

Type Reduction Operators for Interval Type-2 Defuzzification

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Abstract

Fuzzy sets are an important approach to model uncertainty. Defuzzification maps fuzzy sets to non-fuzzy (crisp) values. Type-2 fuzzy sets model uncertainty in the degree of membership in a fuzzy set. Type-2 defuzzification maps type-2 fuzzy sets to non-fuzzy values. Type reduction maps type-2 fuzzy sets to type-1 fuzzy sets, in order to make type-2 defuzzification easier and to implement more efficient type-2 defuzzification algorithms.

This paper is a first step towards a theoretical foundation of the emerging field of type reduction. Five mathematical properties of type reduction are defined, and two existing type reduction methods (Nie-Tan and uncertainty weight) are examined with respect to our five properties. Furthermore, two new type reduction methods are proposed: consistent linear type reduction and consistent quadratic type reduction. All our five properties are satisfied by consistent quadratic type reduction.

Keywords: defuzzification, type-2 fuzzy sets

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1. Introduction

Fuzzy sets [29] are an important approach to model and process uncertain information. Fuzzy information processing (for example in fuzzy rule based systems [14, 26]) usually yield fuzzy sets as outputs. If non-fuzzy outputs are needed (for example in control applications [7, 18]), these can be obtained from fuzzy outputs using *defuzzification*. Many defuzzification methods have been proposed during the last decades, for an overview see [19, 20, 25], and the mathematical properties of defuzzification have been studied in [24].

Interval type-2 fuzzy sets [13, 15, 30] are extensions of (type-1) fuzzy sets, where the degrees of membership in a fuzzy set may be uncertain. Interval type-2 fuzzy information processing yields interval type-2 fuzzy sets as outputs, which can be converted to non-fuzzy outputs using interval type-2 defuzzification. One of the most popular methods for interval type-2 defuzzification is the Karnik-Mendel algorithm [11]. The mathematical properties of interval type-2 defuzzification have been studied in [22].

Recently, alternative methods for interval type-2 defuzzification have been proposed that are based on *type reduction*. Type reduction of an interval type-2 fuzzy set means a mapping to a type-1 fuzzy set which then can be defuzzified using type-1 defuzzification techniques. Examples are the Nie-Tan (NT) [17] and the uncertainty weight (UW) method [23]. The relation between type-1 defuzzification, type-2 defuzzification, and type reduction is illustrated in Fig. 1.

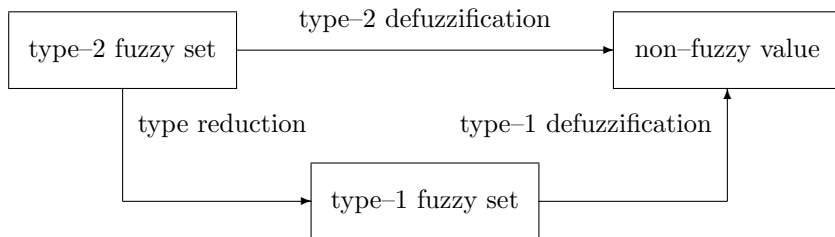


Figure 1: Type-1 and type-2 defuzzification and type reduction.

The NT and UW methods (and also the CQTR and CLTR methods proposed in this paper) are different from the Karnik–Mendel approach. These methods reduce each interval type–2 fuzzy set to a type–1 fuzzy set upfront, and then perform ordinary type–1 defuzzification. These methods can therefore also be viewed as a tool to design type–1 membership functions in scenarios with different levels of uncertainty for different decision options.

Type–reduction based interval type–2 defuzzification reduces interval type–2 fuzzy sets to type–1 fuzzy sets upfront, so we lose some ability to explicitly consider the specific uncertainty of the memberships which is considered an important advantage of interval type–2 fuzzy sets over type–1 fuzzy sets. It was shown however for typical fuzzy controller scenarios that the NT method and its variants yield very good approximations of the Karnik–Mendel algorithm at a reduced computational cost [12, 23], which appears beneficial for a large number of real–world applications.

The emerging field of type reduction needs a theoretical foundation that enables researchers to assess, compare, and develop novel type reduction methods. So the focus of this paper is not just proposing new type reduction formulas. We also propose the required properties (a standard) for designing type reduction methods. As a first step in this direction, this paper defines five mathematical properties of type reduction. There are infinitely many choices for type reduction functions. An important criterion for type reduction functions is (mathematical and computational) simplicity, so we will focus on linear and quadratic functions in this paper. We will derive specific linear and quadratic functions to meet our five proposed properties, which will lead us to propose two new type reduction methods, *consistent linear type reduction (CLTR)* and *consistent quadratic type reduction (CQTR)*.

We start with a brief review of type–1 and interval type–2 defuzzification, including the NT and UW methods (Section 2). Then we define five mathematical properties for type reduction (Section 3) and evaluate the properties of the NT and UW methods (Sections 4 and 5). Further, we propose the new CLTR and CQTR type reduction methods and discuss their properties (Sections

6 and 7). Finally, we summarize the properties of all considered methods, give some illustrative examples (Section 8), and present our conclusions and ideas for future research (Section 9).

2. Type-1 and Type-2 Defuzzification

A *type-1* fuzzy set [29] over a universe X is specified by a membership function $u : X \rightarrow [0, 1]$, so $u(x)$ quantifies the degree of membership of any $x \in X$ in the type-1 fuzzy set. Type-1 defuzzification is a function d that maps any type-1 fuzzy membership function to one representative crisp (non-fuzzy) value in X .

$$d(u(x)) \in X \tag{1}$$

One of the most popular type-1 defuzzification methods is the *centroid* defined as

$$d_C(u(x)) = \frac{\int_X u(x) \cdot x \, dx}{\int_X u(x) \, dx} \tag{2}$$

which represents the (horizontal) position of the center of gravity (first moment) of the area between the membership function and the x coordinate (abscissa). Many other methods for type-1 defuzzification have been proposed in the literature. For an overview see [19, 20, 25].

