A General Formula for Impulse-Invariant Transformation for Continuous-Time Delta-Sigma Modulators

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Abstract, this paper presents a generalised new formula for impulse-invariant transformation which can be used to convert an nth-order Discrete-Time (DT) ΔΣ modulator to an nth-order equivalent Continuous-Time (CT) ΔΣ modulator. Impulse-invariant transformation formulas have been published in many open literature articles for s-domain to z-domain conversion and vice-versa. However, some of the published works contain omissions and oversights. To verify the newly derived formulas, very many designs of varying orders have been tested and a representative 4th-order single-loop DT ΔΣ modulator converted to an equivalent CT ΔΣ modulator through the new formulas are presented in this paper. The simulation results confirm that the CT ΔΣ modulator which has been derived by these formulas works in accordance with the initial DT specifications without any noticeable degradation in performance in comparison to its original DT ΔΣ modulator prototype.

Index Terms — Impulse-Invariant Transformation, Delta-Sigma Modulator, Continuous-Time, Discrete-Time.

I. INTRODUCTION

The ΔΣ modulators are widely used in audio applications and portable devices to achieve high resolution analog-to-digital conversion for relatively low-bandwidth signals by using the oversampling and the noise-shaping techniques. CT ΔΣ modulators have drawn a lot of attention from analog designers over the last decade due to their potential to operate at higher clock frequencies in comparison to their DT counterparts. Sampling requirements are relaxed in the CT ΔΣ modulators because the sampling is inside their loop and any sampling error is shaped by their Noise-Transfer Function (NTF). The CT ΔΣ modulators have an implicit anti-aliasing filter in their forward loop filter. However, CT ΔΣ modulators suffer from several drawbacks: excess loop delay, jitter sensitivity and RC time constant variations.

One way to convert a DT ΔΣ modulator to an equivalent CT ΔΣ modulator is through the use of the impulse-invariant transformation [1]-[6]. A DT ΔΣ modulator and a CT ΔΣ modulator are shown in Figure 1, and are said to be equivalent when their quantizer inputs are equal at the sampling instants.

$$q(n) = q_c(t)|_{t=nT} \quad \text{for all n} \tag{1}$$

Where $q(n)$ and $q(n)$ are the quantizer inputs of the DT and

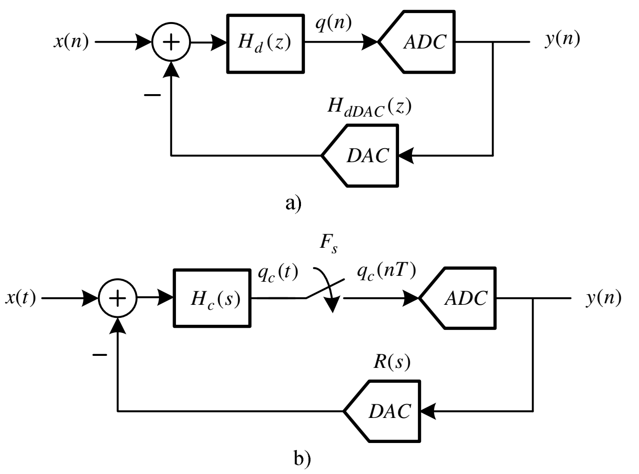


Figure 1: The block diagrams of a) The DT ΔΣ modulator and b) The CT ΔΣ modulator.

CT ΔΣ modulators and T is the clock period of the ΔΣ modulators. This condition would be fulfilled if the impulse responses of the open-loop filter of the CT and DT ΔΣ modulators were equal at the sampling times. As a result (1) translates directly into (2):

$$\mathcal{Z}^{-1}\{H_{dDAC}(z)H_d(z)\} = \mathcal{L}^{-1}\{R(s)H_c(s)\}\big|_{t=nT} \tag{2}$$

Because $H_{dDAC}(z) = 1$, equation (2) can be simplified to give (3):

$$\mathcal{Z}^{-1}\{H_d(z)\} = \mathcal{L}^{-1}\{H_{cDAC}(s)H_c(s)\}|_{t=nT} \tag{3}$$

