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AN OPTIMAL ORDER MULTILEVEL PRECONDITIONER WITH RESPECT TO PROBLEM AND DISCRETIZATION PARAMETERS

O. Axelsson and S. Margenov

An optimal order multilevel preconditioner with respect to problem and discretization parameters

by O.Axelsson[*]and S. Margenov[†]

Abstract

Preconditioners based on various multilevel extensions of two-level finite element methods lead to iterative methods which have an optimal order computational complexity with respect to the size (or discretization parameter) of the system. The methods can be on block matrix factorized form, recursively extended via certain matrix polynomial approximations of the arising Schur complement matrices or on additive, i.e. block diagonal form using stabilizations of the condition number at certain levels.The resulting spectral equivalence holds uniformly with respect to jumps in the coefficients of the differential operator and for arbitrary triangulations. Such methods were first presented by Axelsson and Vassilevski in the late 80s.

An important part of the algorithm is the treatment of the systems with the diagonal block matrix, which arise on each finer level and corresponds to the added degrees of freedom on that level.This block is well-conditioned for model type problems but becomes increasingly ill-conditioned when the coefficient matrix becomes more anisotropic or, equivalently, when the mesh aspect ratio increases.

In the paper two methods are presented to approximate this matrix also leading to a preconditioner with spectral equivalence bounds which hold uniformly with respect to both the problem and discretization parameters. The same holds therefore also for the preconditioner to the global matrix.

1   Introduction

In many problems in mathematical modelling in natural sciences, engineering and in other areas as well where second order boundary value problems must be solved numerically, large scale linear systems arise which frequently must be solved a number of times for each modelling case.

Often, the arising systems are severely ill-conditioned due to some problem parameters taking near limit values. Examples of such parameters are ratio of coefficient jumps, anisotropy, aspect ratio of the mesh and domain geometry, Poisson ratio for nearly incompressible materials etc. Furthermore, the condition number may increase rapidly

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when the discretization mesh is refined (due to both a smaller mesh parameter and possibly irregularity of the mesh elements). In finding a good solution method one must preferably search for efficient preconditioners for the, parameter free, conjugate gradient iterative solution method.

The method to be presented is a block matrix approximate factorization preconditioning of the algebraic multilevel iteration, AMLI type. It is based on two or multilevel finite element meshes and can handle arbitrary coefficient jumps on the coarsest mesh used (which, by itself can be quite fine) and also ratio of anisotropy, using newly developed finite element based preconditioners for the block corresponding to the added nodes. The condition number is bounded for any ratio of coefficient jumps and anisotropy.

Algebraic multilevel preconditioners were first presented in [10, 11] and are multilevel extension of the two-level methods in [13] and [7]. Here block matrix approximate factorizations were considered and it was shown that that by recursively extending the two level method using certain matrix polynomial approximations of the arising Schur complement matrices, one can derive a preconditioning with a condition number which is bounded independent on the number of levels and on jumps in the coefficients, assuming the coarsest mesh used had no jumps inside any element. Similarly, preconditioners in additive form, i.e., using block diagonal preconditioners, but with stabilization at certain levels (see [4]) were developed with the same properties.

In the above methods the block matrix corresponding to the on each level added degrees of freedom gets increasingly ill-conditioned with increasing degree of anisotropy. Until recently, no efficient generally applicable method to handle this problem has been given. In [16], a preconditioner to this matrix in multiplicative form and in [9] an element by element preconditioner in additive form were suggested. The first method considered either x- or y-dominated anisotropy while the latter considered the general case with arbitrary coefficients in the differential operator. It was shown that the preconditioner is spectrally equivalent to the given matrix with bounds which holds uniformly in the number of levels and in the coefficients of the operator.

In the present paper we consider possible improvements of these methods. In particular it is shown that for a new element by element preconditioner in multiplicative form, an significant improvement in the condition number can be achieved.

The remainder of the paper is organized as follows: In section 2 we survey the major results for multiplicative and additive preconditioners in algebraic multilevel form. In Section 3 we recall some basic results for element by element analysis while Section 4 deals with the construction of the new preconditioners for the block diagonal matrix corresponding to the added degrees of freedom on each level.

2 Multilevel preconditioning methods for elliptic boundary value problems

2.1 Variational formulation

Consider the variational formulation of an elliptic boundary value problem,

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) = f \quad \text{in } \Omega$$

with proper boundary conditions, i.e., seek $u \in V = H_0^1(\Omega)$ such that

$$a(u,v) = \int_{\Omega} fv, \quad \text{for all } v \in V,$$

where

$$a(u,v) = \int_{\Omega} \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}, \quad u, v \in V,$$

and it is assumed that $[a_{ij}]$ is a symmetric and positive definite s.p.d. matrix. For its numerical solution a finite element method is used, i.e., one seeks $u_h \in V_h \subset V$ such that

$$a(u_h, v_h) = \sum_{e \in T} a_e^{(h)}(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h, \tag{1}$$

where

$$a_e^{(h)} = \int_e \sum_{i,j=1}^{2} a_{ij}^{(e)} \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j},$$

T denotes a set of triangles in a proper partitioning of the domain Ω (which is assumed to be polygonal for simplicity). Further h is a corresponding mesh size parameter and V_h is the FEM space. In this paper we restrict the FEM space to piecewise linear basis functions.

