Automated Reasoning for Knot Semigroups and π-orbifold Groups of Knots

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Abstract. The paper continues the first author's research which shows that automatic reasoning is an effective tool for establishing properties of algebraic constructions associated with knot diagrams. Previous research considered involutory quandles (also known as keis) and quandles. This paper applies automated reasoning to knot semigroups, recently introduced and studied by the second author, and π-orbifold groups of knots. We test two conjectures concerning knot semigroups (specifically, conjectures aiming to describe knot semigroups of diagrams of the trivial knot and knot semigroups of 4-plat knot diagrams) on a large number of examples. These experiments enable us to formulate one new conjecture. We discuss applications of our results to a classical problem of the knot theory, determining whether a knot diagram represents the trivial knot.

1 Main Definitions

Knot theory is an important part of topology because knots are, in a sense, simplest three-dimensional objects. Studying two-dimensional knot diagrams and studying algebraic constructions arising from knots are two of the most important techniques of knot theory [17]. Frequently (as in this paper) these two approaches are combined. This paper uses automated reasoning to improve our understanding of some known and some new algebraic constructions related to knots and knot diagrams.

1.1 Arcs and Crossings

By an *arc* we mean a continuous line on a knot diagram from one undercrossing to another undercrossing. For example, consider the knot diagram $\mathfrak{t}$ on Figure 1;

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it has three arcs, denoted by a, b and c. To denote a crossing on a knot diagram we shall use notation $x \dashv y \vdash z$, where x and z are the two arcs terminating at the crossing and y is the arc passing over the crossing. For example, the crossings on diagram $\mathfrak{t}$ are $b \dashv a \vdash c$, $b \dashv a \vdash a$ and $c \dashv a \vdash a$.

1.2 Cancellative Semigroups and Knot Semigroups

A semigroup is called *cancellative* if it satisfies two conditions:

$$\text{if } xz = yz \text{ then } x = y, \text{ and if } xy = xz \text{ then } y = z.$$

For each given knot diagram $\mathfrak{d}$, we define a cancellative semigroup which we call the *knot semigroup* of $\mathfrak{d}$ and denote by $K\mathfrak{d}$; the construction has been introduced and studied in [25]. To define the knot semigroup of a diagram $\mathfrak{d}$, assume that each arc is denoted by a letter. Then at every crossing $x \dashv y \vdash z$, 'read' two defining relations

$$xy = yz \text{ and } yx = zy.$$

The cancellative semigroup generated by the arc letters with these defining relations is the knot semigroup $K\mathfrak{d}$ of $\mathfrak{d}$. For example, on diagram $\mathfrak{t}$ we can read relations $ba = ac$ and $ab = ca$ at the left-top crossing, relations $ba = aa$ and $ab = aa$ at the right-top crossing and relations $ca = aa$ and $ac = aa$ at the bottom crossing. Using these relations, one can deduce equalities of words in $K\mathfrak{t}$. In particular, from $aa = ba = ca$, using cancellation, one can deduce $a = b = c$, that is, all generators are equal to one another; in other words, $K\mathfrak{t}$ is an infinite cyclic semigroup.

1.3 Keis

A *kei* (also known as an *involutory quandle*) is defined as an algebra with one binary operation $\triangleright$ and three axioms

$$a \triangleright a = a,\ (a \triangleright b) \triangleright b = a \text{ and } (a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c).$$

It is useful to know that every group can be considered as a kei with the operation $g \triangleright h = hg^{-1}h$. For a given knot diagram $\mathfrak{d}$, the kei $IQ\mathfrak{d}$ of the knot is a kei generated by the arc letters with defining relations $x \triangleright y = z$ and $z \triangleright y = x$ for each crossing $x \dashv y \vdash z$ of $\mathfrak{d}$. The mnemonic behind notation $x \triangleright y = z$ is expressed in [11]: 'x under y gives z'. The three axioms of a kei directly correspond to the three Reidemeister moves [5].

1.4 π-orbifold Groups and Two-fold Groups

For a given knot diagram $\mathfrak{d}$, the π-orbifold group $O\mathfrak{d}$ of the knot is a group generated by the arc letters with the following relations. For each arc x of the diagram $\mathfrak{d}$, introduce a relation $x^2 = 1$. At every crossing $x \dashv y \vdash z$, introduce a defining relation $xy = yz$ (or, equivalently, $yx = zy$, or $yxy = z$, or $yzy = x$).