The transformation in (3) is the well-known impulse-invariant transformation where $\mathcal{Z}^{-1}$, $\mathcal{L}^{-1}$, $R(s)$, $H_d(z)$ and $H_c(s)$ represent the inverse z-transform, the inverse Laplace transform, the CT DAC transfer function, the DT and the CT loop filters respectively [1],[4]. Depending on the output waveform of the CT DAC, there would be an exact mapping between the DT and the CT ΔΣ modulators. The popular feedback-DAC waveforms have rectangular shapes. The time and frequency (Laplace) domain responses of these waveforms are:

$$r_{(\alpha,\beta)}(t) = \begin{cases} 1, & \alpha T \le t \le \beta T, \quad 0 \le \alpha,\beta \le 1 \\ 0, & otherwise \end{cases} \tag{4}$$

$$R(s) = \frac{e^{-\alpha Ts} - e^{-\beta Ts}}{s} \tag{5}$$

In the cases where $\beta > 1$ the DAC equation is divided into two parts as expressed by (6) and the z-domain equivalents of each part is calculated separately.

$$r_{(\alpha,\beta)}(t) = r_{(\alpha,1)}(t) + r_{(0,\beta)}(t-T) \tag{6}$$

This paper is organized as follows. To set the scene, in section II, the concept of the impulse-invariant transformation is reviewed and a general formula for s-domain to z-domain conversion for ΔΣ modulator applications is derived. In section III, simulation results of the 4th-order CT and DT ΔΣ modulators are both presented and discussed in detail. Finally, conclusions are given in section IV.

II. IMPULSE-INVARIANT TRANSFORMATION

In order to derive the equivalent z-domain transfer function of CT ΔΣ modulators with rectangular DAC waveforms, we shall start with the 1st order s-domain term. Equation (7) is derived by substituting (5) and the 1st order s-domain term into (3) as follows.

$$H_{1d}(z) = \mathcal{Z}\{\mathcal{L}^{-1}\left(\frac{e^{-\alpha Ts} - e^{-\beta Ts}}{s^2}\right)\Big|_{t=nT}\} \tag{7}$$

An auxiliary variable λ is deployed to derive a general formula step by step. Equation (8) is equal to (7) when $\lambda = 0$ [7], [8]:

$$H_{1d}(z) = \mathcal{Z}\{\mathcal{L}^{-1}\left(\frac{e^{-\alpha Ts} - e^{-\beta Ts}}{(s-\lambda)^2}\right)\Big|_{t=nT}\}\Big|_{\lambda=0} = H_1(z)\Big|_{\lambda=0} \tag{8}$$

By using the Laplace transform properties, (8) leads to (9) where $u(t)$ represents a step function [7].

$$\begin{aligned} H_1(z) = \mathcal{Z}\Big\{e^{\lambda t}(&e^{-\alpha\lambda t}(t-\alpha T)u(t-\alpha T) \\ &- e^{-\beta\lambda t}(t-\beta T)u(t-\beta T))\big|_{t=nT}\Big\} \end{aligned} \tag{9}$$

The continuous time variable t in (9) is replaced with nT in (10).

$$\begin{aligned} H_1(z) = \mathcal{Z}\Big\{e^{\lambda nT}(&e^{-\alpha\lambda T}(nT-\alpha T)u(nT-\alpha T) \\ &- e^{-\beta\lambda T}(nT-\beta T)u(nT-\beta T))\big|_{t=nT}\Big\} \end{aligned} \tag{10}$$

The z-transform of (10) is expressed by (11) which results in (12) [7], [8].

$$H_1(z) = T\sum_{n=0}^{+\infty} e^{\lambda nT}(e^{-\alpha\lambda T}(n-\alpha) - e^{-\beta\lambda T}(n-\beta))z^{-n} \tag{11}$$

$$H_1(z) = T\left\{\frac{(1-\alpha)e^{(1-\alpha)\lambda T} - (1-\beta)e^{(1-\beta)\lambda T}}{(z-e^{\lambda T})} + \frac{e^{(2-\alpha)\lambda T} - e^{(2-\beta)\lambda T}}{(z-e^{\lambda T})^2}\right\} \tag{12}$$

It can be proved that (12) can be obtained by calculating the 1st derivative with respect to the variable λ of equation (13).

