Optimal Control and Asymptotics of Stochastic Delay Evolution Equations

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by

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Abstract

This thesis mainly studies stochastic neutral differential equations with delays, which can be studied in the fields of existence, uniqueness, controllability and stability of mild solutions.

In Chapter 1, we give a short introduction for the materials in each chapter. We introduce the new models we developed. In Chapter 2, we begin by introducing some definitions and results. To present the proofs of all the results here would require preparatory background material, which would significantly increase both the size and scope of this dissertation. Although this chapter introduces very important theorems, required proofs are omitted here. However, these related proofs can be found from book in Liu [41] and you can also find most of these basic mathematical concepts and their proofs in many well-known text books such as Pazy [32] and Da Prato and Zabczyk [22] or to be found in the literature reviews.

In Chapter 3, we will generalise the previous theory to consider a stochastic optimal control problem for a class of neutral type stochastic systems, which is very important from both theoretic and practical point of view (see, e.g., [39]). We formulate a stochastic optimal control problem with the aim of maximising the objective functional at a given time horizon $T > 0$. This chapter is organised as follows. In Section 3.2, we formulate the optimal problem with the objective functional as an optimal problem with neutral type for an SDDE both in state

and the control. In Section 3.3, we use a representation result that allows us to "lift" this non-Markovian optimisation problem to a Markovian control problem on a Hilbert space and deal with the general case of delays in the state and in the control and the verification result is given. In Section 3.4, we construct an example of a controlled SDDE, whose HJB equation admits an integral solution. Therefore, there exists an optimal control form for the control problem. In Section 3.5, we establish a linear delay differential equation to obtain solutions. In Section 3.6, we have a summary to state the contribution and development of the chapter.

In Chapter 4, we will concentrate on the existence and uniqueness of the square-mean almost periodic mild solutions. This chapter is organised as follows. In Section 4.2, we review and introduce some concepts, basic properties of square-mean almost periodicity and the proofs of two theorems. In Section 4.3, under some suitable conditions, we prove the existence and uniqueness of square-mean almost periodic mild solutions for some stochastic differential equations driven by Poisson jumps. In Section 4.4, we have a summary to state the contribution and development of the chapter.

In Chapter 5, we study the problem of determining the attracting sets of neutral stochastic partial differential equations driven by α-stable noise with impulses. Therefore, the techniques and methods for the global attracting set and stability for neutral SPDEs driven by α-stable processes with impulses should be developed. This chapter is organised as follows. In Section 5.2, we review and introduce the concepts and basic properties of α-stable processes. In Section 5.3, we study the global attracting set and stability of the stochastic neutral differential equations with impulses. In Section 5.4, we have a summary to state the contribution and development of the chapter.

In Chapter 6, we have a conclusion to summarise the contribution and development of this thesis.

Contents

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Chapter 1

Introduction

In modern society, the modelling of stochastic systems has gained significant attention due to its many applications in physics, economics, finance, engineering, etc. However, there also exist many phenomena, which are characteristics of past dependence, that is, their present value depends not only on the present situation but also on past history. Qualitative properties such as existence, uniqueness, controllability and stability for various stochastic differential systems have been investigated by many authors and have already achieved fruitful results (see for example [45], [46], [41], [14], [53]). On the other hand, it is known that a class of stochastic differential equations with neutral type involve derivatives with delays as well as the function itself. Many interesting results about neutral stochastic delay differential equations have been obtained by many researchers, see, for example, Liu [38] has considered standard optimal control problems for a class of neutral functional differential equations in Banach spaces and it turns out that based on a systematic theory of neutral models, the fundamental solution is constructed and a variation of constants formula of mild solutions is established. Balasubramaniam and Ntouyas [4] have given sufficient conditions for the controllability of a class of stochastic partial functional differential inclusions with infinite delay in an abstract space.

Dynamic stochastic optimisation is the study of dynamical systems subject to random perturbations, and which can be controlled in order to optimise some performance criterion. It arises in decision-making problems under uncertainty. Historically, based on Bellman's and pontryagin's optimality principles, the research on control theory has developed considerably over recent years. The dynamic programming principle (DPP) to a stochastic control problem for Markov processes in continuous-time leads to a nonlinear partial differential equation (PDE), called the Hamilton-Jacobi Bellman (HJB) equation, satisfied by the value function. One typical example of this optimal control problem is introduced by the following controlled SDDE in advertising models [29] of the form:

$$\begin{cases} dy(t) = \Big[a_0 y(t) + \displaystyle\int_{-r}^{0} a_1(\theta) y(t+\theta) + b_0 u(t) + \int_{-r}^{0} b_1(\theta) u(t+\theta) d\theta\Big] dt \\ \qquad\qquad\qquad\qquad + \sigma dB(t), \quad \forall t \in [0,T] \\ y(0) = x_0, y(\theta) = x_1(\theta), u(\theta) = \gamma(\theta), \quad \forall \theta \in [-r, 0], \end{cases}$$

where $a_0 \in \mathbb{R}, a_1(\cdot) \in L^2([-r,0];\mathbb{R}), b_0 \in \mathbb{R}, b_1(\cdot) \in L^2([-r,0];\mathbb{R}), x_1(\cdot) \in L^2([-r,0];\mathbb{R})$ and $\gamma(\cdot) \in L^2([-r,0];\mathbb{R})$.

In this work, the optimal advertising problem as an optimal control problem for an SDDE with delays both in the state and the control is considered. The problem is formulated by lifting this non-Markovian optimisation problem to an infinite-dimensional Markovian control problem without involving delays in a suitable product Hilbert space and solutions are derived in an example.

Motivated by the above works, we aim to consider the following neutral stochastic differential equations with control delays in $\mathbb{R}$:

$$\begin{cases} d\Big[y(t) - \displaystyle\int_{-r}^{0} a(\theta)y(t+\theta)d\theta\Big] = \Big[a_0 y(t) + \displaystyle\int_{-r}^{0} a_1(\theta)y(t+\theta) + b_0 u(t) \\ \qquad\qquad\qquad\qquad + \displaystyle\int_{-r}^{0} b_1(\theta)u(t+\theta)d\theta\Big]dt + \sigma dB(t), \quad \forall t \in [0,T] \\ y(0) = x_0, y(\theta) = x_1(\theta), u(\theta) = \gamma(\theta), \quad \forall \theta \in [-r, 0], \end{cases} \tag{1.1}$$

where the Brownian motion $B(t)$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with $(\mathcal{F}_t)_{t\geq 0}$ being the completion of the filtration generated by $B(t), t \geq 0$. It is assumed that $u(t)$ is an admissible control that belongs to $\mathcal{U} := L^2([0,T];\mathbb{R})$, the space of square integrable non-negative stochastic processes adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$. In addition, we need to assume the following conditions:

i. $a_0 \in \mathbb{R}$;

ii. $a_1(\cdot) \in L^2([-r,0];\mathbb{R})$;

iii. $a(\cdot) \in C^1([-r,0];\mathbb{R})$;

iv. $b_0 \in \mathbb{R}$;

v. $b_1(\cdot) \in L^2([-r,0];\mathbb{R})$;

vi. $x_1(\cdot) \in L^2([-r,0];\mathbb{R})$;

vii. $\gamma(\cdot) \in L^2([-r,0];\mathbb{R})$.

We adopt a method that allows us to "lift" this non-Markovian optimisation problem to an infinite-dimensional Markovian control problem. Let us consider the following abstract SDE on a Hilbert space $\mathcal{H}$ (see Chapter 3), which is equivalent to the SDE (1.1):

$$\begin{cases} dY(t) = (A^*Y(t) + B^*u(t))dt + G^*dB(t) \\ Y(0) = x = (x_0, x_1) \in \mathcal{H}, \end{cases}$$

where the operators A^*, B^*, G^* are defined properly in Chapter 3.

In this chapter, we will generalise the previous theory to consider a stochastic optimal control problem for a class of neutral type stochastic systems, which is very important from both theoretic and practical point of view (see, e.g., [39]). We formulate a stochastic optimal control problem with the aim of maximising the objective functional at a given time horizon $T > 0$.

On the other hand, solutions with recurrence property (e.g. almost periodicity and almost automorphy), which enable us to understand the impact of the noise or stochastic perturbation on the corresponding recurrent motions, are of great concern in the study of stochastic differential equations and random dynamical systems. Periodicity often appears in implicit ways in various phenomena. For example, this is the case when one studies the effects of fluctuating environments on population dynamics. Although people can calculate the periodic fluctuations of environmental parameters in controlled laboratory experiments, almost periodicity is more likely to accurately describe natural fluctuations [23].

Recently, Bezandry and Diagana introduced the concept of square-mean almost periodic stochastic process and applied it to study stochastic differential equations (see [9]). In [10], Bezandry and Diagana proved the existence of almost periodic solutions to some stochastic differential equations. Bezandry and Diagana [11] studied the existence of square-mean almost periodic solutions to some stochastic hyperbolic differential equations with infinite delay. Bezandry and Diagana [12] were concerned with the square-mean almost periodic solutions nonautonomous stochastic differential equations. However, many dynamical systems not only depend on the present states, but also on past states and involve derivative with delays. Therefore, it is necessary to consider the stochastic evolution system with infinite delays and the neutral type as well, see ([40], [44], [**?**], [19]). One typical example is to deal with the existence and uniqueness of square-mean almost periodic solutions to a class of neutral stochastic evolution equations with infinite delay [34] of the form:

$$d(x(t) - G(x(t), x_t)) = (Ax(t) + f(t, x(t), x_t))dt + g(t, x(t), x_t)dW(t), \quad t \in \mathbb{R},$$

where $x_t = x(t+\theta) : -\infty < \theta \leq 0$ can be regarded as a $\mathscr{B}$-valued stochastic process. Assume that $f : \mathbb{R} \times H \times \mathscr{B} \to H$, $g : \mathbb{R} \times H \times \mathscr{B} \to \mathscr{L}_2(K_Q, H)$ and $G : H \times \mathscr{B} \to H_\alpha$ (see Chapter 4).

In addition, Lévy processes are essentially stochastic processes with stationary and independent increments, and they are particular useful, as they can describe discontinuous and dramatic fluctuations in practical situations. Also, Wiener processes and Poisson processes are the important special cases of Lévy processes. Stochastic differential equations with Poisson jumps have become popular in modelling those phenomena arising in the field of economics, where jump processes are widely used to describe the asset and commodity price dynamics (see [18]). However, for stochastic partial differential equations with Poisson jumps and infinite delay, as far as we know, there exist only a few results about the existence and stability of mild solutions. One is referred to ([51], [20], [52]). One typical example is to deal with the existence and uniqueness of square-mean almost periodic solutions to a class of stochastic differential equations with Lévy noise without delays [42] of the form:

$$\begin{aligned} dx(t) = f(t, x(t))dt + g(t, x(t))dW(t) &+ \int_{|z|_U < 1} F(t, x(t-), z)\tilde{N}(dt, dz) \\ &+ \int_{|z|_U \geq 1} G(t, x(t-), z)\tilde{N}(dt, dz), \quad t \in \mathbb{R}, \end{aligned}$$

where F and G are H-valued.

Motivated by the above works, we shall study the existence and uniqueness of square-mean almost periodic solutions to a class of neutral stochastic differential equations with Poisson jumps and infinite delay

$$\begin{aligned} d(x(t) - G(x(t), x_t)) &= (Ax(t) + f(t, x(t), x_t))dt + g(t, x(t), x_t)dW(t) \\ &\quad + \int_H h(t, x(t-), x_{t-}, z)\tilde{N}(dt, dz), \quad t \in \mathbb{R}, \end{aligned}$$

where $x_t = x(t + \theta) : -\infty < \theta \leq 0$ can be regarded as a $\mathscr{B}$-valued stochastic process. Assume that $f : \mathbb{R} \times H \times \mathscr{B} \to H$, $g : \mathbb{R} \times H \times \mathscr{B} \to \mathscr{L}_2(K_Q, H), G : H \times \mathscr{B} \to H_\alpha$ and $h : \mathbb{R} \times H \times \mathscr{B} \times H \to H$, are appropriate mappings for all $t \in \mathbb{R}, z \in H$, which will be specified in Chapter 4. We will prove the existence and uniqueness of square-mean almost periodic mild solutions for some stochastic differential equations driven by Poisson jumps under some suitable conditinos by using methods of semi-group and Banach fixed-point theorem.

From the discussions above, we can see that the stochastic differential evolution equations driven by Brownian motions and Lévy processes have been studied by many researchers. However, since Wiener noise and Poisson-jump noise have arbitrary finite moments, while α-stable noise only has finite p-th moment for $p \in (0, \alpha)$ with $\alpha < 2$. Recently, stochastic differential equations driven by α-stable processes have plenty of applications in physics due to the fact that α-stable noise exhibits the heavy tailed phenomenon. For example, Priola and Zabczyk [50] gave a proper starting point on the investigation of structural properties of stochastic partial differential equations (SPDEs) driven by an additive cylindrical stable noise. Dong, Xu and Zhang [25] studied the invariant measures of stochastic 2D Navier-Stokes equation driven by α-stable processes. Xu studied [61] Ergodicity of the stochastic real Ginzburg-Landau equation driven by α-stable noise and Zhang [67] proved a derivative formula of Bismut-Elworthy-Li's type as well as gradient estimate for stochastic differential equations driven by α-stable noises. One the other hand, Wang [55] derived the gradient estimate for Ornstein-Uhlenbeck jump processes and Wang [58] established so-called Harnack inequalities for SDEs driven by cylindrical α-stable processes. However, there are few papers on the asymptotic behaviour of mild solution of SPDEs driven by α-stable processes, so we shall discuss the stability property of mild solutions of a class of SPDEs driven by α-stable processes to complete the theory. The fact is that α-stable noise only has finite p-th moment for $p \in (0, \alpha)$ and the stochastic evolution does not admit a stochastic differential, which leads to some powerful tools such as the Itô formula being unavailable, then some new methods should be used to overcome the difficulties. It is worthwhile to mention that, Wang and Rao [56] discussed the stability of mild solutions for a class of SPDEs driven by α-stable noises and generalised to deal with the SPDEs driven by subordinated cylindrical Brownian motion and fractional Brownian motion, respectively by the Minkovski inequality.

In addition, many researchers have studied attracting sets of dynamical systems extensively. Xu and Long [60] studied the attracting and quasi-invariant sets of non-autonomous neutral networks with delays. Long, Teng and Xu [43] investigated the global attracting set and stability of stochastic neutral partial functional differential equations with impulses. They first established a new impulsive-integral inequality, which improved the inequality established by Chen [16]. On the other hand, impulsive phenomenon can be found in a wide variety of evolutionary processes, for example, medicine and biology, economics, mechanics, electronics and telecommunications, etc., in which many sudden and abrupt changes occur instantaneously, in the form of impulses. Many interesting results haven been found, e.g., ([66], [47]), etc. One typical example is to consider a class of neutral stochastic partial differential equations driven by α-stable processes on a separable Hilbert space [36] of the form:

$$\begin{cases} d[x(t)-g(t,x(t-r)]=(Ax(t)+f(t,x(t-r)))dt+\sigma(t)dZ(t), t\geq 0, \\ x_0(\cdot)=\phi(\cdot)\in D([-r,0],H), \end{cases}$$

where $r>0$ and A generates a strongly continuous semigroup $S(t)$ or $e^{tA}, t\geq 0$, on H. Assume that $f,g:\mathbb{R}_+\times H\to H$ are two given measurable mappings and $\sigma(t):\mathbb{R}_+\to\mathbb{R}$ is a locally integrable function.

But, to the best of my knowledge, there is no result on the Global attracting set and exponential decay of neutral SPDEs driven by α-stable processes with impulses. Motivated by the above discussions, in Chapter 5, we shall consider the following neutral stochastic partial differential equations driven by an additive α-stable with impulses on a separable Hilbert space H,

$$\begin{cases} d[x(t)-g(t,x(t-r)]=(Ax(t)+f(t,x(t-r)))dt \\ \qquad\qquad +\sigma(t)dZ(t), t\geq 0, t\neq t_k, \\ \Delta x(t_k)=x(t_k^+)-x(t_k^-)=I_k(x(t_k^-)), t=t_k, k=1,2,..., \\ x_0(\cdot)=\phi(\cdot)\in D([-r,0],H), \end{cases}$$

where $r>0$ and A generates a strongly continuous semigroup $S(t)$ or $e^{tA}, t\geq 0$, on H. Assume that $f,g:\mathbb{R}_+\times H\to H$ are two given measurable mappings and $\sigma(t):\mathbb{R}_+\to\mathbb{R}$ is a locally integrable function; the fixed moments of time t_k satisfies $0<t_1<t_2<...<t_k<...$, and $\lim_{k\to\infty} t_k=\infty; x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k, k=1,2,...$, respectively; $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$ represents the jump in the state x at time t_k with I_k determining the size of the jump. We will consider the global attracting set and stability of the neutral stochastic partial differential equations with impulses driven by an additive α-stable with impulses on a separable Hilbert space H.

Chapter 2

Preliminaries

The knowledge of stochastic processes and stochastic analysis has played an important role in the real world. Stochastic differential equations are used to model diverse phenomena such as fluctuating stock prices or physical systems subject to thermal fluctuations, which draw great attentions from researchers to develop the things which are getting more complicated.

In this chapter, we begin by recalling some definitions and results, especially those from functional analysis and theories of stochastic process and stochastic differential equations along with probability theories in infinite dimensions. We introduce mild solutions for stochastic differential equations and investigate the existence and uniqueness of solutions under appropriate assumptions. We introduce and clarify definitions and develop our theory in Hilbert spaces. To present the proofs of all the results here would require preparatory background material, which would significantly increase both the size and scope of this dissertation. Although this chapter introduces very important theorems, required proofs are omitted here. However, these related proofs can be found from book written by Liu [41] and you can also find most of these basic mathematical concepts and their proofs in many well-known text books such as Pazy [32] and Da Prato and

Zabczyk [22] or to be found in the literature reviews.

2.1 Some Results from Functional Analysis

A Banach space $(X, \|\cdot\|_X)$ (real or complex) is a complete normed linear space over ($\mathbb{R}$ or $\mathbb{C}$). If the norm $\|\cdot\|_X$ is induced by an inner product $\langle\cdot,\cdot\rangle_X$, then X is called a Hilbert space. We say that X is separable if there exists a countable set $S \subseteq X$ such that the closure $\bar{S} = X$. For a Hilbert space X, a collection $\{e_i\}$ of elements in X is called an orthonormal set if $\langle e_i, e_i\rangle_X = 1$ for all i, and $\langle e_i, e_j\rangle_X = 0$ if $i \neq j$. If S is an orthonormal set and no other orthonormal set contains S as a proper subset, then S is called an orthonormal basis for X. A Hilbert space X is separable if and only if it has a countable orthonormal basis $\{e_i\}, i = 1, 2, \cdots$.

Example 2.1.1 *(Sobolev space) Let $[a, b]$ be an interval in $\mathbb{R}$ and a differentiable function $f(x)$ of one derivative exists at each point in its domain. Now, we define*

$$W^{1,2}([a,b];X) = \Big\{ f : [a,b] \to X, f(x) \textit{ is differentiable}, \\ \int_a^b \|f(x)\|_X^2 dx < \infty \textit{ and} \int_a^b \|f'(x)\|_X^2 dx < \infty \Big\}.$$

If X is a Hilbert space, then $W^{1,2}([a,b];X)$ is a Hilbert space under the norm

$$\|f\|_{1,2} = \Big(\int_a^b \|f(x)\|_X^2 dx\Big)^{1/2} + \Big(\int_a^b \|f'(x)\|_X^2 dx\Big)^{1/2}, \qquad f \in W^{1,2}([a,b];X),$$

and under the inner product

$$\langle f, g\rangle_{1,2} = \int_a^b \langle f(x), g(x)\rangle_X dx + \int_a^b \langle f'(x), g'(x)\rangle_X dx, \qquad f, g \in W^{1,2}([a,b];X).$$

Definition 2.1.1 *Let X and Y be two Banach spaces and $\mathcal{D}(A)$ a subspace of X. A map $A : \mathcal{D}(A) \subseteq X \to Y$ is called a linear operator if the following relation holds:*

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay \quad \text{for any} \quad x, y \in \mathcal{D}(A), \quad \alpha, \beta \in \mathbb{R} \text{ or } \mathbb{C}.$$

The subspace $\mathcal{D}(A)$ is called the domain of A. If A maps any bounded subsets of $\mathcal{D}(A)$ into bounded subsets of Y, we say A is a bounded linear operator. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators A from X to Y with $\mathcal{D}(A) = X$. In particular, if $X = Y$, we write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. In this case, $\mathcal{L}(X, Y)$ is a Banach space equipped with the operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$ given by

$$\|A\| = \|A\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X \leq 1} \|Ax\|_Y < \infty \quad \text{for any} \quad A \in \mathcal{L}(X, Y).$$

For any linear operator $A : \mathcal{D}(A) \subseteq X \to Y$, we define $\mathcal{R}(A) := \{Ax : x \in \mathcal{D}(A)\}$. It is called the range of the operator A.

Definition 2.1.2 *Let $Y = K$ where $K = \mathbb{R}$ or $\mathbb{C}$. Any $f \in \mathcal{L}(X, K)$ is called a bounded linear functional on X. In this case, we put $X^* = \mathcal{L}(X, K)$, which is a Banach space under the norm $\|\cdot\|_{X^*}$ and X^* is called the dual space of X.*

Theorem 2.1.1 *[63] (Riesz's Theorem) Let X be a Hilbert space, then $X^* = X$. That is, every bounded linear functional f on X can be represented in terms of the inner product by*

$$f(x) = \langle x, z \rangle \quad \text{for any} \quad x \in X,$$

where z is uniquely determined by f and has norm

$$\|z\|_X = \|f\|_{X^*}.$$

For any Banach space, we can further define $X^{**} = (X^*)^*$ and if $X = X^{**}$, X is called reflexive. We can conclude that a Hilbert space X is reflexive.

Definition 2.1.3 *Let X and Y be two Banach spaces. A linear operator $A : \mathcal{D}(A) \subseteq X \to Y$ is said to be closed if whenever*

$$x_n \in \mathcal{D}(A), n \geq 1, \text{ and } \lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} Ax_n = y,$$

then $x \in \mathcal{D}(A)$ and $Ax = y$.

Definition 2.1.4 *Let X and Y be two Banach spaces and a linear operator $A : \mathcal{D}(A) \subseteq X \to Y$ is called densely defined if the closure $\overline{\mathcal{D}(A)} = X$.*

Definition 2.1.5 *Let A be a densely defined linear operator on a Hilbert space X. Then the Hilbert adjoint operator $A^* : X \to X$ is defined by*

$$\langle Ax, y\rangle_X = \langle x, A^*y\rangle_X$$

for any $x \in \mathcal{D}(A), y \in \mathcal{D}(A^*)$. In particular, if A is bounded, the adjoint operator A^* of A exists and is unique and bounded with $\|A^*\| = \|A\|$.

Definition 2.1.6 *Let X be a Hilbert space and a densely defined linear operator $A : \mathcal{D}(A) \subseteq X \to X$ is symmetric if for all $x, y \in \mathcal{D}(A)$, $\langle Ax, y\rangle_X = \langle x, Ay\rangle_X$. A symmetric operator A is called self-adjoint if $\mathcal{D}(A^*) = \mathcal{D}(A)$.*

A linear operator A on the Hilbert space X is called non-negative, denoted by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{D}(A)$. It is called positive if $\langle Ax, x\rangle > 0$ for all $x \in \mathcal{D}(A)$ and coercive if $\langle Ax, x\rangle > c\|x\|_X^2$ for some $c > 0$ and $x \in \mathcal{D}(A)$. A linear operator B is called the square root of A if $B^2 = A$.

Theorem 2.1.2 *Let A be a linear operator on the Hilbert space X. If A is self-adjoint and nonnegative, then it has a unique square root, denote it by $A^{1/2}$, which* *is self-adjoint and nonnegative such that $\mathcal{D}(A) \subset \mathcal{D}(A^{1/2})$. Furthermore, if A is positive, so is $A^{1/2}$.*

Theorem 2.1.3 *Let X be a Hilbert space. Suppose that A is self-adjoint and nonnegative on X. Then A is coercive if and only if it has a bounded inverse $A^{-1} \in \mathcal{L}(X)$. In this case, A^{-1} is self-adjoint and nonnegative.*

Definition 2.1.7 *Let X and Y be two Banach spaces. An operator $A \in \mathcal{L}(X, Y)$ is compact if for any bounded sequence $\{x_n\}_{n\geq 1}$ in X, the sequence $\{Ax_n\}_{n\geq 1}$ has a convergent subsequence in Y.*

Let X be a separable Hilbert space. A linear bounded operator $A \in \mathcal{L}(X, Y)$ is a compact operator if and only if

$$Ax = \sum_{i=1}^{\infty} \lambda_i \langle x, e_i \rangle_X \tilde{e}_i \quad \forall x \in X,$$

where $\{e_i\}_{i\geq 1}$ and $\{\tilde{e}_i\}_{i\geq 1}$ are two orthonormal bases in X and Y, respectively and $\lambda_i \geq 0$ for each $i \geq 1$. The operator A is called trace class if $\sum_{i=1}^{\infty} \lambda_i < \infty$ and A is Hilbert-Schmidt if $\sum_{i=1}^{\infty} \lambda_i^2 < \infty$.

Let $\mathscr{L}_1(X)$ be the family of all trace class operators on X. It can be shown that $\mathscr{L}_1(X)$ is a Banach space under the trace norm $\|A\|_1 := \sum_{i=1}^{\infty} \lambda_i$. The space of Hilbert-Schmidt operators denoted by $\mathscr{L}_2(X)$ has the norm $\|A\|_2 := (\sum_{i=1}^{\infty} \lambda_i^2)^{1/2}$. All the Hilbert-Schmidt operators form a Hilbert space .

In this dissertation, we would use the Banach fixed point theorem or contraction theorem, which concerns mappings of a Banach space into itself. It states sufficient conditions for the existence and uniqueness of a fixed point.

Definition 2.1.8 (Contraction) *Let $(X, \|\cdot\|_X)$ be a Banach space. A bounded linear operator $T : X \to X$ is called a contraction on X if there is a positive* *number* $\alpha < 1$ *such that for all* $x, y \in X$

$$\|Tx - Ty\|_X \leq \alpha \|x - y\|_X.$$

Theorem 2.1.4 (Banach Fixed Point Theorem) *Consider a Banach space* $(X, \|\cdot\|_X)$ *and let* $T : X \to X$ *be a contraction on* X*. Then* T *has a unique fixed point. That is, there exists a unique* $x \in X$ *such that*

$$Tx = x.$$

2.2 C_0-Semigroups

Definition 2.2.1 *A strongly continuous or* C_0*-semigroup* $S(\cdot) : [0, \infty) \to \mathcal{L}(X)$ *is a family of bounded linear operators on a Banach space* X *satisfying:*

(i) $S(0) = I$, *where* I *is the identity operator on* X*;*

(ii) $S(t + s) = S(t)S(s)$ *for all* $t, s \geq 0$;

(iii) $S(t)$ *is strongly continuous, i.e., for any* $x \in X, S(t)x : [0, \infty) \to X$ *is continuous.*

It is known that for any C_0-semigroup $S(t)$ on X, there exist constants $M \geq 1$ and $\mu \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\mu t}, \qquad t \geq 0.$$

In association with the C_0-semigroup $S(t)$, we define a linear operator $A : \mathcal{D}(A) \subseteq X \to X$ by

$$\mathcal{D}(A) = \Big\{ x \in X : \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \text{ exists } \Big\},$$

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t}, \quad x \in \mathcal{D}(A).$$

The operator A is called the infinitesimal generator, or simply generator, of the semigroup $\{S(t)\}_{t \geq 0}$. We frequently write it as $e^{tA}, t \geq 0$.

Suppose that A is linear, but not necessarily bounded, operator on a Banach space X. The resolvent set $\rho(A)$ of A is defined as the set of all complex numbers $\lambda \in \mathbb{C}$ such that $(\lambda I - A)^{-1}$ exists and $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda, A) = (\lambda I - A)^{-1}, \lambda \in \rho(A)$ of bounded linear operators is called the resolvent operator of A. The spectrum of A is defined to be $\sigma(A) = \mathbb{C} \backslash \rho(A)$.

Theorem 2.2.1 *(Hille-Yosida) A linear operator A on a Banach space X is the infinitesimal generator of a C_0-semigroup $S(t), t \geq 0$ if and only if*

1. *A is densely defined and closed;*

2. *the resolvent set $\rho(A)$ of A contains the ray (μ, ∞) and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \mu)^n}, \qquad \text{for } \lambda > \mu, \quad n = 1, 2, ...$$

for some $M > 0$.

Proposition 2.2.1 *Suppose that A generates a C_0-semigroup $e^{tA}, t \geq 0$, on a*

Banach space X. It is valid that if $x \in \mathcal{D}(A)$, then $e^{tA}x \in \mathcal{D}(A)$ and in this case

$$\frac{d}{dt}e^{tA}x = e^{tA}Ax = Ae^{tA}x, \qquad \textit{for all } t \geq 0.$$

Let X be a Banach space and consider the following deterministic linear Cauchy problem on X,

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t), \quad t \geq 0, \\ y(0) = y_0 \in X, \end{cases} \tag{2.1}$$

where A is a linear operator which generates a C_0-semigroup $e^{tA}, t \geq 0$, on X. If $y_0 \in \mathcal{D}(A)$, then by Proposition 2.2.1, we have $e^{tA}y_0 \in \mathcal{D}(A)$ and

$$\frac{d}{dt}(e^{tA}y_0) = Ae^{tA}y_0, \qquad t \geq 0. \tag{2.2}$$

Hence, $y(t) = e^{tA}y_0, t \geq 0$, is a solution of the differential equation (2.1). If $y_0 \notin \mathcal{D}(A)$, the equality (2.2) is not necessarily true. However, for any $y_0 \in X$ it does make sense to define $y(t) = e^{tA}y_0, t \geq 0$, which is called a mild solution of (2.1).