This leads to an algebraic system $Au = f$, where A is s.p.d. We survey here a result showing the existence of a preconditioner C with a condition number bound of $C^{-1}A$ which holds uniformly in the parameter h and the coefficients $[a_{ij}]$, i.e. in ratio of anisotropy or shape of elements and jumps in coefficients, if the latter occur only across element edges of the coarsest mesh used. For simplicity, we assume further that the coefficients are constant on each coarse mesh element. (By proper balancing the computational costs of the method used on the coarse mesh with the cost on the finest mesh one finds easily that the coarsest mesh can be quite fine by itself while still allowing an optimal order of computational complexity even if a simple iterative method is used on the coarse mesh, see [8] for further details.) In addition, it allows for efficient computations of the action of C^{-1}.

We consider then a sequence of finite element matrices partitioned in two by two block form

$$A^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix}, \quad k = 0, 1, \ldots, k_0 - 1$$

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where $A = A^{(0)}$, k_0 is the level number for the finest mesh and $A^{(0)}$ is the coarsest mesh matrix. The order of $A^{(k+1)}$ is n_{k+1} and the order of $A_{22}^{(k)}$ equals n_k, i.e. that of $A^{(k)}$. Here $A^{(k+1)}$ is either given on hierarchical basis function form, $\widehat{A}^{(k+1)}$, in which case $\widehat{A}_{22}^{(k)} = A^{(k)}$, or in standard basis form $A^{(k)}$. The latter is sparser than $\widehat{A}^{(k)}$ and it is therefore preferable to use it. Note that $\widehat{A}_{11}^{(k+1)} = A_{11}^{(k+1)}$. This matrix corresponds to the added degrees of freedom on mesh level k_0.

2.2 A recursive block diagonal preconditioner

In this subsection, we consider matrices on hierarchical basis form. For $\widehat{A}^{(k+1)}$ it is efficient to use a block diagonal preconditioner,

$$D^{(k+1)} = \begin{bmatrix} B_{11}^{(k+1)} & 0 \\ 0 & M^{(k)} \end{bmatrix}$$

where $B_{11}^{(k+1)}$ is a preconditioner to $A_{11}^{(k+1)}$ and $M^{(k)}$ is a preconditioner to $A^{(k)}$. The following theorem shows a bound of the condition number of $D^{(k+1)^{-1}} \widehat{A}^{(k+1)}$. It involves the CBS-constant γ which is the cosine of the angle between the finite element subspace $V_1^{(k+1)}$ of added basis functions and subspace $V_2^{(k)}$ of the coarse mesh basis functions. For further details regarding choices of V_1, V_2 see [11] and [7]. Below $A \geq B$ means that $A - B$ is positive semidefinite.

Theorem 1 *(see [7]) Let B_{11}, M be preconditioners to A_{11} and A_{22}, respectively and*

$$b_1 B_{11} \leq A_{11} \leq b_0 B_{11}, \quad (\text{for all } v_1 \in V_1)$$

and

$$a_1 M \leq A_{22} \leq a_0 M, \quad (\text{for all } v_2 \in V_2).$$

Then

a) cond $\left\{ \begin{bmatrix} B_{11}^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right\} \leq \frac{1+\gamma}{1-\gamma} \left(\frac{2}{1+\gamma} \right)^2 \frac{1}{2}(a_0 + b_0)) \cdot \frac{1}{2} \left(\frac{1}{a_1} + \frac{1}{b_1} \right)$

where

$$\gamma = \sup_{\substack{u \in V_1 \\ v \in V_2}} \frac{a(u,v)}{\{a(u,u)a(v,v)\}^{\frac{1}{2}}}.$$

b) For $a_0 \geq b_0$ and $a_1 \leq b_1$ it holds cond $\leq \frac{1+\gamma}{1-\gamma} \frac{a_0}{a_1}$.

This preconditioner can be readily extended by recursion to a k_0-level matrix,

$$D^{(k_0,0)} = \begin{bmatrix} B_{11}^{(k_0)} & & & 0 \\ & \ddots & & \\ & & B_{11}^{(1)} & \\ 0 & & & A^{(0)} \end{bmatrix}$$

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where $A^{(0)}$ is the standard basis function matrix on the coarsest mesh level. It can be seen (see [3]) that the condition number of $D^{(k_0,0)^{-1}}\widehat{A}^{(k_0)}$ becomes bounded by a recursive expression derived from Theorem 1. If $b_0 \leq a_0$, $b_1 \geq a_1$ on each level, this becomes bounded by $\left(\frac{1+\gamma}{1-\gamma}\right)^{k_0}$. Using certain matrix polynomials on some properly chosen levels or similarly using an inner iteration method, one can bound the condition number so that it doesn't grow with the number of levels. In the actual implementation of the iteration method one can use relation (6) (see next subsection) in order to avoid dealing with the less sparse hierarchical basis matrices.

2.3   Block matrix factorized preconditioner

Consider now a block matrix factorized preconditioner.
The exact block matrix factorization of $A^{(k+1)}$ is

$$A^{(k+1)} = \left[\begin{array}{cc} A_{11}^{(k+1)} & 0 \\ A_{21}^{(k+1)} & I_2^{(k)} \end{array}\right] \left[\begin{array}{cc} I_1^{(k)} & A_{11}^{(k+1)^{-1}} A_{12}^{(k+1)} \\ 0 & S_A^{(k)} \end{array}\right]$$

where $I_1^{(k)}, I_2^{(k)}$ are unit matrices of corresponding orders and $S_2^{(k)}$ is the Schur complement matrix

$$S_A^{(k)} = A_{22}^{(k+1)} - A_{21}^{(k+1)} A_{11}^{(k+1)^{-1}} A_{12}^{(k+1)}.$$

It can be readily shown (see e.g. [10]) that the Schur complements for the hierarchical and the standard matrices are identical. We are primarily interested in using the standard basis function matrix. Since $S^{(k)}$ is normally a full matrix we must approximate it with a sparse matrix. In general, $A_{11}^{(k+1)}$ must also be approximated. The resulting preconditioning method is of AMLI type, i.e.