An interval-fuzzy set [3, 9, 10] or closed interval type-2 fuzzy set [16] is specified by two membership functions: a lower membership function $\underline{u} : X \rightarrow [0, 1]$ and an upper membership function $\bar{u} : X \rightarrow [0, 1]$, where

$$\underline{u}(x) \leq \bar{u}(x) \tag{3}$$

for all $x \in X$. Interval type-2 fuzzy sets can be used to model uncertainty about a membership function or to model risk in decision processes [21].

A popular interval type-2 defuzzification method is the *Karnik-Mendel (KM)* algorithm [11] that considers all (type-1) membership functions $u(x)$ between the lower and upper membership functions $\underline{u}(x)$ and $\bar{u}(x)$, respectively,

$$\underline{u}(x) \leq u(x) \leq \bar{u}(x) \tag{4}$$

For one dimensional universes $X \subseteq \mathbb{R}$, KM computes the lower and upper centroids that can be written as

$$\underline{d}_C(\underline{u}(x), \bar{u}(x)) = \inf_{\underline{u}(x) \leq u(x) \leq \bar{u}(x)} d_C(u(x)) \quad (5)$$

$$\bar{d}_C(\underline{u}(x), \bar{u}(x)) = \sup_{\underline{u}(x) \leq u(x) \leq \bar{u}(x)} d_C(u(x)) \quad (6)$$

and returns the average of these

$$d_{\text{KM}}(\underline{u}(x), \bar{u}(x)) = \frac{\underline{d}_C(\underline{u}(x), \bar{u}(x)) + \bar{d}_C(\underline{u}(x), \bar{u}(x))}{2} \quad (7)$$

The Karnik–Mendel algorithm is iterative and computationally intensive, so several enhanced methods and efficient implementations have been proposed in the literature [6, 27, 28].

Nie and Tan [17] have introduced an alternative way to interval type–2 defuzzification. The idea is to transform the lower and upper membership functions $\underline{u}(x)$ and $\bar{u}(x)$ to one single (type–1) membership function $u(x)$, and then apply type–1 defuzzification, for example the centroid method, to $u(x)$. The conversion of the lower and upper interval type–2 membership functions to one single type–1 membership function is called *type reduction*. Type reduction for interval type–2 membership functions can be viewed as an application of a two–dimensional fusion function [1, 4]. In their original paper, Nie and Tan [17] proposed the *Nie–Tan (NT)* type reduction formula

$$u_{\text{NT}}(x) = \frac{\underline{u}(x) + \bar{u}(x)}{2} \quad (8)$$

which converts an interval type–2 fuzzy set to a type–1 fuzzy set by taking the average of each respective upper and lower membership grades.

Recently, Runkler *et al.* [23] presented an alternative type reduction formula called the *uncertainty weight (UW)* method

$$u_{\text{UW}}(x) = \frac{1}{2}(\underline{u}(x) + \bar{u}(x)) \cdot (1 + \underline{u}(x) - \bar{u}(x))^\alpha \quad (9)$$

with a parameter $\alpha > 0$ that quantifies to which degree uncertainty is taken into account in defuzzification. Rather than simply taking the average of each

respective upper and lower membership grades (which is used in the Nie–Tan method), the uncertainty weight method multiplies this average by the weight factor $(1 + \underline{u}(x) - \bar{u}(x))^\alpha$, which is zero for maximum uncertainty ($\underline{u}(x) = 0$, $\bar{u}(x) = 1$) and one for minimum uncertainty ($\underline{u}(x) = \bar{u}(x)$), and which increases linearly with the uncertainty for $\alpha = 1$, less than linearly for all $\alpha < 1$ and more than linearly for all $\alpha > 1$. The experiments in [12, 23] indicated that both NT and UW can be used to yield good approximations of KM with low computational effort. Fig. 2 shows the defuzzification results of the KM, NT, and UW ($\alpha = 1$) methods for an example of a triangular interval type-2 fuzzy set. The vertical solid lines indicate the defuzzification results for the three methods: $d_{\text{KM}} = 0.4650$, $d_{\text{NT}} = 0.4667$, and $d_{\text{UW}} = 0.4727$, which are all very similar for this example. The dashed curves at the NT and UW graphs are the results of type reduction. The KM graph does not show any dashed curve because KM directly produces a non-fuzzy value from a type-2 fuzzy set, without type reduction to a type-1 fuzzy set. NT type reduction yields a (type-1) membership function that passes right through the middle of the grey area limited by the upper and lower interval type-2 membership functions. In this example, UW type reduction yields a membership function partially lying outside the grey area. The observation of this behavior was one of the main motivations to define the properties of type reduction (most specifically the so-called type-2 consistency property) which are introduced in the following section.

3. Properties of Type Reduction

This section follows a similar approach as early work on properties of type-1 defuzzification operators [24] and their extension to interval type-2 defuzzification operators [22]. Based on intuitive requirements we define five mathematical properties of type reduction methods for interval type-2 defuzzification and examine relations between these properties. This list of properties is not intended to be complete, but should serve as a basis for discussion and extension. The

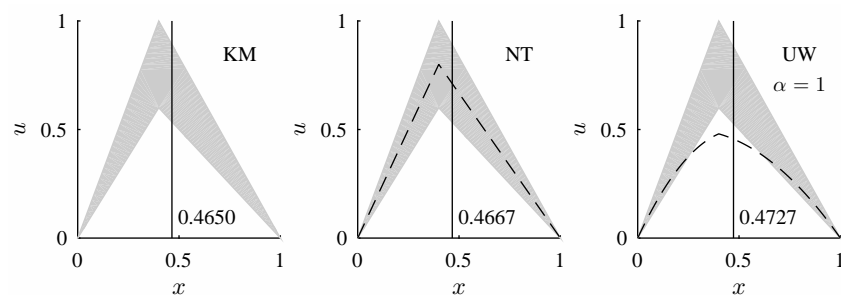


Figure 2: Defuzzification of a triangular interval type-2 fuzzy set using the KM, NT, and UW ($\alpha = 1$) methods.

purpose of these properties is to evaluate, compare, and develop type reduction methods.