Definition 2.2.2 *Let $e^{tA}, t \geq 0$, be a C_0-semigroup on a Banach space X with the generator $A : \mathcal{D}(A) \subseteq X \to X$.*

(i) The semigroup $e^{tA}, t \geq 0$, is called compact if for any $t \in (0, \infty)$, the operator $e^{tA} \in \mathcal{L}(X)$ is compact.

(ii) The semigroup $e^{tA}, t \geq 0$, is called analytic if it admits an extension e^{zA} on $z \in \Delta_\theta := \{z \in \mathbb{C} : |arg\ z| < \theta\}$ for some $\theta \in (0, \pi]$, such that $z \to e^{zA}$ is analytic on Δ_θ and satisfies:

(a) $e^{(z_1+z_2)A} = e^{z_1A}e^{z_2A}$ for any $z_1, z_2 \in \Delta_\theta$;

(b) $\lim_{\Delta_{\bar\theta}\ni z\to 0}\|e^{zA}x - x\|_X = 0$ *for all* $x \in X$ *and* $0 < \bar\theta < \theta$.

Let us define fractional powers of certain unbounded linear operators and study some of their properties. Let $A : \mathcal{D}(A) \subseteq X \to X$ be the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ in H. Suppose that $0 \in \rho(A)$ is the resolvent set of A, then, for $\alpha \in (0, 1]$, it is possible to define the fractional power $(-A)^\alpha$ as a closed linear operator on its domain $\mathcal{D}((-A)^\alpha)$. Furthermore, the subspace $\mathcal{D}((-A)^\alpha)$ is dense in H, and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|_H, \quad x \in \mathcal{D}((-A)^\alpha),$$

defines a norm in $\mathcal{D}((-A)^\alpha)$. We let $H_\alpha = \mathcal{D}((-A)^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$.

We need the following assumption.

(A1) Let $-A$ be a densely defined closed linear operator for which

$$\rho(-A) \supset S = \{\lambda : 0 < \omega < |arg\lambda| \leq \pi\} \cup V$$

and

$$\|R(\lambda, A)\| \leq \frac{M}{1 + |\lambda|} \text{ for } \lambda \in S,$$

where V is a neighborhood of 0.

For $0 < \alpha < 1$, we can define

$$(-A)^\alpha = \frac{sin\pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(tI - A)^{-1}dt, \quad 0 < \alpha < 1.$$

Definition 2.2.3 *Let $-A$ satisfy (A1) with $\omega < \pi/2$. For every $\alpha > 0$, we define*

$$(-A)^{\alpha} = ((-A)^{-\alpha})^{-1}.$$

For $\alpha = 0, (-A)^{\alpha} = I$.

Lemma 2.2.1 *Suppose $0 \in \rho(A)$, then we know that there exist constants $M \geq 1, \lambda > 0$, for every $0 < \beta \leq 1$,*

(1) we have for each $x \in D(-A)^{\alpha}$,

$$S(t)(-A)^{\alpha}x = (-A)^{\alpha}S(t)x;$$

(2) there exists $M_{\beta} > 0$ such that

$$\|(-A)^{\beta}S(t)\| \leq M_{\beta}t^{-\beta}e^{-\lambda t}, \quad t > 0;$$

(3) for any $\beta \in [0, 1]$,

$$\|(-A)^{-\beta}\| \leq C.$$

2.3 Probability Theory and Stochastic Processes

Let Ω be a non-empty set and $\mathcal{F}$ a collection of subsets of Ω. We call $\mathcal{F}$ a σ-algebra if the following hold:

(1) $\emptyset \in \mathcal{F}$, where $\emptyset$ is the empty set;

(2) $A \in \mathcal{F} \Rightarrow A^{c} \in \mathcal{F}$, where $A^{c} = \Omega - A$ is the complement of A in Ω;

(3) $\{A_i\}_{i\geq 1} \subset \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space and the elements of $\mathcal{F}$ are called measurable. A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a mapping $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying that

(1) $\mathbb{P}(\Omega) = 1$, and

(2) (countable additivity) for any disjoint sequence $\{A_i\}_{i\geq 1} \subset \mathcal{F}$ (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$), then

$$\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Suppose that $\mathcal{C}$ is a collection of subsets of Ω, then there exists a smallest σ-algebra $\sigma(\mathcal{C})$ on Ω which contains $\mathcal{C}$. Hence, this $\sigma(\mathcal{C})$ is called the σ-algebra generated by $\mathcal{C}$. If $\Omega = \mathbb{R}^d$ and $\mathcal{C}$ is the collection of all open sets in $\mathbb{R}^d$, then $\sigma(\mathcal{C})$ is called the Borel σ-algebra, denote it by $\mathcal{B}(\mathbb{R}^d)$ and the elements of $\mathcal{B}(\mathbb{R}^d)$ are called Borel sets and any measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called a Borel measure.

If $(\Omega, \mathcal{F})$ and $(S, \mathcal{B}(S))$ are two measurable spaces, then a mapping ξ from Ω into S such that the set $\{\omega \in \Omega : \xi \in A\} = \{\xi \in A\}$ belongs to $\mathcal{F}$ for arbitrary $A \in \mathcal{B}(S)$ is called measurable from $(\Omega, \mathcal{F})$ into $(S, \mathcal{B}(S))$. Hence, $\mathcal{B}(S)$ is a Borel σ-algebra on S, where S is a complete metric space. If ξ is a measurable mapping from $(\Omega, \mathcal{F})$ into $(S, \mathcal{B}(S))$ or an S-valued random variable and $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$, then we will denote by $\mathbb{D}_\xi(\cdot)$ the image of $\mathbb{P}$ under the mapping ξ:

$$\mathbb{D}_\xi(A) = \mathbb{P}\{\omega \in \Omega : \xi(\omega) \in A\}, \quad \forall A \in \mathcal{B}(S).$$

It may be shown that $\mathbb{D}_\xi(\cdot)$ is a probability measure which is called the distribution or the law of ξ.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, we set $\bar{\mathcal{F}} = \{A \subset \Omega : \exists B, C \in \mathcal{F}$ such that $B \subset A \subset C, \mathbb{P}(B) = \mathbb{P}(C)\}$. Then $\bar{\mathcal{F}}$ is a σ-algebra and is called the completion of $\mathcal{F}$. If $\mathcal{F} = \bar{\mathcal{F}}$, then probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete. In general, if $\mathcal{F}$ is not complete, we can extend $\mathbb{P}$ to $\bar{\mathcal{F}}$ by defining $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C)$ for $A \in \bar{\mathcal{F}}$, where $B, C \in \mathcal{F}$ with $B \subset A \subset C$ and $\mathbb{P}(B) = \mathbb{P}(C)$. In this way, $(\Omega, \mathcal{F}, \mathbb{P})$ becomes a complete probability space. A family $\{\mathcal{F}_t\}, t \geq 0$, for which each $\{\mathcal{F}_t\}$ is a sub-σ-field of $\mathcal{F}$ and forms an increasing family of σ-fields, is called a filtration of $\mathcal{F}$. With this $\{\mathcal{F}_t\}, t \geq 0$, one can associate another filtration by setting σ-fields $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$ for $t \geq 0$. We say that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is normal or satisfies the usual conditions if $\mathcal{F}_{t+} = \mathcal{F}_t$ for each $t \geq 0$, that is, the filtration is a right continuous increasing family and contains all $\mathbb{P}$-null sets of $\mathcal{F}$.

Now assume that $S = H$ is a separable Hilbert space with norm $\|\cdot\|_H$ and ξ is an H-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. We can define the integral $\int_\Omega \xi(\omega)\mathbb{P}(d\omega)$ of ξ with respect to the probability measure $\mathbb{P}$. We often denote it by $\mathbb{E}(\xi)$, which is called the expectation. The integral defined in this way is called a Bochner's integral. We denote by $L^p(\Omega, \mathcal{F}, \mathbb{P}; H), p \in [1, \infty)$, the set of all equivalence classes of H-valued random variables with respect to equivalent relation of almost sure equality. Then it can be verified that $L^p(\Omega, \mathcal{F}, \mathbb{P}; H), p \in [1, \infty)$, equipped with the norm

$$\|\xi\|_p = (\mathbb{E}\|\xi\|_H^p)^{1/p}, \quad p \in [1, \infty), \quad \xi \in L^p(\Omega, \mathcal{F}, \mathbb{P}; H),$$

is a Banach space. If Ω is an interval $[0, T], \mathcal{F} = \mathcal{B}([0, T])$ and $\mathbb{P}$ is the usual Lebesgue measure L/T on $[0, T]$ for $L^p([0, T], \mathcal{B}([0, T]), L/T; H), 0 \leq T \leq \infty$, we also write $L^p([0, T]; H)$.

Next, we introduce some useful results.

(1) Hölder inequality (for $p = 2$ it is called Cauchy Schwarz's inequality)

$$\|\mathbb{E}(\xi\zeta)\| \leq (\mathbb{E}\|\xi\|^p)^{1/p}(\mathbb{E}\|\zeta\|^q)^{1/q},$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ for any $\xi \in L^p(\Omega; H), \zeta \in L^q(\Omega; H)$.

(2) Minkowski's inequality

$$(\mathbb{E}\|\xi + \zeta\|^p)^{1/p} \leq (\mathbb{E}\|\xi\|^p)^{1/p} + (\mathbb{E}\|\zeta\|^p)^{1/p},$$

where $p \geq 1$ for any $\xi, \zeta \in L^p(\Omega; H)$.

Theorem 2.3.1 *(Monotonic convergence theorem): If $\{\xi_n\}$ is an increasing sequence of nonnegative random variables, then*

$$\lim_{n\to\infty} \mathbb{E}\{\xi_n\} = \mathbb{E}\{\lim_{n\to\infty} \xi_n\}.$$

Theorem 2.3.2 *(Dominated convergence theorem): Let $p \geq 1, \{\xi_n\} \subset L^p(\Omega; H)$ and $\zeta \in L^p(\Omega; \mathbb{R})$. Assume that $\|\xi_n\|_H \leq \zeta$ almost surely and $\{\xi_n\}$ converges to ξ in probability. Then $\xi \subset L^p(\Omega; H)$, $\{\xi_n\}$ converges to ξ in L^p, and*

$$\lim_{n\to\infty} \mathbb{E}\{\xi_n\} = \mathbb{E}\{\lim_{n\to\infty} \xi_n\} = \mathbb{E}\{\xi\}.$$

Lemma 2.3.1 *(Fatou's Lemma): If the random variable's ξ_n satisfy $\xi_n \geq \zeta$ almost surely ($\zeta \in L^p(\Omega; \mathbb{R})), \forall n$, we have*

$$\mathbb{E}\{\liminf_{n\to\infty} \xi_n\} \leq \liminf_{n\to\infty} \mathbb{E}\{\xi_n\}.$$

If the random variable's ξ_n satisfy $\xi_n \leq \zeta$ almost surely ($\zeta \in L^p(\Omega; \mathbb{R})), \forall n$, we *have*

$$\mathbb{E}\{\limsup_{n\to\infty} \xi_n\} \geq \limsup_{n\to\infty} \mathbb{E}\{\xi_n\}.$$

An arbitrary family $M = \{M(t)\}, t \geq 0$, of H-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a stochastic process. Sometimes, we also write $M(t,\omega)$ or M_t in place of $M(t)$ for all $t \geq 0$. A stochastic process M is called measurable if the mapping $M(\cdot,\cdot) : \mathbb{R}_+ \times \Omega \to H$ is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{F}$-measurable. Let $\{\mathcal{F}_t\}, t \geq 0$, be an increasing family of sub-σ-fields of $\mathcal{F}$. The process M is called $\{\mathcal{F}_t\}_{t\geq 0}$-adapted if $M(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$. Clearly, if $\sigma\{M(t)\}_{t\geq 0}$ is the family of σ-fields generated by $M = \{M(t)\}_{t\geq 0}$, M is $\sigma\{M(t)\}_{t\geq 0}$-adapted. For any $\omega \in \Omega$, the function $M(\cdot, \omega)$ is called a path or trajectory of M. Let $\mathcal{P}$ denote the smallest σ-algebra on $\mathbb{R}_+ \times \Omega$ with respect to every left continuous process is a measurable function of (t, ω). A stochastic process is said to be predictable if the process regarded as a function of (t, ω) is $\mathcal{P}$-measurable.

Definition 2.3.1 *Suppose $M = \{M(t)\}, t \geq 0$, is an H-valued process and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration of $\mathcal{F}$. The process M is said to be progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if for every $t \geq 0$, the mapping*

$$[0,t] \times \Omega \to H, \qquad (s,\omega) \to M(s,\omega),$$

is $\mathcal{B}([0,t]) \times \mathcal{F}_t$-measurable.

Definition 2.3.2 *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration of $\mathcal{F}$. A mapping $\tau : \Omega \to [0, \infty]$ is called a stopping time with respect to $\{\mathcal{F}_t\}, t \geq 0$, if*

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \text{ for each } t \geq 0.$$

The σ-field of events before τ, denoted by $\mathcal{F}_\tau$ is defined as

$$\mathcal{F}_\tau = \Big\{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for every } t \ge 0\Big\}.$$

Theorem 2.3.3 *Let $M = \{M(t)\}, t \ge 0$, be an H-valued progressively measurable process with respect to $\{\mathcal{F}_t\}_{t\ge 0}$, and let τ be a finite stopping time. Then the random variable X_τ is $\mathcal{F}_\tau$-measurable.*

Theorem 2.3.4 *(Fubini Theorem): Let $M(t)$ be an H-valued measurable stochastic process.*

(1) If $\mathbb{E}\{M(t)\}$ exists for all t, then it is measurable as a function of t;

(2) if $\int_a^b \mathbb{E}\|M(t)\|_H dt < \infty$ for all $a < b$,

$$\int_a^b \mathbb{E}\{M(t)\}dt = \mathbb{E}\Big\{\int_a^b M(t)dt\Big\}.$$

If $\mathbb{E}\|M(t)\|_H < \infty$ for all $t \ge 0$, then the process is called integrable.

Proposition 2.3.1 *Assume that H is a separable Hilbert space. Let ξ be a Bochner integral H-valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. There exists a unique, up to a set of $\mathbb{P}$-probability zero, integrable H-valued random variable ζ, which is $\mathcal{G}$-measurable such that*

$$\int_A \xi d\mathbb{P} = \int_A \zeta d\mathbb{P}, \quad \forall A \in \mathcal{G}.$$

This random variable ζ is denoted by $\mathbb{E}(\xi \mid \mathcal{G})$, which is called the conditional expectation of ξ given $\mathcal{G}$.

An integrable and adapted H-valued process $M(t), t \geq 0$, is said to be a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if

$$\mathbb{E}(M(t) \mid \mathcal{F}_s) = M(s), \quad \mathbb{P} - a.s.$$

for arbitrary $t, s \in T, t \geq s$.

A real-valued integrable and adapted process $M(t), t \geq 0$ is said to be a sub-martingale (resp. a supermartingale) if

$$\mathbb{E}(M(t) \mid \mathcal{F}_s) \geq M(s) \quad (\text{ resp. } \mathbb{E}(M(t) \mid \mathcal{F}_s) \leq M(s)), \quad \mathbb{P} - a.s.$$

2.4 Wiener Processes and Stochastic Integral

Let K be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_K$. A probability measure $\mathbb{N}$ on $(K, \mathcal{B}(K))$ is called Gaussian if for arbitrary $u \in K$, there exist numbers $\mu \in \mathbb{R}, \sigma > 0$, such that

$$\mathbb{N}\{x \in K : \langle u, x \rangle_K \in A\} = N(\mu, \sigma)(A), \quad A \in \mathcal{B}(\mathbb{R}),$$

where $N(\mu, \sigma)$ is the standard one dimensional normal distribution with mean μ and variance σ. If $\mathbb{N}$ is Gaussian, there exist an element $m \in K$ and a nonnegative self-adjoint operator $Q \in \mathscr{L}_1(K)$, the family of all trace class operators in K, such that the characteristic function of $\mathbb{N}$ is given by

$$\int_K e^{i\langle \lambda, x \rangle_K} \mathbb{N}(dx) = e^{i\langle \lambda, m \rangle_K - \frac{1}{2}\langle Q\lambda, \lambda \rangle_K}, \quad \lambda \in K.$$

Therefore, the measure $\mathbb{N}$ is uniquely determined by m and Q and denoted by $\mathbb{N}(m, Q)$. In particular, in this case, we call m the mean and Q the covariance operator of $\mathbb{N}(m, Q)$.

The proofs for this section can be founded in Chapter 4, [22] .

For a self-adjoint and nonnegative operator $Q \in \mathcal{L}(K)$, we assume that there exists an orthonormal basis $\{e_k\}_{k\geq 1}$ in K, and a bounded sequence of positive number λ_k such that

$$Qe_k = \lambda_k e_k, \qquad k = 1, 2, \cdots .$$

A stochastic process $W(t), t \geq 0$ on K is called Q-Wiener process if

(i) $W(0) = 0$;

(ii) $W(t)$ has continuous trajectories;

(iii) the law $\mathbb{D}_{W(t)-W(s)} = \mathbb{N}(0, (t-s)Q)$ for all $t \geq s \geq 0$.

If the trace Tr Q is finite, then W is genuine Wiener process. It is possible that Tr $Q = \infty$, e.g., $Q = I$, and in this case we call W a cylindrical Wiener process.

Assume that the probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $W(t), t \geq 0$, be a Q-Wiener process on K which is assumed to be adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and for every $t > s \geq 0$ the increments $W(t) - W(s)$ are independent of $\{\mathcal{F}_s\}$. Then, $W(t), t \geq 0$, is a continuous martingale relative to $\{\mathcal{F}_t\}_{t\geq 0}$ and W has the following representation:

$$W(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} w^i(t) e_i, \quad t \geq 0,$$

where $(\lambda_i \geq 0, i \in \mathbb{N}_+)$ are the eigenvalues of Q with the corresponding eigenvectors $(e_i, i \in \mathbb{N}_+)$, and $(w^i(t), i \in \mathbb{N}_+)$ is a sequence of independent real-valued one-dimensional standard Brownian motions.

We introduce the subspace $K_Q = \mathcal{R}(Q^{1/2}) \subset K$, the range of $Q^{1/2}$, which is a

Hilbert space endowed with the inner product

$$\langle u, v\rangle_{K_Q} = \langle Q^{-1/2}u, Q^{-1/2}v\rangle_K \quad \text{for any } u, v \in K_Q.$$

Let $\mathscr{L}_2(K_Q, H)$ denote the space of all Hilbert-Schmidt operators from K_Q into H, then $\mathscr{L}_2(K_Q, H)$ becomes a separable Hilbert space under the inner product

$$\langle L, P\rangle_{\mathscr{L}_2(K_Q,H)} = Tr[LQP^*] \quad \text{for any } L, P \in \mathscr{L}_2(K_Q, H).$$

For arbitrarily given $T \geq 0$, let $\Phi(t, \omega), t \in [0, T]$, be an $\mathscr{L}_2(K_Q, H)$-valued process. We define the following norm for arbitrary $t \in [0, T]$,

$$\|\Phi\|_t := \Big\{\mathbb{E}\int_0^t Tr\Big[\Phi(s)Q\Phi(s)^*\Big]ds\Big\}^{\frac{1}{2}}.$$

In particular, we denote all $\mathscr{L}_2(K_Q, H)$-valued measurable processes, adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying $\|\Phi\|_T < \infty$ by $\mathcal{U}^2([0, T]; \mathscr{L}_2(K_Q, H))$.

The stochastic integral $\int_0^t \Phi(s)dW(s) \in H, t \geq 0$, may be defined for all $\Phi \in \mathcal{U}^2([0, T]; \mathscr{L}_2(K_Q, H))$ by

$$\int_0^t \Phi(s)dW(s) \overset{L^2}{=} \lim_{n\to\infty}\sum_{i=1}^n \int_0^t \sqrt{\lambda_i}\Phi(s)e_i dw^i(s), \quad t \in [0, T]. \tag{2.3}$$

By the definition of stochastic integrals and using standard limiting procedure, we can establish some useful properties of stochastic integrals.

Proposition 2.4.1 *For arbitrary $T \geq 0$, assume that $\Phi(\cdot) \in \mathcal{U}^2([0, T]; \mathcal{L}_2(K_Q, H))$. Then the stochastic integral $\displaystyle\int_0^t \Phi(s)dW(s)$ is a continuous, square integrable H-valued martingale on $[0, T]$. Moreover,*

$$\mathbb{E}\Big\|\int_0^t \Phi(s)dW(s)\Big\|_H^2 = \|\Phi\|_t^2, \qquad t \in [0, T].$$

Lévy Processes, Let $Z = \{Z(t) : t \geq 0\}$ be an H-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that it has independent increments if for each $n \in \mathbb{N}$ and each $0 \leq t_1 < t_2 \leq ... < t_{n+1} < \infty$, the random variables $(Z_{t_{j+1}} - Z_{t_j}, 1 \leq j \leq n)$ are independent and that it has stationary increments if each $Z_{t_{j+1}} - Z_{t_j}$ and $Z_{t_{j+1}-t_j} - Z(0)$ has the same distribution.

We say that Z is a Lévy process if

1. $Z(0) = 0$ a.s;
2. $Z(t)$ has independent and stationary increments;
3. Z is stochastically continuous, i.e., for any $\delta > 0$ and for all $s \geq 0$

$$\lim_{t \to s} \mathbb{P}(\|Z(t) - Z(s)\|_H > \delta) = 0.$$

Related to the Lévy process Z, we have the following Lévy-Khintchine formula (see, e.g., [1]),

$$\mathbb{E}\Big(e^{i\langle h, Z(t)\rangle_H}\Big) = e^{t\eta_{b,Q,\nu}(h)}, \quad \forall t \geq 0 \text{ and } h \in H,$$

with the exponent

$$\begin{aligned} \eta_{b,Q,\nu}(h) &= i\langle b, h\rangle_H - \frac{1}{2}\langle h, Qh\rangle_H \\ &\quad + \int_H \Big[e^{i\langle h,x\rangle_H} - 1 - i\langle h, x\rangle_H \cdot I_{\|x\|_H<1}(x)\Big]\nu(dx), \end{aligned} \tag{2.4}$$

where $b \in H, Q$ is a positive, self-adjoint and trace class operator on H, and ν is called a Lévy measure on H satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_H \min(1, \|x\|_H^2)\nu(dx) < \infty,$$

We use the symbol $I_E(x)$ to denote the characteristic function on set $E \subset H$, i.e., $I_E(x) = 1$ if $x \in E$ and $I_E(x) = 0$ if $x \notin E$. The triple (b, Q, ν) is called the characteristics of the process Z and the mapping $\eta_{b,Q,\nu}(h)$ is called the characteristic exponent of Z.

It can be proved that Lévy process has a càdlàg version. If Z is a Lévy process on H, we write $\Delta Z(t) = Z(t) - Z(t-)$ for all $t \geq 0$ where $Z(t-) := \lim_{s \uparrow t} Z(s)$. We then obtain a counting Poisson random measure N on $H \backslash \{0\}$ by

$$N(t, E) := \#\{0 \leq s \leq t : \Delta Z(s) \in E\} < \infty, \quad t \geq 0,$$

almost surely for any $E \in \mathcal{B}(H \backslash \{0\})$. Here $\#$ is the counting and $\mathcal{B}(H \backslash \{0\})$ is the Borel σ-field on $H \backslash \{0\}$. Now we denote by $\tilde{N}(t, dx)$ the associated compensating Poisson random martingale measure by

$$\tilde{N}(t, dx) := N(t, dx) - t\nu(dx).$$

Let $\mathcal{O} \in \mathcal{B}(H \backslash \{0\})$ and $\mathcal{V}^2([0, T] \times \mathcal{O}; H)$ denote the space of all predictable processes $L : [0, T] \times \mathcal{O} \times \Omega \to H$ with

$$\int_0^T \int_{\mathcal{O}} \mathbb{E}\|L(t, x)\|_H^2 \nu(dx) dt < \infty.$$

Then we can define the random finite sum

$$\int_0^T \int_{\mathcal{O}} L(t, x) N(dt, dx) = \sum_{0 \leq t \leq T} L(t, \Delta Z(t)) I_{\mathcal{O}}(\Delta Z(t)),$$

which enables us to define the stochastic integral

$$\int_0^T \int_{\mathcal{O}} L(t, x) \tilde{N}(dt, dx) := \int_0^T \int_{\mathcal{O}} L(t, x) N(dt, dx) - \int_0^T \int_{\mathcal{O}} L(t, x) \nu(dx) dt.$$

It is known that

$$\int_0^T \int_{\mathcal{O}} L(t,x)\tilde{N}(dt,dx), t \geq 0,$$

is an H-valued square-integrable martingale satisfying

$$\mathbb{E}\Big(\Big\| \int_0^T \int_{\mathcal{O}} L(t,x)\tilde{N}(dt,dx)\Big\|_H^2\Big) \leq \kappa \int_0^T \int_{\mathcal{O}} \mathbb{E}\|L(t,x)\|_H^2 \nu(dx)dt, \qquad (2.5)$$

where $\kappa > 0$ for all $T \geq 0$ and $\forall x \in H$.

The Lévy-Itô decomposition theorem on a separable Hilbert space H was introduced in [3] as follows:

Theorem 2.4.1 *Suppose that $Z(t), t \geq 0$, is a càdlàg H-valued Lévy process with characteristic exponent given by (2.4), then for each $t \geq 0$,*

$$Z(t) = bt + W_Q(t) + \int_{\|x\|_H<1} x\tilde{N}(t,dx) + \int_{\|x\|_H \geq 1} xN(t,dx),$$

where $W_Q(t)$ is a Q-Wiener process, independent of N.

Let $Z(t), t \geq 0$, be a càdlàg H-valued Lévy process and assume that J is a measurable function from $\mathbb{R}_+$ to $\mathcal{L}(H)$ such that the mapping $t \to \|J\|$ is locally square integrable. Now we define the stochastic integral

$$\int_0^t J(s)dZ(s) \quad \forall t \geq 0.$$

We use the Lévy-Itô decomposition theorem (2.6) to write

$$\begin{aligned}\int_0^t J(s)dZ(s) &= \int_0^t J(s)bds + \int_0^t J(s)dW_Q(s) + \int_0^t \int_{\|x\|_H<1} xJ(s)\tilde{N}(ds,dx) \\ &\quad + \int_0^t \int_{\|x\|_H \geq 1} J(s)xN(ds,dx).\end{aligned}$$

2.5 SDEs and Solutions

The theory of stochastic differential equations in Hilbert spaces is a nautral generalisation of the classic finite dimensional stochastic differential equations (SDEs) introduced by Itô. Readers are referred to Da Prato and Zabczyk [22] for more details. Here, we only analyse a formulation how one can regard a SPDE as some SDE in some Hilbert spaces.

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n, n \in \mathbb{N}_+$, with smooth boundary $\partial\mathcal{O}$. Consider the following initial-boundary value problem for the randomly heat equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) = \displaystyle\sum_{i=1}^{n} \frac{\partial^2 y}{\partial x_i^2}(t,x) + \frac{\partial W}{\partial t}(t,x), \quad t \geq 0, \quad x \in \mathcal{O}, \\ y(0,x) = y_0(x), \quad x \in \mathcal{O}; \quad y(t,x) = 0, \quad t \geq 0, \quad x \in \partial\mathcal{O}, \end{cases} \tag{2.6}$$

where $W(t,x)$ is a standard Wiener random field.

We consider the solution for this stochastic differential equation as a stochastic process indexed by time t with values in a space of functions of spatial variable x, say $L^2(\mathcal{O};\mathbb{R})$. Here, we can use some knowledge from functional analysis to develop a stochastic process theory on a Hilbert space.

Let $H = L^2(\mathcal{O};\mathbb{R})$. Assume that the initial condition $y_0 \in H = L^2(\mathcal{O};\mathbb{R})$ and let $W(t), t \geq 0$, be a Q-Wiener process on H, then we may reformulate (2.6) into the form:

$$\begin{cases} dy(t) = Ay(t)dt + dW(t), \quad t \geq 0, \\ y(0) = y_0 \in L^2(\mathcal{O};\mathbb{R}), \end{cases}$$

where A is $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ in (2.6).

Now we consider the following non-linear stochastic system on a Hilbert space

H,

$$\begin{cases} dy(t) = (Ay(t) + F(t, y(t))dt + B(t, y(t))dW(t), \quad t \geq 0, \\ y(0) = y_0 \in H, \end{cases} \tag{2.7}$$

where A is the infinitesimal generator of a C_0-semigroup $e^{tA}, t \geq 0$, of bounded linear operators on the Hilbert space H. The coefficients $F(\cdot, \cdot)$ and $B(\cdot, \cdot)$ are two nonlinear measurable mappings from $[0, T] \times H$ into H and $\mathscr{L}_2(K_Q, H)$, respectively.