$$H_1(z) = \frac{\partial}{\partial\lambda}\left(\frac{e^{(1-\alpha)\lambda T} - e^{(1-\beta)\lambda T}}{z-e^{\lambda T}}\right) \tag{13}$$

By substituting $\lambda = 0$ into (12) the z-domain equivalent of the 1st order s-domain term is expressed by (14).

$$H_{1d}(z) = T\left(\frac{\beta-\alpha}{z-1}\right) \tag{14}$$

The z-domain equivalent of the 2nd order s-domain term is derived by repeating all steps in the process mentioned above as follows.

$$H_{2d}(z) = \mathcal{Z}\{\mathcal{L}^{-1}\left(\frac{e^{-\alpha Ts} - e^{-\beta Ts}}{s^3}\right)\Big|_{t=nT}\} \tag{15}$$

$$H_{2d}(z) = \mathcal{Z}\{\mathcal{L}^{-1}\left(\frac{e^{-\alpha Ts} - e^{-\beta Ts}}{(s-\lambda)^3}\right)\Big|_{t=nT}\}\Big|_{\lambda=0} = H_2(z)\Big|_{\lambda=0} \tag{16}$$

$$\begin{aligned} H_2(z) = \mathcal{Z}\Big\{e^{\lambda t}(&\frac{e^{-\alpha\lambda t}}{2}(t-\alpha T)^2u(t-\alpha T) \\ &- \frac{e^{-\beta\lambda t}}{2}(t-\beta T)^2u(t-\beta T))\Big|_{t=nT}\Big\} \end{aligned} \tag{17}$$

The z-transform of (17) is given by (18) which leads to (19) [7], [8].

$$H_2(z) = \frac{T^2}{2}\sum_{n=0}^{+\infty} e^{\lambda nT}(e^{-\alpha\lambda T}(n-\alpha)^2 - e^{-\beta\lambda T}(n-\beta)^2)z^{-n} \tag{18}$$

$$\begin{aligned} H_2(z) = \frac{T^2}{2}\Big\{&\frac{(1-\alpha)^2e^{(1-\alpha)\lambda T} - (1-\beta)^2e^{(1-\beta)\lambda T}}{(z-e^{\lambda T})} \\ &+ \frac{(3-2\alpha)e^{(2-\alpha)\lambda T} - (3-2\beta)e^{(2-\beta)\lambda T}}{(z-e^{\lambda T})^2} \\ &+ \frac{2e^{(3-\alpha)\lambda T} - 2e^{(3-\beta)\lambda T}}{(z-e^{\lambda T})^3}\Big\} \end{aligned} \tag{19}$$

The 2nd derivative of equation (20) with respect to the variable λ is equal to (19).

$$H_2(z) = \frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\left(\frac{e^{(1-\alpha)\lambda T} - e^{(1-\beta)\lambda T}}{z-e^{\lambda T}}\right) \tag{20}$$

Substituting $\lambda = 0$ into (19) gives (21) which is the z-domain equivalent of the 2nd order s-domain term.

$$H_{2d}(z) = T^2\frac{[\beta(\beta-9)-\alpha(\alpha-9)]z + (\beta^2-\alpha^2)}{2(z-1)^2} \tag{21}$$

Finally, the above-mentioned process is performed all over again for the 3rd and 4th order s-domain terms which are listed in Table I. To obtain the kth order s-domain term, the impulse-invariant transformation is written in (22).

$$H_{kd}(z) = \mathcal{Z}\{\mathcal{L}^{-1}\left(\frac{e^{-\alpha Ts} - e^{-\beta Ts}}{s^{k+1}}\right)\Big|_{t=nT}\} \tag{22}$$

Table I: The CT-to-DT transformation for rectangular DAC waveforms.

