suited as the optimization index than (4). It does not require phase information about the spectrum of the input signal. In fact we have even more flexibility here. Note that the vector $\mathbf{a}$ that minimizes (5) does not change if this cost is multiplied by a positive number. Therefore we can replace $|\mathrm{U}(\nu)|^2$ in (5) with some function $\mathrm{W}(\nu)$ such that the ratio $|\mathrm{U}(\nu)|^2/\mathrm{W}(\nu)$ is a positive constant for $\nu \in [-0.5, 0.5]$. Of course in practice we use only an approximation of such perfect $\mathrm{W}(\nu)$. Ultimately the problem of designing FIR filter for jittered environment is reduced to minimization of

$$J \doteq \int_{-0.5}^{0.5} \mathrm{W}(\nu)\mathrm{E}\left\{\left|\mathrm{H}(\nu) - \sum_{n=-N}^{N} a_n \exp(\mathrm{j}2\pi\nu(-n+\gamma_{t-n})\right|^2\right\}\mathrm{d}\nu. \tag{6}$$

Cost (6) resembles very much weighted least squares (WLS) cost that is widely used for designing FIR filters not subjected to the input clock jitter [6], [7]. Function $\mathrm{W}(\nu)$ is analogous to the weight function used in WLS cost.

Simple analysis shows that (6) can be expressed as

$$J = \int_{-0.5}^{0.5} \mathrm{W}(\nu)\Big[\mathrm{H}(\nu)\mathrm{H}^*(\nu) - 2\mathbf{a}^{\mathrm{T}}\,\mathrm{H}^*(\nu)\mathbf{z}(\nu)\mathrm{E}\{e^{j2\pi\nu\gamma}\} + \mathbf{a}^{\mathrm{T}}\left(\mathbf{I} + \mathbf{X}(\nu)\mathrm{E}\{e^{j2\pi\nu(\gamma_1-\gamma_2)}\}\right)\mathbf{a}\Big]\mathrm{d}\nu \tag{7}$$

where $\mathbf{z}(\nu) = \left[e^{j2\pi\nu N}, e^{j2\pi\nu(N-1)}, \dots, e^{-j2\pi\nu N}\right]^T$, $\mathbf{I}$ is a unity matrix of size $2N+1$ and $\mathbf{X}(\nu)$ is a $(2N+1)\times(2N+1)$ matrix with all diagonal elements equal to zero and off-diagonal elements being $\mathbf{X}(\nu)_{lk} = e^{j2\pi\nu(k-l)}$. Note that $\chi(\nu) = \mathrm{E}\{e^{j2\pi\nu\gamma}\}$ is the characteristic function of the jitter and $\mathrm{E}\{e^{j2\pi\nu(\gamma_1-\gamma_2)}\}$ is the characteristic function of the difference of random variables representing jitters at two different time instants. By using standard properties of characteristic functions we obtain $\mathrm{E}\{e^{j2\pi\nu(\gamma_1-\gamma_2)}\} = \chi(\nu)\chi^*(\nu)$. Now, let $\alpha \doteq \sqrt{\int_{-0.5}^{0.5} 1/\left(\chi(\nu)\chi^*(\nu)\right)\mathrm{d}\nu}$, and $\mathrm{W}_0(\nu) \doteq \alpha^2\,\mathrm{W}(\nu)\chi(\nu)\chi^*(\nu)$. We can rewrite (7) as

$$\begin{aligned} J = &\int_{-0.5}^{0.5} \mathrm{W}_0(\nu)\frac{\mathrm{H}(\nu)\mathrm{H}^*(\nu)}{\alpha^2\chi(\nu)\chi^*(\nu)}\mathrm{d}\nu \\ &- 2\mathbf{a}^{\mathrm{T}}\int_{-0.5}^{0.5} \mathrm{W}_0(\nu)\frac{\mathrm{H}^*(\nu)\mathbf{z}(\nu)}{\alpha^2\chi^*(\nu)}\mathrm{d}\nu \\ &+ \mathbf{a}^{\mathrm{T}}\int_{-0.5}^{0.5} \mathrm{W}_0(\nu)\left(\frac{\mathbf{I}}{\alpha^2\chi(\nu)\chi^*(\nu)} + \frac{\mathbf{X}(\nu)}{\alpha^2}\right)\mathrm{d}\nu\,\mathbf{a} \end{aligned} \tag{8}$$

or equivalently

$$J = \mathbf{a}^{\mathrm{T}}\mathbf{P}\mathbf{a} - 2\mathbf{a}^{\mathrm{T}}\mathbf{r} + s \tag{9}$$

where definitions of $\mathbf{P}$, $\mathbf{r}$ and s can be easily obtained by comparing (8) and (9). The solution to the problem of minimizing (9) has a standard form and is given by

$$\hat{\mathbf{a}} = \mathbf{P}^{-1}\mathbf{r}. \tag{10}$$

Note that (10) gives the vector of filter coefficients that is optimal in the sense of criterion (7).