Definition 2.5.1 *Let $T \geq 0$ and an $\{\mathcal{F}_t\}_{t \geq 0}$-adapted stochastic process $y(t), t \in [0, T]$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is called a mild solution of (2.7) if it satisfies*

$$\mathbb{P}\Big\{ \int_0^T \|y(t)\|_H^2 dt < \infty \Big\} = 1, \tag{2.8}$$

$$\mathbb{P}\Big\{ \int_0^T \Big(\|F(t, y(t))\|_H + \|B(t, y(t))\|^2_{\mathscr{L}_2(K_Q, H)} \Big) dt < \infty \Big\} = 1, \tag{2.9}$$

and

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} F(s, y(s)) ds + \int_0^t e^{(t-s)A} B(s, y(s)) dW(s), \quad t \in [0, T],$$

for any $y_0 \in H$ almost surely.

By the Banach fixed-point theorem, we can establish an existence and uniqueness theorem of mild solutions for (2.7). Precisely, we suppose that for any $y, z \in H$

and $t \in [0,T]$,

$$\begin{aligned} &\|F(t,y) - F(t,z)\|_H + \|B(t,y) - B(t,z)\|_{\mathscr{L}_2(K_Q,H)} \leq \alpha(T)\|y - z\|_H, \\ &\|F(t,y) - B(t,y)\|^2_{\mathscr{L}_2(K_Q,H)} \leq \beta(T)(1 + \|y\|^2_H), \end{aligned} \tag{2.10}$$

where $\alpha(T) > 0$ and $\beta(T) > 0$.

Theorem 2.5.1 *Let $T \geq 0$ and assume that condition (2.10) holds. Then there exists a unique mild solution $y \in C([0,T]; L^p(\Omega; H))$ to (2.7). Moreover if $\mathbb{E}\|y_0\|^p_H < \infty, p \geq 2$, then the solution y satisfies*

$$\mathbb{E}\Big(\sup_{0 \leq t \leq T} \|y(t, y_0)\|^p_H \Big) < \infty, \quad p \geq 2.$$

Chapter 3

Stochastic Optimal Control Problem with Neutral Type and Control Delays

3.1 Introduction

In the classical case, many random phenomena are described by stochastic differential equations, such as the evolution of stock prices. However, there also exist many phenomena which are characteristics of past dependence, that is, their present value depends not only on the present situation but also on the past history. Such models may be identified as stochastic differential delay equations arising in a wide range of applications in physics, biology, engineering, economics and finance. For instance, let us mention the influence of the ocean in a coupled atmospheric ocean model of the climate, see, e.g., [59], or population growth where the non null finite information transmission times may lead to delay.

Recently, the optimal control problem of deterministic infinite dimensional systems has attracted a lot of attentions (see, e.g., [8], [21], [35], [48], and references cited therein). For stochastic systems without memory, the same or similar problems have been considered by many researchers, e.g., [27], [33], in which it is clearly enough to consider only the state control i.e., $b_1(\cdot) \equiv 0$ in (3.1) of the equations under investigation. Apart from this, we also need to deal with time delays in the control: this is interesting from the practical point of view and new mathematical difficulties arise in the problem. In [29] and [30], a class of stochastic optimal control problems were considered. The state equation is a stochastic delay differential equation. One typical example of this problem introduces the optimal control of delay equations arising in advertising models.

On the other hand, it is known that the neutral type effects in which the class of stochastic equations involve derivatives with delays as well as the function itself exist widely. Many interesting results about neutral type to stochastic delay differential differential equations have been obtained by many authors, see, for example, Liu [38] has considered standard optimal control problems for a class of neutral functional differential equations in Banach spaces and it turns out that based on a systematic theory of neutral models, the fundamental solution is constructed and a variation of constants formula of mild solutions is established. Balasubramaniam and Ntouyas [4] have given sufficient conditions for the controllability of a class of stochastic partial functional differential inclusions with infinite delay in an abstract space with the help of the Leray-Schnauder nonlinear alternative. The problem is formulated by lifting this non-Markovian optimization problem to an infinite-dimensional Markovian control problem without involving delays in a suitable product Hilbert space and the solutions are derived in an explicit example.

In this chapter, we will generalise the previous theory to consider a stochastic optimal control problem for a class of neutral type stochastic systems, which is very important from both theoretic and practical point of view (see, e.g., [39]). We formulate a stochastic optimal control problem aiming at maximising the objective functional at a given time horizon $T > 0$.

This chapter is organised as follows. In Section 3.2, we formulate the optimal problem with the objective functional as an optimal problem with neutral type for an SDDE both in state and the control. In Section 3.3, we use a representation result that allows us to "lift" this non-Markovian optimisation problem to an Hilbert space-valued Markovian control problem and deal with the general case of delays in the state and in the control and the verification result is given. In Section 3.4, we construct an example of a controlled SDDE in the state and in the control, whose HJB equation admits an integral solution. Therefore, there exists an optimal control form for the control problem. In Section 3.5, we calculate solutions by a linear delay differential equation. In Section 3.6, we have a summary to state the contribution and development of the chapter.

3.2 Model

Let $r > 0$ and $L^2([-r,0];\mathbb{R})$ be the space of all $\mathbb{R}$-valued equivalent classes of measurable functions $\gamma(\cdot) : [-r,0] \to \mathbb{R}$ such that $\int_{-r}^0 |\gamma(\theta)|_{\mathbb{R}}^2 d\theta < \infty$. We also denote by $W^{1,2}([-r,0];\mathbb{R})$ the Sobolev space of all $\mathbb{R}$-valued functions $x(\cdot)$ on $[-r,0]$ such that $x(\cdot)$ and its derivatives belong to $L^2([-r,0];\mathbb{R})$. We consider the following stochastic differential equations with neutral type and control delay on $\mathbb{R}$.

$$\begin{cases} d\Big[y(t) - \displaystyle\int_{-r}^0 a(\theta)y(t+\theta)d\theta\Big] = \Big[a_0 y(t) + \displaystyle\int_{-r}^0 a_1(\theta)y(t+\theta)d\theta + b_0 u(t) \\ \qquad\qquad\qquad\qquad + \displaystyle\int_{-r}^0 b_1(\theta)u(t+\theta)d\theta\Big]dt + \sigma dB(t), \quad \forall t \in [0,T], \\ y(0) = x_0, y(\theta) = x_1(\theta), u(\theta) = \gamma(\theta), \quad \forall \theta \in [-r,0], \end{cases} \tag{3.1}$$

where the Brownian motion $B(t)$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with $(\mathcal{F}_t)_{t\geq 0}$ being the completion of the filtration generated by $B(t), t \geq 0$. It is assumed that $u(t)$ is an admissible control that belongs to $\mathcal{U} := L^2([0,T];\mathbb{R}^+)$, the space of square integrable nonnegative stochastic processes adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$. In addition, we need to assume the following conditions:

i. $a_0 \in \mathbb{R}$;

ii. $a_1(\cdot) \in L^2([-r,0];\mathbb{R})$;

iii. $a(\cdot) \in C^1([-r,0];\mathbb{R})$;

iv. $b_0 \in \mathbb{R}$;

v. $b_1(\cdot) \in L^2([-r,0];\mathbb{R})$;

vi. $x_1(\cdot) \in L^2([-r,0];\mathbb{R})$;

vii. $\gamma(\cdot) \in L^2([-r,0];\mathbb{R}^+)$.

Our aim is to study the optimal control problem for (3.1). Setting $x := (x_0, x_1(\cdot)) \in X$ and denoting by $y(t,x,u(\cdot)), t \in [0,T]$, a solution of (3.1). The objective functional is given as follows:

$$J(x,u(\cdot)) = \mathbb{E}^{s,x}\big[\varphi(Y(T,x,u(\cdot))) + \int_0^T h(u(t))dt\big], \tag{3.2}$$

where φ_0 is a concave utility function, which is twice continuously differentiable and satisfies $\varphi_0''(x) < 0$ for all $x \in \mathbb{R}$, and h_0 is a convex cost function, which is twice continuously differentiable and satisfies $h_0''(x) > 0$ for all $x \in \mathbb{R}$. Moreover, h_0 is superlinear at infinity, i.e.

$$\lim_{x\to+\infty} \frac{h_0(x)}{x} = +\infty,$$

and the dynamics of y is determined by (3.1).

Let us also define the value function V for this problem as follows:

$$V(t,x) = \sup_{u \in \mathcal{U}} J(t,x;u).$$

We say that $u^* \in \mathcal{U}$ is an optimal strategy if it is satisfies

$$V(t,x) = J(t,x;u^*).$$

The problem that we will deal with is the maximisation of the objective functional J over all admissible strategies $u \in \mathcal{U}$ and the characterisation of the value function V and of the optimal strategy u^*.

3.3 Equivalent Infinite-Dimensional Markovian Representation

In this section, we shall adapt the approach of Vinter and Kwong [54] to the stochastic case to recast SDDE (3.1) as an abstract SDE on a product Hilbert space $\mathcal{H}$ to reformulate the optimal control problem.

Let $\mathcal{H}$ be a product Hilbert space defined as

$$\mathcal{H} = \mathbb{R} \times L^2([-r,0];\mathbb{R}),$$

with inner product

$$\langle x, y \rangle_{\mathcal{H}} = x_0 y_0 + \int_{-r}^{0} x_1(\theta) y_1(\theta) d\theta,$$

and norm

$$\|x\|_{\mathcal{H}} = \Big(|x_0|^2 + \int_{-r}^{0} |x_1(\theta)|^2 d\theta\Big)^{1/2},$$

for all $x = (x_0, x_1), y = (y_0, y_1) \in \mathcal{H}$, that is, x_0 and $x_1(\cdot)$ denote the $\mathbb{R}$-valued and the $L^2([-r, 0]; \mathbb{R})$-valued components, respectively.

We start by considering the deterministic delay differential equation with neutral type on $\mathbb{R}$,

$$\begin{cases} d\Big[y(t) - \int_{-r}^{0} a(\theta)y(t+\theta)d\theta\Big] = a_0 y(t) + \int_{-r}^{0} a_1(\theta)y(t+\theta)d\theta, \quad \forall t > 0, \\ y(0) = x_0, y(\theta) = x_1(\theta), x = (x_0, x_1(\cdot)) \in \mathcal{H}. \end{cases} \tag{3.3}$$

The mild solution $y(t)$ of (3.3) requires us to introduce a C_0-semigroup on product Hilbert space $\mathcal{H}$. Now, we define a mapping $S(t), t \geq 0$, associated with $y(t)$ by

$$S(t)(x_0, x_1(\cdot)) = (y(t), y(t + \cdot)), \quad \forall t \geq 0.$$

Moreover, $Y(t) = (y(t), y_t(\cdot))$, where $y_t(\cdot) = y(t + \cdot), t \geq 0$, is the $\mathcal{H}$-valued mild solution of an abstract equation without delays

$$\begin{cases} dY(t) = AY(t)dt \\ Y(0) = x \in \mathcal{H}. \end{cases}$$

Here the operator A is the infinitesimal generator of the strongly continuous C_0-semigroup $S(t), t \geq 0$ on the Hilbert space $\mathcal{H}$ (see the proof in Liu [37]) as follows:

$$A : (x_0, x_1(\cdot)) \to \Big(a_0 x_0 + \int_{-r}^{0} a(\theta)x_1'(\theta)d\theta + \int_{-r}^{0} a_1(\theta)x_1(\theta)d\theta, x_1'(\theta)\Big) \tag{3.4}$$

and the domain of A is given by

$$\mathcal{D}(A) = \{(x_0, x_1(\cdot)) \in \mathbb{R} \times W^{1,2}([-r, 0]; \mathbb{R}) : x_0 = x_1(0)\}$$

for any $x = (x_0, x_1) \in \mathcal{D}(A)$.

Moreover, we recall the adjoint operator A^* which is proved in Liu [39], in which, the operator $A^* : \mathcal{D}(A^*) \subset \mathcal{H} \to \mathcal{H}$ generates a C_0-semigroup e^{tA^*}.

Theorem 3.3.1 *The operator A^* of the C_0-semigroup $e^{tA^*} : \mathcal{D}(A^*) \subset \mathcal{H} \to \mathcal{H}$ is given by: for almost all $\theta \in [-r, 0]$,*

$$A^* : (x_0, x_1(\theta)) \to \Big(a_0 x_0 + x_1(0) + a(0)x_0, a_1(\theta)x_0 - \frac{d}{d\theta}\Big[x_1(\theta) - a(\theta)x_0\Big]\Big),$$

and the domain of A^* is given by

$$\mathcal{D}(A^*) = \{(x_0, x_1(\cdot)) \in \mathcal{H}, x_1(\cdot)+a(\cdot)x_0 \in W^{1,2}([-r, 0]; \mathbb{R}), x_1(-r)+a(-r)x_0 = 0\}.$$

Moreover, we need to define the bounded linear control operator $B^* : U \to \mathcal{H}$ as

$$B^* : u \to (b_0 u, b_1(\cdot)u),$$

where $U := \mathbb{R}^+$ and the elements $(b_0, b_1(\cdot)) \in \mathcal{H}$.

Finally $G^* : \mathbb{R} \to \mathcal{H}$ is defined as

$$G^* : x_0 \to (\sigma x_0, 0), \quad \forall x_0 \in \mathbb{R}.$$

We adopt a method that allows us to "lift" this non-Markovian optimisation problem to an infinite-dimensional Markovian control problem. Let us consider the following abstract SDE on the Hilbert space that is equivalent to the SDE

(3.1):

$$\begin{cases} dY(t) = (A^*Y(t) + B^*u(t))dt + G^*dB(t) \\ Y(0) = x = (x_0, x_1) \in \mathcal{H}, \end{cases} \tag{3.5}$$

where A^* is given as in Theorem 3.2.1.

In Da Prato and Zabczyk [22], it is known that the equation (3.5) has exactly one mild solution, which is given by the variation of constants formula

$$Y(t) = e^{tA^*}x + \int_0^t e^{(t-s)A^*}B^*u(s)ds + \int_0^t e^{(t-s)A^*}G^*dB(s).$$

We now relate the solution of the delayed differential equation (3.1) to the mild solution of the abstract evolution equation (3.5) when the initial condition on the abstract evolution equation is appropriately chosen.

Proposition 3.3.1 *[39] Let $Y(t) = (Y_0(t), Y_1(t+\theta)) \in \mathcal{H}, \theta \in [-r, 0]$, be the mild solution of the abstract evolution equation (3.5) with arbitrary initial data $Y(0) = x \in \mathcal{H}$ and control $u \in \mathcal{U}$. Then, for $t \geq 0$, one has the relation*

$$Y_1(t)(\theta) = \int_{-r}^{\theta} [a_1(s) + a'(s)]Y_0(t+s-\theta)ds + \int_{-r}^{\theta} b_1(s)u(t+s-\theta)ds, \theta \in [-r, 0].$$

Moreover, consider the equation (3.5) with initial

$$\bar{x} = (\bar{x}_0, \bar{x}_1) := \Big(x_0, \int_{-r}^{\theta} [a_1(s)+a'(s)]x_1(s-\theta)ds + \int_{-r}^{\theta} b_1(s)u(s-\theta)ds, \theta \in [-r, 0]\Big),$$

then there is the equality

$$Y_0(t, \bar{x}) = y(t, x), \qquad t \in [0, T], \tag{3.6}$$

where $y(t, x), t \geq -r$, is the unique mild solution of the equation (3.1) with initial

$x = (x_0, x_1) \in \mathcal{H}$.

Using this equivalence result, we can now give a Markovian reformulation on the product Hilbert space $\mathcal{H}$ of the problem of maximising (3.2). Since we want to use the dynamic programming approach, from now on we let the initial time vary, denote it by s with $0 \le s \le T$.

The state space is $\mathcal{H} = \mathbb{R} \times L^2([-r, 0]; \mathbb{R})$, the control space is $U := \mathbb{R}^+$ and the control strategy is $u(\cdot) \in \mathcal{U}$. The state equation is (3.5) with initial condition at s as follows

$$\begin{cases} dY(t) = (A^*Y(t) + B^*u(t))dt + G^*dB(t) \\ Y(s) = x \in \mathcal{H}, \end{cases} \tag{3.7}$$

and its unique mild solution with initial data (s, x) and the control strategy $u(\cdot)$, will be denoted by $Y(\cdot; s, x, u(\cdot))$, so (3.2) is equivalent to

$$J(s, x, u(\cdot)) = \mathbb{E}^{s,x}\big[\varphi(Y(T, s, x, u(\cdot))) + \int_s^T h(u(t))dt\big],$$

where the function $h : U \to \mathbb{R}$ and $\varphi : \mathcal{H} \to \mathbb{R}$ are defined as

$$h(u) = -h_0(u),$$
$$\varphi(x_0, x_1(\cdot)) = \varphi_0(x_0).$$

Our aim is to maximise the objective function $J(s, x; u(\cdot))$ over all $u(\cdot) \in \mathcal{U}$. We also define the value function V for this problem as

$$V(s, x) = \sup_{u(\cdot) \in \mathcal{U}} J(s, x; u(\cdot)).$$

Moreover, we shall say that $u^* \in \mathcal{U}$ is an optimal strategy if it is such that

$$V(s,x) = J(s,x;u^*(\cdot)).$$

According to the dynamical programming approach, we need first to characterise the value function V as the unique solution of the following HJB equation

$$\begin{cases} v_t + \frac{1}{2}Tr(Q^* v_{xx}) + \langle A^* x, v_x\rangle + H_0(v_x) = 0, \\ v(T,x) = \varphi(x), \quad x \in \mathcal{H}, \quad T \geq 0, \end{cases} \tag{3.8}$$

where $Q^* = G^* G$, and

$$H_0(p) = \sup_{u \in U}(\langle B^* u, p\rangle_{\mathcal{H}} + h(u)), \quad p \in \mathcal{H}.$$

In general, it is hard to solve the equation (3.8) with x defined in a Hilbert space and obtain regular solutions of the HJB equation (3.8) by using the existing theory. But in this case, we only consider the situation that the regular solutions of the HJB equation exist. Here we define two solutions of a HJB equation.

Definition 3.3.1 *A function v is said to be*

i. A classical solution of the HJB equation (3.8) if $v \in C^{1,2}([0,T] \times \mathcal{H})$ and v satisfies (3.8) pointwise;

ii. An integral solution if $v \in C^{0,2}([0,T] \times \mathcal{H})$, and moreover, for $t \in [0,T]$ and $x \in \mathcal{D}(A^*)$, we have

$$\varphi(x) - v(t,x) + \int_t^T \Big[\frac{1}{2}Tr(Q^* v_{xx}(t,x)) + \langle A^* x, v_x(s,x)\rangle + H_0(v_x(s,x))\Big] ds = 0. \tag{3.9}$$

In addition, we use the verification theorem (see the proof in [29]) to find the value function V and the optimal control u^*.

Theorem 3.3.2 (Verification Theorem) *Let v be an integral solution of the HJB (3.8) and let V be the value function of the optimal control problem. Then*

(1) $v \geq V$ on $[0,T] \times \mathcal{H}$;

(2) if a control $u^* \in \mathcal{U}$ is such that, at starting point (t,x),

$$\begin{aligned} H_0(v_x(s,Y(s))) &= \sup_{u \in \mathcal{U}} \langle B^* u, v_x(s,Y(s)) \rangle + h(u) \\ &= \langle B^* u^*(s), v_x(s,Y(s)) \rangle + h(u^*(s)), \end{aligned}$$

for almost every $s \in [t,T], \mathbb{P}$-a.s., then this control is optimal and $v(t,x) = V(t,x)$;

(3) if we know a priori that $V = v$, then (2) is a necessary (and sufficient) condition of optimality.

3.4 An explicit case

In this section, we study the optimal control problem by an specific example with a linear function φ and a quadratic function h.

In particular, we assume that

$$h(u) = -\beta u^2, \quad \text{and} \quad \varphi(x) = \varphi_0(x_0) = \gamma x_0,$$

with $\beta, \gamma > 0$.

We define the bounded linear control operator $B^* : U \to \mathcal{H}$ as

$$B^* : u \to (b_0 u, b_1(\cdot) u),$$

where $U := \mathbb{R}^+$ and the elements $(b_0, b_1(\cdot)) \in \mathcal{H}$.

Let $H_{CV}(p, u)$ be defined by

$$H_{CV}(p, u) = \langle B^* u, p \rangle + h(u) = \langle B^*, p \rangle_{\mathcal{H}} u - \beta u^2, \quad p \in \mathcal{H}.$$

Then

$$H_0(p) = \sup_{u \in U} H_{CV}(p, u) = \begin{cases} \dfrac{\langle B^*, p \rangle^2}{4\beta}, & \langle B^*, p \rangle \geq 0, \\ 0, & \langle B^*, p \rangle < 0, \end{cases} \tag{3.10}$$

or equvalently,

$$H_0(p) = \frac{(\langle B^*, p \rangle^+)^2}{4\beta}.$$

We guess a solution of the HJB equation (3.8) of the form

$$v(t, x) = \langle \mu(t), x \rangle + c(t), \qquad t \in [0, T], x \in \mathcal{H}, \tag{3.11}$$

where $\mu(\cdot) = (\mu_0(\cdot), \mu_1(\cdot)) : [0, T] \to \mathcal{H}$ and $c(\cdot) : [0, T] \to \mathbb{R}$. Hence, for $t \in [0, T]$ and $x \in \mathcal{H}$, we assume that all objects are well defined, and

$$v_t(t, x) = \langle \mu'(t), x \rangle + c'(t), \tag{3.12}$$

$$v_x(t, x) = \mu(t), \tag{3.13}$$

$$v_{xx} = 0. \tag{3.14}$$

Then, by substituting (3.12), (3.13) and (3.14) into (3.8), we obtain

$$\begin{cases} \langle \mu'(t), x\rangle + c'(t) + \langle A^* x, \mu(t)\rangle + \dfrac{(\langle B^*, \mu(t)\rangle^+)^2}{4\beta} = 0, \quad \forall t \in [0,T), x \in \mathcal{D}(A^*), \\ \langle \mu(T), x\rangle + c(T) = \gamma x_0, \quad \forall x \in \mathcal{H}. \end{cases} \tag{3.15}$$

Assume that $\mu(t) \in \mathcal{D}(A)$ for all $t \in [0,T]$, so (3.15) is equivalent to

$$\begin{cases} \langle \mu'(t), x\rangle + \langle x, A\mu(t)\rangle = 0, \quad t \in [0,T), \\ \mu(T) = (\gamma, 0), \end{cases} \tag{3.16}$$

and

$$\begin{cases} c'(t) + \dfrac{(\langle B^*, \mu(t)\rangle^+)^2}{4\beta} = 0, \quad t \in [0,T), \\ c(T) = 0. \end{cases} \tag{3.17}$$

Then it implies

$$\begin{cases} \mu'(t) + A\mu(t) = 0, \quad t \in [0,T), \\ \mu(T) = (\gamma, 0). \end{cases} \tag{3.18}$$

Recalling (3.4), we obtain that (3.18) is equivalent to

$$\begin{cases} \mu_0'(t) + a_0\mu_0(t) + \displaystyle\int_{-r}^0 a(\theta)\frac{\partial \mu_1(t,\theta)}{\partial\theta}d\theta + \int_{-r}^0 a_1(\theta)\mu_1(t,\theta)d\theta = 0, \quad t \in [0,T), \\ \mu_0(T) = \gamma, \end{cases} \tag{3.19}$$

and

$$\begin{cases} \dfrac{\partial \mu_1(t,\theta)}{\partial t} + \dfrac{\partial \mu_1(t,\theta)}{\partial \theta} = 0, & t \in [0,T), \theta \in [-r,0), \\ \mu_1(T,\theta) = 0, & \theta \in [-r,0), \\ \mu_1(t,0) = \mu_0(t), & t \in [0,T]. \end{cases} \tag{3.20}$$

The solution of (3.20) is given by

$$\mu_1(t,\theta) = \mu_0(t-\theta) I_{[0,T]}(t-\theta), \tag{3.21}$$

from which we can solve the equation (3.19) to obtain $\mu_0(\cdot)$. Hence,

$$v(t,x) = \langle \mu(t), x \rangle + c(t)$$

is an integral solution of of HJB equation (3.8). Since $v \in C^{0,2}([0,T] \times \mathcal{H})$, which is twice differentiable in x and it satisfies the hypotheses of Verification Theorem. Moreover, the optimal strategy is

$$u^*(t) = \frac{\langle B^*, v_x(t) \rangle^+}{2\beta} = \frac{\langle B^*, \mu(t) \rangle^+}{2\beta}, \qquad t \in [0,T].$$

Hence, by the Theorem 3.3.2, $u^*(\cdot)$ is optimal.

3.5 Example with solutions

Now we extend the analysis of this specific situation to a rather explicit solution of the optimal control problem, which could be solved numerically by solving a linear ODE with delay. In particular, let $\mu = (\mu_0, \mu_1)$ be the solution of (3.11).

Let us consider the system (3.1) with $a(\cdot)$ $=a$ and $a_1(\cdot) = 0$, precisely, the following controlled stochastic differential equation with neutral type

$$\begin{cases} d\Big[y(t) - \int_{-r}^{0} ay(t+\theta)d\theta\Big] = \Big[a_0 y(t) + \int_{-r}^{0} b_1(\theta)u(t+\theta)d\theta\Big]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad +\sigma dB(t), \quad \forall t \in [0,T], \\ y(0) = x_0, y(\theta) = x_1(\theta) \in L^2([-r,0];\mathbb{R}), u(\theta) = \gamma(\theta), \qquad \theta \in [-r,0], \end{cases} \tag{3.22}$$

Now the equation (3.19) has become

$$\begin{cases} \mu_0'(t) + a_0\mu_0(t) + a\int_{-r}^{0} \frac{\partial \mu_1(t,\theta)}{\partial\theta} d\theta = 0, \quad t \in [0,T) \\ \mu_0(T) = \gamma, \end{cases} \tag{3.23}$$

where a, a_0 are constants.

Step 1: For $t \in [T-r,T]$, we need to consider the solutions in two cases.

(i) If $t \in [T-r,T]$ and $t-\theta \notin [0,T]$, $\mu_1(t,\theta) = 0$, and (3.23) is equivalent to

$$\begin{cases} \mu_0'(t) + a_0\mu_0(t) = 0, \quad t \in [T-r,T], \\ \mu_0(T) = \gamma. \end{cases} \tag{3.24}$$

Multiplying $e^{a_0 t}$ on both sides of the first equation in (3.24), we obtain

$$e^{a_0 t}\mu_0'(t) + a_0 e^{a_0 t}\mu_0(t) = 0. \tag{3.25}$$

Integrating the equation (3.25) on the interval $[t,T], t \in [T-r,T]$, we get

$$\int_t^T (\mu_0(u)e^{a_0 u})' du = 0, \quad t \in [T-r,T],$$

and further

$$\mu_0(T)e^{a_0 T} - \mu_0(t)e^{a_0 t} = 0.$$

Hence, on the interval $T - r \le t \le T$, the function $\mu_0(t)$ is given uniquely by

$$\mu_0(t) = \gamma e^{a_0(T-t)}, \quad t \in [T-r, T]. \tag{3.26}$$

(ii) If $t \in [T-r, T]$ and $t - \theta \in [0, T]$, $\mu_1(t,\theta) = \mu_0(t-\theta)$, then

$$\begin{aligned} a\int_{-r}^{0} \frac{\partial \mu_1(t,\theta)}{\partial \theta} d\theta &= a\int_{-r}^{0} I_{t-\theta\in[0,T]} \frac{\partial \mu_1(t,\theta)}{\partial \theta} d\theta \\ &= a\int_{t-T}^{0} (\mu_0(t-\theta)'_\theta d\theta \\ &= a(\mu_0(t) - \mu_0(t-t+T)) \\ &= a(\mu_0(t) - \mu_0(T)) \\ &= a(\mu_0(t) - \gamma). \end{aligned}$$

Then (3.23) reduces to

$$\begin{cases} \mu_0'(t) + a_0\mu_0(t) + a(\mu_0(t) - \gamma) = 0, & t \in [T-r, T), \\ \mu_0(T) = \gamma. \end{cases}$$

This is an ordinary differential equation with respect to t, which has the form

$$\begin{cases} \mu_0'(t) + (a + a_0)\mu_0(t) = a\gamma, & t \in [T-r, T) \\ \mu_0(T) = \gamma. \end{cases} \tag{3.27}$$

where a, a_0 are constants. Multiplying $e^{(a_0+a)t}$ on both sides of the first equation of (3.27), we obtain

$$e^{(a_0+a)t}\mu_0'(t) + (a_0+a)e^{(a_0+a)t}\mu_0(t) = ae^{(a_0+a)t}\gamma.$$

Integrating on the interval $[t, T]$,

$$\int_t^T (\mu_0(u)e^{(a_0+a)u})'du = a\int_t^T e^{(a_0+a)u}\gamma du,$$

then

$$\mu_0(T)e^{(a_0+a)T} - \mu_0(t)e^{(a_0+a)t} = a\int_t^T e^{(a_0+a)u}\gamma du.$$

Rearranging this equation, we obtain

$$\begin{aligned}
\mu_0(t) &= \gamma e^{(a_0+a)(T-t)} - a\int_t^T e^{(a_0+a)(u-t)}\gamma du \\
&= \gamma e^{(a_0+a)(T-t)} - \frac{a\gamma}{a_0+a}\int_t^T e^{(a_0+a)(u-t)}d(a_0+a)u \\
&= \gamma e^{(a_0+a)(T-t)} - \frac{a\gamma}{a_0+a}(e^{(a_0+a)(T-t)} - e^{(a_0+a)(t-t)}) \\
&= \gamma e^{(a_0+a)(T-t)} - \frac{a\gamma}{a_0+a}(e^{(a_0+a)(T-t)} - 1) \\
&= \frac{a_0\gamma}{a_0+a}e^{(a_0+a)(T-t)} + \frac{a\gamma}{a_0+a}, \quad t \in [T-r, T].
\end{aligned}$$

On the interval $t \in [T-r, T]$, the function $\mu_0(t)$ is given uniquely by

$$\mu_0(t) = \frac{a_0\gamma}{a_0+a}e^{(a_0+a)(T-t)} + \frac{a\gamma}{a_0+a}. \tag{3.28}$$

Hence, on the interval $t \in [T-r, T]$, the function $\mu_0(t)$ is given uniquely by

$$(\mu_0(t), \mu_1(t)) = (\gamma e^{a_0(T-t)}, 0) \quad \text{for } t \in [T-r, T] \text{ and } t-\theta \notin [0, T],$$

and

$$(\mu_0(t),\mu_1(t)) = \Big(\frac{a_0\gamma}{a_0+a}e^{(a_0+a)(T-t)} + \frac{a\gamma}{a_0+a}, \frac{a_0\gamma}{a_0+a}e^{(a_0+a)(T-(t-\theta))} + \frac{a\gamma}{a_0+a}\Big),$$

for any $t \in [T-r,T]$ and $t-\theta \in [0,T]$.