$$C^{(k+1)} = \left[\begin{array}{cc} B_{11}^{(k+1)} & 0 \\ A_{21}^{(k+1)} & I_2^{(k)} \end{array}\right] \left[\begin{array}{cc} I_1^{(k)} & B_{11}^{(k+1)^{-1}} A_{12}^{(k+1)} \\ 0 & S_B^{(k)} \end{array}\right]$$

where $B_{11}^{(k+1)}$ is a preconditioner to $A_{11}^{(k+1)}$ and $S_B^{(k)}$ to $S_A^{(k)}$, which are assumed to satisfy

$$\beta^{-1} v_1^T A_{11}^{(k+1)} v_1 \leq v_1^T B_{11}^{(k+1)} v_1 \leq v_1^T A_{11}^{(k+1)} v_1 \quad \text{for all } v_1 \in \mathcal{R}^{n_{k+1}-n_k} \tag{2}$$

$$\eta^{-1} v_2^T S_A v_2 \leq v_2^T S_B v_2 \leq v_2^T S_{\widetilde{A}} v_2 \quad \text{for all } v_2 \in \mathcal{R}^{n_k} \tag{3}$$

where $\widetilde{A} = \left[\begin{array}{cc} B_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right]$, $S_{\widetilde{A}} = A_{22} - A_{21} B_{11}^{-1} A_{12}$ and $\beta \geq 1$, $\eta \geq 1$.
Finally, we assume that

$$\alpha^{-1} v_2^T S_A v_2 \leq v_2^T S_{\widetilde{A}} v_2 \leq v_2^T S_A v_2, \quad \text{for all } v_2 \in \mathcal{R}^{n_k}. \tag{4}$$

Here the right inequality follows directly from the right inequality (2). The left inequality is assumed to be sharp. Since $\eta^{-1} v_2^T S_A v_2 \leq v_2^T S_{\widetilde{A}} v_2$ for all v_2 it follows that $\alpha \leq \eta$.
As has been shown in [12], the following condition number bound holds.

Theorem 2 *Let (2) - (4) hold. Then*

$$\kappa^{-1} v^T A v \le v^T B v \le v^T A v, \quad \text{for all } v, \qquad \text{where} \quad \kappa \le \eta\beta$$

Hence, the condition number is bounded by the product of the condition numbers for $B_{11}^{-1} A_{11}$ and $S_B^{-1} S_A$. However, as shown in [12], (see also [18]) η depends in general on β unless hierarchical basis functions are used. (Note that $S_{\widetilde{A}}$ is involved in (3) and $S_{\widetilde{A}}$ depends heavily on B_{11}.) Loosely speaking, it must hold that $v_2^T S_B v_2 \simeq v_2^T S_A v_2$, $v_1^T B_{11} v_1 \simeq v_1^T A_{11} v_1$ and $v_1 = A^{-1} A_{12} v_2 \simeq B_1^{-1} A_{12} v_2$ when v_2 is a "smooth" vector. In order to avoid this limitation in choosing B_{11} one can introduce the following perturbations of the off-diagonal blocks in the preconditioner. This important trick was found already in [11].

The preconditioner takes now the form

$$M^{(k+1)} = \begin{bmatrix} B_{11}^{(k+1)} & 0 \\ \widetilde{A}_{21}^{(k+1)} & S_B^{(k)} \end{bmatrix} \begin{bmatrix} I_1^{(k+1)} & B_{11}^{(k+1)^{-1}} \widetilde{A}_{12}^{(k)} \\ 0 & I_2^{(k)} \end{bmatrix}$$

where

$$\begin{aligned} \widetilde{A}_{12}^{(k+1)} &= A_{12}^{(k+1)} + \left(A_{11}^{(k+1)} - B_{11}^{(k+1)} \right) J_{12}^{(k+1)} \\ \widetilde{A}_{21}^{(k+1)} &= A_{21}^{(k+1)} + J_{12}^{(k+1)^T} \left(A_{11}^{(k+1)} - B_{11}^{(k+1)} \right). \end{aligned} \tag{5}$$

Here $J_{12}^{(k+1)}$ is an interpolation matrix which transforms the components of the current coarse vector to the new components of the vector on the next finer level. The reason for perturbing the off-diagonal block matrices as done in (5) is that in this way

$$\widehat{M}^{(k+1)} \equiv J^{(k+1)^T} M^{(k+1)} J^{(k+1)} \tag{6}$$

where

$$J^{(k+1)} = \begin{bmatrix} I_1^{(k)} & J_{12}^{(k+1)} \\ 0 & I_2^{(k)} \end{bmatrix},$$

takes the form

$$\widehat{M}^{(k+1)} = \begin{bmatrix} B_{11}^{(k+1)} & \widehat{A}_{12}^{(k+1)} \\ \widehat{A}_{21}^{(k+1)} & S_B^{(k)} + \widehat{A}_{21}^{(k+1)} B_{11}^{(k+1)^{-1}} \widehat{A}_{12}^{(k+1)} \end{bmatrix},$$

which follows from an elementary computation. Here $\widehat{A}_{12}^{(k+1)} = A_{12} + A_{11} J_{12}^{(k+1)}$ is the off-diagonal block in the hierarchical basis function matrix

$$\widehat{A}^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & \widehat{A}_{12}^{(k+1)} \\ \widehat{A}_{21}^{(k+1)} & A^{(k)} \end{bmatrix}.$$