Obviously, $O\mathfrak{d}$ is a factor group of $K\mathfrak{d}$. Denote the generating set of $O\mathfrak{d}$, that is, the set of arcs of $\mathfrak{d}$, by A, and consider the natural homomorphism from the free semigroup A^+ onto $O\mathfrak{d}$. It is easy to see that for each element g of $O\mathfrak{d}$, either only words of an odd length are mapped to g or only words of an odd length are mapped to g. Accordingly, let us say that g is an element of an odd (even) length in the former (latter) case. A subgroup of $O\mathfrak{d}$ consisting of elements of an even length is called the fundamental group of the 2-fold branched cyclic cover space of a knot [26, 20]; we shall shorten this name to the *two-fold group of a knot*, and shall denote the group by $T\mathfrak{d}$.

1.5 Putting These Constructions Together

Consider a simple example. The π-orbifold group $O\mathfrak{t}_3$ of the trefoil knot diagram $\mathfrak{t}_3$ is the dihedral group D_3. The group D_3 naturally splits into two types of elements: 3 rotations and 3 reflections. The rotations form a subgroup of D_3, which is the group $T\mathfrak{t}_3$, and which happens to be isomorphic to $\mathbb{Z}_3$. The reflections are in a one-to-one correspondence with the arcs of the trefoil knot diagram, and the subkei of D_3 consisting of reflections is $IQ\mathfrak{t}_3$. Generalising this example, one can notice that every group $O\mathfrak{d}$ splits into the subgroup $T\mathfrak{d}$ consisting of elements of an even length and a subkei consisting of elements of an odd length; this subkei is related to (and in many natural examples is isomorphic to) the kei of the knot $IQ\mathfrak{d}$.

1.6 Other Constructions: Knot Groups and Quandles

If one considers the diagram of a knot as an oriented curve, that is, in the context of a specific prescribed direction of travel along the curve, another pair of algebraic constructions can be introduced (whose definitions we shall skip, because they are not directly related to the topic of the paper). One of them is the *knot group*, which is historically the first and the best known construction (see, for example, Section 6.11 in [9] or Chapter 11 in [16]). The other is the *quandle* (also known as a *distributive groupoid*) of a knot; see, for example, [18]. These two constructions are 'larger' than the ones we consider in the sense that the π-orbifold group is a factor group of the knot group, and the kei is a factor kei of the quandle.

2 Trivial Knots

Trivial knots can be characterised via algebaic constructions associated with them, as the following results show.

Fact 1. *The following are equivalent:*

- *A knot is trivial.*
- *The two-fold group of the knot is trivial [26, 20].*
- *The kei of the knot is trivial [26].*

Fig. 1. Knot diagrams $\mathfrak{t}$ and $\mathfrak{t}_3$

– *The group of the knot is trivial [3].*
– *The quandle of the knot is trivial [11].*

Since the π-orbifold group of a knot is 'sandwiched' between the two-fold group of the knot and the group of the knot, the result also holds for π-orbifold groups.

The unusually simple structure of $K\mathfrak{t}$ in the example in Section 1 may be related to the fact that $\mathfrak{t}$ is a diagram of the trivial knot: it is easy to see that $\mathfrak{t}$ is not really 'knotted'. A general conjecture was formulated in [25]:

Conjecture 1. A knot diagram $\mathfrak{d}$ is a diagram of the trivial knot if and only if $K\mathfrak{d}$ is an infinite cyclic semigroup.

When Conjecture 1 is fully proved, it will be a natural addition to the list of results in Fact 1. In this paper we test Conjecture 1 on a series of knot diagrams and check how efficient the technique suggested by it is at detecting trivial knots. We conduct three types of computational experiments related to Conjecture 1:

– There are well-known examples of complicated diagrams of the trivial knot. Given one of these diagrams $\mathfrak{d}$, we prove that $K\mathfrak{d}$ is cyclic.
– We consider a number of standard diagrams of non-trivial knots. For each of these diagrams $\mathfrak{d}$, we prove that $K\mathfrak{d}$ is not cyclic or, equivalently, that $O\mathfrak{d}$ is not trivial.
– We can construct complicated diagrams of the trefoil, the simplest non-trivial knot, by considering the sum of the standard trefoil diagram $\mathfrak{t}_3$ and of one the complicated diagrams of the trivial knot. Given a complicated diagram $\mathfrak{d}$ of the trefoil, we check how efficiently automated reasoning proves that $K\mathfrak{d}$ is not cyclic.

For comparison, in [6] using automated reasoning was proposed for unknot detection and experiments with proving and disproving triviality of $IQ\mathfrak{d}$ were conducted. Yet another technique was used in [7]: the problem of checking if a knot is trivial was reduced to comparing factor quandles of the knot with families of pre-computed quandles, and this procedure, in its turn, was reduced to SAT solving.

2.1 How to Test if a Knot Semigroup is Cyclic

Consider a knot diagram $\mathfrak{d}$ with n arcs $a_1, \ldots, a_n$. Let $R_{\mathfrak{d}}$ be the set of relations read on the crossings of $\mathfrak{d}$, as defined in Section 1.