Let $\kappa(t) := \mu_0(t), \quad t \in [T-r,T]$. Once $(\mu_0(t),\mu_1(t)(\cdot))$ is known on interval $[T-r,T]$, the function $\kappa(t)$ on interval $[T-r,T]$ can be used to obtain $\mu_0(t)$ on $[T-2r,T-r]$.

Step 2: For $t \in [T-2r,T-r]$, it is clearly that $t-\theta \in [0,T]$. Then $\mu_1(t,\theta) = \mu_0(t-\theta)$, and

$$\begin{aligned}
a\int_{-r}^{0}\frac{\partial\mu_1(t,\theta)}{\partial\theta}d\theta &= a\int_{-r}^{0} I_{t-\theta\in[0,T]}\frac{\partial\mu_1(t,\theta)}{\partial\theta}d\theta \\
&= a\int_{-r}^{0}(\mu_0(t-\theta)'_\theta d\theta \\
&= a(\mu_0(t)-\mu_0(t+r)).
\end{aligned}$$

Then (3.23) reduces to

$$\begin{cases}
\mu_0'(t) + a_0\mu_0(t) + a(\mu_0(t)-\mu_0(t+r)) = 0, \quad t \in [T-2r,T-r], \\
\mu_0(T) = \gamma.
\end{cases}$$

This is a linear delay differential equation which has the form

$$\begin{cases}
\mu_0'(t) + (a+a_0)\mu_0(t) = a\mu_0(t+r), \quad t \in [T-2r,T-r], \\
\mu_0(T) = \gamma,
\end{cases} \tag{3.29}$$

where a, a_0 and r are constants with $r > 0$.

Multiplying $e^{(a_0+a)t}$ on both sides of the first equation of (3.29), we obtain

$$e^{(a_0+a)t}\mu_0'(t) + (a_0+a)e^{(a_0+a)t}\mu_0(t) = ae^{(a_0+a)t}\mu_0(t+r).$$

Integrating on the interval $[t, T-r], t \in [T-2r, T-r]$,

$$\int_t^{T-r} (\mu_0(u)e^{(a_0+a)u})' du = a\int_t^{T-r} e^{(a_0+a)u}\mu_0(u+r)du,$$

then

$$\mu_0(T-r)e^{(a_0+a)(T-r)} - \mu_0(t)e^{(a_0+a)t} = a\int_t^{T-r} e^{(a_0+a)u}\mu_0(u+r)du.$$

Rearranging this equation, we have

$$\mu_0(t) = \kappa(T-r)e^{(a_0+a)(T-t-r)} - a\int_t^{T-r} e^{(a_0+a)(s-t)}\kappa(s+r)ds. \tag{3.30}$$

Since $T-r \in [T-r, T]$ and $s+r \in [T-r, T]$ for $s \in [t, T-r]$, we can derive the values of $\kappa(T-r)$ and $\kappa(s+r)$ from equation (3.28) in Step 1.

Step 3: For $t \in [T-3r, T-2r]$, multiplying $e^{(a_0+a)t}$ on both sides of the first equation of (3.29), we obtain

$$e^{(a_0+a)t}\mu_0'(t) + (a_0+a)e^{(a_0+a)t}\mu_0(t) = ae^{(a_0+a)t}\mu_0(t+r).$$

Integrating on the interval $[t, T-2r], t \in [T-3r, T-2r]$,

$$\int_t^{T-2r} (\mu_0(s)e^{(a_0+a)s})' ds = a\int_t^{T-2r} e^{(a_0+a)s}\mu_0(s+r)ds,$$

then

$$\mu_0(T-2r)e^{(a_0+a)(T-2r)} - \mu_0(t)e^{(a_0+a)t} = a\int_t^{T-2r} e^{(a_0+a)s}\mu_0(s+r)ds,$$

and

$$\mu_0(t) = \mu_0(T-2r)e^{(a_0+a)(T-t-2r)} - a\int_t^{T-2r} e^{(a_0+a)(s-t)}\mu_0(s+r)ds. \qquad (3.31)$$

Here $\mu_0(T-2r)$ and $\mu_0(s+r), s \in [t, T-2r], t \in [T-3r, T-2r]$ are given in Step 2.

Hence, on the interval $T-3r \leq \ t \leq T-2r$, the function $\mu_0(t)$ is given uniquely by

$$\mu_0(t) \ = \ \mu_0(T-2r)e^{(a_0+a)(T-t-2r)} - a\int_t^{T-2r} e^{(a_0+a)(s-t)}\mu_0(s+r)ds.$$

According the methods of steps, we can derive the unique solution $\mu_0(t)$ on the interval $T-4r \leq \ t \leq T-3r$, which is

$$\mu_0(t) \ = \ \mu_0(T-3r)e^{(a_0+a)(T-t-3r)} - a\int_t^{T-3r} e^{(a_0+a)(s-t)}\mu_0(s+r)ds.$$

We can conclude that, in general, the explicit solution $\mu_0(t)$ on the interval $T-nr \leq \ t \leq T-(n-1)r$, may be written in this form

$$\mu_0(t) \ = \ \mu_0(T-(n-1)r)e^{(a_0+a)(T-t-(n-1)r)} - a\int_t^{T-(n-1)r} e^{(a_0+a)(s-t)}\mu_0(s+r)ds.$$

3.6 Summary

In this chapter, we made the first attempt to study solutions of stochastic delay differential equations with neutral type. Our work extended the work of Gozzi and Marinelli (2006) where the optimal control solutions cannot be solved explicitly. In addition, we also discussed the system with neutral type which has not yet been discussed in the context of stochastic delay differential equations in terms of optimal control problem. Finally, we obtained solutions in an explicit example.

Chapter 4

Almost Periodic Solutions for Neutral Stochastic Evolution Equations with Poisson Jumps and Infinite Delay

4.1 Introduction

Stochastic evolution differential equations have attracted much attention because of their applications in many areas such as physics, population dynamics, electrical engineering, medicine biology, ecology and other areas of science and engineering. Qualitative properties such as existence, uniqueness, controllability and stability for various stochastic differential systems have been investigated by many authors and have already achieved fruitful results (see for example [45], [46], [41], [14], [53]). In particular, solutions with recurrence property (e.g. almost periodicity and almost automorphy), which enable us to understand the impact of the noise or stochastic perturbation on the corresponding recurrent motions, are of great concern in the study of stochastic differential equations and random dynamical systems. The existence of almost periodic solutions for deterministic differential equations has been considerably investigated in a lots of publications. To be specific, Abbas and Bahuguna [2] studied the almost periodic solutions of neutral functional differential equations in Banach spaces. Diagana, Mahop, N'Guerekata, Toni [24] discussed the existence and uniqueness of pseudo almost periodic solutions to some classes of semilinear differential equations and applications.

Recently, Bezandry and Diagana introduced the concept of square-mean almost periodic stochastic process and applied it to the study of stochastic differential equations (see [9]). In [10], Bezandry and Diagana proved the existence of almost periodic solutions to some stochastic differential equations. Bezandry and Diagana [11] studied the existence of square-mean almost periodic solutions to some stochastic hyperbolic differential equations with infinite delay. Bezandry and Diagana [12] were concerned with the square-mean almost periodic solutions nonautonomous stochastic differential equations. However, many dynamical systems not only depend on the present states, but also on past states and involve derivative with delays. Therefore, it is necessary to talk about the stochastic evolution system with infinite delays and neutral type as well, see ([40], [44], [**?**], [19]).

In addition, Poisson processes are essentially stochastic processes with stationary and independent increments. Stochastic differential equations with Poisson jumps have become popular in modelling the phenomena arising in such field, as economics, where jump processes are widely used to describe the asset and commodity price dynamics (see [18]). However, as for stochastic partial differential equations with Poisson jumps and infinite delay, as for as we know, there exist only a few results about the existence and stability of mild solution. Readers are referred to ([51], [20], [52]).

Motivated by the above works by using the method of semigroups and Banach fixed point theorems. The main purpose of this chapter is to study the existence and uniqueness of square-mean almost periodic solutions to a class of neutral stochastic differential equations with Poisson jumps and infinite delay

$$d(x(t) - G(x(t), x_t)) = (Ax(t) + f(t, x(t), x_t))dt + g(t, x(t), x_t)dW(t) \\ + \int_H h(t, x(t-), x_{t-}, z)\tilde{N}(dt, dz), \quad t \in \mathbb{R}.$$

We assume some conditions to make sure the existence and uniqueness of square-mean almost periodic solutions.

This chapter is organised as follows. In Section 4.2, we review and introduce some concepts, basic properties of square-mean almost periodicity and the proofs of two theorems. In Section 4.3, under some suitable conditions, we prove the existence and uniqueness of square-mean almost periodic mild solutions for some stochastic differential equations driven by Poisson jumps. In Section 4.4, we have a summary to state the contribution and development of the chapter.

4.2 Almost Periodicity

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space. An axiomatic definition of the phase space $\mathscr{B}$ is introduced by Hale and Kato, see [32].

Definition 4.2.1 *The axioms of the phase space $\mathscr{B}((-\infty, 0], H)$ (simply denoted by $\mathscr{B}$) are defined for continuous functions mapping from $(-\infty, 0]$ into H endowed with a norm $\|\cdot\|_{\mathscr{B}}$, and $\mathscr{B}$, satisfying the following conditions:*

(1) For any $T > 0$, if $x : (-\infty, T] \to H$, is continuous on $[0, T]$ and $x_0 \in \mathscr{B}$,

then, for every $t \in [0,T]$, the following properties hold:

(a) $x_t(\cdot) := x(t+\cdot) \in \mathscr{B}$*;*

(b) $\|x(t)\|_H \le K\|x_t\|_{\mathscr{B}}, \quad \forall t > 0$, *where $K > 0$ is a constant;*

(c) $\|x_t\|_{\mathscr{B}} \le M_0 \sup_{0\le s\le t} \|x(s)\|_H$, *where $M_0 > 0$ is a constant.*

(2) The space $\mathscr{B}$ is complete.

For a Hilbert space $(H, \|\cdot\|)$, we denote by $L^2(\Omega, H)$ the Hilbert space of all H-valued random variable ξ such that

$$\mathbb{E}\|\xi(\omega)\|^2 = \int_\Omega \|\xi(\omega)\|^2 \mathbb{P}(d\omega) < \infty.$$

For $\xi \in L^2(\Omega, H)$, let

$$\|\xi(\omega)\|_2 = \Big(\int_\Omega \|\xi(\omega)\|^2 \mathbb{P}(d\omega)\Big)^{1/2}.$$

Definition 4.2.2 *A stochastic process $x : \mathbb{R}\times\Omega \to H$ is said to be L^2-continuous if for any $s \in \mathbb{R}$,*

$$\lim_{t\to s} \mathbb{E}\|x(t) - x(s)\|_H^2 = 0.$$

Definition 4.2.3 *An L^2-continuous stochastic process $x : \mathbb{R}\times\Omega \to H$ satisfying $\mathbb{E}\|x(t)\|_H^2 < \infty$ for any $t \in \mathbb{R}$ is said to be square-mean almost periodic if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ and an L^2-continuous stochastic process $\tilde{x} : \mathbb{R}\times\Omega \to H$such that*

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\|x(t+s_n) - \tilde{x}(t)\|_H^2 = 0.$$

The collection of all square-mean almost periodic stochastic processes $x : \mathbb{R} \times \Omega \to H$ will be denoted by $AP(\mathbb{R} \times \Omega; H)$.

Definition 4.2.4 *A function $f : \mathbb{R} \times H \times \mathscr{B} \to H$, is said to be square-mean almost periodic in $t \in \mathbb{R}$, uniformly for $(x, y) \in \mathbb{K}$, where $\mathbb{K} \subset H \times \mathscr{B}$ is compact if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ and a function $\tilde{f} : \mathbb{R} \times H \times \mathscr{B} \to H$such that*

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|f(t + s_n, x, y) - \tilde{f}(t, x, y)\|_H^2 = 0.$$

The collection of all square mean almost periodic functions $f : \mathbb{R}\times H\times\mathscr{B} \to H$ will be denoted by $AP(\mathbb{R} \times H \times \mathscr{B}; H)$.

Definition 4.2.5 *A function $h : \mathbb{R} \times H \times \mathscr{B} \times H \to H$, is said to be Poisson square-mean almost periodic in $t \in \mathbb{R}$, uniformly for $(x, y) \in \mathbb{K}$, where $\mathbb{K} \subset H\times\mathscr{B}$ is compact if h satisfies:*

$$\int_H \|h(t, x, y, z) - h(t', x, y, z)\|_H^2 \nu(dz) \to 0 \qquad as \quad t' \to t, t \in \mathbb{R}, \tag{4.1}$$

and for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ and a function $\tilde{h} : \mathbb{R} \times H \times \mathscr{B} \times H \to H, (t, x, y, z) \mapsto h(t, x, y, z)$ satisfying (4.1) and satisfies

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_H \|h(t + s_n, x, y, z) - \tilde{h}(t, x, y, z)\|_H^2 \nu(dz) = 0.$$

The collection of all Poisson almost periodic functions $h : \mathbb{R}\times H\times\mathscr{B}\times H \to H$ will be denoted by $PAP(\mathbb{R} \times H \times \mathscr{B} \times H; H)$.

The proof for Proposition 4.2.1 is similar to [57].

Proposition 4.2.1 *If $h, h_1, h_2 : \mathbb{R}\times H\times\mathscr{B}\times H \to H$ are Poisson almost periodic* *functions in* $t \in \mathbb{R}$*, uniformly for* $(x, y) \in \mathbb{K}$*, then*

1. $h_1 + h_2$ *is Poisson almost periodic.*

2. λh *is Poisson almost periodic for every scalar* λ.

3. *For any compact subset* $\mathbb{K} \subset H \times \mathscr{B}$*, there exists a constant* $M > 0$ *such that*

$$\sup_{t\in\mathbb{R}} \int_H \|h(t, x, y, z)\|_H^2 \nu(dz) \leq M.$$

By the proposition above, the following proposition can be obtained .

Proposition 4.2.2 *If* $f, f_1, f_2 : \mathbb{R} \times H \times \mathscr{B} \to H$ *are all square-mean almost periodic functions in* $t \in \mathbb{R}$*, uniformly for* $(x, y) \in \mathbb{K}$*,*

1. $f_1 + f_2$ *is square-mean almost periodic.*

2. λf *is square-mean almost periodic for every scalar* λ.

3. *For any compact subset* $\mathbb{K} \subset H \times \mathscr{B}$*, there exists a constant* $N > 0$ *such that*

$$\sup_{t\in\mathbb{R}} \|f(t, x, y)\|_H^2 \leq N.$$

Proposition 4.2.3 $AP\left(\mathbb{R} \times \Omega; H\right)$ *is a Banach space which is equipped with the norm*

$$\|x\|_\infty =: \sup_{t\in\mathbb{R}} \|x(t)\|_2 = \sup_{t\in\mathbb{R}} (\mathbb{E}\|x(t)\|_H^2)^{\frac{1}{2}},$$

for $x \in AP\left(\mathbb{R} \times \Omega; H\right)$.

The proof of Proposition 4.2.3 is similar to [34] with minor modifications.

Theorem 4.2.1 *Let $\mathbb{K} \subset H \times \mathscr{B}$ be a compact set and the function $(t, x, y) \to F(t, x, y) : \mathbb{R} \times H \times \mathscr{B} \to H$ be square-mean almost periodic in $t \in \mathbb{R}$, uniformly for $(x, y) \in \mathbb{K}$. Furthermore, there exists a constant $K > 0$ such that*

$$\|F(t, x, y) - F(t, \tilde{x}, \tilde{y})\|_H^2 \leq K(\|x - \tilde{x}\|_H^2 + \|y - \tilde{y}\|_{\mathscr{B}}^2),$$

for $t \in \mathbb{R}$, uniformly for $(x, y), (\tilde{x}, \tilde{y}) \in H \times \mathscr{B}$, then for any square-mean almost periodic stochastic process $\phi : \mathbb{R} \times \Omega \to H$ with $\phi_t \in \mathscr{B}, t \in \mathbb{R}$, the stochastic process $t \to F(t, \phi(t), \phi_t)$ is square-mean almost periodic.

Proof: Let $\{s'_n\}$ be a sequence of real numbers. Assume that $D(t) = F(t, \phi(t), \phi_t)$, where $\phi_t = \{\phi(t + \theta) : -\infty < \theta \leq 0\}$ is regarded as $\mathscr{B}$-valued stochastic process. Consider the function $\tilde{D}(t) : \mathbb{R} \times H \times \mathscr{B} \to H$ given by $\tilde{D}(t) := \tilde{F}(t, \tilde{\phi}(t), \tilde{\phi}_t)$. Since the process $\phi(t)$ is square-mean almost periodic, there exists a subsequence $\{s_n\}$ of $\{s'_n\}$ and a continuous process $\tilde{\phi} : \mathbb{R} \times \Omega \to H$ such that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\|\phi(t + s_n) - \tilde{\phi}(t)\|_H^2 = 0,$$

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\|\phi_{t+s_n} - \tilde{\phi}_t\|_{\mathscr{B}}^2 = 0,$$

and

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\|F(t + s_n, x, y) - \tilde{F}(t, x, y)\|_H^2 = 0.$$

Note that

$$\begin{aligned} F(t + s_n, \phi(t + s_n), \phi_{t+s_n}) - \tilde{F}(t, \tilde{\phi}(t), \tilde{\phi}_t) &= F(t + s_n, \phi(t + s_n), \phi_{t+s_n}) \\ &\quad - F(t + s_n, \tilde{\phi}(t), \tilde{\phi}_t) + F(t + s_n, \tilde{\phi}(t), \tilde{\phi}_t) - \tilde{F}(t, \tilde{\phi}(t), \tilde{\phi}_t). \end{aligned}$$

Hence, we have

$$
\begin{aligned}
& \mathbb{E}\|D(t+s_n) - \tilde{D}(t)\|_H^2 \\
= \ & \mathbb{E}\|F(t+s_n, \phi(t+s_n), \phi_{t+s_n}) - \tilde{F}(t, \tilde{\phi}(t), \tilde{\phi}_t)\|^2 \\
\leq \ & 2\mathbb{E}\|F(t+s_n, \phi(t+s_n), \phi_{t+s_n}) - F(t+s_n, \tilde{\phi}(t), \tilde{\phi}_t)\|^2 \\
+ \ & 2\mathbb{E}\|F(t+s_n, \tilde{\phi}(t), \tilde{\phi}_t) - \tilde{F}(t, \tilde{\phi}(t), \tilde{\phi}_t)\|^2.
\end{aligned}
\tag{4.2}
$$

Letting $n \to \infty$ and using Definition 4.2.4 , we have from (4.2) that

$$
\begin{aligned}
& \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\|D(t+s_n) - \tilde{D}(t)\|_H^2 \\
\leq \ & 2K \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}(\|\phi(t+s_n) - \tilde{\phi}(t)\|_H^2 + \|\phi_{t+s_n} - \tilde{\phi}_t\|_{\mathscr{B}}^2) \\
+ \ & 2 \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\|F(t+s_n, \tilde{\phi}(t), \tilde{\phi}_t) - \tilde{F}(t, \tilde{\phi}(t), \tilde{\phi}_t)\|^2 \\
= \ & 0.
\end{aligned}
$$

Thus, the stochastic process $F(t, \phi(t), \phi_t)$ is square-mean almost periodic. $\square$

Theorem 4.2.2 *Let $\mathbb{K} \subset H \times \mathscr{B}$ be a compact set and the function $(t, x, y, z) \to h(t, x, y, z) : \mathbb{R} \times H \times \mathscr{B} \times H \to H$ be Poisson square-mean almost periodic in $t \in \mathbb{R}$, uniformly for $(x, y) \in \mathbb{K}$. Furthermore, there exists a constant $K > 0$ such that*

$$
\int_H \|h(t, x, y, z) - h(t, \tilde{x}, \tilde{y}, z)\|_H^2 \nu(dz) \leq K(\|x - \tilde{x}\|_H^2 + \|y - \tilde{y}\|_{\mathscr{B}}^2)
$$

for all $(x, y), (\tilde{x}, \tilde{y}) \in H \times \mathscr{B}$ and for each $t \in \mathbb{R}$. Then for any almost periodic stochastic process $\phi : \mathbb{R} \times \Omega \to H$ with $\phi_t \in \mathscr{B}, t \in \mathbb{R}$, the stochastic process $t \to h(t, \phi(t), \phi_t, z)$ is square-mean Poisson almost periodic.

Proof: Let $\{s'_n\}$ be a sequence of real numbers. let $U(t) = h(t, \phi(t), \phi_t, z)$, where $\phi_t = \{\phi(t+\theta) : -\infty < \theta \leq 0\}$ is regarded as $\mathscr{B}$-valued stochastic process. Consider the function $\tilde{U}(t) : \mathbb{R}\times H\times\mathscr{B}\times H \to H$ given by $\tilde{U}(t) := \tilde{h}(t, \tilde{\phi}(t), \tilde{\phi}_t, z)$. Since the process $\phi(t)$ is square-mean almost periodic and h is Poisson almost periodic, there exists a subsequence $\{s_n\}$ of $\{s'_n\}$ and a continuous process $\tilde{\phi} : \mathbb{R}\times\Omega \to H$ such that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\|\phi(t+s_n) - \tilde{\phi}(t)\|_H^2 = 0,$$

and

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\|\phi_{t+s_n} - \tilde{\phi}_t\|_{\mathscr{B}}^2 = 0,$$

and

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_H \mathbb{E}\|h(t+s_n, x, y, z) - \tilde{h}(t, x, y, z)\|_H^2 \nu(dz) = 0,$$

uniformly in $(x, y) \in \mathbb{K}, z \in H$.

Note that

$$\begin{aligned} h(t+s_n, \phi(t+s_n), \phi_{t+s_n}, z) - \tilde{h}(t, \tilde{\phi}(t), \tilde{\phi}_t, z) &= h(t+s_n, \phi(t+s_n), \phi_{t+s_n}, z) \\ &\quad - h(t+s_n, \tilde{\phi}(t), \tilde{\phi}_t, z) + h(t+s_n, \tilde{\phi}(t), \tilde{\phi}_t, z) - \tilde{h}(t, \tilde{\phi}(t), \tilde{\phi}_t, z). \end{aligned}$$

Hence, we have

$$\begin{aligned}
& \int_H \mathbb{E}\|U(t+s_n)-\tilde{U}(t)\|_H^2\nu(dz) \\
= & \int_H \mathbb{E}\|h(t+s_n,\phi(t+s_n),\phi_{t+s_n},z)-\tilde{h}(t,\tilde{\phi}(t),\tilde{\phi}_t,z)\|^2\nu(dz) \\
\leq & \ 2\int_H \mathbb{E}\|h(t+s_n,\phi(t+s_n),\phi_{t+s_n},z)-h(t+s_n,\tilde{\phi}(t),\tilde{\phi}_t,z)\|^2\nu(dz) \\
+ & \ 2\int_H \mathbb{E}\|h(t+s_n,\tilde{\phi}(t),\tilde{\phi}_t,z)-\tilde{h}(t,\tilde{\phi}(t),\tilde{\phi}_t,z)\|^2\nu(dz).
\end{aligned} \tag{4.3}$$

Letting $n\to\infty$ and using Definition 4.2.5, we have from (4.3) that

$$\begin{aligned}
& \lim_{n\to\infty}\sup_{t\in\mathbb{R}}\int_H \mathbb{E}\|U(t+s_n)-\tilde{U}(t)\|_H^2\nu(dz) \\
\leq & \ 2K\lim_{n\to\infty}\sup_{t\in\mathbb{R}}\int_H \mathbb{E}(\|\phi(t+s_n)-\tilde{\phi}(t)\|^2+\|\phi_{t+s_n}-\tilde{\phi}_t\|^2)\nu(dz) \\
+ & \ 2\lim_{n\to\infty}\sup_{t\in\mathbb{R}}\int_H \mathbb{E}\|h(t+s_n,\tilde{\phi}(t),\tilde{\phi}_t,z)-\tilde{h}(t,\tilde{\phi}(t),\tilde{\phi}_t,z)\|^2\nu(dz) \\
= & \ 0.
\end{aligned}$$

Thus, the stochastic process $h(t,\phi(t),\phi_t,z)$ is square-mean Poisson almost periodic. □

4.3 Existence and Uniqueness of Almost Periodic Solutions

In this section, we study the existence and uniqueness of square-mean almost periodic solutions for neutral stochastic functional differential equations with infinite delay and Poisson jumps. Consider the following stochastic differential equation in H:

$$d(x(t) - G(x(t), x_t)) = (Ax(t) + f(t, x(t), x_t))dt + g(t, x(t), x_t)dW(t) + \int_H h(t, x(t-), x_{t-}, z)\tilde{N}(dt, dz), \quad t \in \mathbb{R}, \tag{4.4}$$

where $x_t = x(t+\theta) : -\infty < \theta \leq 0$ can be regarded as a $\mathscr{B}$-valued stochastic process. Assume that $f : \mathbb{R} \times H \times \mathscr{B} \to H$, and $g : \mathbb{R} \times H \times \mathscr{B} \to \mathscr{L}_2(K_Q, H)$ and $h : \mathbb{R} \times H \times \mathscr{B} \times H \to H$, are appropriate mappings for all $t \in \mathbb{R}, z \in H$, which will be specified later.

Definition 4.3.1 *An $\mathcal{F}_t$-measurable stochastic process $x(t), t \in \mathbb{R}$ is called the mild solution for (4.4) if*

1. *$x(t)$ is adapted to $\mathcal{F}_t$ and x_t is a $\mathscr{B}$-valued stochastic process;*
2. *$\int_{-\infty}^{T} \|x(u)\|_H^2 du < \infty$ almost surely for any $T \geq 0$;*
3. *for any $a \in \mathbb{R}$ and $t \geq a$, $x(t)$ satisfies the following integral equation:*

$$\begin{aligned} x(t) &= S(t-a)(x(a) - G(x(a), x_a)) + G(x(t-a), x_{t-a}) \\ &\quad + \int_a^t AS(t-s)G(x(s), x_s)ds \\ &\quad + \int_a^t S(t-s)f(s, x(s), x_s)ds + \int_a^t S(t-s)g(s, x(s), x_s)dW(s) \\ &\quad + \int_a^t S(t-s)\int_H h(s, x(s-), x_{s-}, z)\tilde{N}(ds, dz). \end{aligned} \tag{4.5}$$

In what follows, we need the following assumptions:

(A) Assume that A is the infinitesimal generator of an analytic semigroup $S(t)_{t\geqslant 0}$, of bounded linear operator on H, satisfying

$$\|S(t)\| \leq Me^{-\gamma t}, \quad t \geq 0 \tag{4.6}$$

for some $\gamma > 0, M > 0$.