Hence $\widehat{M}^{(k+1)}$ can be considered as a preconditioner to $\widehat{A}^{(k+1)}$ and the extreme eigenvalues of $M^{(k+1)^{-1}} A^{(k+1)}$ equal those of $\widehat{M}^{(k+1)^{-1}} \widehat{A}^{(k+1)}$, since

$$\sup_v \frac{v^T A^{(k+1)} v}{v^T M^{(k+1)} v} = \sup_{\widehat{v}} \frac{\widehat{v}^T \widehat{A}^{(k+1)} \widehat{v}}{\widehat{v}^T \widehat{M}^{(k+1)} \widehat{v}}, \qquad \inf_v \frac{v^T A^{(k+1)} v}{v^T M^{(k+1)} v} = \inf_{\widehat{v}} \frac{\widehat{v}^T \widehat{A}^{(k+1)} \widehat{v}}{\widehat{v}^T \widehat{M}^{(k+1)} \widehat{v}}$$

Since the off-diagonal blocks in $\widehat{M}^{(k+1)}$ equal those in $\widehat{A}^{(k+1)}$ the estimate of the extreme eigenvalues of $\widehat{M}^{(k+1)^{-1}}\widehat{A}^{(k+1)}$ can be readily done. As shown in [11], see also [18], if

$$v_1^T A_{11}^{(k+1)} v_1 \leq v_1^T B_{11}^{(k+1)} v_1 \leq (1+b)v_1^T A_{11}^{(k+1)} v_1, \quad \text{for all } v_1 \in \mathcal{R}^{n_{k+1}-n_k}$$

and

$$v_2^T A^{(k)} v_2 \leq v_2^T S_B v_2 \leq (1+d)v_2^T A^{(k)} v_2, \quad \text{for all } v_2 \in \mathcal{R}^{n_k}$$

then

$$\text{cond}\left(M^{(k+1)^{-1}} A^{(k+1)}\right) \leq \frac{1+b+d}{1-\gamma^2}.$$

Both the additive and multiplicative methods can be extended recursively replacing S_B with a matrix polynomial approximation

$$\widehat{M}^{(k)} = [I - P_\nu(M^{(k)^{-1}} A^{(k)})]^{-1} A^{(k)}$$

where $P_\nu(0) = 1$ and P_ν is small on the interval of the eigenvalues of $M^{(k)^{-1}} A^{(k)}$. The best approximation is by a shifted and scaled Chebyshev polynomial, see [10]. In this way, the condition number can be stabilized, i.e. bounded by a number which does not depend on the number of levels. The polynomial doesn't have to be the same on each level.

Remark 1 *There are some restrictions on ν (lower and upper bounds) to obtain an optimal order, $O(n)$ of computational complexity. As has been shown in [10], both conditions can be met by applying, if necessary, the stabilization only on certain levels.*

It remains now to construct approximations $B_{11}^{(k+1)}$ to $A_{11}^{(k+1)}$ which is the major topic of the paper.

3 Element by element analysis of multilevel preconditioners

A crucial part of the analysis of multilevel iteration methods is the analysis of the behaviour of the constant γ in the strengthened Cauchy-Bunyakowski-Schwarz inequality,

$$a(u,v) \leq \gamma \{a(u,u)a(v,v)\}^{\frac{1}{2}}, \quad \text{for all } u \in V_1, \ v \in V_2$$

where V_1, V_2 are defined in Section 2. As shown in [7], see also [14], [5], equivalently we can analyze γ from the inequality

$$(1-\gamma)(a(u,u)+a(v,v)) \leq a(u,v) \leq (1+\gamma)(a(u,u)+a(v,v)), \quad \text{for all } u \in V_1, \ v \in V_2. \tag{7}$$

The inequality (7) corresponds to a block diagonal preconditioner. More generally, the following result holds.

Lemma 1 *Let* $A = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix}$ *be a symmetric and positive definite matrix partitioned in blocks consistent with a vector partitioning* $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. *Let* γ, $0 < \gamma < 1$ *be the smallest constant for which* $v^T A w \leq \gamma (v^T A v)^{\frac{1}{2}} (w^T A w)^{\frac{1}{2}}$, *for all* $v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$ *holds. Then*

(i) $\quad \gamma^2 = \sup_{v_i} \left\{ v_i^T A_{ij} A_j^{-1} A_{ji} v_i / v_i^T A_i v_i \right\}$

(ii) $\quad (1 - \gamma)(v^T A v + w^T A w) \leq v^T A w \leq (1 + \gamma)(v^T A v + w^T A w)$, *for all* v, w.

(iiia) $v_1^T S_A^{(1)} v_1 \geq (1 - \gamma^2) v_1^T A_1 v_1$, *for all* v_1

(iiib) $v_2^T S_A^{(2)} v_2 \geq (1 - \gamma^2) v_2^T A_2 v_2$, *for all* v_2

where $S_A^{(i)} = A_i - A_{ij} A_j^{-1} A_{ji}$, $i \neq j$, $i, j = 1, 2$ *and the inequalities are sharp.*

Proof. Part (i) follows directly from the definition of γ. Further

$$(v + w)^T A (v + w) = v^T A v + 2 v^T A w + w^T A w,$$

so part (ii) follows from part (i),

$$|v^T A w| \leq \gamma \{ v^T A v w^T A w \}^{\frac{1}{2}}$$

and the inequality $2ab \leq a^2 + b^2$. Further, to show that part (i) implies parts (iii, a, b) we note that for any ξ, $\gamma \leq \xi \leq \gamma^{-1}$ the generalized arithmetic-geometric mean inequality $2ab \leq \xi a^2 + \xi^{-1} b^2$, $a, b \geq 0$, with $a = v^T A v$, $b = w^T A w$ implies

$$(v + w)^T A (v + w) = v^T A v + 2 v^T A w + w^T A w \geq (1 - \xi \gamma) v^T A v + (1 - \xi^{-1} \gamma) w^T A w.$$