The equational theory of the knot semigroup $K\mathfrak{d}$ is $E_{K\mathfrak{d}} = E_{cs} \cup R_{\mathfrak{d}}$, where E_{cs} is the set of equational axioms of cancellative semigroups (see them listed explicitly in Subsection 2.2).

By Birkhoff's completeness theorem, two words are equal if and only if their equality can be proved by equational reasoning [1, 10]; hence the following statement follows (a similar statement for keis is formulated as Proposition 1 in [6]).

Proposition 1. *A knot semigroup $K\mathfrak{d}$ of a diagram $\mathfrak{d}$ with n arcs $a_1, \ldots, a_n$ is cyclic if and only if $E_{K\mathfrak{d}} \vdash \wedge_{i=1\ldots n-1}(a_i = a_{i+1})$, where $\vdash$ denotes derivability in the equational logic, or, equivalently in the first-order logic with equality.*

Proposition 1 suggests a practical way for experimental testing of Conjecture 1. Given a knot diagram $\mathfrak{d}$ (represented, for example, by its Gauss code [21]), translate it into knot semigroup presentation $R_{\mathfrak{d}}$ and further into its equational theory $E_{K\mathfrak{d}}$. Then apply an automated theorem prover and disprover to the problem $E_{K\mathfrak{d}} \vdash \wedge_{i=1\ldots n-1}(a_i = a_{i+1})$.

Note that if a *complete* prover is used (that is, given a valid formula it eventually produces a proof), the described procedure constitutes a semi-decision algorithm: if a knot semigroup is cyclic then this fact will be eventually established. Most common procedure for disproving is a finite model building [4], which, given a formula, builds a finite model for the formula's negation, thereby refuting the original formula. Usually it is possible to ensure *finite completeness* of a model builder: given a formula, it eventually produces a finite model refuting it, providing such a model exists. In general, however, due to undecidability of first-order logic, no complete disproving procedure is available. In particular, sometimes only infinite models refuting an invalid formula exist; then the model builder cannot build a model, even it is a complete finite model builder.

2.2 Cyclic Knot Semigroups

We applied automated theorem prover Prover9[3] [19] to several well-known diagrams of the trivial knot, and it has successfully proved that the knot semigroup is cyclic in each case. To illustrate the approach we present the proof for the simple diagram $\mathfrak{t}$ in Section 1. The task specification for Prover9 is divided into assumptions and goals parts. The assumptions part includes cancellative semigroups axioms E_{cs}:

```
(x * y ) * z = x * (y * z).
x * y = x * z -> y = z.
y * x = z * x -> y = z.
```

and defining relations for diagram $\mathfrak{t}$:

[3] We have chosen Prover9 and model builder Mace4 (below), primarily to be able to compare efficiency of automated reasoning with semigroups with that for involutory quandles in [6], where the same systems were used. Otherwise the choice is not very essential and any other automated first order (dis)provers could be used instead.

```
a * a = a * c. c * a = a * b.
b * a = a * a. a * a = c * a.
a * c = b * a. a * b = a * a.
```

The goals part is

```
(a = b) & (b = c).
```

For this task Prover9 produces the proof of length 14 in 0.05s. Table 1 presents the results for several well-known diagrams of the trivial knot. Time for the proof search grows with the size of the diagram. The diagram Ochiai,II (45 crossings) is a distinctive outlier: for some reason, the proof search for it took more than 8000s, comparing with 368s for Haken Gordian diagram with 141 crossings. We do not understand the reasons of why Ochiai,II diagram is so difficult for the automated proof. We are planning to explore this case further and to apply various automated provers and strategies to it.

Table 1. Proving that semigroups of diagrams of the trivial knot are cyclic

Name of unknot	Reference	# of crossings	Time, s
Culprit	[13]	10	0.4
Goerlitz	[12]	11	2.5
Thistlethwaite	[24]	15	6.1
Ochiai, I	[22]	16	14.85
Freedman	[23]	32	38.2
Ochiai, II	[22]	45	8458.6
Ochiai, III	[22]	55	195.2
Haken Gordian	[15]	141	368

2.3 Non-cyclic Knot Semigroups: Small Knots

We applied an automated model builder Mace4 [19] to all standard knot diagrams with up to 9 crossings (a table defining these knots can be found, for example, as Appendix 1 of [17]). The word *model* in this case means a finite non-cyclic factor semigroup of the diagram's knot semigroup. It is useful to note that since every finite cancellative semigroup is a group, Mace4 actually finds a group model. We illustrate the approach by considering the simplest untrivial knot, the trefoil knot (diagram $\mathfrak{t}_3$ in Section 1, entry 3_1 in Table 2). The task specification for Mace4 includes the cancellative semigroup axioms (as in Subsection 2.2) and the defining relations for the knot semigroup of the trefoil knot:

```
a * b = b * c. b * c = c * a. c * a = a * b.
b * a = c * b. c * b = a * c. a * c = b * a.
```

The goal to disprove is

```
(a = b) & (b = c).
```

Mace4 disproves the goal by finding a model in which both the cancellative semigroup axioms and the defining relations are satisfied, but at the same time, the goal statement is false. The model found by Mace4 is the dihedral group D_3, which is the knot's π-orbifold group, as discussed in Subsection 1.5.