Here, we only consider *pointwise* type reduction methods, i.e. for the computation of a type-1 membership degree $u(x)$ we only consider the lower and upper interval type-2 membership degrees $\underline{u}(x)$ and $\bar{u}(x)$ at the same point x , but not the membership degree $\underline{u}(y)$ and $\bar{u}(y)$ at any other point $y \neq x$. Therefore, we can omit the argument x and simply write each type reduction method as a function $f : [0, 1]^2 \rightarrow [0, 1]$ with

$$u = f(\underline{u}, \bar{u}) \quad (10)$$

So we will always consider the type-1 and type-2 memberships of only one single element at a time. If we want to use the function f to reduce the lower and upper interval type-2 membership functions $\underline{u}(x)$ and $\bar{u}(x)$ to a type-1 membership function $u(x)$, then we have to apply the function f to each $x \in X$. The graph of each type reduction function f can be visualized in a three-dimensional diagram with the coordinate axes \underline{u} , \bar{u} , and u . We will show such diagrams for each of the discussed type reduction functions in the corresponding Sections 4-7.

3.1. Type-1 Consistency

If the lower and upper interval type-2 memberships are equal, then we obtain the special case of type-1 memberships.

$$\underline{u} = \bar{u} \tag{11}$$

In this case there no uncertainty about the membership, and we expect the type reduction to yield

$$\Rightarrow u = \underline{u} = \bar{u} \tag{12}$$

This means that if the uncertainty disappears, the interval type-2 fuzzy model will reduce to a conventional type-1 fuzzy model. This property corresponds to the (only) requirement that was used to derive the popular Karnik–Mendel method. It also corresponds to the notion of idempotency, which is very well known in algebra and in the field of aggregation functions [2].

Definition 1. A type reduction function f is *type-1 consistent* if and only if

$$f(u, u) = u \text{ for all } u \in [0, 1] \tag{13}$$

3.2. Type-2 Consistency

Let us assume arbitrary lower and upper interval type-2 memberships $\underline{u}, \bar{u} \in [0, 1]$ with

$$\underline{u} \leq \bar{u} \tag{14}$$

A common interpretation of this situation is that there is uncertainty about the membership, and the membership is at least \underline{u} and at most \bar{u} .

$$\underline{u} \leq u \leq \bar{u} \tag{15}$$

This means that the effective (type-1) memberships produced by type reduction will never be outside the interval of uncertainty specified for the type-2 memberships, never lower than the lower bound, and never higher than the upper bound.

Definition 2. A type reduction function f is *type-2 consistent* if and only if

$$\underline{u} \leq f(\underline{u}, \bar{u}) \leq \bar{u} \text{ for all } \underline{u}, \bar{u} \in [0, 1] \text{ with } \underline{u} \leq \bar{u} \quad (16)$$

Theorem 1. *If a function is type-2 consistent, then it is type-1 consistent.*

Proof. Trivial. □

Type-2 consistency is related to the notion of an averaging function (a function which is between the minimum and the maximum), but it is not the same because we are not assuming symmetry here. Type reduction is also closely related to the idea of ignorance functions and entropy for interval-valued fuzzy sets [5]. We are planning to investigate in a future project if adding a symmetry constraint could yield a generalized definition of type reduction in the context of averaging functions, and we want to have a closer look at the relation between type reduction and ignorance functions.

3.3. (Strict) Uncertainty Conformity

Let us assume that the membership is around $u^* \in [0, 1]$ with an uncertainty of $\pm\Delta u$, where

$$u^* - \Delta u \geq 0, \quad u^* + \Delta u \leq 1 \quad (17)$$

so we obtain

$$\underline{u} = u^* - \Delta u, \quad \bar{u} = u^* + \Delta u \quad (18)$$

For $\Delta u = 0$ we have the type-1 case, and for any type-1 consistent function f we obtain

$$u = f(u^*, u^*) = u^* \quad (19)$$

If we increase Δu , then the average membership u^* stays the same, but the uncertainty around u^* increases. If we have two objects $x_1, x_2 \in X$ both with memberships around u^* , one with uncertainty Δu_1 and the other with a higher uncertainty Δu_2 , $\Delta u_1 \leq \Delta u_2$, then we may want to give the first (i.e. more certain) object x_1 a larger weight than x_2 , $u(x_1) > u(x_2)$. This leads us to require that u should be (strictly) monotonically decreasing with Δu . If a

type reduction operator has this property, then for two options with the same average membership but with different levels of uncertainty (spread between the lower and upper type-2 memberships) the option with lower uncertainty will be preferred over the option with higher uncertainty, so the degree of uncertainty is reflected in the type reduction process.

Definition 3. A type reduction function f is *uncertainty conform* if and only if for any $u^* \in [0, 1]$ and any $\Delta u_1, \Delta u_2 \geq 0$, $\Delta u_1 < \Delta u_2$ with

$$u^* - \Delta u_2 \geq 0, \quad u^* + \Delta u_2 \leq 1 \quad (20)$$

the following condition holds:

$$f(u^* - \Delta u_1, u^* + \Delta u_1) \geq f(u^* - \Delta u_2, u^* + \Delta u_2) \quad (21)$$

Definition 4. A type reduction function f is *strictly uncertainty conform* if and only if for any $u^* \in [0, 1]$ and any $\Delta u_1, \Delta u_2 \geq 0$, $\Delta u_1 < \Delta u_2$ with

$$u^* - \Delta u_2 \geq 0, \quad u^* + \Delta u_2 \leq 1 \quad (22)$$

the following condition holds:

$$f(u^* - \Delta u_1, u^* + \Delta u_1) > f(u^* - \Delta u_2, u^* + \Delta u_2) \quad (23)$$

Lemma 1. *The preconditions of Definitions 3 and 4 imply $\Delta u \leq 0.5$.*

Proof.

$$\text{For } u^* \in [0, 0.5] : \quad u^* - \Delta u \geq 0 \quad \Rightarrow \quad \Delta u \leq u^* \leq 0.5 \quad (24)$$

$$\text{For } u^* \in [0.5, 1] : \quad u^* + \Delta u \leq 1 \quad \Rightarrow \quad \Delta u \leq 1 - u^* \leq 0.5 \quad (25)$$

So for $u^* \in [0, 1]$ we have $\Delta u \leq 0.5$. □

Theorem 2. *If a function is strictly uncertainty conform, then it is uncertainty conform.*

Proof. Trivial. □

3.4. Ignorance of Indifference

The most uncertain interval type-2 membership values are

$$\underline{u} = 0, \quad \bar{u} = 1 \quad (26)$$

In this case we have no information about the membership at all, so we may want to completely ignore such indifferent objects

$$\Rightarrow u = 0 \quad (27)$$

If a type reduction operator has this property, then options with complete uncertainty will receive zero memberships and hence will receive weight zero in the defuzzification process.