(B) The function $f \in AP(\mathbb{R} \times H \times \mathscr{B}, H)$, and there exists a constant $M_f > 0$ such that for any $(x, y), (\tilde{x}, \tilde{y}) \in H \times \mathscr{B}$ and $t \in \mathbb{R}$,

$$\|f(t, x, y) - f(t, \tilde{x}, \tilde{y})\|_H \leq M_f(\|x - \tilde{x}\|_H + \|y - \tilde{y}\|_{\mathscr{B}}). \tag{4.7}$$

(C) The function $g \in AP(\mathbb{R} \times H \times \mathscr{B}, \mathscr{L}_2(K_Q, H))$, and there exists a constant $M_g > 0$ such that for any $(x, y), (\tilde{x}, \tilde{y}) \in H \times \mathscr{B}$ and $t \in \mathbb{R}$,

$$\|g(t, x, y) - g(t, \tilde{x}, \tilde{y})\|_{\mathscr{L}_2(K_Q, H)} \leq M_g(\|x - \tilde{x}\|_H + \|y - \tilde{y}\|_{\mathscr{B}}). \tag{4.8}$$

(D) There exists a constant $\alpha \in (\frac{1}{2}, 1)$ and a constant $M_G > 0$ such that the mapping $G \in AP(H \times \mathscr{B}, H_\alpha)$ and for any $(x, y), (\tilde{x}, \tilde{y}) \in H \times \mathscr{B}$,

$$\|(-A)^\alpha G(x, y) - (-A)^\alpha G(\tilde{x}, \tilde{y})\|_H \leq M_G(\|x - \tilde{x}\|_H + \|y - \tilde{y}\|_{\mathscr{B}}). \tag{4.9}$$

(E) The function $h \in PAP(\mathbb{R} \times H \times \mathscr{B} \times, H)$, and there exists a constant $M_h > 0$ such that for any $(x, y, z), (\tilde{x}, \tilde{y}, z) \in H \times \mathscr{B} \times H$ and $t \in \mathbb{R}$,

$$\int_H \|h(t, x, y, z)\nu(dz) - h(t, \tilde{x}, \tilde{y}, z)\|_H^2 \nu(dz) \leq M_h(\|x - \tilde{x}\|_H^2 + \|y - \tilde{y}\|_{\mathscr{B}}^2). \tag{4.10}$$

(A') The function $\tilde{f} \in AP(\mathbb{R} \times H \times \mathscr{B}, H)$, and there exists a constant $M_f > 0$ such that for any $(x, y), (\tilde{x}, \tilde{y}) \in H \times \mathscr{B}$ and $t \in \mathbb{R}$,

$$\|\tilde{f}(t, x, y) - \tilde{f}(t, \tilde{x}, \tilde{y})\|_H \leq M_f(\|x - \tilde{x}\|_H + \|y - \tilde{y}\|_{\mathscr{B}}).$$

(B') The function $\tilde{g} \in AP(\mathbb{R} \times H \times \mathscr{B}, \mathscr{L}_2(K_Q, H))$, and there exists a constant

$M_g > 0$ such that for any $(x, y), (\tilde{x}, \tilde{y}) \in H \times \mathscr{B}$ and $t \in \mathbb{R}$,

$$\|\tilde{g}(t, x, y) - \tilde{g}(t, \tilde{x}, \tilde{y})\|_{\mathscr{L}_2(K_Q, H)} \leq M_g(\|x - \tilde{x}\|_H + \|y - \tilde{y}\|_{\mathscr{B}}).$$

(C') There exists a constant $\alpha \in (\frac{1}{2}, 1)$ and a constant $M_G > 0$ such that the mapping $G \in AP(H \times \mathscr{B}, H_\alpha)$ and for any $(x, y), (\tilde{x}, \tilde{y}) \in H \times \mathscr{B}$,

$$\|(-A)^\alpha \tilde{G}(x, y) - (-A)^\alpha \tilde{G}(\tilde{x}, \tilde{y})\|_H \leq M_G(\|x - \tilde{x}\|_H + \|y - \tilde{y}\|_{\mathscr{B}}).$$

(D') The function $\tilde{h} \in PAP(\mathbb{R} \times H \times \mathscr{B} \times, H)$, and there exists a constant $M_h > 0$ such that for any $(x, y, z), (\tilde{x}, \tilde{y}, z) \in H \times \mathscr{B} \times H$ and $t \in \mathbb{R}$,

$$\int_H \|\tilde{h}(t, x, y, z) - \tilde{h}(t, \tilde{x}, \tilde{y}, z)\|_H^2 \nu(dz) \leq M_h(\|x - \tilde{x}\|_H^2 + \|y - \tilde{y}\|_{\mathscr{B}}^2).$$

Theorem 4.3.1 *Suppose that* $(A) - (E)$ *hold. Then (4.4) has a unique square-mean almost periodic mild solution whenever*

$$\frac{4M_f^2 M^2(1 + M_0)}{\gamma^2} + 4M_{1-\alpha}^2 M_G^2(1 + M_0) \cdot \frac{\Gamma(2\alpha - 1)}{\gamma^{2\alpha}} + \frac{2M_g^2 M^2(1 + M_0)}{\gamma}$$
$$+ \frac{2M_h M^2 \kappa(1 + M_0)}{\gamma} < 1.$$

Proof: Consider a mapping $\mathscr{L}$ on the Banach space $AP(\mathbb{R}\times\Omega;H)$ defined by

$$
\begin{aligned}
(\mathscr{L}x)(t) \quad =: \quad & S(t-a)[x(a)-G(x(a),x_a)]+G(x(t-a),x_{t-a}) \\
& +\int_a^t S(t-s)f(s,x(s),x_s)ds \\
& +\int_a^t AS(t-s)G(x(s),x_s)ds+\int_a^t S(t-s)g(s,x(s),x_s)dW(s) \\
& +\int_a^t S(t-s)\int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz) \\
=: \quad & S(t-a)[x(a)-G(x(a),x_a)]+G(x(t-a),x_{t-a})+I_1x(t)+I_2x(t) \\
& +I_3x(t)+I_4x(t), \quad \forall x\in AP(\mathbb{R}\times\Omega;H).
\end{aligned} \tag{4.11}
$$

We want to show that $\mathscr{L}x(t)\in AP(\mathbb{R}\times\Omega;H)$ for any $x\in AP(\mathbb{R}\times\Omega;H)$.

Step 1. The L^2-continuity of $\mathscr{L}x(t)$.

We first verify that $G(x(t-a),x_{t-a})$ is L^2-continuous in $t\in\mathbb{R}$. From condition (D), we have

$$
\begin{aligned}
& \mathbb{E}\|G(x,y)-G(\tilde{x},\tilde{y})\|_H \\
= \quad & \mathbb{E}\|(-A)^{-\alpha}\cdot(-A)^{\alpha}[G(x,y)-G(\tilde{x},\tilde{y})]\|_H \\
\leq \quad & \mathbb{E}\|(-A)^{-\alpha}\|\cdot\|(-A)^{\alpha}[G(x,y)-G(\tilde{x},\tilde{y})]\|_H \\
\leq \quad & C\mathbb{E}\|(-A)^{\alpha}G(x,y)-(-A)^{\alpha}G(\tilde{x},\tilde{y})\|_H \\
\leq \quad & CM_G\mathbb{E}(\|x-\tilde{x}\|_H+\|y-\tilde{y}\|_{\mathscr{B}}).
\end{aligned} \tag{4.12}
$$

Since $G\in AP(H\times\mathscr{B};H_\alpha)$ and $x\in AP(\mathbb{R}\times\Omega;H)$, we have by Theorem 4.2.1 that $G(x(t-a),x_{t-a})\in AP(H\times\mathscr{B};H_\alpha)$, and

$$
\mathbb{E}\|G(x(t+r-a),x_{t+r-a})-G(x(t-a),x_{t-a})\|_H^2\to 0, \quad as \quad r\to 0.
$$

Hence, we just show that $G(x(t-a),x_{t-a})$ is L^2-continuous.

Next we verify that $I_1x(t)$ is L^2-continuous in $t \in \mathbb{R}$. Letting $r - s = -u$, we have that for any $t \in \mathbb{R}$,

$$
\begin{aligned}
R_1(t) &:= \mathbb{E}\|I_1x(t+r) - I_1x(t)\|_H^2 \\
&= \mathbb{E}\Big\| \int_a^{t+r} S(t+r-s)f(s, x(s), x_s)ds - \int_a^t S(t-s)f(s, x(s), x_s)ds \Big\|_H^2 \\
&= \mathbb{E}\Big\| \int_{a-r}^{t} S(t-u)f(u+r, x(u+r), x_{u+r})du - \int_a^t S(t-s)f(s, x(s), x_s)ds \Big\|_H^2 \\
&= \mathbb{E}\Big\| \int_{a-r}^{a} S(t-s)f(s+r, x(s+r), x_{s+r})ds \\
&\quad + \int_a^t S(t-s)f(s+r, x(s+r), x_{s+r})ds - \int_a^t S(t-s)f(s, x(s), x_s)ds \Big\|_H^2.
\end{aligned}
$$

By the relation that $(a+b)^2 \leq 2a^2 + 2b^2$, for any $a, b \in \mathbb{R}$, we have for any $t \in \mathbb{R}$ that

$$
\begin{aligned}
R_1(t) &\leq 2\mathbb{E}\Big\| \int_{a-r}^{a} S(t-s)f(s+r, x(s+r), x_{s+r})ds \Big\|_H^2 \\
&\quad +2\mathbb{E}\Big\| \int_a^t S(t-s)f(s+r, x(s+r), x_{s+r})ds - \int_a^t S(t-s)f(s, x(s), x_s)ds \Big\|_H^2 \\
&= 2\mathbb{E}\Big\| \int_{a-r}^{a} S(t-s)f(s+r, x(s+r), x_{s+r})ds \Big\|_H^2 \\
&\quad +2\mathbb{E}\Big\| \int_a^t S(t-s)[f(s+r, x(s+r), x_{s+r}) - f(s, x(s), x_s)]ds \Big\|_H^2,
\end{aligned}
$$

which, by Hölder inequality, further implies that for any $t \in \mathbb{R}$,

$$\begin{aligned}
R_1(t) &\leq 2\mathbb{E}\Big\| \int_{a-r}^{a} S(t-s) f(s+r, x(s+r), x_{s+r}) ds \Big\|_H^2 \\
&\quad +2\mathbb{E}\Big(\int_a^t \|S(t-s)\| \|f(s+r, x(s+r), x_{s+r}) - f(s, x(s), x_s)\|_H ds \Big)^2 \\
&\leq 2\mathbb{E}\Big\| \int_{a-r}^{a} S(t-s) f(s+r, x(s+r), x_{s+r}) ds \Big\|_H^2 \\
&\quad +2\mathbb{E}\Big(\int_a^t \|S(t-s)\|^2 ds \cdot \int_a^t \|f(s+r, x(s+r), x_{s+r}) - f(s, x(s), x_s)\|_H^2 ds \Big).
\end{aligned} \tag{4.13}$$

On the other hand, letting $t - s = u$, we have

$$\begin{aligned}
&\int_a^t \|S(t-s)\|^2 ds = -\int_{t-a}^{0} \|S(u)\|^2 du = \int_0^{t-a} \|S(u)\|^2 du \\
&\leq \int_0^{\infty} \|S(u)\|^2 du \leq \int_0^{\infty} M^2 e^{-2\gamma u} du \leq \frac{M^2}{2\gamma}.
\end{aligned} \tag{4.14}$$

Substituting (4.14) into (4.13), we have for any $t \in \mathbb{R}$,

$$\begin{aligned}
R_1(t) &\leq 2\mathbb{E}\Big\| \int_{a-r}^{a} S(t-s) f(s+r, x(s+r), x_{s+r}) ds \Big\|_H^2 \\
&\quad +2\frac{M^2}{2\gamma}\mathbb{E}\Big(\int_a^t \|f(s+r, x(s+r), x_{s+r}) - f(s, x(s), x_s)\|_H^2 ds \Big) \\
&= 2\mathbb{E}\Big\| \int_{a-r}^{a} S(t-s) f(s+r, x(s+r), x_{s+r}) ds \Big\|_H^2 \\
&\quad +\frac{M^2}{\gamma}\Big(\int_a^t \mathbb{E}\|f(s+r, x(s+r), x_{s+r}) - f(s, x(s), x_s)\|_H^2 ds \Big).
\end{aligned}$$

Since $f \in AP(\mathbb{R} \times H \times \mathscr{B}; H)$ and $x \in AP(\mathbb{R} \times \Omega; H)$, we have by Theorem 4.2.1 that $f(t, x(t), x_t) \in AP(\mathbb{R} \times H \times \mathscr{B}; H)$ and

$$\mathbb{E}\|f(s+r, x(s+r), x_{s+r}) - f(s, x(s), x_s)\|_H^2 \to 0, \quad as \quad r \to 0.$$

By dominated convergence theorem, letting $r \to 0$, we have

$$\int_a^t \mathbb{E}\|f(s+r, x(s+r), x_{s+r}) - f(s, x(s), x_s)\|_H^2 ds \to 0. \tag{4.15}$$

On the other hand, it is easy to see that for any $t \geq a$,

$$\mathbb{E}\Big\| \int_{a-r}^a S(t-s) f(s+r, x(s+r), x_{s+r}) ds \Big\|_H^2 \to 0, \quad as \quad r \to 0.$$

Hence, we just show that $I_1 x(t)$ is L^2-continuous.

Next we verify that $I_2 x(t)$ is L^2-continuous in $t \in \mathbb{R}$. To this end, for any $t \in \mathbb{R}$, we have by letting $r - s = -u$ that

$$\begin{aligned}
R_2(t) \; &:= \; \mathbb{E}\|I_2 x(t+r) - I_2 x(t)\|_H^2 \\
&= \; \mathbb{E}\Big\| \int_a^t AS(t+r-s) G(x(s), x_s) ds - \int_a^t AS(t-s) G(x(s), x_s) ds \Big\|_H^2 \\
&= \; \mathbb{E}\Big\| \int_{a-r}^t AS(t-u) G(x(u+r), x_{u+r}) du \\
&\quad - \int_a^t AS(t-s) G(x(s), x_s) ds \Big\|_H^2 \\
&= \; \mathbb{E}\Big\| \int_{a-r}^a AS(t-s) G(x(s+r), x_{s+r}) ds \\
&\quad + \int_a^t (-A)^{1-\alpha} (-A)^\alpha S(t-s) G(x(s+r), x_{s+r}) ds \\
&\quad - \int_a^t (-A)^{1-\alpha} (-A)^\alpha S(t-s) G(x(s), x_s) ds \Big\|_H^2.
\end{aligned}$$

By the relation that $(a+b)^2 \leq 2a^2+2b^2$, for any $a,b\in\mathbb{R}$, we have for any $t\in\mathbb{R}$

$$\begin{aligned}
R_2(t) &\leq 2\mathbb{E}\Big\|\int_{a-r}^{a} AS(t-s)G(x(s+r),x_{s+r})ds\Big\|_H^2 \\
&\quad +2\mathbb{E}\Big\|\int_a^t (-A)^{1-\alpha}(-A)^{\alpha}S(t-s)G(x(s+r),x_{s+r})ds \\
&\quad -\int_a^t (-A)^{1-\alpha}(-A)^{\alpha}S(t-s)G(x(s),x_s)ds\Big\|_H^2 \\
&= 2\mathbb{E}\Big\|\int_{a-r}^{a} AS(t-s)G(x(s+r),x_{s+r})ds\Big\|_H^2 \\
&\quad +2\mathbb{E}\Big\|\int_a^t (-A)^{1-\alpha}S(t-s)\Big[(-A)^{\alpha}G(x(s+r),x_{s+r}) \\
&\quad -(-A)^{\alpha}G(x(s),x_s)\Big]ds\Big\|_H^2 \\
&\leq 2\mathbb{E}\Big\|\int_{a-r}^{a} AS(t-s)G(x(s+r),x_{s+r})ds\Big\|_H^2 \\
&\quad +2\mathbb{E}\Big(\int_a^t \|(-A)^{1-\alpha}S(t-s)\|\|(-A)^{\alpha}G(x(s+r),x_{s+r}) \\
&\quad -(-A)^{\alpha}G(x(s),x_s)\|_H ds\Big)^2.
\end{aligned}$$

By using Hölder inequality and Lemma 2.2.1 (2), we have that for any $t\in\mathbb{R}$,

$$\begin{aligned}
R_2(t) &\leq 2\mathbb{E}\Big\|\int_{a-r}^{a} AS(t-s)G(x(s+r),x_{s+r})ds\Big\|_H^2 \\
&\quad +2\mathbb{E}\Big(\int_a^t \|(-A)^{1-\alpha}S(t-s)\|^2 ds\cdot\int_a^t \|(-A)^{\alpha}G(x(s+r),x_{s+r}) \\
&\quad -(-A)^{\alpha}G(x(s),x_s)\|_H^2 ds\Big) \\
&\leq 2\mathbb{E}\Big\|\int_{a-r}^{a} AS(t-s)G(x(s+r),x_{s+r})ds\Big\|_H^2 \\
&\quad +2M_{1-\alpha}^2\int_a^t e^{-2\gamma(t-s)}(t-s)^{2(\alpha-1)}ds\cdot\int_a^t \mathbb{E}\|(-A)^{\alpha}G(x(s+r),x_{s+r}) \\
&\quad -(-A)^{\alpha}G(x(s),x_s)\|_H^2 ds.
\end{aligned} \tag{4.16}$$

On the other hand, letting $t-s=u$, we have

$$\begin{aligned}
& \int_a^t e^{-2\gamma(t-s)}(t-s)^{2(\alpha-1)}ds = -\int_{t-a}^0 e^{-2\gamma u}u^{2(\alpha-1)}du \\
= & \int_0^{t-a} e^{-2\gamma u}u^{2(\alpha-1)}du \le \int_0^{\infty} e^{-2\gamma u}u^{2(\alpha-1)}du \\
= & \frac{1}{2\gamma}\int_0^{\infty} e^{-2\gamma u}u^{2(\alpha-1)}d(2\gamma u). \\
= & \frac{1}{2\gamma}\int_0^{\infty} e^{-s}(\frac{s}{2\gamma})^{2(\alpha-1)}ds = \frac{1}{(2\gamma)^{1+2(\alpha-1)}}\int_0^{\infty} e^{-s}s^{2(\alpha-1)}ds \\
= & \frac{1}{(2\gamma)^{2\alpha-1}}\cdot\Gamma(2\alpha-1).
\end{aligned} \tag{4.17}$$

Hence, from (4.16) and (4.17), we have for any $t\in\mathbb{R}$,

$$\begin{aligned}
R_2(t) \le\ & 2\mathbb{E}\Big\|\int_{a-r}^a AS(t-s)G(x(s+r),x_{s+r})ds\Big\|_H^2 \\
& +2M_{1-\alpha}^2\cdot\frac{\Gamma(2\alpha-1)}{(2\gamma)^{2\alpha-1}}\cdot\int_a^t \mathbb{E}\|(-A)^\alpha G(x(s+r),x_{s+r}) \\
& -(-A)^\alpha G(x(s),x_s)ds\|_H^2.
\end{aligned}$$

Since $x\in AP(\mathbb{R}\times\Omega;H)$, we have by using condition (D) that

$$\begin{aligned}
& \mathbb{E}\|(-A)^\alpha G(x(s+r),x_{s+r})-(-A)^\alpha G(x(s),x_s)\|_H^2 \\
\le\ & M_G\mathbb{E}(\|x(s+r)-x(s)\|_H^2+\|x_{s+r}-x_s\|_{\mathscr{B}}^2)\to 0, \quad as \quad r\to 0.
\end{aligned}$$

Hence, by dominated convergence theorem when $r\to 0$, we have

$$\int_a^t \mathbb{E}\|(-A)^\alpha G(x(s+r),x_{s+r})-(-A)^\alpha G(x(s),x_s)\|_H^2 ds\to 0.$$

On the other hand, it is easy to see that for any $t \geq a$,

$$\mathbb{E}\Big(\int_{a-r}^{a} \|AS(t-s)G(x(s+r), x_{s+r})\|_H ds\Big)^2 \to 0, \quad \text{as } r \to 0.$$

Hence, we just show that $I_2x(t)$ is L^2-continuous.

Next we verify that $I_3x(t)$ is L^2-continuous in $t \in \mathbb{R}$. To this end, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}
R_3(t) &:= \mathbb{E}\|I_3x(t+r) - I_3x(t)\|_H^2 \\
&= \mathbb{E}\Big\| \int_a^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s) - \int_a^t S(t-s)g(s,x(s),x_s)dW(s)\Big\|_H^2 \\
&= \mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s) + \int_a^t S(t+r-s)g(s,x(s),x_s)dW(s) \\
&\quad - \int_a^t S(t-s)g(s,x(s),x_s)dW(s)\Big\|_H^2 \\
&= \mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s) \\
&\quad + \int_a^t [S(t+r-s) - S(t-s)]g(s,x(s),x_s)dW(s)\Big\|_H^2 \\
&\leq 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s)\Big\|_H^2 \\
&\quad + 2\mathbb{E}\Big\| \int_a^t [S(t+r-s) - S(t-s)]g(s,x(s),x_s)dW(s)\Big\|_H^2.
\end{aligned}$$

By using isometry property of stochastic integral, we have for any $t \in \mathbb{R}$,

$$\begin{aligned}
R_3(t) &\leq 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s)\Big\|_H^2 \\
&\quad + 2\int_a^t \mathbb{E}\|[S(t+r-s) - S(t-s)]g(s,x(s),x_s)\|_{\mathscr{L}_2(K_Q,H)}^2 ds \\
&= 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s)\Big\|_H^2 \\
&\quad + 2\int_a^t \mathbb{E}\|[S(t-s)S(r) - S(t-s)]g(s,x(s),x_s)\|_{\mathscr{L}_2(K_Q,H)}^2 ds.
\end{aligned}$$

Then, by the property of C_0-semigroup, we have for any $t \in \mathbb{R}$,

$$\begin{aligned} R_3(t) &\leq 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s)\Big\|_H^2 \\ &\quad +2\int_a^t \mathbb{E}\|(S(t-s)(S(r)-I))g(s,x(s),x_s)\|^2_{\mathscr{L}_2(K_Q,H)}ds \\ &\leq 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s)\Big\|_H^2 \\ &\quad +2\int_a^t \mathbb{E}\|S(t-s)\|^2\cdot\|(S(r)-I)g(s,x(s),x_s)\|^2_{\mathscr{L}_2(K_Q,H)}ds. \end{aligned} \tag{4.18}$$

By using condition (A), we have

$$\|S(t-s)\|^2 \leq M^2 e^{-2\gamma(t-s)} \leq M^2, \text{ for any } t \geq s. \tag{4.19}$$

Substituting (4.19) into (4.18), we have for any $t \in \mathbb{R}$,

$$\begin{aligned} R_3(t) &\leq 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s)\Big\|_H^2 \\ &\quad +2M^2\Big(\int_a^t \mathbb{E}\|(S(r)-I)g(s,x(s),x_s)\|^2_{\mathscr{L}_2(K_Q,H)}ds\Big). \end{aligned}$$

By the property of strong continuity of C_0-semigroup of $S(t), t \geq 0$, we have for $s \in \mathbb{R}$,

$$\mathbb{E}\|(S(r)-I)g(s,x(s),x_s)\|^2_{\mathscr{L}_2(K_Q,H)} \to 0, \quad as \quad r \to 0.$$

By dominated convergence theorem when $r \to 0$,

$$\int_a^t \mathbb{E}\|(S(r)-I)g(s,x(s),x_s)\|^2_{\mathscr{L}_2(K_Q,H)}ds \to 0.$$

On the other hand, it is easy to see that for any $t \geq a$,

$$\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)g(s,x(s),x_s)dW(s)\Big\|_H^2 \to 0, \quad as \quad r \to 0.$$

Hence, we just show that $I_3x(t)$ is L^2-continuous.

Finally, we verify that $I_4x(t)$ is L^2-continuous in $t \in \mathbb{R}$. To this end, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}
R_4(t): &= \mathbb{E}\|I_4x(t+r) - I_4x(t)\|_H^2 \\
&= \mathbb{E}\Big\| \int_a^{t+r} S(t+r-s)\int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz) \\
&\quad - \int_a^t S(t-s)\int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz)\Big\|_H^2 \\
&= \mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)\int_H h(s,x(s),x_s,z)\tilde{N}(du,dz) \\
&\quad + \int_a^t S(t+r-s)\int_H h(s,x(s),x_s,z)\tilde{N}(du,dz) \\
&\quad - \int_a^t S(t-s)\int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz)\Big\|_H^2 \\
&\leq 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)\int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz)\Big\|_H^2 \\
&\quad + 2\mathbb{E}\Big\| \int_a^t S(t+r-s) - S(t-s)\int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz) \\
&= 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s)\int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz)\Big\|_H^2 \\
&\quad + 2\mathbb{E}\Big\| \int_a^t S(t-s)(S(r)-I)\int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz)\Big\|_H^2.
\end{aligned}$$

By using the isometry property of the compensating Poisson random measure

(2.5), we have that for $t \geq a$,

$$
\begin{aligned}
R_4(t) &\leq 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s) \int_H h(s, x(s), x_s, z)\tilde{N}(ds, dz)\Big\|_H^2 \\
&\quad +2\kappa \int_a^t \int_H \mathbb{E}\|S(t-s)(S(r)-I)h(s, x(s), x_s, z)\|_H^2 \nu(dz)ds \\
&\leq 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s) \int_H h(s, x(s), x_s, z)\tilde{N}(ds, dz)\Big\|_H^2 \\
&\quad +2\kappa \int_a^t \mathbb{E}\|S(t-s)\|^2 \|(S(r)-I) \int_H h(s, x(s), x_s, z)\|_H^2 \nu(dz)ds.
\end{aligned}
$$

By using condition (A), we have

$$
\|S(t-s)\|^2 \leq M^2 e^{-2\gamma(t-s)} \leq M^2, \text{ for any } t \geq s.
$$

Hence, we have for any $t \in \mathbb{R}$,

$$
\begin{aligned}
R_4(t) &\leq 2\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s) \int_H h(s, x(s), x_s, z)\tilde{N}(ds, dz)\Big\|_H^2 \\
&\quad +2\kappa M^2 \int_a^t \int_H \mathbb{E}\|(S(r)-I)h(s, x(s), x_s, z)\|_H^2 \nu(dz)ds.
\end{aligned}
$$

By the property of strong continuity of C_0-semigroup of $S(t), t \geq 0$ and dominated convergence theorem, we have for $s \in \mathbb{R}$,

$$
\int_H \mathbb{E}\|(S(r)-I)h(s, x(s), x_s)\|_H^2 \nu(dz) \to 0, \quad as \quad r \to 0,
$$

and

$$
\int_a^t \int_H \mathbb{E}\|(S(r)-I)h(s, x(s), x_s, z)\|_H^2 \nu(dz)ds \to 0, \quad as \quad r \to 0.
$$

On the other hand, it is easy to see that for any $t \geq a$,

$$\mathbb{E}\Big\| \int_t^{t+r} S(t+r-s) \int_H h(s, x(s), x_s, z)\tilde{N}(ds, dz)\Big\|_H^2 \to 0, \qquad \text{as } r \to 0.$$

Hence, $I_4 x(t)$ is L^2-continuous in t.

Step 2. $\mathscr{L}x(t) \in AP(\mathbb{R} \times \Omega; H)$ **for any** $x \in AP(\mathbb{R} \times \Omega; H)$**.** Denote by $AP(\mathbb{R}\times\Omega; H)$ the Banach space of all L^2-continuous square-mean almost periodic mappings from $\mathbb{R}$ to H endowed with the norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} (\mathbb{E}\|x(t)\|_H^2)^{\frac{1}{2}}.$$

Let us consider the mild solution of (4.4) given by

$$\begin{aligned} x(t) : \quad = \quad & S(t-a)[x(a) - G(x(a), x_a)] + G(x(t-a), x_{t-a}) \\ & + \int_a^t S(t-s)f(s, x(s), x_s)ds \\ & + \int_a^t AS(t-s)G(x(s), x_s)ds + \int_a^t S(t-s)g(s, x(s), x_s)dW(s) \\ & + \int_a^t S(t-s) \int_H h(s, x(s), x_s, z)\tilde{N}(ds, dz), \end{aligned} \tag{4.20}$$

for all $t \geq a, x \in AP(\mathbb{R} \times \Omega; H)$.

Note that the process for any $t \in \mathbb{R}$,

$$\begin{aligned} x(t) : \quad = \quad & \int_{-\infty}^t S(t-s)f(s, x(s), x_s)ds + \int_{-\infty}^t AS(t-s)G(x(s), x_s)ds \\ & + \int_{-\infty}^t S(t-s)g(s, x(s), x_s)dW(s) \\ & + \int_{-\infty}^t S(t-s) \int_H h(s, x(s), x_s, z)\tilde{N}(ds, dz) \end{aligned} \tag{4.21}$$

is a mild solution of (4.4). Indeed, as $a \to -\infty$ in (4.20), and by the property of exponential stability, (4.21) satisfies (4.20). Therefore, we just need to prove the desired result for the stochastic process (4.21).