Table 2 shows the results for all standard knot diagrams with up to 9 crossings. For each diagram we list the size of the model found and the time taken to find this model. The results presented in non-bold font are obtained by running Mace4 with the default iterative search strategy; that is, the search for a model starts with the size 2; if no model is found by an exhaustive search of models of a certain size, the size is increased by 1 and the search continues. Thus, assuming correctness of Mace4, entries in non-bold font represent smallest possible models. In all these cases the size of the model is two times the size of a smallest kei model computed in [6]; this observation has led us to formulating the following conjecture.

Conjecture 2. Consider a knot diagram $\mathfrak{d}$. Suppose the kei of $\mathfrak{d}$ has a factor kei of size n. Then the semigroup $K\mathfrak{d}$ has a factor semigroup of size $2n$.

To add some more details regarding the conjecture, the smallest semigroup model is frequently the knot's π-orbifold group, which is frequently (see Fact 2) a dihedral group, and the size of a dihedral group is two times the size of the corresponding *dihedral kei* (that is, the kei consisting of reflections), which is then the smallest kei model of the same knot diagram (Proposition 2 in [6], Theorem 3 in [7]). In some other cases (for example, 8_{19} in Table 2, which is not a 4-plat), the knot's π-orbifold group is not a dihedral group, but the smallest semigroup model, which is is a factor group of the knot's π-orbifold group, happens to be isomorphic to the dihedral group D_3. We don't know what happens to smallest model sizes when the smallest semigroup model is not a dihedral group and the smallest kei model is not a dihedral kei.

Table 2 contains remarks related to Conjecture 2. The entries in bold font represent the diagrams for which the default iterative strategy of Mace4 has failed to find a model in 50000s. In this case we used Conjecture 2 to guess a possible model size. These entries further split into three categories:

(1) the size is given with a mark ^{a}, meaning the search has been completed successfully for this particular size, predicted by Conjecture 2; the conjecture is confirmed, but the model found is not necessarily minimal;

(2) the size is given with a mark ^{b}, meaning the search has been done for increasing model sizes and ended successfully, but this was not an exhaustive search, as a time limit was imposed on search for each size; Conjecture 2 is confirmed, but the model found is not necessarily minimal;

(3) the size is given with a question mark and the time is given as N/F for 'not found', meaning neither search strategy has succeeded to find a model in 50000s; an estimated model size is given as predicted by Conjecture 2.

It is interesting to note that we could not find a model of the predicted size 30 in any entry in the table, as Mace4 search has timed out, although for larger values up to 46, Mace4 was able to find a model of predicted size. It might be just a coincidence, but 30 is the only value in the table which is not two multiplied by a prime number.

Table 2. Models for the standard knot diagrams with at most 9 crossings.