Letting here $\xi = \gamma$ shows that

$$v_1^T S_A^{(1)} v_1 = \inf_w (v + w)^T A (v + w) \geq (1 - \gamma^2) v^T A_1 v$$

which is $(iiia)$. Similarly letting $\xi = \gamma^{-1}$ $(iiib)$ follows. The relation in (i) and the sharpness of the estimates follows by considering $\tilde{v}_i^T \tilde{A}_{ij} \tilde{v}_j \leq \gamma (\tilde{v}_i^T \tilde{v}_i \tilde{v}_j^T \tilde{v}_j)^{\frac{1}{2}}$, where $\tilde{v}_i = A_i^{\frac{1}{2}} v_i$, $\tilde{A}_{ij} = A_i^{-\frac{1}{2}} A_{ij} A_j^{-\frac{1}{2}}$ and repeating the above for the matrix $\tilde{A} = \begin{bmatrix} I_1 & \tilde{A}_{12} \\ \tilde{A}_{21} & I_2 \end{bmatrix}$. $\blacksquare$

Next we make the well-known observation that

$$a(u, v) = \sum_{e \in \mathcal{T}} a_e(u, v),$$

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where $a_e(u,v)$ is the element contribution. This is the basis for the assembly process in finite element methods. If the preconditioner to the global stiffness matrix is constructed in the same way by assembly of local stiffness matrices, it follows therefore that the analysis of the corresponding condition number can be done for the element matrices. This observation was first done in [6] and tremendously simplifies the analysis of preconditioners for finite element matrices.

We make next the last basic observation, which enables the analysis of finite element matrices for an arbitrary linear form (1). We show that the analysis for an arbitrary finite element triangle (e) with coordinates (x_i, y_i), $i = 1, 2, 3$ can be done on the reference triangle $(\tilde{e})$, with coordinates (0,0), (1,0), (0,1). Transforming the finite element function between these triangles, the element bilinear form becomes (see e.g. [4]),

$$a_e(u,v) = a_{\tilde{e}}(\tilde{u}, \tilde{v}) =$$

$$\int_{\tilde{e}} \left[\frac{\partial \tilde{u}}{\partial \tilde{x}}, \frac{\partial \tilde{u}}{\partial \tilde{y}} \right] \left[\begin{array}{cc} (x_2-x_1) & (y_2-y_1) \\ (x_3-x_1) & (y_2-y_1) \end{array} \right] \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \left[\begin{array}{cc} (x_2-x_1) & (x_3-x_1) \\ (y_2-y_1) & (y_3-y_1) \end{array} \right] \left[\begin{array}{c} \frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{\partial \tilde{u}}{\partial \tilde{y}} \end{array} \right] d\tilde{x}d\tilde{y},$$

where $0 < \tilde{x}, \tilde{y} < 1$, i.e., it takes the form

$$\tilde{a}_e(\tilde{u}, \tilde{v}) = \int_{\tilde{e}} \sum_{i,j} \tilde{a}_{ij} \frac{\partial \tilde{u}}{\partial \tilde{x}_i} \frac{\partial \tilde{v}}{\partial \tilde{x}_j} \, d\tilde{x} \, d\tilde{y},$$

where $\tilde{x}_1 = \tilde{x}$, $\tilde{x}_2 = \tilde{y}$ and where the coefficients $\tilde{a}_{ij}$ depend on both the coordinates in e (or, equivalently, the angles in e) and the coefficients a_{ij} in the differential operator. We conclude that it suffices for the analysis of the CBS-constant γ and as we shall see, also of the condition number of $B_{11}^{-1} A_{11}$ to consider (7) for the reference triangle and arbitrary coefficients $[a_{ij}]$, or alternatively, for the operator $-\Delta$ and an arbitrary triangle e, see [4], respectively [15]. A computation (see [4]) shows that $\gamma^2 < \frac{2}{3}$ for any coefficients $[a_{ij}]$ and any triangulation. In the next section it will be seen that the construction and analysis of preconditioners for matrix $A_{11}^{(k+1)}$ can be done elementwise also.

4 Optimal order preconditioners of $A_{11}^{(k+1)}$

In this section we describe two algorithms for construction of optimal order preconditioners $B_{11}^{(k+1)}$ to the matrices $A_{11}^{(k+1)}$ which are required within the AMLI methods under consideration. For both algorithms the condition numbers are bounded for all levels, i.e.

$$\kappa \left(B_{11}^{(k+1)^{-1}} A_{11}^{(k+1)} \right) = O(1) \quad k = k_0, \dots, \ell - 1$$

where the constant in the estimate is independent of the initial triangulation and the coefficients $a_{ij}(x)$ of the differential operator. The construction and the analysis of the preconditioners $B_{11}^{(k+1)}$ are based on a macroelement-by-macroelement assembling procedure.