We show that $\mathscr{L}x(t)$ is square-mean almost periodic whenever $x \in AP(\mathbb{R} \times \Omega; H)$. Since f, g, G are square-mean almost periodic and h is Poisson square-mean almost periodic, then for an arbitrary sequence $\{s'_n\}$ of real numbers there exists a subsequence $\{s_n\}$ of $\{s'_n\}$ and certain functions $\tilde{f}, \tilde{g}, \tilde{G}$ and $\tilde{h}$ such that

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \|f(s+s_n, x(s+s_n), x_{s+s_n}) - \tilde{f}(s, \tilde{x}(s), \tilde{x}_s)\|_H^2 = 0, \tag{4.22}$$

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \mathbb{E}\|(-A)^\alpha G(x(s+s_n), x_{s+s_n}) - (-A)^\alpha \tilde{G}(\tilde{x}(s), \tilde{x}_s)\|_H^2 = 0, \tag{4.23}$$

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \mathbb{E}\|g(s+s_n, x(s+s_n), x_{s+s_n}) - \tilde{g}(s, \tilde{x}(s), \tilde{x}_s)\|_{\mathscr{L}_2(K_Q,H)}^2 = 0, \tag{4.24}$$

and

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \int_H \mathbb{E}\|h(s+s_n, x(s+s_n), x_{s+s_n}, z) - \tilde{h}(s, \tilde{x}(s), \tilde{x}_s, z)\|_H^2 \nu(dz) = 0. \tag{4.25}$$

Let $\tilde{x}(t)$ satisfy the integral equation for any $t \in \mathbb{R}$,

$$\begin{aligned} \tilde{x}(t) : &= \int_{-\infty}^t S(t-s)\tilde{f}(s, \tilde{x}(s), \tilde{x}_s)ds + \int_{-\infty}^t AS(t-s)\tilde{G}(\tilde{x}(s), \tilde{x}_s)ds \\ &\quad + \int_{-\infty}^t S(t-s)\tilde{g}(s, \tilde{x}(s), \tilde{x}_s)dW(s) \\ &\quad + \int_{-\infty}^t S(t-s) \int_H \tilde{h}(s, \tilde{x}(s), \tilde{x}_s, z)\tilde{N}(ds, dz). \end{aligned}$$

Note that for any $t \in \mathbb{R}$,

$$\begin{aligned}
& \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\|x(t+s_n)-\tilde{x}(t)\|_H^2 \\
\leq\ & 4\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t+s_n} S(t+s_n-s)f(s,x(s),x_s)ds - \int_{-\infty}^{t} S(t-s)\tilde{f}(s,\tilde{x}(s),\tilde{x}_s)ds\Big\|_H^2 \\
& +4\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t+s_n} AS(t+s_n-s)G(x(s),x_s)ds - \int_{-\infty}^{t} AS(t-s)\tilde{G}(\tilde{x}(s),\tilde{x}_s)ds\Big\|_H^2 \\
& +4\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t+s_n} S(t+s_n-s)g(s,x(s),x_s)dW(s) \\
& - \int_{-\infty}^{t} S(t-s)\tilde{g}(s,\tilde{x}(s),\tilde{x}_s)dW(s)\Big\|_H^2 \\
& +4\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t+s_n} S(t+s_n-s)\int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz) \\
& - \int_{-\infty}^{t} S(t-s)\int_H \tilde{h}(s,\tilde{x}(s),\tilde{x}_s,z)\tilde{N}(ds,dz)\Big\|_H^2 \\
:=\ & 4J_1(t)+4J_2x(t)+4J_3x(t)+4J_4x(t).
\end{aligned}$$

Firstly, we show that $J_1(t)$ is square-mean almost periodic when $x \in AP(\mathbb{R}\times\Omega;H)$.

$$\begin{aligned}
J_1(t) &= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t+s_n} S(t+s_n-s)f(s,x(s),x_s)ds - \int_{-\infty}^{t} S(t-s)\tilde{f}(s,\tilde{x}(s),\tilde{x}_s)ds\Big\|_H^2 \\
&= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} S(t-u)f(u+s_n,x(u+s_n),x_{u+s_n})du \\
&\quad - \int_{-\infty}^{t} S(t-s)\tilde{f}(s,\tilde{x}(s),\tilde{x}_s)ds\Big\|_H^2 \\
&= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s)f(s+s_n,x(s+s_n),x_{s+s_n})ds \\
&\quad - \int_{-\infty}^{t} S(t-s)\tilde{f}(s,\tilde{x}(s),\tilde{x}_s)ds\Big\|_H^2 \\
&\leq \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big(\int_{-\infty}^{t} \|S(t-s)\|\|f(s+s_n,x(s+s_n),x_{s+s_n})-\tilde{f}(s,\tilde{x}(s),\tilde{x}_s)\|_H ds\Big)^2 \\
&\leq \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big(\int_{-\infty}^{t} Me^{-\frac{\gamma(t-s)}{2}}\|e^{-\frac{\gamma(t-s)}{2}}[f(s+s_n,x(s+s_n),x_{s+s_n}) \\
&\quad -\tilde{f}(s,\tilde{x}(s),\tilde{x}_s)]\|_H ds\Big)^2.
\end{aligned}$$

By using Hölder inequality, we have from (4.14) that for any $t \in \mathbb{R}$,

$$\begin{aligned}
J_1(t) &\leq \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} M^2 e^{-\gamma(t-s)} ds \cdot \int_{-\infty}^{t} e^{-\gamma(t-s)} \mathbb{E}\|f(s+s_n, x(s+s_n), x_{s+s_n}) \\
&\quad - \tilde{f}(s, \tilde{x}(s), \tilde{x}_s)\|_H^2 ds \\
&= \frac{M^2}{\gamma} \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} d(-\gamma(t-s)) \cdot \int_{-\infty}^{t} e^{-\gamma(t-s)} \mathbb{E}\|f(s+s_n, x(s+s_n), x_{s+s_n}) \\
&\quad - \tilde{f}(s, \tilde{x}(s), \tilde{x}_s)\|_H^2 ds \\
&= \frac{M^2}{\gamma} \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} \mathbb{E}\|f(s+s_n, x(s+s_n), x_{s+s_n}) - \tilde{f}(s, \tilde{x}(s), \tilde{x}_s)\|_H^2 ds \\
&\leq \frac{M^2}{\gamma} \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} \sup_{s\in\mathbb{R}} \mathbb{E}\|f(s+s_n, x(s+s_n), x_{s+s_n}) - \tilde{f}(s, \tilde{x}(s), \tilde{x}_s)\|_H^2 ds \\
&\leq \frac{M^2}{\gamma} \lim_{n\to\infty} \sup_{s\in\mathbb{R}} \mathbb{E}\|f(s+s_n, x(s+s_n), x_{s+s_n}) - \tilde{f}(s, \tilde{x}(s), \tilde{x}_s)\|_H^2 \\
&\quad \cdot \Big(\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} ds \Big).
\end{aligned} \tag{4.26}$$

On the other hand, letting $t - s = u$, we have

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} ds &= -\sup_{t\in\mathbb{R}} \int_{+\infty}^{0} e^{-\gamma u} du \\
&= -\sup_{t\in\mathbb{R}} \int_{0}^{+\infty} e^{-\gamma u} du \\
&= -\frac{1}{\gamma} \sup_{t\in\mathbb{R}} \int_{0}^{+\infty} e^{-\gamma u} d(-\gamma u) = \frac{1}{\gamma}.
\end{aligned} \tag{4.27}$$

Hence, from (4.26), (4.27) and version of (4.22), we have for any $t \in \mathbb{R}$,

$$\begin{aligned}
J_1(t) &\leq \frac{M^2}{\gamma^2} \lim_{n\to\infty} \sup_{s\in\mathbb{R}} \mathbb{E}\|f(s+s_n, x(s+s_n), x_{s+s_n}) - \tilde{f}(s, \tilde{x}(s), \tilde{x}_s)\|_H^2 \\
&= 0.
\end{aligned}$$

This implies that $J_1(t)$ is square-mean almost periodic in $t \in \mathbb{R}$.

Next, we show that $J_2(t)$ is square-mean almost periodic when $x \in AP(\mathbb{R} \times \Omega; H)$.

$$\begin{aligned} J_2(t) &= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t+s_n} AS(t+s_n-s)G(x(s),x_s)ds \\ &\quad - \int_{-\infty}^{t} AS(t-s)\tilde{G}(\tilde{x}(s),\tilde{x}_s)ds\Big\|_H^2 \\ &= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} AS(t-u)G(x(u+s_n),x_{u+s_n})du \\ &\quad - \int_{-\infty}^{t} AS(t-s)\tilde{G}(\tilde{x}(s),\tilde{x}_s)ds\Big\|_H^2. \end{aligned}$$

Since $-A = (-A)^{1-\alpha}(-A)^{\alpha}$, so we have for any $t \in \mathbb{R}$,

$$\begin{aligned} J_2(t) &= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} (-A)^{1-\alpha}(-A)^{\alpha}S(t-s)G(x(s+s_n),x_{s+s_n})ds \\ &\quad - \int_{-\infty}^{t} (-A)^{1-\alpha}(-A)^{\alpha}S(t-s)\tilde{G}(\tilde{x}(s),\tilde{x}_s)ds\Big\|_H^2 \\ &= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} (-A)^{1-\alpha}S(t-s)(-A)^{\alpha}G(x(s+s_n),x_{s+s_n})ds \\ &\quad - \int_{-\infty}^{t} (-A)^{1-\alpha}S(t-s)(-A)^{\alpha}\tilde{G}(\tilde{x}(s),\tilde{x}_s)ds\Big\|_H^2 \\ &= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} (-A)^{1-\alpha}S(t-s)\Big[(-A)^{\alpha}G(x(s+s_n),x_{s+s_n}) \\ &\quad -(-A)^{\alpha}\tilde{G}(\tilde{x}(s),\tilde{x}_s)\Big]ds\Big\|_H^2 \\ &\le \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big(\int_{-\infty}^{t} \|(-A)^{1-\alpha}S(t-s)\| \|(-A)^{\alpha}G(x(s+s_n),x_{s+s_n}) \\ &\quad -(-A)^{\alpha}\tilde{G}(\tilde{x}(s),\tilde{x}_s)\|_H ds\Big)^2. \end{aligned}$$

By using Lemma 2.2.1 (2), we have that for any $t \in \mathbb{R}$,

$$\begin{aligned}
J_2(t) \;\leq\;& \lim_{n\to\infty}\sup_{t\in\mathbb{R}} \mathbb{E}\Big(\int_{-\infty}^{t} M_{1-\alpha}e^{-\gamma(t-s)}(t-s)^{(\alpha-1)}\|(-A)^{\alpha}G(x(s+s_n),x_{s+s_n}) \\
& -(-A)^{\alpha}\tilde{G}(\tilde{x}(s),\tilde{x}_s)\|_H ds\Big)^2 \\
=\;& \lim_{n\to\infty}\sup_{t\in\mathbb{R}} \mathbb{E}\Big(\int_{-\infty}^{t} M_{1-\alpha}e^{-\frac{\gamma(t-s)}{2}}(t-s)^{(\alpha-1)}\Big\|e^{-\frac{\gamma(t-s)}{2}}\Big[(-A)^{\alpha}G(x(s+s_n),x_{s+s_n}) \\
& -(-A)^{\alpha}\tilde{G}(\tilde{x}(s),\tilde{x}_s)\Big]\Big\|_H ds\Big)^2.
\end{aligned}$$

Then by using Hölder inequality, we have that for any $t \in \mathbb{R}$,

$$\begin{aligned}
J_2(t) \;\leq\;& \lim_{n\to\infty}\sup_{t\in\mathbb{R}} M_{1-\alpha}^2\int_{-\infty}^{t} e^{-\gamma(t-s)}(t-s)^{2(\alpha-1)}ds\,\cdot \\
& \int_{-\infty}^{t} e^{-\gamma(t-s)}\mathbb{E}\|(-A)^{\alpha}G(x(s+s_n),x_{s+s_n})-(-A)^{\alpha}\tilde{G}(\tilde{x}(s),\tilde{x}_s)\|_H^2 ds.
\end{aligned} \tag{4.28}$$

On the other hand, letting $t-s=u$, we have

$$\begin{aligned}
& \int_{-\infty}^{t} e^{-\gamma(t-s)}(t-s)^{2(\alpha-1)}ds \\
=\;& -\int_{+\infty}^{0} e^{-\gamma u}u^{2(\alpha-1)}du \\
=\;& \int_{0}^{+\infty} e^{-\gamma u}u^{2(\alpha-1)}du = \frac{1}{\gamma}\int_{0}^{\infty} e^{-\gamma u}u^{2(\alpha-1)}d(\gamma u) \\
=\;& \frac{1}{\gamma}\int_{0}^{\infty} e^{-s}(\frac{s}{\gamma})^{2(\alpha-1)}ds = \frac{1}{\gamma^{1+2(\alpha-1)}}\int_{0}^{\infty} e^{-s}s^{2(\alpha-1)}ds \\
=\;& \frac{1}{\gamma^{\alpha-1}}\cdot\Gamma(2\alpha-1).
\end{aligned} \tag{4.29}$$

Hence, from (4.28), (4.29) and version of (4.23), we have for any $t \in \mathbb{R}$,

$$
\begin{aligned}
J_2(t) \quad \leq \quad & M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha-1}} \cdot \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} \mathbb{E}\|(-A)^\alpha G(x(s+s_n), x_{s+s_n}) \\
& -(-A)^\alpha \tilde{G}(\tilde{x}(s), \tilde{x}_s)\|_H^2 ds \\
\leq \quad & M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha-1}} \cdot \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} \sup_{s\in\mathbb{R}} \mathbb{E}\|(-A)^\alpha G(x(s+s_n), x_{s+s_n}) \\
& -(-A)^\alpha \tilde{G}(\tilde{x}(s), \tilde{x}_s)\|_H^2 ds \\
\leq \quad & M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha-1}} \cdot \lim_{n\to\infty} \sup_{s\in\mathbb{R}} \mathbb{E}\|(-A)^\alpha G(x(s+s_n), x_{s+s_n}) \\
& -(-A)^\alpha \tilde{G}(\tilde{x}(s), \tilde{x}_s)\|_H^2 \cdot \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\gamma(t-s)} ds \\
\leq \quad & M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha-1}} \cdot \frac{1}{\gamma} \cdot \lim_{n\to\infty} \sup_{s\in\mathbb{R}} \mathbb{E}\|(-A)^\alpha G(x(s+s_n), x_{s+s_n}) \\
& -(-A)^\alpha \tilde{G}(\tilde{x}(s), \tilde{x}_s)\|_H^2,
\end{aligned}
$$

which by (4.23) is equivalent to

$$
\begin{aligned}
J_2(t) \quad = \quad & M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha}} \cdot \lim_{n\to\infty} \sup_{s\in\mathbb{R}} \mathbb{E}\|(-A)^\alpha G(x(s+s_n), x_{s+s_n}) \\
& -(-A)^\alpha \tilde{G}(\tilde{x}(s), \tilde{x}_s)\|_H^2 \\
= \quad & 0.
\end{aligned}
$$

This implies that $J_2(t)$ is square-mean almost periodic in $t \in \mathbb{R}$.

Next, we show that $J_3(t)$ is square-mean almost periodic for all $t \in \mathbb{R}$. Let $\tilde{W}(u) = W(u+u_0) - W(u_0)$, for all $u_0 \in \mathbb{R}$. By Proposition and definition of Q-Wiener process, it is easy to know that $\tilde{W}(u)$ is also a Q-Wiener process and has the same distribution as $W(u+s_n)-W(s_n)$. Here, we have for any $t\in\mathbb{R}$,

$$\begin{aligned}
J_3(t) &= \lim_{n\to\infty}\sup_{t\in\mathbb{R}}\mathbb{E}\Big\|\int_{-\infty}^{t+s_n} S(t+s_n-s)g(s,x(s),x_s)dW(s) \\
&\quad -\int_{-\infty}^{t} S(t-s)\tilde{g}(s,\tilde{x}(s),\tilde{x}_s)dW(s)\Big\|_H^2 \\
&= \lim_{n\to\infty}\sup_{t\in\mathbb{R}}\mathbb{E}\Big\|\int_{-\infty}^{t} S(t-u)g(u+s_n,x(u+s_n),x_{u+s_n})dW(u+s_n) \\
&\quad -\int_{-\infty}^{t} S(t-s)\tilde{g}(s,\tilde{x}(s),\tilde{x}_s)dW(s)\Big\|_H^2 \\
&= \lim_{n\to\infty}\sup_{t\in\mathbb{R}}\mathbb{E}\Big\|\int_{-\infty}^{t} S(t-u)g(u+s_n,x(u+s_n),x_{u+s_n})d(W(u+s_n)-dW(s_n)) \\
&\quad -\int_{-\infty}^{t} S(t-s)\tilde{g}(s,\tilde{x}(s),\tilde{x}_s)dW(s)\Big\|_H^2 \\
&= \lim_{n\to\infty}\sup_{t\in\mathbb{R}}\mathbb{E}\Big\|\int_{-\infty}^{t} S(t-s)g(s+s_n,x(s+s_n),x_{s+s_n})d\tilde{W}(s) \\
&\quad -\int_{-\infty}^{t} S(t-s)\tilde{g}(s,\tilde{x}(s),\tilde{x}_s)dW(s)\Big\|_H^2 \\
&= \lim_{n\to\infty}\sup_{t\in\mathbb{R}}\mathbb{E}\Big\|\int_{-\infty}^{t} S(t-s)[g(s+s_n,x(s+s_n),x_{s+s_n})-\tilde{g}(s,\tilde{x}(s),\tilde{x}_s)]dW(s)\Big\|_H^2.
\end{aligned}$$

By the isometry property of Q-Wiener process, we have for any $t\in\mathbb{R}$,

$$\begin{aligned}
J_3(t) &\leq \lim_{n\to\infty}\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\mathbb{E}\|S(t-s)[g(s+s_n,x(s+s_n),x_{s+s_n}) \\
&\quad -\tilde{g}(s,\tilde{x}(s),\tilde{x}_s)]\|^2_{\mathscr{L}_2(K_Q,H)}ds \\
&\leq \lim_{n\to\infty}\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\mathbb{E}\|S(t-s)\|^2\|g(s+s_n,x(s+s_n),x_{s+s_n}) \\
&\quad -\tilde{g}(s,\tilde{x}(s),\tilde{x}_s)\|^2_{\mathscr{L}_2(K_Q,H)}ds.
\end{aligned}$$

By condition (A), we have

$$\|S(t-s)\|^2\leq M^2e^{-2\gamma(t-s)},\ \text{for any } t\geq s.$$

Hence, we have for any $t \in \mathbb{R}$,

$$\begin{aligned}
J_3(t) &\leq M^2 \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\gamma(t-s)} \mathbb{E}\|g(s+s_n, x(s+s_n), x_{s+s_n}) \qquad (4.30)\\
&\quad -\tilde{g}(s, \tilde{x}(s), \tilde{x}_s)\|^2_{\mathscr{L}_2(K_Q,H)} ds \\
&\leq M^2 \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\gamma(t-s)} \sup_{s\in\mathbb{R}} \mathbb{E}\|g(s+s_n, x(s+s_n), x_{s+s_n}) \\
&\quad -\tilde{g}(s, \tilde{x}(s), \tilde{x}_s)\|^2_{\mathscr{L}_2(K_Q,H)} ds \\
&= M^2 \lim_{n\to\infty} \sup_{s\in\mathbb{R}} \mathbb{E}\|g(s+s_n, x(s+s_n), x_{s+s_n}) \\
&\quad -\tilde{g}(s, \tilde{x}(s), \tilde{x}_s)\|^2_{\mathscr{L}_2(K_Q,H)} \Big(\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\gamma(t-s)} ds \Big).
\end{aligned} \tag{4.31}$$

On the other hand, letting $t - s = u$, we have

$$\begin{aligned}
& \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\gamma(t-s)} ds = -\sup_{t\in\mathbb{R}} \int_{+\infty}^{0} e^{-2\gamma u} du = \sup_{t\in\mathbb{R}} \int_{0}^{+\infty} e^{-2\gamma u} du \\
= \ & -\frac{1}{2\gamma} \sup_{t\in\mathbb{R}} \int_{0}^{+\infty} e^{-2\gamma u} d(-2\gamma u) = \frac{1}{2\gamma}.
\end{aligned} \tag{4.32}$$

Therefore, we further have by (4.30), (4.32) and version of (4.24) for any $t \in \mathbb{R}$,

$$\begin{aligned}
J_3(t) &\leq \frac{M^2}{2\gamma} \lim_{n\to\infty} \sup_{s\in\mathbb{R}} \mathbb{E}\|g(s+s_n, x(s+s_n), x_{s+s_n}) \\
&\quad -\tilde{g}(s, \tilde{x}(s), \tilde{x}_s)\|^2_{\mathscr{L}_2(K_Q,H)} \\
&= 0.
\end{aligned}$$

This implies that $J_3(t)$ is square-mean almost periodic in $t \in \mathbb{R}$.

Finally, we show that $J_4(t)$ is square-mean almost periodic for all $t \in \mathbb{R}$. Let $u = s - s_n, \bar{N}(u, dz) = \tilde{N}(u + u_0, dz) - \tilde{N}(u_0, dz)$, for any $u_0 \in \mathbb{R}$. Note that $\bar{N}(u)$ is also a Lévy process and has the same distribution as $\tilde{N}(u+s_n) - \tilde{N}(s_n)$.

Hence, we have for all $t \in \mathbb{R}$,

$$\begin{aligned}
J_4(t) &= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t+s_n} S(t+s_n-s) \int_H h(s,x(s),x_s,z)\tilde{N}(ds,dz) \\
&\quad - \int_{-\infty}^{t} S(t-s) \int_H \tilde{h}(s,\tilde{x}(s),\tilde{x}_s,z)\tilde{N}(ds,dz)\Big\|_H^2 \\
&= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} S(t-u) \int_H h(u+s_n,x(u+s_n),x_{u+s_n},z)\tilde{N}(d(u+s_n),dz) \\
&\quad - \int_{-\infty}^{t} S(t-s) \int_H \tilde{h}(s,\tilde{x}(s),\tilde{x}_s,z)\tilde{N}(ds,dz)\Big\|_H^2 \\
&= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} S(t-u) \int_H h(u+s_n,x(u+s_n),x_{u+s_n},z)\tilde{N}(d(u+s_n)-ds_n,dz) \\
&\quad - \int_{-\infty}^{t} S(t-s) \int_H \tilde{h}(s,\tilde{x}(s),\tilde{x}_s,z)\tilde{N}(ds,dz)\Big\|_H^2 \\
&= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s) \int_H h(s+s_n,x(s+s_n),x_{s+s_n},z)\bar{N}(ds,dz) \\
&\quad - \int_{-\infty}^{t} S(t-s) \int_H \tilde{h}(s,\tilde{x}(s),\tilde{x}_s,z)\tilde{N}(ds,dz)\Big\|_H^2 \\
&= \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s) \int_H [h(s+s_n,x(s+s_n),x_{s+s_n},z) \\
&\quad -\tilde{h}(s,\tilde{x}(s),\tilde{x}_s,z)]\tilde{N}(ds,dz)\Big\|_H^2.
\end{aligned}$$

By using the isometry property of the compensating Poisson random measure (2.5), we have that for $t \in \mathbb{R}$,

$$\begin{aligned}
J_4(t) &\leq \kappa \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \int_H \mathbb{E}\|S(t-s)h(s+s_n,x(s+s_n),x_{s+s_n},z) \\
&\quad -\tilde{h}(s,\tilde{x}(s),\tilde{x}_s,z)\|_H^2 \nu(dz)ds \\
&\leq \kappa \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \|S(t-s)\|^2 \int_H \mathbb{E}\|h(s+s_n,x(s+s_n),x_{s+s_n},z) \\
&\quad -\tilde{h}(s,\tilde{x}(s),\tilde{x}_s,z)\|_H^2 \nu(dz)ds.
\end{aligned} \tag{4.33}$$

By condition (A), we have

$$\|S(t-s)\|^2 \leq M^2 e^{-2\gamma(t-s)}, \text{ for any } t \geq s. \tag{4.34}$$

Hence, we further have by (4.33), (4.34) and version of (4.25) for any $t \in \mathbb{R}$,

$$\begin{aligned}
J_4(t) &\leq M^2\kappa \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\gamma(t-s)} \int_H \mathbb{E}\|h(s+s_n, x(s+s_n), x_{s+s_n}, z) \\
&\quad -\tilde{h}(s, \tilde{x}(s), \tilde{x}_s, z)\|_H^2 \nu(dz)ds \\
&\leq M^2\kappa \lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\gamma(t-s)} \sup_{s\in\mathbb{R}} \int_H \mathbb{E}\|h(s+s_n, x(s+s_n), x_{s+s_n}, z) \\
&\quad -\tilde{h}(s, \tilde{x}(s), \tilde{x}_s, z)\|_H^2 \nu(dz)ds \\
&= M^2\kappa \lim_{n\to\infty} \sup_{s\in\mathbb{R}} \int_H \mathbb{E}\|h(s+s_n, x(s+s_n), x_{s+s_n}, z) \\
&\quad -\tilde{h}(s, \tilde{x}(s), \tilde{x}_s, z)\|_H^2 \nu(dz)\Big(\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-2\gamma(t-s)} ds\Big) \\
&= \frac{M^2\kappa}{2\gamma} \lim_{n\to\infty} \sup_{s\in\mathbb{R}} \int_H \mathbb{E}\|h(s+s_n, x(s+s_n), x_{s+s_n}, z) \\
&\quad -\tilde{h}(s, \tilde{x}(s), \tilde{x}_s, z)\|_H^2 \nu(dz) \\
&= 0.
\end{aligned}$$

This implies that $J_4(t)$ is square-mean almost periodic in $t \in \mathbb{R}$.

By above discussions, it is clear that $\mathscr{L}$ maps $AP(\mathbb{R}\times\Omega; H)$ into $AP(\mathbb{R}\times\Omega; H)$ itself.

Step 3. $\mathscr{L}$ is a contraction mapping and has a unique fixed point. Assume that $(\mathscr{L}x)(t)$ and $(\mathscr{L}y)(t)$ are defined as in (4.11), respectively. By the relation that $(a+b+c+d)^2 \leq 4a^2 + 4b^2 + 4c^2 + 4d^2$, for any $a, b, c, d \in \mathbb{R}$, we have for any $t \in \mathbb{R}$ that

$$
\begin{aligned}
& \mathbb{E}\|(\mathscr{L}x)(t) - (\mathscr{L}y)(t)\|_H^2 \\
= \; & \mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s) f(s, x(s), x_s) ds + \int_{-\infty}^{t} AS(t-s) G(x(s), x_s) ds \\
& + \int_{-\infty}^{t} S(t-s) g(s, x(s), x_s) dW(s) + \int_{-\infty}^{t} S(t-s) \int_H h(s, x(s), x_s, z) \tilde{N}(ds, dz) \\
& - \Big(\int_{-\infty}^{t} S(t-s) f(s, y(s), y_s) ds + \int_{-\infty}^{t} AS(t-s) G(y(s), y_s) ds \\
& + \int_{-\infty}^{t} S(t-s) g(s, y(s), y_s) dW(s) + \int_{-\infty}^{t} S(t-s) \int_H h(s, y(s), y_s, z) \tilde{N}(ds, dz) \Big) \Big\|_H^2 \\
= \; & \mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s)[f(s, x(s), x_s) - f(s, y(s), y_s)] ds \\
& + \int_{-\infty}^{t} AS(t-s)[G(x(s), x_s) - G(y(s), y_s)] ds \\
& + \int_{-\infty}^{t} S(t-s)[g(s, x(s), x_s) - g(s, y(s), y_s)] dW(s) \\
& + \int_{-\infty}^{t} S(t-s) \int_H [h(s, x(s), x_s, z) - h(s, y(s), y_s, z)] \tilde{N}(ds, dz) \Big\|_H^2 \\
\leq \; & 4\mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s)[f(s, x(s), x_s) - f(s, y(s), y_s)] ds \Big\|_H^2 \\
& + 4\mathbb{E}\Big\| \int_{-\infty}^{t} AS(t-s)[G(x(s), x_s) - G(y(s), y_s)] ds \Big\|_H^2 \\
& + 4\mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s)[g(s, x(s), x_s) - g(s, y(s), y_s)] dW(s) \Big\|_H^2 \\
& + 4\mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s) \int_H [h(s, x(s), x_s, z) - h(s, y(s), y_s, z)] \tilde{N}(ds, dz) \Big\|_H^2 \\
:= \; & A_1(t) + A_2(t) + A_3(t) + A_4(t).
\end{aligned}
$$

From the previous definition , we know $AP(\mathbb{R} \times \Omega, H)$ is a Banach space equipped with the norm

$$\|x\|_\infty = \sup_{s \in \mathbb{R}} (\mathbb{E}\|x(s)\|_H^2)^{\frac{1}{2}}.$$

Then, for every $x, y \in AP(\mathbb{R} \times \Omega, H)$, we have

$$\|x - y\|_\infty^2 = \sup_{s \in \mathbb{R}} \mathbb{E}\|x(s) - y(s)\|_H^2.$$

We first evaluate $A_1(t)$, by using (4.6), (4.7) and Hölder inequality, we have for any $t \in \mathbb{R}$,

$$\begin{aligned}
A_1(t) &= 4\mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s)[f(s, x(s), x_s) - f(s, y(s), y_s)]ds \Big\|_H^2 \\
&\leq 4\mathbb{E}\Big(\int_{-\infty}^{t} \|S(t-s)[f(s, x(s), x_s) - f(s, y(s), y_s)]\|_H ds \Big)^2 \\
&\leq 4\mathbb{E}\Big(\int_{-\infty}^{t} \|S(t-s)\| \|[f(s, x(s), x_s) - f(s, y(s), y_s)]\|_H ds \Big)^2 \\
&\leq 4\mathbb{E}\Big(\int_{-\infty}^{t} M e^{-\frac{\gamma(t-s)}{2}} \cdot \|e^{-\frac{\gamma(t-s)}{2}} f(s, x(s), x_s) - f(s, y(s), y_s)\|_H ds \Big)^2 \\
&\leq 4 \int_{-\infty}^{t} M^2 e^{-\gamma(t-s)} ds \cdot \int_{-\infty}^{t} e^{-\gamma(t-s)} \mathbb{E}\|f(s, x(s), x_s) - f(s, y(s), y_s)\|_H^2 ds \\
&= \frac{4M^2}{\gamma} \int_{-\infty}^{t} e^{-\gamma(t-s)} \mathbb{E}\|f(s, x(s), x_s) - f(s, y(s), y_s)\|_H^2 ds \\
&\leq \frac{4M^2}{\gamma} \int_{-\infty}^{t} e^{-\gamma(t-s)} \sup_{s \in \mathbb{R}} \mathbb{E}\|f(s, x(s), x_s) - f(s, y(s), y_s)\|_H^2 ds \\
&\leq \frac{4M^2}{\gamma} \sup_{s \in \mathbb{R}} \mathbb{E}\|f(s, x(s), x_s) - f(s, y(s), y_s)\|_H^2 \Big(\int_{-\infty}^{t} e^{-\gamma(t-s)} ds \Big) \\
&\leq \frac{4M_f^2 M^2}{\gamma} \sup_{s \in \mathbb{R}} \mathbb{E}(\|x(s) - y(s)\|_H^2 + \|x_s - y_s\|_{\mathscr{B}}^2) \Big(\int_{-\infty}^{t} e^{-\gamma(t-s)} ds \Big) \\
&= \frac{4M_f^2 M^2}{\gamma^2} \sup_{s \in \mathbb{R}} \mathbb{E}(\|x(s) - y(s)\|_H^2 + \|x_s - y_s\|_{\mathscr{B}}^2) \\
&\leq \frac{4M_f^2 M^2 (1 + M_0)}{\gamma^2} \sup_{s \in \mathbb{R}} \mathbb{E}\|x(s) - y(s)\|_H^2 \\
&= \frac{4M_f^2 M^2 (1 + M_0)}{\gamma^2} \|x - y\|_\infty^2.
\end{aligned}$$