Knot	3_1	4_1	5_1	5_2	6_1	6_2	6_3	7_1
Size	6	10	10	14	6	22	26	14
Time	0.01	0.45	0.30	3.54	0.06	297	1362	5.02
Knot	7_2	7_3	7_4	7_5	7_6	7_7	8_1	8_2
Size	22	26	6	34	**38**^{a}	6	26	**34**^{a}
Time	339	1378	0.05	20193	**5715**	0.08	1484	**285**
Knot	8_3	8_4	8_5	8_6	8_7	8_8	8_9	8_{10}
Size	**34**^{a}	**38**^{a}	6	**46**^{a}	**46**^{a}	10	10	6
Time	**247**	**1350**	0.05	**2569**	**2684**	0.53	1.15	0.08
Knot	8_{11}	8_{12}	8_{13}	8_{14}	8_{15}	8_{16}	8_{17}	8_{18}
Size	6	**58?**	**58?**	**62?**	6	10	**74?**	6
Time	0.12	**N/F**	**N/F**	**N/F**	0.08	2.71	**N/F**	0.09
Knot	8_{19}	8_{20}	8_{21}	9_1	9_2	9_3	9_4	9_5
Size	6	6	6	6	6	**38?**	6	**46**^{b}
Time	0.06	0.06	0.05	0.28	0.22	**N/F**	0.20	**20316**
Knot	9_6	9_7	9_8	9_9	9_{10}	9_{11}	9_{12}	9_{13}
Size	6	**58?**	**62?**	**62?**	6	6	10	**74?**
Time	0.34	**N/F**	**N/F**	**N/F**	0.39	0.19	13.10	**N/F**
Knot	9_{14}	9_{15}	9_{16}	9_{17}	9_{18}	9_{19}	9_{20}	9_{21}
Size	**74?**	6	6	6	**82?**	**82?**	**82?**	**86?**
Time	**N/F**	0.05	0.33	0.16	**N/F**	**N/F**	**N/F**	**N/F**
Knot	9_{22}	9_{23}	9_{24}	9_{25}	9_{26}	9_{27}	9_{28}	9_{29}
Size	**30?**	6	6	**30?**	**94?**	14	6	6
Time	**N/F**	0.09	0.09	**N/F**	**N/F**	65	0.05	0.12
Knot	9_{30}	9_{31}	9_{32}	9_{33}	9_{34}	9_{35}	9_{36}	9_{37}
Size	**30?**	10	**118?**	**122?**	6	6	**30?**	6
Time	**N/F**	2.51	**N/F**	**N/F**	0.09	0.28	**N/F**	0.17
Knot	9_{38}	9_{39}	9_{40}	9_{41}	9_{42}	9_{43}	9_{44}	9_{45}
Size	6	10	6	14	14	**26**^{a}	**30?**	**30?**
Time	0.20	6.35	0.20	117	50.22	**365**	**N/F**	**N/F**
Knot	9_{46}	9_{47}	9_{48}	9_{49}				
Size	6	6	6	10				
Time	0.37	0.09	0.05	9.47				

2.4 Non-cyclic Knot Semigroups: Sums of Knots

We applied automated finite model builder Mace4 to the sums of all named trivial knot diagrams from Table 1 with the trefoil diagram in order to test whether a suitable model can be found by automated reasoning. When applied to the largest Haken Gordian diagram Mace4 generated an error message[4]. For all other sample diagrams the model of the expected size 6 was found (the same model as in the example presented in Subsection 2.3). The iterative search starting with models of size 2 did not always work because larger diagrams timed out at a 1500s limit. However, the search through models of size 6 has found a model in under 0.2s in all cases. The results are shown in Table 3.

Table 3. Search for models for the sum of a trivial knot and the trefoil

Name of unknot	# of crossings in the sum	Started with size 6 Time,s	Started with size 2 Time,s
Culprit	13	0.09	0.45
Goerlitz	14	0.06	1.39
Thistlethwaite	18	0.03	> 1500
Ochiai, I	19	0.08	3.92
Freedman	35	0.09	> 1500
Ochiai, II	48	0.14	> 1500
Ochiai, III	58	0.12	> 1500
Haken Gordian	144	n/a	n/a

2.5 Efficiency Comparison

Our experiments demonstrate that automated reasoning using knot semigroups can be applied for unknot detection (providing that Conjecture 1 is true), but it is not as efficient as automated reasoning using keis or quandles.

As to recognising a diagram of the trivial knot, automated reasoning on keis does it in under 1s for all diagrams in Table 1 (reported in [6]), except Ochiai's unknots and Haken unknot. The sharpest difference is the Ochiai, II diagram, whose kei is proved to be trivial by Prover9 in under 4s, as compared with more than 8000s for semigroups. As to Haken unknot, the kei is proved to be trivial in about 15s, as compared with 368s for semigroups.

As to finding models for non-trivial knots, Mace 4 using knot semigroups (or π-orbifold groups) has reported time out (50000s) on 23 out of 84 diagrams in Table 2. The corresponding kei models were found in [6] for all 9-crossing diagrams with the average time 28.6s.

An even more efficient automated reasoning procedure for detecting trivial knots has been obtained in [7] by considering quandles and reducing the problem of finding a finite factor quandle by to SAT. One reason why detecting trivial knots with keis and quandles is more efficient in practice than with semigroups (or π-orbifold groups) is because in many natural examples, the smallest factor kei of the knot kei is two times smaller than the smallest factor group of the knot semigroup, as discussed in Conjecture 2.

[4] `Fatal Error: mace4: domain_element too big`

3 4-plats

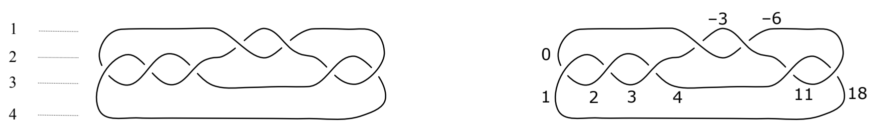

Fig. 2. A 4-plat and labelling its arcs

A 4-plat knot diagram is a braid with 4 strands whose ends on the left-hand side and the right-hand side are connected to form one closed curve, as in the example shown on Figure 2 (taken from [25]). 4-plat knots, that is, knots represented by 4-plat diagrams, form an important class of knots and are also known as 2-bridge knots and rational knots. Now we shall introduce some concepts which we need to formulate Conjecture 3 which aims to describe knot semigroups of 4-plat diagrams.