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Let us consider two consecutive levels of uniform refinement (k) and $(k + 1)$. They correspond to the triangulations $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ where each element of $\mathcal{T}_k$ is divided into four congruent triangles of $\mathcal{T}_{k+1}$. We call the union of these four triangles *macroelement* $E \in \mathcal{T}_{k+1}$ (see Figure 1).
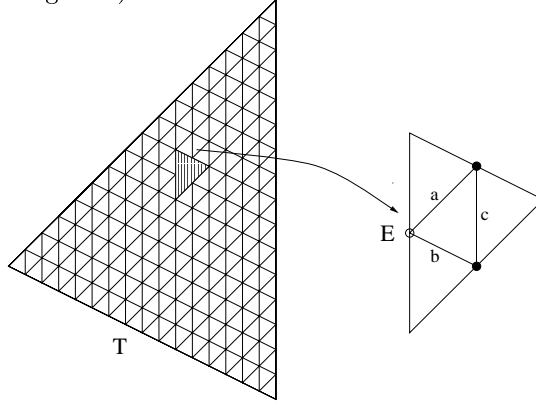

Figure 1: Four levels of uniform refinement of $T \in \mathcal{T}_0$ and macroelement $E \in \mathcal{T}_3$.

Following the standard FEM assembling procedure we can write $A_{11}^{(k+1)}$ in the form

$$A_{11}^{(k+1)} = \sum_{E \in \mathcal{T}_{k+1}} L_E^{(k+1)}{}^T A_{11:E}^{(k+1)} L_E^{(k+1)}, \tag{8}$$

where $L_E^{(k+1)}$ stands for the restriction mapping of the global vector of unknowns to the local one corresponding to the macroelement E. Accounting for the general form of the element stiffness matrix corresponding to $T \in \mathcal{T}_0$ we get the following simple presentation of $A_{11:E}^{(k+1)}$, see e.g. [5].

$$A_{11:E}^{(k+1)} = 2\, r_T \begin{bmatrix} a_T + b_T + c_T & -c_T & -b_T \\ -c_T & a_T + b_T + c_T & -a_T \\ -b_T & -a_T & a_T + b_T + c_T \end{bmatrix} \tag{9}$$

where r_T depends on the shape of $T \in \mathcal{T}_0$ and on the related coefficients of the differential operator.

In what follows we will simplify the notations omitting the argument and the subscript T. This will not lead to any confusion as all the constructions we will introduce are local, that is they are within one and the same element of the initial triangulation $T \in \mathcal{T}_0$. Now without loss of generality we assume that $|a| \leq b \leq c$. This follows from the following relations.

Lemma 2 *Let $\theta_1, \theta_2, \theta_3$ be the angles in an arbitrary triangle. Then if $a = \cot\theta_1$, $b = \cot\theta_2$, $c = \cot\theta_3$ it holds*

(i) $\quad a = \frac{1-bc}{b+c}$

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(ii)   If $\theta_1 \geq \theta_2 \geq \theta_3$ then $|a| \leq b \leq c$

(iii) $a + b > 0$.

*Proof.*Since $a = \cot(\pi - (\theta_2 + \theta_3))$ elementary trigonometric relations show that

$$a = -\cot(\theta_2 + \theta_3) = \frac{1 - \cot\theta_2 \cot\theta_3}{\cot\theta_2 + \cot\theta_3},$$

which is part (i). To prove part (ii), note that if $\theta_1 \leq \frac{\pi}{2}$, then $\theta_1 \geq \theta_2 \geq \theta_3$ shows that $0 < a \leq b \leq c$. If the triangle is obtuse, i.e. $\theta_1 > \frac{\pi}{2}$, then $\theta_2 + \theta_3 < \frac{\pi}{2}$ and it follows that $a < 0$ and

$$|a| = \frac{\cot\theta_2 \cot\theta_3 - 1}{\cot\theta_2 + \cot\theta_3} = \cot\theta_2 \frac{\cot\theta_3 - 1/\cot\theta_2}{\cot\theta_3 + \cot\theta_2} < \cot\theta_2 = b.$$

Finally, $a + b = \sin(\theta_1 + \theta_2)/(\sin\theta_1 \sin\theta_2) > 0$.   ∎
Then

$$A_{11:E}^{(k+1)} = 2\ r\ c \begin{bmatrix} \alpha + \beta + 1 & -1 & -\beta \\ -1 & \alpha + \beta + 1 & -\alpha \\ -\beta & -\beta & \alpha + \beta + 1 \end{bmatrix}, \tag{10}$$

where $\alpha = a/c$, $\beta = b/c$. Taking into account, that $a = \cot\theta_T^{(1)}$, $b = \cot\theta_T^{(2)}$, and $c = \cot\theta_T^{(3)}$ where $\theta_T^{(1)} + \theta_T^{(2)} + \theta_T^{(3)} = \pi$ are the angles of some auxiliary triangle depending on $T \in \mathcal{T}_0$ and on the corresponding coefficients $a_{ij}(T)$ of the differential operator (see e.g. [5]), we get that $(\alpha, \beta) \in D$ where

$$D = \{(\alpha, \beta) \in \mathcal{R}^2 : -\frac{1}{2} < \alpha \leq 1,\ 0 < \beta \leq 1,\ \alpha + \beta > 0,\ and\ |\alpha| \leq \beta\}. \tag{11}$$
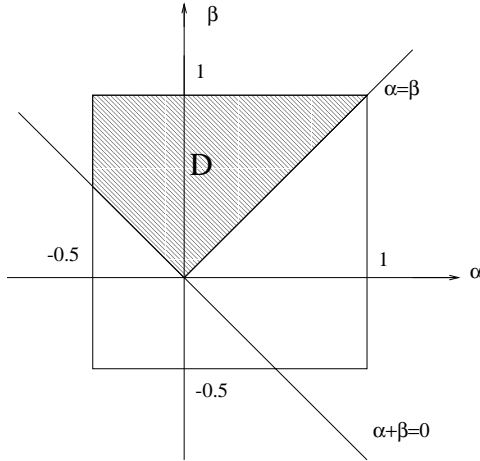

Figure 2: Domain of parameters (α, β).

The next pure algebraic inequality will also be used in the following two subsections.