3. PERFORMANCE LIMITATION OF JITTERED FILTERS

It can be proven for jitter-free case that when $N \to \infty$ then, if FIR filter (1) is optimal in WLS sense its frequency response becomes identical with the target frequency response for all frequencies apart from a finite number of discontinuity points. This statement is true if the target frequency response and the weight function satisfy some not very restrictive conditions. In other words, by allowing sufficient number of taps in the filter (and consequently introducing sufficiently long input-output delay in the filter implementation) we can achieve arbitrarily good performance of the filter. Now we are going to investigate if similar property characterizes optimal FIR filters with jitter. For the purpose of this analysis we confine our discussions in this and the following section to the case when $\mathrm{W}_0(\nu) = 1$. Note first that

$$\int_{-0.5}^{0.5} \frac{1}{\alpha^2 \chi(v)\chi^*(v)} \mathrm{d}v = 1 \quad (11)$$

and

$$\int_{-0.5}^{0.5} \mathbf{X}(v)\,\mathrm{d}v = \mathbf{0} \quad (12)$$

Therefore when $W_0(v)=1$ we have $\mathbf{P}=\mathbf{I}$ and solution (10) can be put as

$$\hat{\mathbf{a}} = \frac{1}{\alpha^2}\int_{-0.5}^{0.5} \frac{H^*(v)}{\chi^*(v)} \mathbf{z}(v)\,\mathrm{d}v \quad (13)$$

It is interesting to note that exactly the same vector of coefficients can be obtained by solving a problem of designing Least-Squares optimal FIR filter (for implementation in jitter-free environment) whose frequency response approximates

$$H_{\prime\prime}(v) = \frac{1}{\alpha^2}\frac{H(v)}{\chi(v)}. \quad (14)$$

By exploiting this analogy even further we can prove that if function $H_{\prime\prime}(v)$ is analytic (i.e. its values are identical with the values of its own Fourier series) then

$$\lim_{N\to\infty} \mathbf{a}^T \mathbf{z}(v) = \frac{1}{\alpha^2}\frac{H(v)}{\chi(v)}. \quad (15)$$

When we apply Parseval's theorem to (15) we get

$$\lim_{N\to\infty} \mathbf{a}^T \mathbf{a} = \int_{-0.5}^{0.5} \frac{1}{\alpha^4}\frac{H(v)H^*(v)}{\chi(v)\chi^*(v)}\mathrm{d}v. \quad (16)$$

Now we can assess asymptotic quality of the optimal FIR filter with jitter. The cost J is a monotonically decreasing function of N. Therefore the best achievable performance is $J_{best} = \lim_{N\to\infty} J$. By combining (8), (13), (15) and (16) we get

$$J_{best} = \frac{\alpha^2 - 1}{\alpha^4}\int_{-0.5}^{0.5} \frac{H(v)H^*(v)}{\chi(v)\chi^*(v)}\mathrm{d}v \quad (17)$$

It is easy to check that $\alpha = 1$ occurs only in jitter-free case. Otherwise we have $\alpha > 1$. Therefore, when jitter is present, the right hand side of (17) is always positive. It is clearly visible from (17) that the performance of the filter with jittered input is limited since cost J can never reach 0.

4. NUMERICAL EXAMPLES

To illustrate how the input jitter affects asymptotic quality of the filter consider design of an all-pass filter, i.e. a filter for which $|H(v)| = 1$. In this case (17) simplifies to

$$\lim_{N\to\infty} J = 1 - \frac{1}{\alpha^2}. \quad (18)$$

We arbitrarily choose two types of jitter to see how they affect quality of filtering. The first one is uniformly distributed jitter whose p.d.f. is given by

$$f(\gamma) = \begin{cases} \frac{1}{2\Delta} & \text{when } |\gamma| < \Delta \\ 0 & \text{otherwise} \end{cases}. \quad (19)$$

Second jitter is discrete-valued characterized by the following probabilities

$$P(\gamma) = \begin{cases} p & \text{if } |\gamma| = \Delta \\ 1-2p & \text{if } \gamma = 0 \\ 0 & \text{otherwise} \end{cases}. \quad (20)$$

Note that in (19) and (20) Δ is the maximum value of the jitter and therefore it may vary between 0 and 0.5. Probability p in (20) may also take values only between 0 and 0.5.

To calculate the values of (18) in both considered cases we need to obtain jitters' characteristic functions and then the appropriate values of α.

In the case of uniformly distributed jitter we have

$$\chi_{\prime\prime}(v) = \int_{-\Delta}^{\Delta} \frac{1}{2\Delta} e^{j2\pi v\gamma}\,\mathrm{d}\gamma = \frac{\sin(2\pi v\Delta)}{2\pi v\Delta} \quad (21)$$

and

$$\alpha_{\prime\prime} = \int_{-0.5}^{0.5} \frac{4\pi^2 v^2 \Delta^2}{\sin^2(\pi v\Delta)}\mathrm{d}v. \quad (22)$$

Figure 1 shows the plot of J_{best} as a function of Δ.