Definition 5. A type reduction function f ignores indifference if and only if

$$f(0, 1) = 0 \quad (28)$$

4. Properties of the Nie–Tan Method

The Nie–Tan (NT) method [17] is specified by the type reduction function

$$f_{\text{NT}}(\underline{u}, \bar{u}) = \frac{\underline{u} + \bar{u}}{2} \quad (29)$$

In this section we examine f_{NT} with respect to the five properties defined in the previous section. The properties of the NT and all other methods considered in this paper will be summarized in section 8.

Theorem 3. *The Nie–Tan type reduction function is type-1 consistent.*

Proof.

$$f_{\text{NT}}(u, u) = \frac{u + u}{2} = u \text{ for all } u \in [0, 1] \quad \square \quad (30)$$

Theorem 4. *The Nie–Tan type reduction function is type-2 consistent.*

Proof. For all $\underline{u}, \bar{u} \in [0, 1]$ with $\underline{u} \leq \bar{u}$

$$f_{\text{NT}}(\underline{u}, \bar{u}) = \frac{\underline{u} + \bar{u}}{2} \geq \frac{\underline{u} + \underline{u}}{2} = \underline{u} \quad (31)$$

$$f_{\text{NT}}(\underline{u}, \bar{u}) = \frac{\underline{u} + \bar{u}}{2} \leq \frac{\bar{u} + \bar{u}}{2} = \bar{u} \quad \square \quad (32)$$

Theorem 5. *The Nie–Tan type reduction function is uncertainty conform.*

Proof. For any $u^* \in [0, 1]$ and any $\Delta u \geq 0$ we obtain

$$f_{\text{NT}}(u^* - \Delta u, u^* + \Delta u) = \frac{u^* - \Delta u + u^* + \Delta u}{2} = u^* \quad (33)$$

so for $\Delta u_1 \geq \Delta u_2$

$$f_{\text{NT}}(u^* - \Delta u_1, u^* + \Delta u_1) = f_{\text{NT}}(u^* - \Delta u_2, u^* + \Delta u_2) = u^* \quad (34)$$

$$\Rightarrow f_{\text{NT}}(u^* - \Delta u_1, u^* + \Delta u_1) \geq f_{\text{NT}}(u^* - \Delta u_2, u^* + \Delta u_2) \quad \square \quad (35)$$

Theorem 6. *The Nie–Tan type reduction function is not strictly uncertainty conform.*

Proof. From (34) follows

$$f_{\text{NT}}(u^* - \Delta u_1, u^* + \Delta u_1) \not\asymp f_{\text{NT}}(u^* - \Delta u_2, u^* + \Delta u_2) \quad \square \quad (36)$$

Theorem 7. *The Nie–Tan type reduction function does not ignore indifference.*

Proof.

$$f_{\text{NT}}(0, 1) = \frac{0 + 1}{2} = 0.5 \neq 0 \quad \square \quad (37)$$

Fig. 3 shows the graph of the NT function over the triangle $(\underline{u}, \bar{u}) \in [0, 1]^2$ with $\underline{u} \leq \bar{u}$. The graph of the NT function is a triangle with the edge points $(\underline{u}, \bar{u}, u) = (0, 0, 0)$, $(1, 1, 1)$, and $(0, 1, 0.5)$, which contradicts ignorance of indifference. The graph of the NT function is linear with a slope of 0.5 along both the \underline{u} and \bar{u} axes.

5. Properties of the Uncertainty Weight Method

Runkler *et al.* proposed the uncertainty weight (UW) method [23] specified by the type reduction function

$$f_{\text{UW}}(\underline{u}, \bar{u}) = \frac{1}{2}(\underline{u} + \bar{u}) \cdot (1 + \underline{u} - \bar{u})^\alpha \quad (38)$$

with the parameter $\alpha > 0$. Obviously, UW converges to NT as α approaches 0.

$$\lim_{\alpha \rightarrow 0} f_{\text{UW}} = f_{\text{NT}} \quad (39)$$

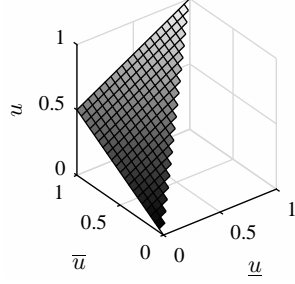


Figure 3: Graph of the NT function.

In this section we examine f_{UW} with respect to the five properties defined in this paper.

Theorem 8. *The uncertainty weight type reduction function is type-1 consistent.*

Proof.

$$f_{\text{UW}}(u, u) = \frac{1}{2}(u + u) \cdot (1 + u - u)^\alpha = u \text{ for all } u \in [0, 1] \quad \square \quad (40)$$

Theorem 9. *The uncertainty weight type reduction function is not generally (i.e. for any $\alpha > 0$) type-2 consistent.*

Proof. Let for example $\underline{u} = 0.5$, $\bar{u} = 1$. Then

$$f_{\text{UW}}(0.5, 1) = \frac{1}{2}(0.5 + 1) \cdot (1 + 0.5 - 1)^\alpha = 0.5 \frac{3}{2^{\alpha+1}} \not\geq 0.5 = \underline{u}$$

$$\text{for } 2^{\alpha+1} > 3 \Rightarrow \alpha > \frac{\log 3}{\log 2} - 1 \approx 0.585 \quad \square \quad (41)$$

Theorem 10. *The uncertainty weight type reduction function is strictly uncertainty conform.*

Proof.