Let $B \subseteq \mathbb{Z}_n$ for some fixed positive integer n. By the alternating sum of a word $b_1 b_2 b_3 b_4 \dots b_k \in B^+$ we shall mean the value of the expression $b_1 - b_2 + b_3 - b_4 + \dots + (-1)^{k+1} b_k$ calculated in $\mathbb{Z}_n$. We shall say that two words $u, v \in B^+$ are in relation $\sim$ if and only u and v have the same length and the same alternating sum. It is obvious that $\sim$ is a congruence on B^+. We denote the factor semigroup $B^+/{\sim}$ by $AS(\mathbb{Z}_n, B)$ and call it an *alternating sum semigroup* [25]. For example, consider an alternating sum semigroup with letters $B = \{0, 1, 2, 3, 4, 11, 14\}$ in the arithmetic modulo $n = 17$. In this semigroup we have $3 \cdot 1 \cdot 4 \cdot 14 = 1 \cdot 11 \cdot 4 \cdot 2$ because $3 - 1 + 4 - 14 = 1 - 11 + 4 - 2 = 5 \bmod 17$.

To assign useful numerical labels[5] to the arcs of a 4-plat diagram $\mathfrak{d}$, label the two leftmost arcs by 0 and 1, as on the example in Figure 2. To distinguish between arcs and their labels, we shall denote the label of an arc x by b_x. Propagate the labelling as follows: moving from the left to the right on the diagram, at each crossing $x \dashv y \vdash z$, let $b_z = 2b_y - b_x$. After we have done this at every crossing, two arcs will get two labels each: in our example, the top-right arc on the diagram is labelled -6 and 11, and the bottom-right arc on the diagram is labelled 1 and 18. Considering either of the two equalities $-6 = 11$ or $1 = 18$, we conclude that we should treat the labels as numbers in the arithmetic modulo 17 (hence, for convenience, -3 can be rewritten as 14). Given the modulus n ($n = 17$ in our example) and the set of arc labels B ($B = \{0, 1, 2, 3, 4, 11, 14\}$ in our example), consider an alternating sum semigroup $AS(\mathbb{Z}_n, B)$.

Proposition 2. *$AS(\mathbb{Z}_n, B)$ produced using the procedure above is a factor semigroup of $K\mathfrak{d}$.*

Proof. Consider a mapping from $K\mathfrak{d}$ to $AS(\mathbb{Z}_n, B)$ induced by the rule $x \mapsto b_x$, where x is an arc. The knot semigroup $K\mathfrak{d}$ is defined by relations stating that at

[5] The described procedure is a version of so-called Fox coloring [17]. Note that in general, labels of some distinct arcs may coincide.

each crossing $x \dashv y \vdash z$ we have $xy = yz$ and $yx = zy$. The two corresponding equalities are satisfied in $AS(\mathbb{Z}_n, B)$: indeed, words $b_x b_y$ and $b_y b_z$ both have length 2; the alternating sum of $b_x b_y$ is $b_x - b_y$, and since $b_z = 2b_y - b_x$, the alternating sum of $b_y b_z$ is also $b_x - b_y$; therefore, $b_x b_y = b_y b_z$; similarly, $b_\mathrm{y} b_x = b_z b_y$.

The following result[6] is first proved as Proposition 3.2 in [2], or see [14]; the idea originates from [8].

Fact 2. *A knot is a* 4*-plat knot if and only if its* π*-orbifold group is dihedral.*

Generalising the 'only if' part of Fact 2 to knot semigroups, we can state the following conjecture (first formulated in [25], after having described knot semigroups of some subclasses of the class of 4-plat knots):

Conjecture 3. The homomorphism from $K\mathfrak{d}$ onto $AS(\mathbb{Z}_n, B)$ described in Proposition 2 is an isomorphism.

3.1 Defining Relations for an Alternative Sum Semigroup

In Subsection 3.2 we present experiments which use automated reasoning to prove Conjecture 3 for a number of 4-plats. Proving the isomorphism becomes possible if $AS(\mathbb{Z}_n, B)$ is redefined using defining relations. In this subsection we introduce an algorithm for finding a finite list of defining relations for an alternating sum semigroup.

Below we assume that set B contains 0; this is merely a convenience for simpler notation, and all statements can be rewritten to use another element of B instead of 0. All words below are assumed to be words over the alphabet B.