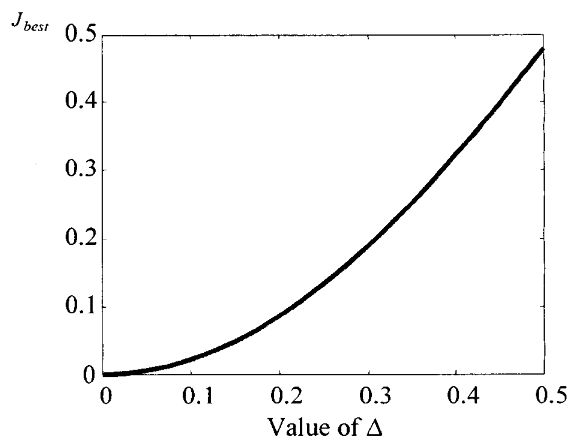

Figure 1. Minimum value of cost for all-pass filters as a function of maximum jitter – uniformly distributed p.d.f. case.

For the discrete-valued jitter we have the following characteristic function

$$\chi_d(v) = 1 - 2p + 2p\cos(2\pi v\Delta). \quad (23)$$

Therefore

$$\alpha_d = \int_{-0.5}^{0.5} \frac{1}{(1-2p+2p\cos(2\pi v\Delta))^2}\mathrm{d}v. \quad (24)$$

Figure 2 shows the plot of J_{best} as a function of p and Δ.

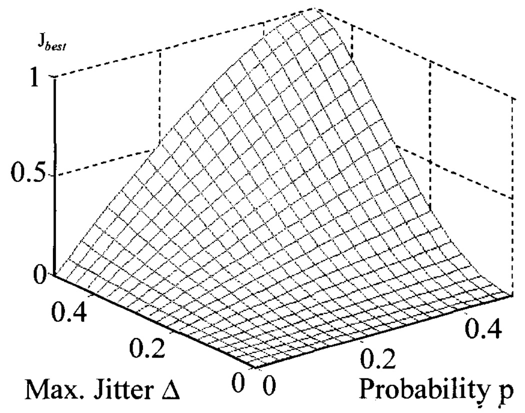

Figure 2. Minimum value of cost for all-pass filters as a function of maximum jitter and probability p - discrete p.d.f. case.

5. SIMULATION

In this section we show how much it is possible to gain by taking properly into account the presence of jitter. Let the signal to be filtered consist of four unity-amplitude sinusoids whose normalized frequencies are 0.1, 0.2, 0.3 and 0.4. We design two 121 tap, half-band, high-pass filters. Ideally, these filters should pass only two higher frequency sinusoids. The first filter is designed for jitter-free environment using classical least-squares approach (i.e. WLS method with weight being 1 for all frequencies). The other filter is optimized to reduce influence of jitter. This is achieved by minimizing criterion (7) with $W(\nu)=1$. The jitter in this example is described by (20) with $\Delta = 0.45$ and $p = 0.5$. Both filters are simulated to operate on the same set of jittered samples. Their outputs are compared against perfectly filtered signal, i.e. the sum of high frequency sinusoids delayed by 60 sample periods. In each case we calculate mean squared error (MSE). For the first filter MSE was 0.683 while in the second case MSE was reduced to 0.595 – improvement of nearly 15%. The filtering error for each simulated case is plotted in Figure 3.

6. SUMMARY

This paper presents a method of designing FIR filters for filtering signals that, due to clock jitter, are sampled on non-uniform time grid. The proposed method is, to some extent, a modification of WLS approach that is often used in classical designs of FIR filters. The design process of the filter is only slightly more complicated than the WLS design. Moreover, since the proposed filters can be implemented with use of tapped delay lines with fixed values of the taps, the cost of implementation of the proposed filters is exactly the same as for those designed with classical methods.

The approach we discussed here allows reducing deterioration of filter performance caused by clock jitter. At the moment, simulation experiments show that for wide-band designs the mean squared error can be bettered by up to 20%. The research performed so far clearly indicates that there exists a theoretical limit on how much improvement the filter can bring. In future, further investigation will be carried out to explore in full the potential and limitations created by this new technique of designing digital filters.

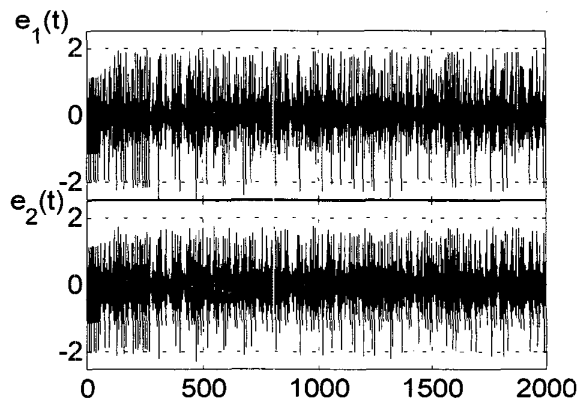

Figure 3. Simulation results - filtering errors $e_1(t)$ - filtering error when jitter is ignored $e_2(t)$ - filtering error when jitter is compensated (15% improvement).

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