Next, we evaluate $A_2(t)$. By Lemma 2.2.1 (2), we have for any $t \in \mathbb{R}$,

$$\begin{aligned}
A_2(t) &= 4\mathbb{E}\Big\|\int_{-\infty}^{t} AS(t-s)[G(x(s),x_s)-G(y(s),y_s)]ds\Big\|_H^2 \\
&= 4\mathbb{E}\Big\|\int_{-\infty}^{t} (-A)^{1-\alpha}(-A)^{\alpha}S(t-s)[G(x(s),x_s)-G(y(s),y_s)]ds\Big\|_H^2 \\
&= 4\mathbb{E}\Big\|\int_{-\infty}^{t} (-A)^{1-\alpha}S(t-s)(-A)^{\alpha}[G(x(s),x_s)-G(y(s),y_s)]ds\Big\|_H^2 \\
&= 4\mathbb{E}\Big(\int_{-\infty}^{t} \|(-A)^{1-\alpha}S(t-s)\|\|(-A)^{\alpha}[G(x(s),x_s)-G(y(s),y_s)]\|_H ds\Big)^2 \\
&\leq 4\mathbb{E}\Big(\int_{-\infty}^{t} M_{1-\alpha}(t-s)^{\alpha-1}e^{-\gamma(t-s)}\|(-A)^{\alpha}[G(x(s),x_s)-G(y(s),y_s)]\|_H ds\Big)^2 \\
&= 4\mathbb{E}\Big(\int_{-\infty}^{t} M_{1-\alpha}e^{-\frac{\gamma(t-s)}{2}}(t-s)^{(\alpha-1)}\Big\|e^{-\frac{\gamma(t-s)}{2}}\Big[(-A)^{\alpha}G(x(s),x_s) \\
&\quad -(-A)^{\alpha}G(y(s),y_s)\Big]\Big\|_H ds\Big)^2.
\end{aligned}$$

By using Hölder inequality and (4.29), we have that for any $t \in \mathbb{R}$,

$$\begin{aligned}
A_2(t) &\leq 4M_{1-\alpha}^2\int_{-\infty}^{t} e^{-\gamma(t-s)}(t-s)^{2(\alpha-1)}ds \cdot \int_{-\infty}^{t} e^{-\gamma(t-s)}\mathbb{E}\|(-A)^{\alpha}G(x(s),x_s) \\
&\quad -(-A)^{\alpha}G(y(s),y_s)\|_H^2 ds \\
&\leq 4M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha-1}} \cdot \int_{-\infty}^{t} e^{-\gamma(t-s)}\mathbb{E}\|(-A)^{\alpha}G(x(s),x_s)-(-A)^{\alpha}G(y(s),y_s)\|_H^2 ds \\
&\leq 4M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha-1}} \cdot \int_{-\infty}^{t} e^{-\gamma(t-s)}\sup_{s\in\mathbb{R}}\mathbb{E}\|(-A)^{\alpha}G(x(s),x_s) \\
&\quad -(-A)^{\alpha}G(y(s),y_s)\|_H^2 ds \\
&\leq 4M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha-1}} \cdot \sup_{s\in\mathbb{R}}\mathbb{E}\|(-A)^{\alpha}G(x(s),x_s)-(-A)^{\alpha}G(y(s),y_s)\|_H^2 \\
&\quad \cdot\Big(\int_{-\infty}^{t} e^{-\gamma(t-s)}ds\Big) \\
&= 4M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha-1}} \cdot \frac{1}{\gamma} \cdot \sup_{s\in\mathbb{R}}\mathbb{E}\|(-A)^{\alpha}G(x(s),x_s)-(-A)^{\alpha}G(y(s),y_s)\|_H^2 \\
&= 4M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha}} \cdot \sup_{s\in\mathbb{R}}\mathbb{E}\|(-A)^{\alpha}G(x(s),x_s)-(-A)^{\alpha}G(y(s),y_s)\|_H^2.
\end{aligned}$$

Then, by using (4.9), we have for any $t \in \mathbb{R}$,

$$
\begin{aligned}
A_2(t) &\leq 4M_G^2 M_{1-\alpha}^2 \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha}} \cdot \sup_{s\in\mathbb{R}} \mathbb{E}(\|x(s)-y(s)\|_H^2 + \|x_s - y_s\|_{\mathscr{B}}^2) \\
&= 4M_G^2 M_{1-\alpha}^2 (1+M_0) \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha}} \cdot \sup_{s\in\mathbb{R}} \mathbb{E}\|x(s)-y(s)\|_H^2 \\
&= 4M_G^2 M_{1-\alpha}^2 (1+M_0) \cdot \frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha}} \|x-y\|_\infty^2.
\end{aligned}
$$

For $A_3(t)$, using isometry identity and (4.6), we obtain that for any $t \in \mathbb{R}$,

$$
\begin{aligned}
A_3(t) &= 4\mathbb{E}\Big\| \int_{-\infty}^t S(t-s)[g(s,x(s),x_s) - g(s,y(s),y_s)]dW(s)\Big\|_H^2 \\
&\leq 4\int_{-\infty}^t \mathbb{E}\|S(t-s)[g(s,x(s),x_s) - g(s,y(s),y_s)]\|_{\mathscr{L}_2(K_Q,H)}^2 ds \\
&= 4\mathbb{E}\Big(\int_{-\infty}^t \|S(t-s)[g(s,x(s),x_s) - g(s,y(s),y_s)]\|_{\mathscr{L}_2(K_Q,H)}^2 ds\Big) \\
&\leq 4\Big(\int_{-\infty}^t \|S(t-s)\|^2 \mathbb{E}\|g(s,x(s),x_s) - g(s,y(s),y_s)\|_{\mathscr{L}_2(K_Q,H)}^2 ds\Big) \\
&\leq 4\Big(\int_{-\infty}^t \|S(t-s)\|^2 \sup_{s\in\mathbb{R}} \mathbb{E}\|g(s,x(s),x_s) - g(s,y(s),y_s)\|_{\mathscr{L}_2(K_Q,H)}^2 ds\Big) \\
&\leq 4M^2 \sup_{s\in\mathbb{R}} \mathbb{E}\|g(s,x(s),x_s) - g(s,y(s),y_s)\|_{\mathscr{L}_2(K_Q,H)}^2 \Big(\int_{-\infty}^t e^{-2\gamma(t-s)} ds\Big).
\end{aligned}
$$

Then, by using (4.8), we have for any $t \in \mathbb{R}$,

$$
\begin{aligned}
A_3(t) &\leq 4M_g^2 M^2 \sup_{s\in\mathbb{R}} \mathbb{E}(\|x(s)-y(s)\|_H^2 + \|x_s - y_s\|_{\mathscr{B}}^2)\Big(\int_{-\infty}^t e^{-2\gamma(t-s)} ds\Big) \\
&= \frac{4M_g^2 M^2}{2\gamma}(1+M_0) \sup_{s\in\mathbb{R}} \mathbb{E}\|x(s)-y(s)\|_H^2 \\
&= \frac{2M_g^2 M^2 (1+M_0)}{\gamma} \|x-y\|_\infty^2.
\end{aligned}
$$

Finally, for $A_4(t)$, by using (4.6), (4.10) and the properties of Poisson random measures, we have for any $t \in \mathbb{R}$,

$$
\begin{aligned}
A_4(t) &= 4\mathbb{E}\Big\| \int_{-\infty}^{t} S(t-s) \int_H [h(s,x(s),x_s,z) - h(s,y(s),y_s,z)]\tilde{N}(ds,dz)\Big\|_H^2 \\
&\le 4\kappa\Big(\int_{-\infty}^{t} \int_H \mathbb{E}\|S(t-s)[h(s,x(s),x_s,z) - h(s,y(s),y_s,z)]\|_H^2 \nu(dz)ds\Big) \\
&\le 4\kappa\Big(\int_{-\infty}^{t} \int_H \mathbb{E}\|S(t-s)\|^2 \|h(s,x(s),x_s,z) - h(s,y(s),y_s,z)\|_H^2 \nu(dz)ds\Big) \\
&\le 4M^2\kappa \int_{-\infty}^{t} e^{-2\gamma(t-s)} \int_H \mathbb{E}\|h(s,x(s),x_s,z) \\
&\quad -h(s,y(s),y_s,z)\|_H^2 \nu(dz)ds \\
&\le 4M^2\kappa \int_{-\infty}^{t} e^{-2\gamma(t-s)} \sup_{s\in\mathbb{R}} \int_H \mathbb{E}\|h(s,x(s),x_s,z) - h(s,y(s),y_s,z)\|_H^2 \nu(dz)ds \\
&= 4M^2\kappa \sup_{s\in\mathbb{R}} \int_H \mathbb{E}\|h(s,x(s),x_s,z) - h(s,y(s),y_s,z)\|_H^2 \nu(dz) \\
&\quad \cdot\Big(\int_{-\infty}^{t} e^{-2\gamma(t-s)} ds\Big) \\
&= \frac{4M^2\kappa}{2\gamma} \sup_{s\in\mathbb{R}} \int_H \mathbb{E}\|h(s,x(s),x_s,z) - h(s,y(s),y_s,z)\|_H^2 \nu(dz) \\
&\le \frac{2M_h M^2\kappa}{\gamma} \sup_{s\in\mathbb{R}} \mathbb{E}(\|x(s)-y(s)\|_H^2 + \|x_s - y_s\|_{\mathscr{B}}^2) \\
&= \frac{2M_h M^2\kappa(1+M_0)}{\gamma} \sup_{s\in\mathbb{R}} \mathbb{E}\|x(s)-y(s)\|_H^2 \\
&= \frac{2M_h M^2\kappa(1+M_0)}{\gamma} \|x-y\|_\infty^2.
\end{aligned}
$$

Thus, by combining $A_1(t), A_2(t), A_3(t)$ and $A_4(t)$, it follows that in Banach space $AP(\mathbb{R}\times\Omega, H)$, we have for $t \in \mathbb{R}$,

$$
\begin{aligned}
&\mathbb{E}\|(\mathscr{L}x)(t) - (\mathscr{L}y)(t)\|_H^2 \\
\le\ & \Big(\frac{4M_f^2 M^2(1+M_0)}{\gamma^2} + 4M_{1-\alpha}^2 M_G^2(1+M_0)\cdot\frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha}} + \frac{2M_g^2M^2(1+M_0)}{\gamma} \\
&+ \frac{2M_hM^2\kappa(1+M_0)}{\gamma}\Big)\|x-y\|_\infty^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|(\mathscr{L}x)-(\mathscr{L}y)\|_\infty^2 = \sup_{t\in\mathbb{R}} \mathbb{E}\|(\mathscr{L}x)-(\mathscr{L}y)\|_H^2 \\
\leq\ & \Big(\frac{4M_f^2M^2(1+M_0)}{\gamma^2} + 4M_{1-\alpha}^2M_G^2(1+M_0)\cdot\frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha}} + \frac{2M_g^2M^2(1+M_0)}{\gamma} \\
&+\frac{2M_hM^2\kappa(1+M_0)}{\gamma}\Big)\|x-y\|_\infty^2.
\end{aligned}
$$

Let

$$
\begin{aligned}
&\frac{4M_f^2M^2(1+M_0)}{\gamma^2} + 4M_{1-\alpha}^2M_G^2(1+M_0)\cdot\frac{\Gamma(2\alpha-1)}{\gamma^{2\alpha}} + \frac{2M_g^2M^2(1+M_0)}{\gamma} \\
&+\frac{2M_hM^2\kappa(1+M_0)}{\gamma} < 1,
\end{aligned}
$$

as we know that $\mathscr{L}$ is a contraction mapping. Therefore, by the contraction mapping principle, $\mathscr{L}$ has a unique fixed point $x(t)$, which obviously means that it is the unique square-mean almost periodic mild solution to equation (4.4). The proof is completed. □

4.4 Summary

In this chapter, we made the first attempt to study the square-mean almost periodic solutions for a class of neutral stochastic evolution equations with Poisson jumps and infinite delay. Our work extended that of Li, Liu and Luo (2014) where the neutral stochastic evolution equation without Poisson jumps is investigated. We also extended that of Wang and Liu (2012) where the infinite delay for a class of stochastic differential equation with Lévy process is not studied. In addition, we discussed the existence and uniqueness of the square-mean almost periodic solutions for the stochastic evolution system with Poisson jumps.

Chapter 5

Global Attracting Set and Stability of Neutral SPDEs Driven by α-Stable Processes with Impulses

5.1 Introduction

Th stability of stochastic partial differential equations (SPDEs) driven by Brownian motions or Lévy processes have been well established. Especially, the study of stochastic neutral functional differential equations have received a great deal of attention in recent year. For example, Bao and Yuan [7] extended the stochastic stabilization problems of PDEs that is perturbed by Lévy noise from finite dimension to infinite dimension. Bao and Hou [5] extended the existence and uniqueness of mild solutions to a class of general stochastic neutral partial functional differential equations under non-Lipschitz conditions. Caraballo, Real and Taniguchi [15] investigated the exponential stability and ultimate boundedness of the solutions to a class of neutral stochastic semilinear partial delay differential equations. Yuan and Bao [64] focused on the path wise stability of mild solutions for a class of stochastic partial differential equations which are driven by switching-diffusion processes with jumps.

However, such restriction clearly rules out the interesting α-stable processes since Wiener noise and Poisson-jump noise have arbitrary finite moments, while α-stable noise only has finite p-th moment for $p \in (0, \alpha)$ with $\alpha < 2$. Recently, stochastic equations driven by α-stable processes have plenty of applications in physics due to the fact that α-stable noise exhibits the heavy tailed phenomenon, e.g., Priola and Zabczyk [50] gave a proper starting point on the investigation of structural properties of stochastic partial differential equations (SPDEs) driven by an additive cylindrical stable noise. Dong, Xu and Zhang [25] studied the invariant measures of stochastic 2D Navier-Stokes equation driven by α-stable processes, Xu studied [61] Ergodicity of the stochastic real Ginzburg-Landau equation driven by α-stable noise and Zhang [67] proved a derivative formula of Bismut-Elworthy-Li's type as well as gradient estimate for stochastic differential equations driven by α-stable noises. One the other hand, Wang [55] derived the gradient estimate for Ornstein-Uhlenbeck jump processes and Wang [58] established so-called Harnack inequalities for SDEs driven by cylindrical α-stable processes. However, there are few papers on the asymptotic behaviour of mild solution of SPDEs driven by α-stable processes, so we shall discuss the stability property of mild solutions of a class of SPDEs driven by α-stable processes to complete the theory. The fact is that α-stable noise only has finite p-th moment for $p \in (0, \alpha)$ and the stochastic evolution does not admit a stochastic differential , which leads to some powerful tools such as the Itô formula being unavailable, then some new methods should be used to overcome the difficulties. It is worthwhile to mention that, Wang and Rao [56] discussed the stability of mild solutions for a class of SPDEs driven by α-stable noises and generalized to deal with the SPDEs driven by subordinated cylindrical Brownian motion and fractional Brownian motion, respectively by the Minkovski inequality and Zang and Li [65] proved the existence and uniqueness of the mild solution to a class of neutral SPDEs.

In addition, attracting sets of dynamical systems have been studied extensively by many researchers. Xu and Long [60] studied the attracting and quasi-invariant sets of non-autonomous neutral networks with delays. Xu and Xu [62] considered the P-attracting and p-invariant sets for a class of impulsive stochastic functional differential equations. Long, Teng and Xu [43] investigated the global attracting set and stability of stochastic neutral partial functional differential equations with impulses. They first established a new impulsive-integral inequality, which improved the inequality established by Chen [16]. On the other hand, impulsive phenomenon can be found in a wide variety of evolutionary processes, for example, medicine and biology, economics, mechanics, electronics and telecommunications, etc., in which many sudden and abrupt changes occur instantaneously, in the form of impulses. Many interesting results haven been found, e.g., ([66], [47]). But to the best of my knowledge, there are no results on the Global attracting set and exponential stability of neutral SPDEs driven by α-stable processes with impulses. On the basis of this, this chapter is devoted to the discussion of this problem. The problem of determining the attracting sets of neutral stochastic partial differential equations driven by α-stable noise with impulses is more complicated. Therefore, the techniques and methods for the global attracting set and exponential stability of neutral SPDEs driven by α-stable processes with impulses should be developed.

Motivated by the above discussions, we shall consider the following neutral stochastic partial differential equations driven by an additive α-stable with impulses on a separable Hilbert space H,

$$
\begin{cases}
d[x(t) - g(t, x(t-r)] = (Ax(t) + f(t, x(t-r)))dt \\
\qquad\qquad +\sigma(t)dZ(t), t \geq 0, t \neq t_k, \\
\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), t = t_k, k = 1, 2, ..., \\
x_0(\cdot) = \phi(\cdot) \in D([-r, 0], H),
\end{cases} \tag{5.1}
$$

where $r > 0$ and A generates a strongly continuous semigroup $S(t)$ or $e^{tA}, t \geq 0$, on H. Assume that $f, g : \mathbb{R}_+ \times H \to H$ are two given measurable mappings and $\sigma(t) : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function; $I_K : H \to H$ is a measurable mapping for $k = 1, 2, ...$; the fixed moments of time t_k satisfies $0 < t_1 < t_2 < ... < t_k < ...$, and $\lim_{k\to\infty} t_k = \infty; x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k, k = 1, 2, ...$, respectively; $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k with I_k determining the size of the jump.

This chapter is organised as follows. In Section 5.2, we review and introduce the concept and basic property of α-processes. In Section 5.3, we consider the global attracting set and stability of the neutral stochastic differential equations with impulses. In Section 5.4, we have a summary to state the contribution and development of the chapter.

5.2 α-stable processes

Recall that X is a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and takes values in $\mathbb{R}$ with probability law p_X. Its characteristic function $\phi_X : \mathbb{R} \to \mathbb{C}$ is defined by

$$
\begin{aligned}
\phi_X(u) &= \mathbb{E}(e^{i(u,X)}) = \int_\Omega e^{i(u,X(\omega))}\mathbb{P}(d\omega) \\
&= \int_\mathbb{R} e^{i(u,y)} p_X(dy),
\end{aligned}
$$

for each $u \in \mathbb{R}$. Particularly, a real-valued stochastic process $\{X(t) : t \geq 0\}$ is called an Lévy α-stable process if

1. $X(0) = 0$ a.s;
2. $X(t)$ has independent and stationary increments;
3. $\phi_{X(t)}(u) = e^{\eta(t,u)} = e^{t\eta(1,u)}$, for each $u \in \mathbb{R}, t \geq 0$,

where $\eta(1, \cdot)$ is the Lévy symbol of $X(1)$ and $X(1)$ is uniquely determined by its characteristic function involved with four parameters: $\alpha \in (0, 2)$, the index of stability; $\beta \in [-1, 1]$, the skewness parameter; $\sigma \in (0, \infty)$, the scale parameter and $\mu \in (-\infty, \infty)$, the shift. We call η strictly α-stable whenever $u = 0$, and in addition, if $\beta = 0$, η is said to be symmetric α-stable.

Theorem 5.2.1 *A real-valued random variable X is α-stable if and only if there exist $\sigma > 0, -1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that for all $u \in \mathbb{R}$:*

(1) when $\alpha = 2$,

$$\phi_X(u) = exp(i\mu u - \frac{1}{2}\sigma^2 u^2);$$

(2) when $\alpha \neq 1, 2$,

$$\phi_X(u) = exp\Big(i\mu u - \sigma^\alpha |u|^\alpha \Big[1 - i\beta sgn(u) tan\Big(\frac{\pi\alpha}{2}\Big)\Big]\Big);$$

(3) when $\alpha = 1$,

$$\phi_X(u) = exp\Big(i\mu u - \sigma|u|\Big[1 - i\beta\frac{2}{\pi} sgn(u) log(|u|)\Big]\Big).$$

It can be shown that $\mathbb{E}(X^2) < \infty$ if and only if $\alpha = 2$ (i.e. X is Gaussian) and that $\mathbb{E}(|X|) < \infty$ if and only if $1 < \alpha \leq 2$. For more details on α-stable processes, we refer to [1].

Let $Z(t)$ be a cylindrical α-stable process, $\alpha \in (0,2)$, defined by

$$Z(t) := \sum_{m=1}^{\infty} \beta_m Z_m(t) e_m. \tag{5.2}$$

Here $\{e_m\}_{m\geq 1}$ is an orthonormal basis of H, $\{Z_m(t)\}_{m\geq 1}$ is a group of i.i.d. real-valued, symmetric α-stable Lévy processes with $\alpha > 1$ defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and $\{\beta_m\}_{m\geq 1}$ is a sequence of positive numbers which denote the intensity of the noise so that the series (5.2) is well-defined in a proper sense.

5.3 Global attracting set and stability

Throughout this chapter, we use the following notations. Let $(H, \|\cdot\|)$ be a real separable Hilbert space. Recall that a function $f : [-r, 0] \to H$ is called the càdlàg if it is right-continuous and has finite left-hand limits. Denote by $D([-r, 0], H)$ the space of all H-valued càdlàg functions defined on $[-r, 0]$, equipped with the uniform norm $\|\phi\|_D := \sup_{-r\leq s\leq 0} \|\phi(s)\|, \phi \in D([-r, 0], H)$.

In this section, we shall consider the global attracting set of the neutral stochastic differential equation with impulses (5.1). We first give the following definition of mild solutions to equation (5.1).

Definition 5.3.1 *An $\mathcal{F}_t$-adapted càdlàg H-valued stochastic process $x(t), t \geq 0$, is called the mild solution for (5.1) if it has the following properties:*

1. $x_0(\cdot) = \phi(\cdot) \in D([-r, 0]; H)$;
2. $\int_0^T \|x(u)\|_H^2 du < \infty$ *almost surely;*
3. *for each* $t_k, x(t_k^+) = \lim_{t\to t_{k+}} x(t)$ *exists and* $x(t_k^-) = \lim_{t\to t_{k-}} x(t)$ *exists;*

4. for arbitrary $t \geq 0, x(t)$ satisfied the following integral equation:

$$\begin{aligned} x(t) &= S(t)[\phi(0) + g(0, \phi(-r))] - g(t, x(t-r)) - \int_0^t AS(t-s)g(s, x(s-r))ds \\ &\quad + \int_0^t S(t-s)f(s, x(s-r))ds + \int_0^t S(t-s)\sigma(s)dZ(s) \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)). \end{aligned} \tag{5.3}$$

Here for the solution process $\{x(t)\}_{t\geq -r}$ with initial value $\phi \in D([-r, 0]; H)$, we put $x_t(\phi) := \{x(t+\theta; \phi) : -r \leq \theta \leq 0\}$ for all $t \geq 0$. Quite frequently, stochastic process $\{x_t(\phi)\}_{t\geq -r}$ is called the *segment process* of $\{x(t, \phi)\}_{t\geq -r}$.

In what follows, we need the following assumptions:

$(H1)$ The operator $(A, \mathcal{D}(A))$ is a self-adjoint operator on the separable Hilbert space H admitting a discrete spectrum

$-\infty \leftarrow -\lambda_m \leq -\lambda_{m-1} \leq ... \leq -\lambda_2 \leq -\lambda_1 < 0$ with corresponding eigenvector basis $\{e_m\}_{m\geq 1}$ of H and generating an analytic semigroup $S(t), t \geq 0$, such that $\|S(t)\| \leq Me^{-\lambda_1 t}, M \geq 1$ for all $t \geq 0$.

$(H2)$ There exists a positive constant K_1 such that for all $x, y \in H$ and $t \geq 0$,

$$\|f(t,x) - f(t,y)\| \leq K_1\|x-y\|, \qquad \|f(t,x)\| \leq K_1(1 + \|x\|).$$

$(H3)$ There exists a constant $\kappa \in (0, 1)$ and a positive constant K_2 such that and for all $x, y \in H$ and $t \geq 0$,

$$\|(-A)^\kappa g(t,x) - (-A)^\kappa g(t,y)\| \leq K_2\|x-y\|, \qquad g(t, 0) = 0,$$

where $(-A)^\kappa$ is the fractional power of operator $-A$.

$(H4)$ There exists a sequence of positive numbers $q_k(k = 1, 2, ...)$ such that for any $x, y \in H$ and $\sum_{k=1}^{+\infty} q_k < \infty$,

$$\|I_k(x) - I_k(y)\| \leq q_k \|x - y\|, \qquad I_k(0) = 0, k = 1, 2, ...$$

Definition 5.3.2 *Let $p \geq 1$and a set $S \subset H$ is called the p-th global attracting set of (5.1) if for all initial value $\phi(\cdot) \in D([-r, 0], H)$, the solution process $\{x(t, \phi)\}_{t \geq -r}$ of (5.1) converges to S as $t \to \infty$, i.e.,*

$$dist\ (x(t, \phi), S) \to 0 \quad t \to \infty,$$

where $dist\ x(t, S) = \inf_{y \in S} \mathbb{E}\|x - y\|^p, p \geq 1$.

Lemma 5.3.1 *[36] Let Z be a cylindrical α-stable process, $\alpha \in (0, 2)$. Assume that the condition (H1) holds, then for any $t \geq 0$ and $p > 0$,*

$$\mathbb{E}\Big\| \int_0^t S(t-s)\sigma(s)dZ(s)\Big\|^p \leq C_{p,\alpha}\Big(\sum_{k=1}^{\infty} \beta_k^\alpha \int_0^t e^{-\alpha\lambda_k(t-s)}\sigma^\alpha(s)ds\Big)^{\frac{p}{\alpha}} \tag{5.4}$$

where $\{\beta_k\}_{k \geq 1}$ is the sequence given in (5.2) and the constant $C_{p,\alpha} > 0$ depends only on p and α.

