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Socratic Proofs for Propositional Linear-Time Logic

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Abstract:

This paper presents a calculus of Socratic proofs for Propositional Linear-Time Logic (PLTL) and discusses potential automation of its proof search.

1 Introduction

Propositional Linear-Time Logic (PLTL) [6] gained various deductive constructions: axiomatic [5], tableau [11], resolution [4], and natural deduction [1]. In this paper we present a calculus of Socratic proofs for Propositional Linear-Time Logic (PLTL) [6], [3], [7] abbreviated as PLTL$_{\text{SP}}$. The calculus is based upon the hypersequent calculus [2] and it fits into the framework of Socratic proofs by Wisniewski (cf. [8], [10] and [9]).

2 Logic PLTL$_T$

We utilise the language of PLTL which extends the language of Classical Propositional Calculus (CPC) by temporal operators: $\mathcal{U}$ (*until*) $\bigcirc$ (*at the next moment in time*), $\square$ (*always in the future*), and $\diamond$ (*at sometime in the future* or *eventually*). The semantics for the temporal part of the logic PLTL$_T$ is defined in the standard way over linear sequence of states, finite in the past, infinite in the future.

In order to formulate PLTL$_T$ we need to extend the language of PLTL with the following signs: $\vdash$, ?, 1 and 2. Intuitively, $\vdash$ stands for derivability relation and ? is a question-forming operator. The numerals 1 and 2 will be used to encode tree-structure of a Socratic transformation.

There are two disjoint categories of wffs: *declarative* wffs (d-wffs) and *erotetic* wffs (e-wffs), or questions. There are also two types of d-wffs: *atomic* d-wffs and *indexed d-wffs*. Atomic d-wffs are expressions of the form $S \vdash A$, where S is a finite sequence (possibly with repetitions) of PLTL-wffs, and A is a PLTL-wff, and if A is an empty formula, then S is a non-empty sequence. Indexed d-wffs are expressions of the form $S \vdash^n A$ or of the form $T \vdash^n$, where $S \vdash A$ and $T \vdash$ are atomic d-wffs of and n is a sequence of 1's or 2's, starting with 1. E-wffs, or questions are expressions of the form ?(Φ), where Φ is a non-empty finite sequence of indexed atomic d-wffs (*constituents* of Φ).

In the formulation of rules we shall use the following classification of PLTL formulae to α and β types:

α	α_1	α_2
$A \wedge B$	A	B
$\neg(A \vee B)$	$\neg A$	$\neg B$
$\neg(A \rightarrow B)$	A	$\neg B$
$\square A$	A	$\bigcirc \square A$
$\neg \diamond A$	$\neg A$	$\bigcirc \square \neg A$
$\neg(A\mathcal{U}B)$	$\neg B$	$\neg(A \wedge \bigcirc(A\mathcal{U}B))$

β	β_1	β_2	β_1^*
$\neg(A \wedge B)$	$\neg A$	$\neg B$	A
$A \vee B$	A	B	$\neg A$
$A \rightarrow B$	$\neg A$	B	A
$\neg \square A$	$\neg A$	$\bigcirc \diamond \neg A$	A
$\diamond A$	A	$\bigcirc \diamond A$	$\neg A$
$A\mathcal{U}B$	B	$A \wedge \bigcirc(A\mathcal{U}B)$	$\neg B$

Rules for PLTL$_T$:

$$\mathbf{L}_\alpha : \quad \frac{?\,(\Phi; S \,'\, \alpha \,'\, T \vdash^n C; \Psi)}{?\,(\Phi; S \,'\, \alpha_1 \,'\, \alpha_2 \,'\, T \vdash^n C; \Psi)}$$

$$\mathbf{L}_\beta : \quad \frac{?\,(\Phi; S \,'\, \beta \,'\, T \vdash C; \Psi)}{?\,(\Phi; S \,'\, \beta_1 \,'\, T \vdash^{n1} C; S \,'\, \beta_2 \,'\, T \vdash^{n2} C; \Psi)}$$

$$\mathbf{L}_{\neg\neg} : \quad \frac{?\,(\Phi; S \,'\, \neg\neg A \,'\, T \vdash^n C; \Psi)}{?\,(\Phi; S \,'\, A \,'\, T \vdash^n C; \Psi)}$$

$$\mathbf{L}_{\neg\bigcirc} : \quad \frac{?\,(\Phi; S \,'\, \neg \bigcirc A \,'\, T \vdash^n C; \Psi)}{?\,(\Phi; S \,'\, \bigcirc \neg A \,'\, T \vdash^n C; \Psi)}$$

$$\mathbf{R}_\alpha : \quad \frac{?\,(\Phi; S \vdash^n \alpha; \Psi)}{?\,(\Phi; S \vdash^{n1} \alpha_1; S \vdash^{n2} \alpha_2; \Psi)}$$

$$\mathbf{R}_\beta : \quad \frac{?\,(\Phi; S \vdash^n \beta; \Psi)}{?\,(\Phi; S \,'\, \beta_1^* \vdash^n \beta_2; \Psi)}$$

$$\mathbf{R}_{\neg\neg} : \quad \frac{?\,(\Phi; S \vdash^n \neg\neg A; \Psi)}{?\,(\Phi; S \vdash^n A; \Psi)}$$

$$\mathbf{R}_{\neg\bigcirc} : \quad \frac{?\,(\Phi; S \vdash^n \neg \bigcirc A; \Psi)}{?\,(\Phi; S \vdash^n \bigcirc \neg A; \Psi)}$$

If none of the above rules is applicable to a PLTL formula B, such a formula is called *marked*. If all PLTL-formulas within an indexed formula $S \vdash^n A$ are marked, such a formula is called *a state*.