Let us say that a word w is zero-ending if its last letter is 0. For every word w we shall define its canonical form $c(w)$ as the smallest word (relative to the right-to-left dictionary order) which is equal to w in $AS(\mathbb{Z}_n, B)$.

Let us say that a pair of sets of words Y, Z is a *basis* if

1. For every $w \in Y$ its canonical form $c(w)$ is not zero-ending;
2. For every $w \in Z$ its canonical form $c(w)$ is zero-ending;
3. Every word is either contained in Y or has a suffix contained in Z.

Theorem 3. *Suppose* Y, Z *is a basis. Then all equalities of words in* $AS(\mathbb{Z}_n, B)$ *can be deduced*[7] *from defining relations* $w = c(w)$*, where* $w \in Y \cup Z$.

[6] We are grateful to José Montesinos (Universidad Complutense de Madrid), Genevieve Walsh (Tufts University) and Vanni Noferini (University of Essex) for attracting our attention to this result.

[7] Note that here we mean the usual semigroup deduction, not a more complicated one used in cancellative semigroups. It is useful to remind oneself of this, because knot semigroups are defined using a cancellative presentation, and it makes proving equalities of words in knot semigroups more involved.

Proof. We shall use the proof by induction on the length of words. For words of length 1, there is no need to apply the defining relations because none of words is equal to another word. Now assume that all equalities in $AS(\mathbb{Z}_n, B)$ for words shorter than the length we are considering can be deduced from the defining relations. It is sufficient to prove that for each word v we can deduce the equality $v = c(v)$. Three cases are possible:

1. Suppose $c(v)$ is not zero-ending. Consider v as a product of two words $v = v_1v_2$ and assume that $c(v_2)$ is zero-ending; then v is equal to a zero-ending word $v_1c(v_2)$, hence, $c(v)$ is also zero-ending; since it is not so, we conclude that none of suffixes of v has a zero-ending canonical form. Therefore, neither v nor any of its suffixes is contained in in Z. Hence, $v \in Y$, and the equality $v = c(v)$ is one of the defining relations.

2. Suppose $c(v)$ is zero-ending and $v \in Z$. Then the equality $v = c(v)$ is one of the defining relations.

3. Suppose $c(v)$ is zero-ending and $v \notin Z$. Then v is a product of two words $v = v_1v_2$ such that $v_2 \in Z$. Then $c(v_2)$ is zero-ending, and we can represent it as $c(v_2) = t0$ for some word t. Hence, $v = v_1t0$. On the other hand, since $c(v)$ is zero-ending, we can represent it as $c(v) = s0$ for some word s. Thus, $v_1t0 = s0$; since $AS(\mathbb{Z}_n, B)$ is cancellative, $v_1t = s$. This is an equality of two words whose length is less than that of v; thus, by induction, this equality can be deduced from the defining relations. Therefore, the equality $v = c(v)$ can be deduced in the order $v_1v_2 = v_1t0 = s0$, that is, first by applying the defining relation $v_2 = c(v_2)$, and then by applying all the defining relations needed to prove that $v_1t = s$.

Theorem 3 suggests a simple algorithm for building a basis for a given alternating sum semigroup. Consider all words one after another, starting from the shorter ones; as you consider a word w, add it to Y if $c(w)$ is not zero-ending, or to Z if $c(w)$ is zero-ending, or to neither if a suffix of w is contained in Z. When all possible suffixes of longer words have been added to Z, stop. This algorithm will produce a basis; however, it would be nice to have an assurance that the algorithm will terminate; it is provided by the following statement.

Proposition 3. *Every sufficiently long word in an alternating sum semigroup contains a suffix whose canonical form is zero-ending. Hence, each alternating sum semigroup has a finite basis Y, Z.*

Proof. We shall prove that the canonical form of every word w of length $L \geq 2n^2$ is zero-ending. Indeed, there is a letter, say, a, which stands at least at n distinct even positions in w. Notice that in an alternating sum semigroup we have $xyz = zyx$ for any three letters x, y, z. Applying these equalities as needed, move n letters a to positions $L, L-2, \ldots, L-2n+2$; thus, we produce a word w' which is equal to w such that $w' = vax_1ax_2 \cdots ax_{n-1}a$ for some word v and letters $x_1, x_2, \ldots, x_{n-1}$. Notice that in an alternating sum semigroup $AS(\mathbb{Z}_n, B)$ the word w' is equal to $w'' = v0x_10x_2 \cdots 0x_{n-1}0$. The word w'' is equal to w and is zero-ending; therefore, $c(w)$ is zero-ending.