$$f_{\text{UW}}(u^* - \Delta u, u^* + \Delta u)$$

$$= \frac{1}{2}(u^* - \Delta u + u^* + \Delta u) \cdot (1 + u^* - \Delta u - u^* - \Delta u)^\alpha = u^*(1 - 2\Delta u)^\alpha \quad (42)$$

is strictly monotonically decreasing with Δu for $\Delta u \leq 0.5$ (Lemma 1) and for $\alpha > 0$. □

Theorem 11. *The uncertainty weight type reduction function is uncertainty conform.*

Proof. Follows from Theorems 2 and 10. □

Theorem 12. *The uncertainty weight type reduction function ignores indifference.*

Proof.

$$f_{\text{UW}}(0, 1) = \frac{1}{2}(0 + 1) \cdot (1 + 0 - 1)^\alpha = 0 \quad \square \quad (43)$$

Fig. 4 shows the graphs of the UW function for $\alpha \in \{0.2, 1, 5\}$. All three graphs are nonlinear but smooth, and contain the points $(\underline{u}, \bar{u}, u) = (0, 0, 0)$, $(1, 1, 1)$, and $(0, 1, 0)$, corresponding to ignorance of indifference. For $\alpha = 0.2$ (left), the UW function looks very similar to the NT function at Fig. 3, only the very edge at $\underline{u} = 0$ and $\bar{u} = 1$ is bent from $u = 0.5$ to $u = 0$ (ignorance of indifference). Depending on the value of the parameter α the graph of the UW function is bent in different ways. It is monotonically increasing along the \underline{u} axis with increasing ($\alpha = 0.2$) or decreasing slope ($\alpha = 5$). It is however not generally monotonic along the \bar{u} axis. Consider for example the front edge curve at $\underline{u} = 0$ which is distinctly bent up for medium values of $\bar{u} \approx 0.5$ for $\alpha = 0.2$ (left) and $\alpha = 1$ (center). For $\alpha = 5$ (right), the UW function always yields very small memberships $u \approx 0$ except near the main diagonal at $\underline{u} = \bar{u}$, where it yields $u = \underline{u} = \bar{u}$ (type-1 consistency). For $\alpha = 1$ (center), the UW function is quite flat along the \bar{u} axis, and almost linearly increases along the \underline{u} axis, so it approximates the function $u = \underline{u}$.

6. Linear Type Reduction

Besides NT and UW we can construct infinitely many type reduction functions f . For reasons of simplicity we first consider linear functions and derive specific instances of linear functions trying to satisfy our five proposed properties. So we start with the general case of linear functions for type reduction

$$u = a \cdot \underline{u} + b \cdot \bar{u} + c \quad (44)$$

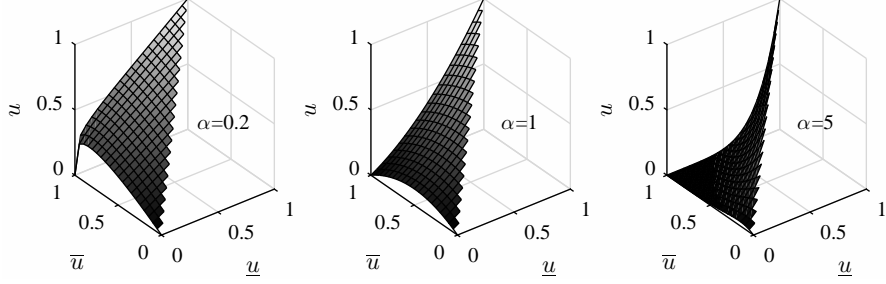


Figure 4: Graphs of the UW function for $\alpha \in \{0.2, 1, 5\}$.

with arbitrary parameters $a, b, c \in \mathbb{R}$. If we require type-1 consistency, then we obtain

$$\underline{u} = 0, \quad \bar{u} = 0, \quad u = 0 \Rightarrow c = 0 \quad (45)$$

$$\underline{u} = 1, \quad \bar{u} = 1, \quad u = 1 \Rightarrow 1 = a + b + c \Rightarrow b = 1 - a \quad (46)$$

and so

$$u = a \cdot \underline{u} + (1 - a) \cdot \bar{u} \quad (47)$$

If we further require type-2 consistency, then we obtain

$$\underline{u} \leq a \cdot \underline{u} + (1 - a) \cdot \bar{u} \Rightarrow (1 - a) \cdot (\bar{u} - \underline{u}) \geq 0 \Rightarrow a \leq 1 \quad (48)$$

$$a \cdot \underline{u} + (1 - a) \cdot \bar{u} \leq \bar{u} \Rightarrow a \cdot (\bar{u} - \underline{u}) \geq 0 \Rightarrow a \geq 0 \quad (49)$$

which leads us to

Definition 6. The *consistent linear type reduction (CLTR)* function is defined as

$$f_{\text{CLTR}}(\underline{u}, \bar{u}) = a \cdot \underline{u} + (1 - a) \cdot \bar{u} \quad (50)$$

with $a \in [0, 1]$

While the Nie-Tan type reduction function computes the plain average of the upper and lower membership grades, the CLTR function uses a convex combination of the upper and lower membership grades, which represents a weighted average with the weight a for the lower membership grades and the weight $1 - a$

for the upper membership grades. For $a = 0$, CLTR will yield the upper membership function; for $a = 0.5$, CLTR will yield the average of the upper and lower membership functions, which is also the result of the NT type reduction function (so the NT type reduction function is a special case of the CLTR function); and for $a = 1$, CLTR will yield the lower membership function. For all other values of $a \in [0, 1]$, CLTR will linearly interpolate between these special cases.

Theorem 13. *The consistent linear type reduction function is type-1 and type-2 consistent.*

Proof. Follows immediately from the requirements for consistent linear type reduction functions (45)–(49). \square

Theorem 14. *The consistent linear type reduction function is uncertainty conform if and only if $a \geq 0.5$.*

Proof.

$$\begin{aligned} f_{\text{CLTR}}(u^* - \Delta u, u^* + \Delta u) \\ = a \cdot (u^* - \Delta u) + (1 - a) \cdot (u^* + \Delta u) = u^* + (1 - 2a) \cdot \Delta u \end{aligned} \quad (51)$$

is monotonically decreasing with Δu for $a \in [0.5, 1]$. \square

Theorem 15. *The consistent linear type reduction function is strictly uncertainty conform if and only if $a > 0.5$.*

Proof. (51) is strictly monotonically decreasing with Δu for $a < (0.5, 1]$. \square

Notice that the NT type reduction function is the only instance of the CLTR function that is uncertainty conform but not strictly uncertainty conform.