The following is a state-prestate rule:

$$\mathbf{S} - \mathbf{P} : \quad \frac{?\,(\Phi; S \vdash^n A; \Psi)}{?\,(\Phi; S^\circ \vdash^n A^\circ; \Psi)}$$

where $S \vdash^n A$ is a state and S° (resp. A°) results from S (resp. A) by replacing all the formulas of the form $\bigcirc B$ with B and deleting all the remaining formulas. Every formula

of the form $S^* \vdash^m A^*$, where n is an initial subsequence of m or m is an initial subsequence of n, is called *a pre-state* (cf. [11]).

Definition 1. *Let* $\mathbf{q} = \langle Q_1, \ldots, Q_r \rangle$ *be a finite sequence of questions of* $\mathbf{P}^*$. *Let* Q_g, Q_{h-1}, Q_h $(1 \le g < h-1 \le r)$ *be elements of the sequence* $\mathbf{q}$. *Let* $S_j \vdash^n A_j$ *be a constituent of* Q_g *and let* $S_k \vdash^m A_k$ *be a constituent of* Q_h *such that* $S_j = S_k$, $A_j = A_k$ *and the sequence* n *is an initial subsequence of the sequence* m. *Let* $S_l \vdash^i A_l$ *be a constituent of* Q_{h-1} *such that* $S_k \vdash^m A_k$ *is obtained from* $S_l \vdash^i A_l$ *by application of a* PT^*-*rule. Then* $S_j \vdash^n A_j, \ldots, S_l \vdash^i A_l$ *form a* loop *(a sequence of atomic d-wffs of* $\mathbf{P}^*$ *... etc.), and* $S_k \vdash^m A_k$ *is called a* loop-generating formula*.

Socratic transformations are sequences of questions that aim at deciding derivability of formuls from sets of formuls.

Definition 2. *A finite sequence* $\langle Q_1, \ldots, Q_r \rangle$ *of questions of* $\mathbf{P}^*$ *is a* Socratic transformation *of* $S \vdash A$ *iff the following conditions hold: (i)*$Q_1 =?(S \vdash^1 A)$*; (ii)*Q_i *(where* $i = 2, \ldots, r$*) results from* Q_{i-1} *by applying a* PT^*-*rule.*

Definition 3. *A constituent* ϕ *of a question* Q_i *is called* successful *iff one of the following holds: (a)* ϕ *is of the form* $T'B'U \vdash^n B$, *or (b)* ϕ *is of the form* $T'B'U'\neg B'W \vdash^n C$, *or (c)* ϕ *is of the form* $T'\neg B'U'B'W \vdash^n C$.

Definition 4. *A Socratic transformation* $\langle Q_1, \ldots, Q_r \rangle$ *of* $S \vdash A$ *is* completed *iff the for each constituent* ϕ *of* Q_r *at least one of the following conditions hold: (a) no rule is applicable to PLTL-formulas in* ϕ, *or (b)* ϕ *is successful, or (c)* ϕ *is a loop-generating formula.*

Definition 5. *A formula* B *is called an* eventuality *in* $S \vdash^n A$ *iff one of the following holds: (i)* B *is a term of* S *and there exists a PLTL-formula* C *such that* $B = \diamond C$, *or (ii) there exists a PLTL-formula* C *such that* $B = A = \square C$.

Definition 6. *A completed Socratic transformation* $\mathbf{q} = \langle Q_1, \ldots, Q_r \rangle$ *is a Socratic proof of* $S \vdash A$ *iff: (a) all the constituents of* Q_n *are successful, or (b) for each nonsuccessful constituent* ϕ *of* Q_n, ϕ *is a loop-generating formula and the loop generated by* ϕ *contains a pre-state with an unfulfilled eventuality.*

The presented system is sound and complete. Proofs of these theorems involve construction of a canonical model with maximal consistent sets of formulae as its states.

3   Examples

In the examples below by highlighting we indicate a formula which is analyzed at the current step. Double underlining of a formula reflects that it is a state. The question following the one containing a state is obtained by state-prestate rule.

Example 1

1. $?(\vdash^1 \boxed{\square p \to p}\,)$
2. $?(\boxed{\square p} \vdash^1 p)$
3. $?(p, \bigcirc\square p \vdash^1 p)$

Example 2

1. $?(\vdash^1 \boxed{\square p \to \bigcirc p}\,)$
2. $?(\boxed{\square p} \vdash^1 \bigcirc p)$
3. $?(p, \bigcirc\square p \vdash^1 \bigcirc p)$
4. $?(\boxed{\square p} \vdash^1 p)$
5. $?(p, \bigcirc\square p \vdash^1 p)$

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