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Lemma 3 *For all $(\alpha, \beta) \in D$ holds the inequality*

$$\frac{\alpha\beta + \alpha + \beta + 1}{(\alpha + \beta + 1)(\alpha + \beta + 2)} > \frac{4}{15} \tag{12}$$

Proof. The inequality is equivalent to

$$4\alpha^2 + 4\beta^2 - 3(\alpha + \beta) - 7\alpha\beta < 7. \tag{13}$$

Introduce the auxiliary function $\psi(\alpha, \beta) = 4\alpha^2 + 4\beta^2 - 3(\alpha + \beta) - 7\alpha\beta$ defined in D (see Figure (2)). From

$$\frac{\partial \psi}{\partial \alpha} = 8\alpha - 7\beta - 3$$

It follows that if ψ has an extremum in some interior point $(\tilde{\alpha}, \tilde{\beta}) \in D$ then $\tilde{\alpha} = (7\tilde{\beta} + 3)/8$. Now we consider

$$\psi(\frac{7\beta + 3}{8}, \beta) = \tilde{\psi}(\beta) = \frac{1}{16}(15\beta^2 - 90\beta - 9)$$

which is strictly decreasing if $0 \leq \beta \leq 1$. This means that $\psi(\alpha, \beta)$ achieves its infimum on the boundary of D. From the expression (12) follows that the extreme values must be taken either for $\alpha < 0$ and $|\alpha|\beta$ *maximum* or for $\alpha = \beta = 1$. This simply leads to

$$\psi_{max} = \psi(-\frac{1}{2}, 1) = 7$$

which completes the proof of the lemma. ∎

The approach used to construct the preconditioners discussed in the next subsections can be summarized as *preserving the links between the mesh nodes along the dominating anisotropy.*

4.1 Additive preconditioning of $A_{11}^{(k+1)}$

The additive preconditioner is defined as follows

$$B_{11}^{(k+1)} = \sum_{E \in \mathcal{T}_{k+1}} L_E^{(k+1)^T} B_{11:E}^{(k+1)} L_E^{(k+1)}. \tag{14}$$

The local matrix $B_{11:E}^{(k+1)}$ is obtained by preserving only the *strongest* off-diagonal entries, i.e., we have

$$B_{11:E}^{(k+1)} = 2\ r\ c \begin{bmatrix} \alpha + \beta + 1 & -1 & 0 \\ -1 & \alpha + \beta + 1 & 0 \\ 0 & 0 & \alpha + \beta + 1 \end{bmatrix}, \tag{15}$$

It is important to note that the so defined matrix $B_{11:E}^{(k+1)}$ has a generalized tridiagonal structure (see [9] and also [17]) which means that the solution of linear systems with

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$B_{11:E}^{(k+1)}$ has a computational cost which is proportional to the related problem size. The structure of the assembled preconditioning matrix $B_{11}^{(k+1)}$ is convenient for a rapid solution. Due to the form of the corresponding element matrices $B_{11:E}^{(k+1)}$, each node is coupled to none, one or at most two neighbors. This means that the coupled nodes form either a single point, a polyline or a polygon. Therefore, there are no cross-points. If we order the nodes along the connectivity lines, we get a block-diagonal form of the matrix $B_{11}^{(k+1)}$, where each block matrix is tridiagonal and corresponds to such a group of coupled nodes. Clearly, each of the blocks can be solved by a direct method with an arithmetic cost proportional to its dimension. Furthermore, an algorithm for ordering the unknowns can also be implemented with such an optimal order of complexity.

To estimate the condition number of the preconditioner (14) we consider the local generalized eigenvalue problem

$$A_{11:E}^{(k+1)} v_E = \lambda_E B_{11:E}^{(k+1)} v_E. \tag{16}$$

The characteristic equation for λ_E is $\det(A_{11:E}^{(k+1)} - \lambda_E B_{11:E}^{(k+1)}) = 0$ which can be written in the form

$$\begin{vmatrix} (\alpha + \beta + 1)\mu_E & -\mu_E & -\beta \\ -\mu_E & (\alpha + \beta + 1)\mu_E & -\alpha \\ -\beta & -\alpha & (\alpha + \beta + 1)\mu_E \end{vmatrix} = 0, \tag{17}$$

where $\mu_E = 1 - \lambda_E$. For the solutions of (17) we get

$$\mu_E^{(1)} = 0, \quad and \quad \left(\mu_E^{(2,3)}\right)^2 = \frac{(\alpha + \beta + 1)(\alpha^2 + \beta^2) + 2\alpha\beta}{(\alpha + \beta + 1)[(\alpha + \beta + 1)^2 - 1]},$$

or, after simplification,

$$\left(\mu_E^{(2,3)}\right)^2 = \frac{\alpha^2 + \beta^2 + \alpha + \beta}{(\alpha + \beta + 1)(\alpha + \beta + 2)} = 1 - 2\frac{\alpha + \beta + 1 + \alpha\beta}{(\alpha + \beta + 1)(\alpha + \beta + 2)}.$$

Hence, applying the inequality (12), it follows that $\left(\mu_E^{(2,3)}\right)^2 < 7/15$, and the local eigenvalue estimate

$$1 - \sqrt{7/15} < \lambda_E < 1 + \sqrt{7/15} \tag{18}$$

holds. Now we are ready to prove the next theorem.

Theorem 3 *The additive preconditioner of $A_{11}^{(k+1)}$ is of optimal order computational complexity with a relative condition number uniformly bounded by*

$$\kappa\left(B_{11}^{(k+1)^{-1}} A_{11}^{(k+1)}\right) < \frac{1}{4}(11 + \sqrt{105}) \approx 5.31. \tag{19}$$

This condition number holds independent on shape and size of each element and on the coefficients in the differential operator.