Theorem 16. *The consistent linear type reduction function ignores indifference if and only if $a = 1$.*

Proof.

$$f_{\text{CLTR}}(0, 1) = a \cdot 0 + (1 - a) \cdot 1 = 1 - a = 0 \quad \Leftrightarrow a = 1 \quad \square \quad (52)$$

Fig. 5 shows the graphs of the CLTR function for $a \in \{0.5, 0.75, 1\}$. All three graphs are triangles through $(\underline{u}, \bar{u}, u) = (0, 0, 0)$ and $(1, 1, 1)$. The third edge of each triangle is at $(0, 1, 1 - a)$. The intersection of all CLTR graphs is the unit main diagonal through $(0, 0, 0)$ and $(1, 1, 1)$ (type-1 consistency). All three graphs are linear with non-negative slopes along \underline{u} and \bar{u} . The slope is zero along \bar{u} for $a = 1$. All these three CLTR instances are uncertainty conform, because $a \in [0.5, 1]$. The left graph ($a = 0.5$) corresponds to the NT function at Fig. 3. The right graph ($a = 1$) corresponds to the only CLTR function that ignores indifference, $u = \underline{u}$, which is approximately equal to the UW function for $\alpha = 1$ (Fig. 4 center). For $a = 1$, the CLTR function always yields $u = \underline{u}$, so the complete information about \bar{u} is lost in this case. Hence, if we want to keep the information about both \underline{u} and \bar{u} , and at the same type satisfy all our five properties, then we have to use a nonlinear type reduction function.

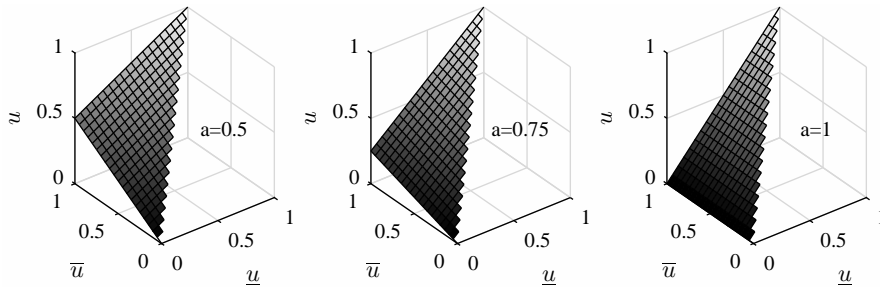


Figure 5: Graphs of the CLTR function for $a \in \{0.5, 0.75, 1\}$.

7. Nonlinear Type Reduction

As shown in the previous section, linear functions for type reduction may satisfy type-1 consistency, type-2 consistency, and (strict) uncertainty conformity, but we are not able to define any non-trivial ($a \neq 1$) linear function for type reduction that ignores indifference. So, this property can only be satisfied by *nonlinear* type reduction functions. In this section we will therefore try to derive a nonlinear type reduction function that is as simple as possible and

should be able to satisfy all five proposed properties.

To find such a function we start with the linear approach (44) and add a nonlinear term. This nonlinear term should be suitable to indicate indifference, so it should contain \underline{u} and \bar{u} . In order to stay type-1 consistent, the nonlinear term should be zero when $\underline{u} = \bar{u}$. So, the nonlinear term could be defined as $g(\bar{u} - \underline{u})$ with a suitable function g where $g(0) = 0$. Here for simplicity we use a quadratic function for g and obtain

$$u = a \cdot \underline{u} + b \cdot \bar{u} + c + d \cdot (\bar{u} - \underline{u})^2 \quad (53)$$

with arbitrary parameters $a, b, c, d \in \mathbb{R}$. If we require type-1 consistency, then we obtain just as in (45) and (46)

$$\underline{u} = 0, \quad \bar{u} = 0, \quad u = 0 \quad \Rightarrow \quad c = 0 \quad (54)$$

$$\underline{u} = 1, \quad \bar{u} = 1, \quad u = 1 \quad \Rightarrow \quad 1 = a + b + c \quad \Rightarrow \quad b = 1 - a \quad (55)$$

and so

$$u = a \cdot \underline{u} + (1 - a) \cdot \bar{u} + d \cdot (\bar{u} - \underline{u})^2 \quad (56)$$

If we further want to ignore indifference, then we obtain

$$\underline{u} = 0, \quad \bar{u} = 1, \quad u = 0 \quad \Rightarrow \quad 0 = 1 - a + d \quad \Rightarrow \quad d = -(1 - a) \quad (57)$$

and so

$$u = a \cdot \underline{u} + (1 - a) \cdot \bar{u} - (1 - a) \cdot (\bar{u} - \underline{u})^2 \quad (58)$$

If we further require type-2 consistency, then we obtain

$$\begin{aligned} \underline{u} &\leq a \cdot \underline{u} + (1 - a) \cdot \bar{u} - (1 - a) \cdot (\bar{u} - \underline{u})^2 \\ &\Rightarrow (1 - a) \cdot (\bar{u} - \underline{u}) \geq (1 - a) \cdot (\bar{u} - \underline{u})^2 \quad \Rightarrow \quad a \leq 1 \end{aligned} \quad (59)$$

and

$$\begin{aligned} a \cdot \underline{u} + (1 - a) \cdot \bar{u} - (1 - a) \cdot (\bar{u} - \underline{u})^2 &\leq \bar{u} \\ &\Rightarrow a \cdot (\bar{u} - \underline{u}) \geq (a - 1) \cdot (\bar{u} - \underline{u})^2 \quad \Rightarrow \quad a \geq 0 \end{aligned} \quad (60)$$

which leads us to

Definition 7. The *consistent quadratic type reduction (CQTR)* function is defined as