s-domain	z-domain equivalent for a rectangular DAC waveform	
	Proposed Formulas	Formulas in [4]
1	$[u(-\alpha T)-u(-\beta T)]+[u(T-\alpha T)-u(T-\beta T)]$	
$\frac{1}{sT}$	$\frac{y_0}{z-1}$ $y_0=\beta-\alpha$	$\frac{y_0}{z-1}$ $y_0=\beta-\alpha$
$\frac{1}{s^2T^2}$	$\frac{y_1z+y_0}{(z-1)^2}$ $y_0=\frac{1}{2}(\beta^2-\alpha^2)$ $y_1=\frac{1}{2}(\beta(2-\beta)-\alpha(2-\alpha))$	$\frac{y_1z+y_0}{(z-1)^2}$ $y_0=\frac{1}{2}(\beta^2-\alpha^2)$ $y_1=\frac{1}{2}(\beta(1-\beta)-\alpha(1-\alpha))$
$\frac{1}{s^3T^3}$	$\frac{y_2z^2+y_1z+y_0}{(z-1)^3}$ $y_0=\frac{1}{6}(\beta^3-\alpha^3)$ $y_1=-\frac{1}{3}(\beta^3-\alpha^3)+\frac{1}{2}(\beta^2-\alpha^2)+\frac{1}{2}(\beta-\alpha)$ $y_2=+\frac{1}{6}(\beta^3-\alpha^3)-\frac{1}{2}(\beta^2-\alpha^2)+\frac{1}{2}(\beta-\alpha)$	$\frac{y_2z^2+y_1z+y_0}{(z-1)^3}$ $y_0=\frac{1}{6}(\beta^3-\alpha^3)$ $y_1=-\frac{1}{3}(\beta^3-\alpha^3)+\frac{1}{2}(\beta^2-\alpha^2)+\frac{1}{2}(\beta-\alpha)$ $y_2=-\frac{1}{6}(\beta^3-\alpha^3)-\frac{1}{2}(\beta^2-\alpha^2)+\frac{1}{2}(\beta-\alpha)$
$\frac{1}{s^4T^4}$	$\frac{y_3z^3+y_2z^2+y_1z+y_0}{(z-1)^3}$ $y_0=\frac{1}{24}(\beta^4-\alpha^4)$ $y_1=-\frac{1}{8}(\beta^4-\alpha^4)+\frac{1}{6}(\beta^3-\alpha^3)+\frac{1}{4}(\beta^2-\alpha^2)+\frac{1}{6}(\beta-\alpha)$ $y_2=+\frac{1}{8}(\beta^4-\alpha^4)-\frac{1}{3}(\beta^3-\alpha^3)+\frac{2}{3}(\beta-\alpha)$ $y_3=-\frac{1}{24}(\beta^4-\alpha^4)+\frac{1}{6}(\beta^3-\alpha^3)-\frac{1}{4}(\beta^2-\alpha^2)+\frac{1}{6}(\beta-\alpha)$	$\frac{y_3z^3+y_2z^2+y_1z+y_0}{(z-1)^3}$ $y_0=\frac{1}{24}(\beta^4-\alpha^4)$ $y_1=-\frac{1}{8}(\beta^4-\alpha^4)+\frac{1}{6}(\beta^3-\alpha^3)+\frac{1}{4}(\beta^2-\alpha^2)+\frac{1}{6}(\beta-\alpha)$ $y_2=+\frac{1}{8}(\beta^4-\alpha^4)-\frac{1}{3}(\beta^3-\alpha^3)+\frac{2}{3}(\beta-\alpha)$ $y_3=-\frac{1}{24}(\beta^4-\alpha^4)+\frac{1}{6}(\beta^3-\alpha^3)-\frac{1}{4}(\beta^2-\alpha^2)+\frac{1}{6}(\beta-\alpha)$
$\frac{1}{s^kT^k}$	$\left.\frac{1}{T^kk!}\frac{\partial^k}{\partial\lambda^k}\left(\frac{e^{(1-\alpha)\lambda T}-e^{(1-\beta)\lambda T}}{z-e^{\lambda T}}\right)\right\|_{\lambda=0}$	

By utilizing the Laplace transform properties, (22) leads to (23) [9].

$$H_2(z)=\mathcal{Z}\left\{e^{\lambda t}(\frac{e^{-\alpha\lambda t}}{k!}(t-\alpha T)^k u(t-\alpha T)-\left.\frac{e^{-\beta\lambda t}}{k!}(t-\beta T)^k u(t-\beta T))\right|_{t=nT}\right\} \quad (23)$$

The z-domain equivalent for the kth order s-domain function is expressed by (24) where k represents the order of the s-domain term.

$$H_{kd}(z)=\left.\left(\frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\left(\frac{e^{(1-\alpha)\lambda T}-e^{(1-\beta)\lambda T}}{z-e^{\lambda T}}\right)\right)\right|_{\lambda=0} \quad (24)$$

The z-domain equivalent for the 1st to 4th and the general kth order s-domain terms for a rectangular DAC waveform are presented in Table I.