3.2 Experiments

To test Conjecture 3, we considered 4-plat knot diagrams with up to 9 crossings; we restricted ourselves to canonical 4-plat knot diagrams (see, for example, Proposition 12.13 in [3] and page 187 in [21]), that is, those in which every crossing is either a clockwise half-twist of strands in positions 1 and 2 or an anticlockwise half-twist of strands in positions 2 and 3, and the arcs on the left-hand side of the diagram connect level 1 to level 2 and level 3 to level 4 (like on the diagram in Subsection 3). Note that a knot semigroup is defined for a diagram and not for a knot, and two diagrams of the same knot can have non-isomorphic semigroups; in our study of Conjecture 3 we consider all individual diagrams. For instance, the number of distinct knots with 9 crossings is 49. However, when we consider canonical 4-plat diagrams with 9 crossings, each of 9 crossings can be in one of two possible positions, between levels 1 and 2 or between levels 2 and 3 on the diagram; in addition to this, the arcs on the right-hand side of the diagram can be connected in two possible ways: level 1 to level 2 and level 3 to level 4, or level 1 to level 4 and level 2 to level 3. Therefore, we generated $2^{9+1} = 1024$ diagrams. Out of these, 664 diagrams were knots, and we confirmed Conjecture 3 for each of them. Other diagrams are links and not knots, and we discard them from consideration; knot semigroups of 4-plat links are not alternating sum semigroups (for example, knot semigroups of a class of 4-plat links is described in Section 6 in [25]).

For each diagram $\mathfrak{d}$ out of these 664 diagrams, we produced an alternating sum semigroup $S = AS(\mathbb{Z}_n, B)$ using the procedure described before Proposition 2 (we wrote a Python script to do this). Semigroup S is, according to Proposition 2, a factor semigroup of the knot semigroup $K\mathfrak{d}$. Thus, to prove that $K\mathfrak{d}$ and S are isomorphic, it is sufficient to show that all defining relations of S are derivable in $E_{K\mathfrak{d}}$, the equational theory of $K\mathfrak{d}$.

We wrote another Python script which finds defining relations for a given alternating sum semigroup S using the procedure described in Subsection 3.1 and outputs the task $E_{K\mathfrak{d}} \vdash E_S$ to be used by Prover9, where E_S means the conjunction of all defining relations of S. The tasks were then passed to Prover9. Due to a large size of E_S, in order to get automated proofs, some tasks had to be split into up to four subtasks $E_{K\mathfrak{d}} \vdash E_S^i$, with $E_S = \cup_i E_S^i$. Eventually all proofs have been obtained with the time limit 1200s for each task.

4 Future Research

We are continuing working of proving Conjecture 1. Its 'if' part follows from the observation below. The 'only if' part is much more difficult to prove.

Proposition 4. *If $K\mathfrak{d}$ is an infinite cyclic semigroup then $\mathfrak{d}$ represents the trivial knot.*

Proof. Indeed, if $K\mathfrak{d}$ is cyclic then its factor group $O\mathfrak{d}$ is isomorphic to $\mathbb{Z}_2$, whose subgroup $T\mathfrak{d}$ is trivial. Since $T\mathfrak{d}$ is trivial, by Fact 1, $\mathfrak{d}$ represents the trivial knot.

Among algebraic constructions listed in Fact 1, computational experiments with keis [6], quandles [7], semigroups and π-orbifold groups (in this paper) have been conducted. Experiments with groups have been only started in [6], and experiments with two-fold groups (which are smaller and may be easier to manipulate) can be conducted in the future.

Using semigroups or π-orbifold groups to prove that a knot diagram represents the trivial knot is a topic for more future research. As discussed in Subsection 2.5, this is not the fastest method of proving that a knot is trivial. However, we have reasons to believe that such proofs, if they are produced by a specialised prover and properly presented, can be more human-readable than others (for example, those based on keis). We shall continue studying such proofs because of new mathematical constructions arising in them and because this is an impressive example of how complicated computer-generated proofs can be made human-readable.

5 Technical Details

We used Prover9 and Mace4 version 0.5 (December 2007) [19] and one of two system configurations:

1) AMD A6-3410MX APU 1.60Ghz, RAM 4 GB, Windows 7 Enterprise when producing Tables 1 and 3 (and results from [6] used in Subsections 2.3 and 2.5);

2) Intel(R) Core(TM) i7-4790 CPU 3.60Ghz, RAM 32 GB, Windows 7 Enterprise when producing Table 2 and results in Section 3.

We have used default iterative Mace4 search strategy, except for the cases explicitly mentioned as using different strategies in 2.3 and 2.4. We have used default search strategies in Prover9, with the following exceptions. For the results presented in 2.2 we have used Knuth-Bendix term ordering (KBO) instead of default choice of LPO (Lexicographic Path Ordering). In order to handle large clauses occurring in the proofs reported in 3.2 we have set `max_weight` (maximum weight of clauses) to 8000.

We have published all computer-generated proofs online[8].

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