$$f_{\text{CQTR}}(\underline{u}, \bar{u}) = a \cdot \underline{u} + (1 - a) \cdot \bar{u} - (1 - a) \cdot (\bar{u} - \underline{u})^2 \quad (61)$$

with $a \in [0, 1]$

The CQTR function (61) is equal to the CLTR function (50) minus the term $(1 - a) \cdot (\bar{u} - \underline{u})^2$ which is zero for minimum uncertainty ($\underline{u}(x) = \bar{u}(x)$) and maximal (equal to $1 - a$) for maximum uncertainty ($\underline{u}(x) = 0, \bar{u}(x) = 1$), and which increases quadratically with the uncertainty. For $a = 0$, CQTR will yield the upper membership function minus the quadratic uncertainty, $\bar{u} - (\bar{u} - \underline{u})^2$; and for $a = 1$, CQTR will simply yield the lower membership function \underline{u} . For all other values of $a \in [0, 1]$, CQTR will linearly interpolate between these two cases.

Theorem 17. *The consistent quadratic type reduction function is type-1 and type-2 consistent, and ignores indifference.*

Proof. Follows immediately from the requirements for consistent linear type reduction functions (54)–(60). \square

Theorem 18. *The consistent quadratic type reduction function is uncertainty conform if and only if $a \geq 0.5$.*

Proof.

$$\begin{aligned} & f_{\text{CQTR}}(u^* - \Delta u, u^* + \Delta u) \\ &= a \cdot (u^* - \Delta u) + (1 - a) \cdot (u^* + \Delta u) - (1 - a) \cdot (2\Delta u)^2 \\ &= u^* + (1 - 2a) \cdot \Delta u - 4(1 - a)\Delta u^2 \end{aligned} \quad (62)$$

is monotonically decreasing with Δu for $a = 1$. For $a < 1$ consider the slope

$$\frac{\partial f_{\text{CQTR}}}{\partial \Delta u} = 1 - 2a - 8(1 - a)\Delta u \quad (63)$$

This slope is negative or zero for any $a < 1$ if $1 - 2a \leq 0 \Rightarrow a \in [0.5, 1)$. So, $a \in [0.5, 1]$. \square

Theorem 19. *The consistent linear type reduction function is strictly uncertainty conform if and only if $a > 0.5$.*

Proof. (62) is strictly monotonically decreasing with Δu for $a = 1$. (63) is negative for any $a < 1$ if $1 - 2a < 0 \Rightarrow a \in (0.5, 1)$. So, $a \in (0.5, 1]$. \square

Fig. 6 shows the graphs of the CQTR function for $a \in \{0.5, 0.75, 1\}$. All three graphs contain the points $(\underline{u}, \bar{u}, u) = (0, 0, 0)$, $(1, 1, 1)$, and $(0, 1, 0)$ (ignorance of indifference). All three graphs are strictly monotonically increasing along \underline{u} with non-increasing slopes. For $a = 1$ (right), the graph is linear; here the CQTR function is equal to the CLTR function for $a = 1$ (Fig. 5 right), $u = \underline{u}$. This is the only intersection point of the CLTR and CQTR function instances. For $a = 0.5$ (left) and $a = 0.75$ (center), the graph of the CQTR function looks somewhat similar to the graph of the UW function for $\alpha = 1$ (Fig. 4 center) but the front edge curve is bent more strongly.

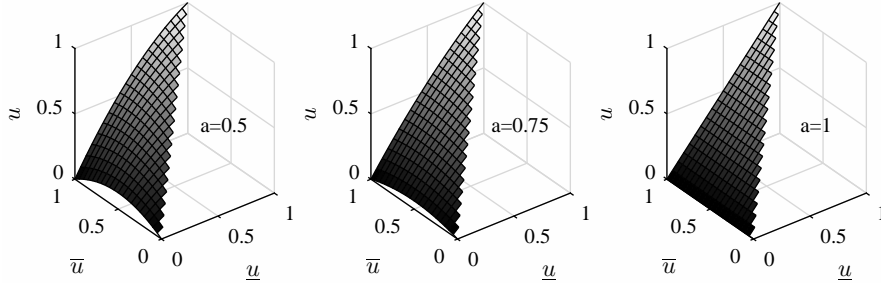


Figure 6: Graphs of the CQTR function for $a \in \{0.5, 0.75, 1\}$.

8. Summary and Examples

The properties of the NT, UW, CLTR, and CQTR functions are summarized in Table 1. The NT function is not strictly uncertainty conform and does not ignore indifference. The UW function is not type-2 consistent. CLTR is uncertainty conform for $a \in [0.5, 1]$ and strictly uncertainty conform for $a \in (0.5, 1]$ but ignores indifference only for $a = 1$. If we ignore this trivial special

case, the only function considered here that satisfies all five properties is CQTR for $a \in (0.5, 1]$.

Table 1: Properties of the NT, UW, CLTR, and CQTR functions.

	NT	UW	CLTR	CQTR
type-1 consistent	●	●	●	●
type-2 consistent	●		●	●
uncertainty conform	●	●	●	●
strictly uncertainty conform		●	●	●
ignores indifference		●	○	●

● only for $a \in (0.5, 1]$ or $[0.5, 1]$, ○ only for $a = 1$

To illustrate the practical relevance of these properties we look at three selected interval type-2 fuzzy sets and present the results for the four different (families of) type reduction functions.