One popular method to compensate the excess loop delay in CT ΔΣ modulators is to deploy negative feedback from the output of the DACs to the input of their quantizers as shown in Figure 2.b [1].

The z-domain equivalent of this feedback ($H_c(s)=1$) is developed and given by (26) as follows.

$$H_{0d}(z)=\mathcal{Z}\left\{\mathcal{L}^{-1}\left.\left(\frac{e^{-\alpha Ts}-e^{-\beta Ts}}{s}\right)\right|_{t=nT}\right\}=\sum_{n=0}^{+\infty}(u(nT-\alpha T)-u(nT-\beta T))z^{-n} \quad (25)$$

$$H_{0d}(z)=u(-\alpha T)-u(-\beta T)+(u(T-\alpha T)-u(T-\beta T))z^{-1} \quad (26)$$

One popular rectangular DAC waveform is the Non-Return-to-Zero (NRZ) one. The z-domain equivalent of the NRZ DAC with $\alpha=\tau_d$ and $\beta=1+\tau_d$ is calculated from (26) and is given by (27).

$$H_{0d}(z)=z^{-1} \quad (27)$$

The newly derived z-domain equivalent formulas can be compared with the formulas in [4] which both are illustrated in Table I. The results of this comparison indicate that y_1 in 2nd-order term and y_2 in 3rd-order term are entirely different. The comparison can be done between the newly mentioned formulas and the ones presented in [1] which show y_1 in 3rd-order term are not the same. What is surprising is that even z-domain equivalent formulas in [1] and [4] are not identical and y_1 in 2nd-order term and y_1 and y_2 in 3rd-order term are completely different.

III. SIMULATION RESULTS

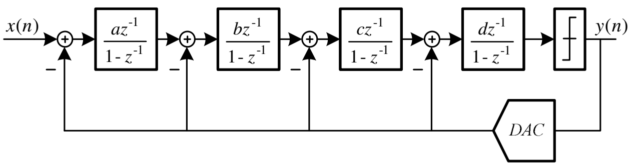
To validate the newly derived formulas presented in Table I, a 4th-order DT ΔΣ modulator with an OverSampling Ratio (OSR) of 64 and 3-bit quantizer has been designed by using the Schreier toolbox and was then converted to its 4th-order CT ΔΣ modulator equivalent with a NonReturn-to-Zero (NRZ) DAC waveform by using DT-to-CT formulas described in Table I. The block diagrams of the 4th-order DT and CT ΔΣ modulator are shown in Figure 2. An extra feedback of f_{c0} is used to compensate the effect of excess loop delay in the CT ΔΣ modulator. The coefficients of the DT ΔΣ modulator are given in (36).

$$\{a, b, c, d\} = \{0.1798, 0.4384, 0.8769, 2.0\} \quad (36)$$

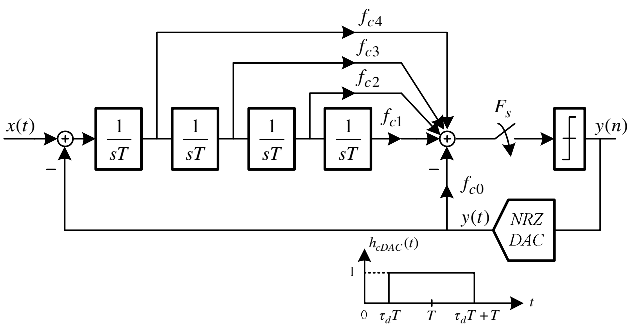
By using Table I the coefficients of the equivalent 4th-order CT ΔΣ modulator with NRZ DAC and $\{\alpha, \beta\} = \{0.2, 1.2\}$ shown in Figure 2.b have been derived and presented in (37).

$$\{f_{c4}, f_{c3}, f_{c2}, f_{c1}, f_{c0}\} = \{1.6189, 1.2266, 0.5892, 0.1382, 0.3\} \quad (37)$$

Both modulators have been simulated by using the Mathworks SIMULINK environment and a sinusoidal input signal with amplitude of 0.7V and a frequency of 61.34 KHz is applied to both modulators in the simulation. The simulation results show that the SNR of the DT and CT ΔΣ modulators are about 130.37dB and 130.21dB respectively with a clock frequency of 80MHz and signal bandwidth of 625 KHz. The output spectra of the DT and CT ΔΣ modulators and their respective in-band noise are approximately the same as shown in Figure 3.