Lemma 5.3.2 *[43] Suppose that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lipschitz continuous function. Let $y : [-r, \infty) \to \mathbb{R}_+$ be a Borel measurable function which is a solution of delay integral inequality*

$$y(t) \leq \begin{cases} g(\|\phi\|_D)e^{-\gamma t} + b_1\|y_t\|_D + b_2 \int_0^t e^{-\gamma(t-s)}\|y_s\|_D ds + \sum_{0<t_k<t} c_k e^{-\gamma(t-t_k)} y(t_k^-) \\ \qquad +J, \quad t \geq 0, \\ \phi(t), \qquad t \in [-r, 0], \end{cases} \tag{5.5}$$

where $\phi \in D([-r, 0], \mathbb{R}_+), \gamma > 0, b_1, b_2$ and J are nonnegative constants. Then for *any* $\phi \in D([-r,0],\mathbb{R}_+)$ *satisfying* $\|\phi\|_D \le K$ *for some constant* $K>0$ *and*

$$b_1 + \frac{b_2}{\gamma} + \sum_{k=1}^{+\infty} c_k := \rho < 1.$$

Then there are constants $\lambda \in (0,\gamma)$ *and* $N \ge K$ *such that*

$$y(t) \le Ne^{-\gamma t} + \frac{J}{1-\rho}, \quad \forall t \ge 0,$$

where λ *and* N *satisfy that*

$$\rho_\lambda := b_1 e^{\lambda r} + \frac{b_2 e^{\lambda r}}{\gamma - \lambda} + \sum_{k=1}^{+\infty} c_k < 1 \quad \textit{and} \quad N \ge \frac{K}{1-\rho_\lambda},$$

or if $b_2 \neq 0$*, that*

$$\rho_\lambda := b_1 e^{\lambda r} + \frac{b_2 e^{\lambda r}}{\gamma - \lambda} + \sum_{k=1}^{+\infty} c_k \le 1 \quad \textit{and} \quad N \ge \frac{(\gamma-\lambda)[K - \frac{b_2 J}{\gamma(1-\rho)}]}{b_2 e^{\lambda r}}.$$

Theorem 5.3.1 *Let* $\phi(\cdot) \in D([-r,0],H)$*. Assume that the conditions (H1) - (H4) are satisfied. Then the set*

$$\mathbb{S} = \Big\{ y \in H : \|y\| \le (\frac{J}{1-\rho})^{\frac{1}{p}} \Big\} \tag{5.6}$$

is a global attracting set of (5.1) provided that the following relations

$$\begin{aligned} \rho : \ &= 6\|(-A)^{-\kappa}\|^p K_2^p + 6M_{1-\kappa}K_2^p \lambda_1^{p(1-\kappa)-\frac{p+q}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} + 12M^p K_1^p \lambda_1^{-\frac{p+q}{q}} \\ &\quad + \sum_{k=1}^{+\infty} 6M^p \Big(\sum_{k=1}^{+\infty} q_k \Big)^{\frac{p}{q}} q_k < 1, \end{aligned} \tag{5.7}$$

where $\frac{1}{p} + \frac{1}{q} = 1, 0^0 = 1,$

and

$$\sup_{t\geq 0}\Big(\sum_{k=1}^{\infty}\beta_k^{\alpha}\int_0^t e^{-\alpha\lambda_k(t-s)}\sigma^{\alpha}(s)ds\Big)<\infty \tag{5.8}$$

hold for $\kappa\in(0,1),\alpha\in(1,2),p\in(1,\alpha)$ *where* $\Gamma(\cdot)$ *is the standard Gamma function,*

$$J=6C_{p,\alpha}\Big(\sum_{k=1}^{\infty}\beta_k^{\alpha}\int_0^t e^{-\alpha\lambda_k(t-s)}\sigma^{\alpha}(s)ds\Big)^{\frac{p}{\alpha}}+12M^pK_1^p\lambda_1^{-\frac{p+q}{q}}$$

and $C_{p,\alpha}>0$ *is the constant given in (5.4).*

Proof: From Remark 1.1 in [6] and Theorem 5.4 in [50], we know that under the conditions (H1)-(H4), (5.7) and (5.8), the equation (5.1) has a unique mild solution. Hence, from (5.3) and the relation that $(a+b+c+d+e+f)^p\leq 6^p(a^p+b^p+c^p+d^p+e^p+f^p)$, for any $a,b,c,d,e,f\in\mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}\|x(t)\|^p &= \mathbb{E}\Big\|S(t)[\phi(0)+g(0,\phi(-r))]-g(t,x(t-r))-\int_0^t AS(t-s)g(s,x(s-r))ds\\
&\quad+\int_0^t S(t-s)f(s,x(s-r))ds+\int_0^t S(t-s)\sigma(s)dZ(s)\\
&\quad+\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k^-))\Big\|^p\\
&\leq 6^p\mathbb{E}\|S(t)[\phi(0)+g(0,\phi(-r))]\|^p+6^p\mathbb{E}\|g(t,x(t-r))\|^p\\
&\quad+6^p\mathbb{E}\Big\|\int_0^t AS(t-s)g(s,x(s-r))ds\Big\|^p+6^p\mathbb{E}\Big\|\int_0^t S(t-s)f(s,x(s-r))ds\Big\|^p\\
&\quad+6^p\mathbb{E}\Big\|\int_0^t S(t-s)\sigma(t)dZ(s)\Big\|^p+6^p\mathbb{E}\|\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k^-))\|^p\\
&:= 6^p(J_1(t)+J_2(t)+J_3(t)+J_4(t)+J_5(t)+J_6(t)),\quad \forall t\geq 0.
\end{aligned}\tag{5.9}$$

It follows from $(H3)$ for any $t \geq 0$ that

$$\begin{aligned}
J_1(t) &= \mathbb{E}\|S(t)[\phi(0) + g(0, \phi(-r))]\|^p \\
&\leq \mathbb{E}\|\phi(0) + g(0, \phi(-r))\|^p \|S(t)\|^p \\
&\leq \mathbb{E}\|\phi(0) + (-A)^{-\kappa}(-A)^{\kappa} g(0, \phi(-r))\|^p M^p e^{-p\lambda_1 t} \\
&\leq \mathbb{E}[\|\phi(0)\| + \|(-A)^{-\kappa}\| \cdot \|(-A)^{\kappa} g(0, \phi(-r))\|]^p M^p e^{-p\lambda_1 t} \\
&\leq \mathbb{E}[\|\phi(0)\| + \|(-A)^{-\kappa}\| K_2 \|\phi(-r)\|]^p M^p e^{-p\lambda_1 t} \\
&\leq \mathbb{E}[\|\phi(0)\| + K_2 \|(-A)^{-\kappa}\| \cdot \|\phi\|_D]^p M^p e^{-p\lambda_1 t} \\
&\leq M^p e^{-p\lambda_1 t} 2^p \mathbb{E}(\|\phi(0)\|^p + K_2^p \|(-A)^{-\kappa}\|^p \cdot \|\phi\|_D^p) \\
&\leq M^p e^{-p\lambda_1 t} 2^p \mathbb{E}(\|\phi\|_D^p + K_2^p \|(-A)^{-\kappa}\|^p \cdot \|\phi\|_D^p) \\
&= 2^p M^p (1 + K_2^p \|(-A)^{-\kappa}\|^p) \|\phi\|_D^p e^{-p\lambda_1 t} \\
&= C^* \|\phi\|_D^p e^{-p\lambda_1 t},
\end{aligned} \tag{5.10}$$

where $C^* = 2^p M^p (1 + K_2^p \|(-A)^{-\kappa}\|^p) > 0$ is a positive constant.

It follows from $(H3)$ for any $t \geq 0$ that

$$\begin{aligned}
J_2(t) &= \mathbb{E}\|g(t, x(t-r))\|^p \\
&= \mathbb{E}\|(-A)^{-\kappa}(-A)^{\kappa} g(t, x(t-r))\|^p \\
&\leq \mathbb{E}[\|(-A)^{-\kappa}\| \|(-A)^{\kappa} g(t, x(t-r))\|]^p \\
&\leq \mathbb{E}[\|(-A)^{-\kappa}\| K_2 \|x(t-r)\|]^p \\
&\leq \|(-A)^{-\kappa}\|^p K_2^p \mathbb{E}\|x(t-r)\|^p \\
&\leq \|(-A)^{-\kappa}\|^p K_2^p \sup_{-r \leq \theta \leq 0} \mathbb{E}\|x(t+\theta)\|^p.
\end{aligned} \tag{5.11}$$

For $J_3(t)$, by using Lemma 2.2.1 (2) and $(H3)$ for any $t \geq 0$, we have

$$\begin{aligned}
J_3(t) &= \mathbb{E}\Big\| \int_0^t AS(t-s)g(s,x(s-r))ds\Big\|^p \\
&= \mathbb{E}\Big\| \int_0^t (-A)^{1-\kappa}S(t-s)(-A)^{\kappa}g(s,x(s-r))ds\Big\|^p \\
&\leq \mathbb{E}\Big(\int_0^t \|(-A)^{1-\kappa}S(t-s)(-A)^{\kappa}g(s,x(s-r))\|ds\Big)^p \\
&\leq \mathbb{E}\Big(\int_0^t \|(-A)^{1-\kappa}S(t-s)\|\|(-A)^{\kappa}g(s,x(s-r))\|ds\Big)^p \\
&\leq \mathbb{E}\Big(\int_0^t \frac{M_{1-\kappa}e^{-\lambda_1(t-s)}}{(t-s)^{1-\kappa}}K_2\|x(s-r)\|ds\Big)^p \\
&= \mathbb{E}\Big(\int_0^t \frac{M_{1-\kappa}e^{-\lambda_1(t-s)\cdot\frac{1}{q}}}{(t-s)^{1-\kappa}}\cdot e^{-\lambda_1(t-s)\cdot\frac{1}{p}}K_2\|x(s-r)\|ds\Big)^p.
\end{aligned}$$

Then, by using Hölder inequality, we have for any $t \geq 0$,

$$\begin{aligned}
J_3(t) &\leq M_{1-\kappa}^p\mathbb{E}\Big(\Big[\int_0^t \frac{e^{-\lambda_1(t-s)\cdot\frac{1}{q}\cdot q}}{(t-s)^{q(1-\kappa)}}ds\Big]^{\frac{1}{q}}\cdot\Big[\int_0^t e^{-\lambda_1(t-s)\cdot\frac{1}{p}\cdot p}K_2^p\|x(s-r)\|^p ds\Big]^{\frac{1}{p}}\Big)^p \\
&= M_{1-\kappa}^p\mathbb{E}\Big(\Big[\int_0^t \frac{e^{-\lambda_1(t-s)}}{(t-s)^{q(1-\kappa)}}ds\Big]^{\frac{1}{q}}\cdot\Big[\int_0^t e^{-\lambda_1(t-s)}K_2^p\|x(s-r)\|^p ds\Big]^{\frac{1}{p}}\Big)^p \\
&= M_{1-\kappa}^p\Big[\int_0^t \frac{e^{-\lambda_1(t-s)}}{(t-s)^{q(1-\kappa)}}ds\Big]^{\frac{p}{q}}\mathbb{E}\Big(\Big[\int_0^t e^{-\lambda_1(t-s)}K_2^p\|x(s-r)\|^p ds\Big]^{\frac{1}{p}}\Big)^p \\
&= M_{1-\kappa}^p\Big[\int_0^t \frac{e^{-\lambda_1(t-s)}}{(t-s)^{q(1-\kappa)}}ds\Big]^{\frac{p}{q}}\mathbb{E}\Big(\int_0^t e^{-\lambda_1(t-s)}K_2^p\|x(s-r)\|^p ds\Big) \\
&= M_{1-\kappa}^p\Big[\int_0^t \frac{\lambda_1^{q(1-\kappa)}e^{-\lambda_1(t-s)}}{[\lambda_1(t-s)]^{q(1-\kappa)}}ds\Big]^{\frac{p}{q}}\cdot\mathbb{E}\Big(\int_0^t e^{-\lambda_1(t-s)}K_2^p\|x(s-r)\|^p ds\Big).
\end{aligned} \tag{5.12}$$

On the other hand, letting $\lambda_1(t-s) = u$, we have

$$\begin{aligned}
&\int_0^t \frac{\lambda_1^{q(1-\kappa)}e^{-\lambda_1(t-s)}}{[\lambda_1(t-s)]^{q(1-\kappa)}}ds = \int_{\lambda_1 t}^0 \frac{\lambda_1^{q(1-\kappa)}e^{-u}}{u^{q(1-\kappa)}}(-\frac{1}{\lambda_1})du \\
&= \int_0^{\lambda_1 t} \frac{\lambda_1^{q(1-\kappa)}e^{-u}}{u^{q(1-\kappa)}}(\frac{1}{\lambda_1})du \\
&\leq \frac{1}{\lambda_1}\int_0^{\infty} \lambda_1^{q(1-\kappa)}e^{-u}u^{-q(1-\kappa)}du.
\end{aligned} \tag{5.13}$$

Then substituting (5.13) into (5.12), and by using the definition of Gamma function, we have for any $t \geq 0$,

$$
\begin{aligned}
J_3(t) &\leq M_{1-\kappa}^p (\lambda_1^{q(1-\kappa)-1})^{\frac{p}{q}} \Big[\int_0^\infty e^{-u} u^{-q(1-\kappa)} du\Big]^{\frac{p}{q}} \cdot \\
&\quad \mathbb{E}\Big(\int_0^t e^{-\lambda_1(t-s)} K_2^p \|x(s-r)\|^p ds\Big) \\
&= M_{1-\kappa}^p \lambda_1^{p(1-\kappa)-\frac{p}{q}} \Big[\int_0^\infty e^{-u} u^{-q(1-\kappa)} du\Big]^{\frac{p}{q}} \cdot \\
&\quad \mathbb{E}\Big(\int_0^t e^{-\lambda_1(t-s)} K_2^p \|x(s-r)\|^p ds\Big) \\
&= M_{1-\kappa}^p \lambda_1^{p(1-\kappa)-\frac{p}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} \cdot \mathbb{E}\Big(\int_0^t e^{-\lambda_1(t-s)} K_2^p \|x(s-r)\|^p ds\Big) \\
&= M_{1-\kappa}^p \lambda_1^{p(1-\kappa)-\frac{p}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} K_2^p \int_0^t e^{-\lambda_1(t-s)} \mathbb{E}\|x(s-r)\|^p ds \\
&\leq M_{1-\kappa}^p K_2^p \lambda_1^{p(1-\kappa)-\frac{p}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} \int_0^t e^{-\lambda_1(t-s)} \sup_{-r\leq\theta\leq 0} \mathbb{E}\|x(s+\theta)\|^p ds,
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

From (H_2) for any $t \geq 0$, we obtain

$$
\begin{aligned}
J_4(t) &= \mathbb{E}\Big\|\int_0^t S(t-s) f(s, x(s-r)) ds\Big\|^p \\
&\leq \mathbb{E}\Big(\int_0^t \|S(t-s)\| \|f(s, x(s-r))\| ds\Big)^p \\
&\leq \mathbb{E}\Big(\int_0^t M e^{-\lambda_1(t-s)} K_1(1 + \|x(s-r)\|) ds\Big)^p \\
&= \mathbb{E}\Big(\int_0^t M e^{-\lambda_1(t-s)\cdot\frac{1}{q}} \cdot e^{-\lambda_1(t-s)\cdot\frac{1}{p}} K_1(1 + \|x(s-r)\|) ds\Big)^p \\
&= M^p \mathbb{E}\Big(\int_0^t e^{-\lambda_1(t-s)\cdot\frac{1}{q}} \cdot e^{-\lambda_1(t-s)\cdot\frac{1}{p}} K_1(1 + \|x(s-r)\|) ds\Big)^p.
\end{aligned}
$$

Then, by using Hölder inequality, we have for any $t \geq 0$,

$$\begin{aligned}
J_4(t) &\leq M^p \mathbb{E}\Big(\Big[\int_0^t e^{-\lambda_1 (t-s)\cdot \frac{1}{q}\cdot q} ds \Big]^{\frac{1}{q}} \cdot \Big[\int_0^t e^{-\lambda_1 (t-s)\cdot \frac{1}{p}\cdot p} K_1^p (1 + \|x(s-r)\|)^p ds \Big]^{\frac{1}{p}} \Big)^p \\
&= M^p \mathbb{E}\Big(\Big[\int_0^t e^{-\lambda_1 (t-s)} ds \Big]^{\frac{1}{q}} \cdot \Big[\int_0^t e^{-\lambda_1 (t-s)} K_1^p (1 + \|x(s-r)\|)^p ds \Big]^{\frac{1}{p}} \Big)^p \\
&= M^p \Big[\int_0^t e^{-\lambda_1 (t-s)} ds \Big]^{\frac{p}{q}} \mathbb{E}\Big(\Big[\int_0^t e^{-\lambda_1 (t-s)} K_1^p (1 + \|x(s-r)\|)^p ds \Big]^{\frac{1}{p}} \Big)^p \\
&= M^p \Big[\int_0^t e^{-\lambda_1 (t-s)} ds \Big]^{\frac{p}{q}} \mathbb{E}\Big(\int_0^t e^{-\lambda_1 (t-s)} K_1^p (1 + \|x(s-r)\|)^p ds \Big) \\
&\leq M^p \Big[\lambda_1^{-1} \cdot \int_0^t e^{-\lambda_1 (t-s)} d[-\lambda_1 (t-s)] \Big]^{\frac{p}{q}} \mathbb{E}\Big(\int_0^t e^{-\lambda_1 (t-s)} K_1^p (1 + \|x(s-r)\|)^p ds \Big) \\
&= M^p \lambda_1^{-\frac{p}{q}} \Big[\int_0^t e^{-\lambda_1 (t-s)} d[-\lambda_1 (t-s)] \Big]^{\frac{p}{q}} \mathbb{E}\Big(\int_0^t e^{-\lambda_1 (t-s)} K_1^p (1 + |x(s-r)\|)^p ds \Big) \\
&\leq M^p \lambda_1^{-\frac{p}{q}} \Big[\int_{-\infty}^t e^{-\lambda_1 (t-s)} d[-\lambda_1 (t-s)] \Big]^{\frac{p}{q}} \cdot \mathbb{E}\Big(\int_0^t e^{-\lambda_1 (t-s)} K_1^p (1 + \|x(s-r)\|)^p ds \Big) \\
&= M^p K_1^p \lambda_1^{-\frac{p}{q}} \int_0^t e^{-\lambda_1 (t-s)} \mathbb{E}(1 + \|x(s-r)\|)^p ds \\
&\leq M^p K_1^p \lambda_1^{-\frac{p}{q}} \int_0^t e^{-\lambda_1 (t-s)} \sup_{-r \leq \theta \leq 0} \mathbb{E}(1 + \|x(s+\theta)\|)^p ds \\
&\leq 2^p M^p K_1^p \lambda_1^{-\frac{p}{q}} \int_0^t e^{-\lambda_1 (t-s)} \Big(1 + \sup_{-r \leq \theta \leq 0} \mathbb{E}\|x(s+\theta)\|^p \Big) ds,
\end{aligned} \tag{5.14}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For $J_5(t)$, from Lemma 5.3.1, we have

$$\mathbb{E}\Big\| \int_0^t S(t-s)\sigma(s) dZ(s) \Big\|^p \leq C_{p,\alpha} \Big(\sum_{k=1}^{\infty} \beta_k^{\alpha} \int_0^t e^{-\alpha \lambda_k (t-s)} \sigma^{\alpha}(s) ds \Big)^{\frac{p}{\alpha}}, \tag{5.15}$$

where the constant $C_{p,\alpha} > 0$ depends only on p and α.

From (H_4) and Hölder inequality for any $t \geq 0$, we obtain

$$
\begin{aligned}
J_6(t) &= \mathbb{E}\| \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-))\|^p \\
&\leq \mathbb{E}\Big(\sum_{0<t_k<t} \|S(t-t_k)\| \|I_k(x(t_k^-))\|\Big)^p \\
&\leq \mathbb{E}\Big(\sum_{0<t_k<t} M e^{-\lambda_1(t-t_k)} \|I_k(x(t_k^-))\|\Big)^p \\
&\leq M^p \mathbb{E}\Big(\sum_{0<t_k<t} q_k e^{-\lambda_1(t-t_k)} \|x(t_k^-)\|\Big)^p \\
&= M^p \mathbb{E}\Big(\sum_{0<t_k<t} q_k^{\frac{1}{q}} \cdot q_k^{\frac{1}{p}} e^{-\lambda_1(t-t_k)} \|x(t_k^-)\|\Big)^p \\
&\leq M^p \mathbb{E}\Big((\sum_{0<t_k<t} (q_k^{\frac{1}{q}})^q)^{\frac{1}{q}} \cdot (\sum_{0<t_k<t} (q_k^{\frac{1}{p}} e^{-\lambda_1(t-t_k)} \|x(t_k^-)\|)^p)^{\frac{1}{p}}\Big)^p \\
&= M^p \Big(\sum_{0<t_k<t} q_k\Big)^{\frac{p}{q}} \cdot \mathbb{E}\Big(\sum_{0<t_k<t} q_k (e^{-\lambda_1(t-t_k)} \|x(t_k^-)\|)^p\Big),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
J_6(t) &\leq M^p \Big(\sum_{0<t_k<t} q_k\Big)^{\frac{p}{q}} \mathbb{E}\Big(\sum_{0<t_k<t} q_k e^{-\lambda_1 p(t-t_k)} \|x(t_k^-)\|^p\Big) \\
&\leq M^p \Big(\sum_{k=1}^{+\infty} q_k\Big)^{\frac{p}{q}} \sum_{0<t_k<t} q_k e^{-\lambda_1 p(t-t_k)} \mathbb{E}\|x(t_k^-)\|^p \\
&\leq M^p \Big(\sum_{k=1}^{+\infty} q_k\Big)^{\frac{p}{q}} \sum_{0<t_k<t} q_k e^{-\lambda_1 p(t-t_k)} \mathbb{E}\|x(t_k^-)\|^p,
\end{aligned}
\tag{5.16}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

By substituting (5.10) to (5.16) into (5.9), we have for any $t \geq 0$,

$$\begin{aligned}
\mathbb{E}\|x(t)\|^p &\leq 6^p C^* \|\phi\|_D^p e^{-p\lambda_1 t} + 6\|(-A)^{-\kappa}\|^p K_2^p \sup_{-r\leq\theta\leq 0} \mathbb{E}\|x(t+\theta)\|^p \\
&\quad +6^p M_{1-\kappa}^p K_2^p \lambda_1^{p(1-\kappa)-\frac{p}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} \int_0^t e^{-\lambda_1(t-s)} \sup_{-r\leq\theta\leq 0} \mathbb{E}\|x(s+\theta)\|^p ds \\
&\quad +12^p M^p K_1^p \lambda_1^{-\frac{p}{q}} \int_0^t e^{-\lambda_1(t-s)} \Big(1 + \sup_{-r\leq\theta\leq 0} \mathbb{E}\|x(s+\theta)\|^p\Big) ds \\
&\quad +6^p C_{p,\alpha} \Big(\sum_{k=1}^{\infty} \beta_k^\alpha \int_0^t e^{-\alpha\lambda_k(t-s)} \sigma^\alpha(s) ds\Big)^{\frac{p}{\alpha}} \\
&\quad +6^p M^p \Big(\sum_{k=1}^{+\infty} q_k\Big)^{\frac{p}{q}} \sum_{0<t_k<t} q_k e^{-\lambda_1 p(t-t_k)} \mathbb{E}\|x(t_k^-)\|^p \\
&\leq 6^p C^* \|\phi\|_D^p e^{-p\lambda_1 t} + 6^p \|(-A)^{-\kappa}\|^p K_2^p \sup_{-r\leq\theta\leq 0} \mathbb{E}\|x(t+\theta)\|^p \\
&\quad +6^p M_{1-\kappa}^p K_2^p \lambda_1^{p(1-\kappa)-\frac{p}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} \int_0^t e^{-\lambda_1(t-s)} \sup_{-r\leq\theta\leq 0} \mathbb{E}\|x(s+\theta)\|^p ds \\
&\quad +12^p M^p K_1^p \lambda_1^{-\frac{p}{q}} \int_0^t e^{-\lambda_1(t-s)} \sup_{-r\leq\theta\leq 0} \mathbb{E}\|x(s+\theta)\|^p ds \\
&\quad +6^p C_{p,\alpha} \Big(\sum_{k=1}^{\infty} \beta_k^\alpha \int_0^t e^{-\alpha\lambda_k(t-s)} \sigma^\alpha(s) ds\Big)^{\frac{p}{\alpha}} + 12^p M^p K_1^p \lambda_1^{-\frac{p}{q}} \int_{-\infty}^t e^{-\lambda_1(t-s)} ds \\
&\quad +6^p M^p \Big(\sum_{k=1}^{+\infty} q_k\Big)^{\frac{p}{q}} \sum_{0<t_k<t} q_k e^{-\lambda_1 p(t-t_k)} \mathbb{E}\|x(t_k^-)\|^p,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\mathbb{E}\|x(t)\|^p &\leq 6^p C^* \|\phi\|_D^p e^{-p\lambda_1 t} + 6^p \|(-A)^{-\kappa}\|^p K_2^p \sup_{-r\leq\theta\leq 0} \mathbb{E}\|x(t+\theta)\|^p \\
&\quad +\Big[6^p M_{1-\kappa}^p K_2^p \lambda_1^{p(1-\kappa)-\frac{p}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} + 12^p M^p K_1^p \lambda_1^{-\frac{p}{q}}\Big] \\
&\quad \times \int_0^t e^{-\lambda_1(t-s)} \sup_{-r\leq\theta\leq 0} \mathbb{E}\|x(s+\theta)\|^p ds \\
&\quad + \sum_{0<t_k<t} 6^p M^p \Big(\sum_{k=1}^{+\infty} q_k\Big)^{\frac{p}{q}} q_k e^{-\lambda_1(t-t_k)} \mathbb{E}\|x(t_k^-)\|^p \\
&\quad +6^p C_{p,\alpha} \Big(\sum_{k=1}^{\infty} \beta_k^\alpha \int_0^t e^{-\alpha\lambda_k(t-s)} \sigma^\alpha(s) ds\Big)^{\frac{p}{\alpha}} + 12^p M^p K_1^p \lambda_1^{-\frac{p+q}{q}}.
\end{aligned} \tag{5.17}$$

Let $y(t) = \mathbb{E}\|x(t)\|^p$ and use Lemma 5.3.2, then we have

$$b_1 := 6^p\|(-A)^{-\kappa}\|^p K_2^p,$$

$$b_2 := 6^p M_{1-\kappa} K_2^p \lambda_1^{p(1-\kappa)-\frac{p}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} + 12^p M^p K_1^p \lambda_1^{-\frac{p}{q}},$$

$$c_k := 6^p M^p \Big(\sum_{k=1}^{+\infty} q_k\Big)^{\frac{p}{q}} q_k,$$

$$J = 6^p C_{p,\alpha}\Big(\sum_{k=1}^{\infty} \beta_k^\alpha \int_0^t e^{-\alpha\lambda_k(t-s)}\sigma^\alpha(s)ds\Big)^{\frac{p}{\alpha}} + 12^p M^p K_1^p \lambda_1^{-(1+\frac{p}{q})}.$$

From Lemma 5.3.2, we know that if $\phi \in D([-r,0],\mathbb{R}_+)$ satisfying $\|\phi\|_D \leq K$ for some constant $K > 0$ and

$$\rho = b_1 + \frac{b_2}{\lambda_1} + \sum_{k=1}^{+\infty} c_k < 1,$$

that is,

$$\begin{aligned}
\rho &= 6^p\|(-A)^{-\kappa}\|^p K_2^p + \frac{6^p M_{1-\kappa} K_2^p \lambda_1^{p(1-\kappa)-\frac{p}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} + 12^p M^p K_1^p \lambda_1^{-\frac{p}{q}}}{\lambda_1} \\
&\quad + \sum_{k=1}^{+\infty} 6^p M^p \Big(\sum_{k=1}^{+\infty} q_k\Big)^{\frac{p}{q}} q_k \\
&= 6^p\|(-A)^{-\kappa}\|^p K_2^p + 6^p M_{1-\kappa} K_2^p \lambda_1^{p(1-\kappa)-\frac{p+q}{q}} [\Gamma(1-q(1-\kappa))]^{\frac{p}{q}} + 12 M^p K_1^p \lambda_1^{-\frac{p+q}{q}} \\
&\quad + \sum_{k=1}^{+\infty} 6^p M^p \Big(\sum_{k=1}^{+\infty} q_k\Big)^{\frac{p}{q}} q_k < 1.
\end{aligned}$$

Therefore, by Lemma 5.3.2, there exist some constants $K > 0$, $\lambda \in (0, \lambda_1)$ and

$N \geq K$ such that

$$\mathbb{E}\|x(t)\|^p \leq Ne^{-\lambda_1 t} + \frac{J}{1-\rho}, \quad \forall t \geq 0,$$

when $t \to \infty$, we have

$$\lim_{t\to\infty} \mathbb{E}\|x(t)\|^p \leq \frac{J}{1-\rho}.$$

Hence, we obtain the global attracting set

$$\mathbb{S} = \Big\{ y \in H : \|y\| \leq (\frac{J}{1-\rho})^{\frac{1}{p}} \Big\}.$$

Therefore, by Definition 5.3.2 we know $\mathbb{S}$ in (5.6) is a global attracting set of the mild solution $\{x(t,\phi)\}, t \geq -r, \phi \in D([-r,0],H)$ to equation (5.1). The proof is complete. □

5.4 Summary

In this chapter, we made the first attempt to study the global attracting set for a class of neutral stochastic evolution equations with impulses. Our work extended that of Li and Liu (2016) where the neutral stochastic functional evolution equation without impulses is investigated. We also extended that of Long, Teng and Xu (2012) where the class of stochastic differential equation driven by Wiener processes rather than α-stable processes.

Chapter 6

Conclusions

This research program focused on the several stochastic delay evolution equations dealing with the optimal control problem and asymptotics for the stochastic systems. Therefore, we have adopted the methods of stochastic analysis and semi-group which help us to study and understand the existence, uniqueness, controllability and stability for various stochastic differential systems. There, we aim to generalise and develop the existing stochastic models based on the some certain assumptions. Clearly, these stochastic delay differential equations defined on Hilbert spaces can also be simplified to do many applications in financial mathematics.

For the first model, we generalise the previous theory to consider a stochastic optimal control problem for a class of neutral stochastic system. We adopt a method that allows us to "lift" this non-Markovian optimisation problem to an infinite-dimensional Markovian control problem. The aim of the stochastic optimal control problem is to maximise the objective functional at a given time horizon $T > 0$. In practice, the explicit solution to this model is not computable. Thus, we establish a linear differential difference equation to obtain the solutions to this model.

On the other hand, solutions with recurrence property (e.g. almost periodicity and almost automorphy) enable us to understand the impact of the noise or stochastic perturbation on the corresponding recurrent motions. For the second model, the neutral stochastic evolution equations with Poisson jumps and infinite delays are considered. We study the existence and uniqueness of the stochastic system, which satisfy the Lipschitz conditions. The constant coefficients with parameters M_f, M_G, M_g and M_h for the model are constrained based on the method of Banach fixed-point theory. Therefore, we have proved the existence and uniqueness of mild solutions.

From the models above, we have seen the stochastic differential evolution equations driven by Brownian motions and Lévy processe. However, since Wiener noise and Poisson-jump noise have arbitrary finite moments, while α-stable noise only has finite p-th moment for $p \in (0, \alpha)$ with $\alpha < 2$. For the third model. we consider the global attracting set and stability of the neutral stochastic partial differential equations with impulses driven by an additive α-stable with impulses on a separable Hilbert space H. Thus, new techniques have been established and developed to a stochastic system driven by α-stable processes. Then, we have proved the existence and uniqueness of mild solutions.

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