Fig. 7 left illustrates type-2 consistency for the interval type-2 membership function (shaded area) computed as

$$\underline{u}(x) = 1.8 \cdot ((x - 0.3) \cdot (x - 0.5) \cdot (x - 0.7) + 0.15) \quad (64)$$

$$\bar{u}(x) = 3.5 \cdot ((x - 0.2) \cdot (x - 0.6) \cdot (x - 1) + 0.25) \quad (65)$$

These polynomial functions represent an example for smooth and continuous membership functions where the individual options possess different levels of preference and also different levels of uncertainty, where preference and uncertainty are not directly correlated. The maximum of the upper membership is at $x \approx 0.3$, but here the uncertainty is quite high. For $x \approx 1$ the upper membership is quite high but the uncertainty is much lower, so this area may be considered as a preferable choice in defuzzification. For this example NT (solid), CLTR ($a = 0.75$, dotted), and CQTR ($a = 0.75$, dash-dotted) yield type-1 membership functions that pass through the shaded area between $\underline{u}(x)$ and $\bar{u}(x)$. Only UW ($\alpha = 1$, dashed) yields a type-1 membership function that partially lies outside

the shaded area. As illustrated in this example, NT, CLTR, and CQTR are type-2 consistent and therefore yield memberships inside the uncertainty range of the type-2 memberships (between the lower and upper type-2 membership), whereas UW is not type-2 consistent and therefore may yield memberships outside this uncertainty range.

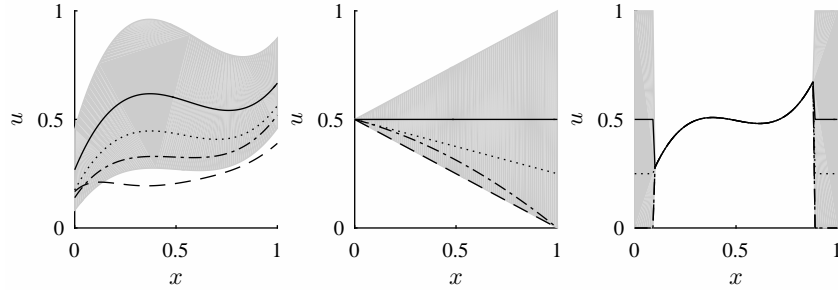


Figure 7: Example applications of type reduction functions: NT (solid), UW ($\alpha = 1$, dashed), CLTR ($a = 0.75$, dotted), CQTR ($a = 0.75$, dash-dotted).

Fig. 7 center illustrates strict uncertainty conformity for the interval type-2 membership function

$$\underline{u}(x) = 0.5 \cdot (1 - x) \quad (66)$$

$$\bar{u}(x) = 0.5 \cdot (1 + x) \quad (67)$$

This is an example where the average degree of preference is constant but where the uncertainty increases linearly with x , from zero at $x = 0$ to maximum uncertainty at $x = 1$, so we may prefer the option with minimum uncertainty in defuzzification. For this example UW ($\alpha = 1$, dashed), CLTR ($a = 0.75$, dotted), and CQTR ($a = 0.75$, dash-dotted) take into account the uncertainty and yield lower type-1 membership values for increasing uncertainty (increasing x). Only NT (solid) ignores the different levels of uncertainty and always yields a constant type-1 membership value of 0.5. As illustrated in this example, UW, CLTR, and CQTR are (for $a \in (0.5, 1]$) strictly uncertainty conform and therefore yield lower type-1 membership when the uncertainty is high, whereas

NT is not strictly uncertainty conform and therefore does not recognize different levels of uncertainty.

Fig. 7 right illustrates ignorance of indifference for the interval type-2 membership function

$$\underline{u}(x) = \begin{cases} 4.5 \cdot ((x - 0.3) \cdot (x - 0.5) \cdot (x - 0.7) + 0.11) & \text{for } x \in [0.1, 0.9] \\ 0 & \text{otherwise} \end{cases} \quad (68)$$

$$\bar{u}(x) = \begin{cases} 4.5 \cdot ((x - 0.3) \cdot (x - 0.5) \cdot (x - 0.7) + 0.11) & \text{for } x \in [0.1, 0.9] \\ 1 & \text{otherwise} \end{cases} \quad (69)$$

This is an example where the uncertainty is zero in a certain interval (here $x \in [0.1, 0.9]$), so here the memberships are of type-1, and outside this interval we have maximum uncertainty, $\underline{u}(x) = 0$ and $\bar{u}(x) = 1$, for $x \in [0, 0.1]$ and for $x \in (0.9, 1]$ (shaded areas), so we may want to ignore the areas of extreme uncertainty in defuzzification. For this example all four methods correctly recognize the type-1 part of this membership function. However, only UW ($\alpha = 1$, dashed) and CQTR ($a = 0.75$, dash-dotted) yield zero type-1 membership values for the two areas of indifference, whereas NT (solid) yields 0.5, and CLTR ($a = 0.75$, dotted) yields 0.25 there. As illustrated in this example, UW and CQTR (for $a \in (0.5, 1]$) ignore indifference and therefore always yield zero membership for options with maximal uncertainty, whereas NT and CLTR (for $a \neq 1$) do not ignore indifference and therefore yield non-zero memberships here.

9. Conclusions

As a first step towards a theoretical foundation of type reduction we have introduced a set of five mathematical properties and illustrated some relations between these properties.

Our experimental studies have proven the merit of objective mathematical criteria for the study of type reduction operators. The Nie-Tan (NT) method is a good choice for type reduction when it is important to be type-2 consistent, but not important to be strictly uncertainty conform or to ignore indifference. The uncertainty weight (UW) method is a good choice when it is important to be

strictly uncertainty conform and to ignore indifference, but when not important to be type-2 consistent.

Further we have proposed two new type reduction methods, consistent linear type reduction (CLTR) and consistent quadratic type reduction (CQTR). CLTR satisfies all our five properties except ignorance of indifference, and CQTR completely satisfies all five properties (for $a \in (0.5, 1]$). So, CLTR and CQTR are considered good alternatives to NT and UW with attractive mathematical properties.

The purpose of this paper is to provide a foundation of the theory of type reduction. Given the width of this field, this paper has to leave many important aspects open for further research, for example: Can we find a real-world application example where satisfying our five properties will improve system performance? How can our list of five properties of type reduction be reasonably refined and/or extended? How can similar properties be defined for other types of (non-interval) type-2 fuzzy sets? How does type reduction relate to averaging and ignorance functions [5]? How can we define properties of type reduction for non-pointwise type reduction, for example in constrained fuzzy sets [8], where a type-2 fuzzy set is generated by wobbling a type-1 fuzzy set along the horizontal axis? What kind of properties can we realize with nonlinear type reduction functions other than UW and CQTR? Where can type reduction be applied beyond defuzzification?

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