(a)



(b)

Figure 2: a) The block diagram of the fourth-order DT ΔΣ modulator and b) The block diagram of the fourth-order CT ΔΣ modulator.

IV. CONCLUSION

In this paper a general and novel formula for impulse invariant transformation is presented. The CT-to-DT conversion formulas for the 1st to 4th order terms are derived and listed in Table I. The 4th-order DT ΔΣ modulator and its 4th-orde CT modulator equivalent which is derived by these formulas both were simulated by using MATLAB. Similar simulation results for both modulators support the validity of the proposed formulas derived and described in this paper.

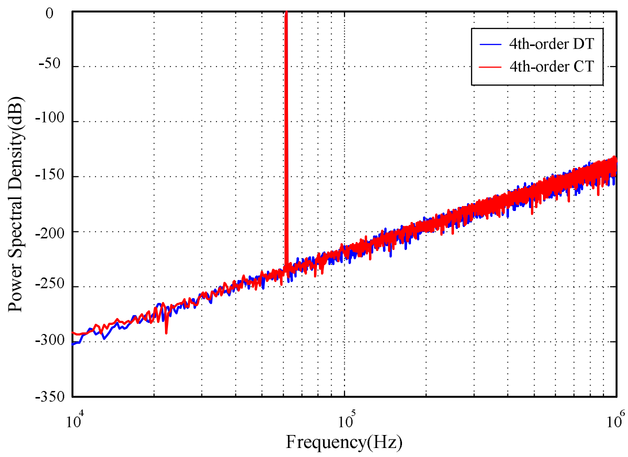


Figure 3: The output spectra of the fourth-order DT and CT ΔΣ modulators for a 61.34 KHz input with a clock frequency of 80MHz.

REFERENCES

[1] J. A. Cherry, and W. M. Snelgrove, "Excess Loop Delay in Continuous-Time Delta-Sigma Modulators," *IEEE Trans. Circuits Syst. II*, vol. 46, no. 4, pp. 376-389, April 1999.

[2] M. Ortmanns, F. Gerfers, and Y. Manoli, "A Case Study on a 2-1-1 Cascaded Continuous-Time Sigma-Delta Modulators, "*IEEE Trans. Circuits Syst. I*, vol. 52, no. 8, pp. 1515-1525, Aug. 2005.

[3] H. Shamsi, O. Shoaei, "Continuous- Time Delta-Sigma Modulators with Arbitrary DAC Waveforms, "*IEEE ISCAS,* pp. 187-190, 2006.

[4] M. Ortmanns and F. Gerfers, "Continuous-Time Sigma-Delta A/D Conversion, "*Berlin: Springer*, 2006.

[5] T. Kim, C. Han and, N. Maghari, "A 7.2mW 75.3dB SNDR 10MHz BW CT Delta-Sigma Modulator Using GM-C Based Noise-Shaped Quantizer and Digital Integrator" *IEEE J. Solid-State Circuits,* vol. 51, no. 8, pp. 1840-1850, 2016.

[6] H. Chae, and M. P. Flynn, "A 69dB SNDR, 25MHz BW, 800 MS/s Continuous-Time Bandpass Delta Sigma Modulator Using Duty-Cycle-Controlled DAC for Low Power and Reconfigurability" *IEEE J. Solid-State Circuits,* vol. 51, no. 3, pp. 649-659, 2016.

[7] E. I. Jury, "Theory and Application of the z-Transform Method," *New York, Robert E. Krieger Publishing Co.,* 1973.

[8] P. P. G. Dyke, "An introduction to Laplace Transforms and Fourier Series," *Great Britain, Springer*, 2004

[9] Benabes, P., Keramat, M., Kielbasa, R., "A methodolgy for designing countinuous-time sigma-delta modulators" Analog. Int. Circuits Signal Process. 123(3), 189–200 June, 2000.