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*Proof.*Applying (18) we get

$$v^T A_{11}^{(k+1)} v = \sum_{E \in \mathcal{T}_{k+1}} v_E^T L_E^{(k+1)^T} A_{11:E}^{(k+1)} L_E^{(k+1)} v_E < \sum_{E \in \mathcal{T}_{k+1}} \lambda_E^{max} v_E^T L_E^{(k+1)^T} B_{11:E}^{(k+1)} L_E^{(k+1)} v_E <$$

$$\left(1 + \sqrt{7/15}\right) v_E^T L_E^{(k+1)^T} B_{11:E}^{(k+1)} L_E^{(k+1)} v_E = \left(1 + \sqrt{7/15}\right) v^T B_{11}^{(k+1)} v$$

and, similarly,

$$v^T A_{11}^{(k+1)} v > \left(1 - \sqrt{7/15}\right) v^T B_{11}^{(k+1)} v.$$

Combining the last two inequalities we complete the proof.

$$\kappa \left(B_{11}^{(k+1)^{-1}} A_{11}^{(k+1)} \right) < \frac{\lambda^{max} \left(B_{11}^{(k+1)^{-1}} A_{11}^{(k+1)} \right)}{\lambda^{min} \left(B_{11}^{(k+1)^{-1}} A_{11}^{(k+1)} \right)} < \frac{1 + \sqrt{7/15}}{1 - \sqrt{7/15}} \qquad \blacksquare$$

Remark 2 *The additive preconditioner $B_{11}^{(k+1)}$ was first introduced in [9] where the above estimate of the condition number was derived using a slightly different approach. The parameter dependent version $B(\sigma)_{11}^{(k+1)}$ in the form*

$$B(\sigma)_{11}^{(k+1)} = B_{11}^{(k+1)} + \sigma R_{11}^{(k+1)}, \qquad \sigma \in [0,1),$$

where $R_{11}^{(k+1)} = diag(-\beta, -\alpha, -(\alpha + \beta))$ can be analyzed numerically, see Table 1. Note that when $\sigma = 1$ the row-sum criterion $B(1)_{11}^{(k+1)} e = A_{11}^{(k+1)} e$ is satisfied where $e^T = (1,1,1)^T$ is the unit vector. We observe that the best relative condition number corresponds to the case $\sigma = 0$. Moreover, for $\sigma \to 1$ the condition number deteriorates when $(\alpha, \beta) \to (-1/2, 1)$.

Table 1: Numerically computed estimates of $\kappa \left(B(\sigma)_{11:E}^{(k+1)^{-1}} A_{11:E}^{(k+1)} \right)$.

σ	0.	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
κ	5.31	5.33	5.39	5.51	5.72	6.09	6.74	7.96	10.64	19.16	176.47

4.2 Multiplicative preconditioning of $A_{11}^{(k+1)}$

In order to define a still stronger preconditioner we consider the multiplicative pre-conditioner $B_{11}^{(k+1)}$ where we partition the nodes corresponding to the block $A_{11}^{(k+1)}$ into two groups where the first one contains the centers of parallelogram superelements Q (see Figure (3)) which are weakly connected in the sense of the relations between the coefficients $|a| \leq b \leq c$. It is important to note that the parallelograms $Q \subset T \in \mathcal{T}_0$, i.e. it is not allowed to be composed by triangles of neighbour elements

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of the coarsest triangulation $\mathcal{T}_0$. With respect to this partitioning, $A_{11}^{(k+1)}$ admits the following two-by-two block-factored form

$$A_{11}^{(k+1)} = \begin{bmatrix} D_{11}^{(k+1)} & F_{11}^{(k+1)} \\ F_{11}^{(k+1)\,T} & E_{11}^{(k+1)} \end{bmatrix} = \begin{bmatrix} D_{11}^{(k+1)} & 0 \\ F_{11}^{(k+1)\,T} & S_{11}^{(k+1)} \end{bmatrix} \begin{bmatrix} I & D_{11}^{(k+1)\,-1}F_{11}^{(k+1)} \\ 0 & I \end{bmatrix},$$
(20)

where $S_{11}^{(k+1)}$ stands for the related Schur complement. We define now $B_{11}^{(k+1)}$ as the symmetric block Gauss-Seidel preconditioner of $A_{11}^{(k+1)}$, i.e.,

$$B_{11}^{(k+1)} = \begin{bmatrix} D_{11}^{(k+1)} & 0 \\ F_{11}^{(k+1)\,T} & E_{11}^{(k+1)} \end{bmatrix} \begin{bmatrix} I & D_{11}^{(k+1)\,-1}F_{11}^{(k+1)} \\ 0 & I \end{bmatrix}.$$
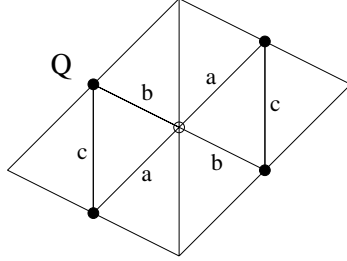(21)

Figure 3: Block partitioning of the nodes of the superelement Q

Since $D_{11}^{(k+1)}$ is a diagonal matrix it follows that the Schur complement $S_{11}^{(k+1)}$ can be assembled from the corresponding superelement Schur complements

$$S_{11:Q}^{(k+1)} = E_{11:Q}^{(k+1)} - F_{11:Q}^{(k+1)\,T} D_{11:Q}^{(k+1)\,-1} F_{11:Q}^{(k+1)}$$

Such a procedure is sometimes called *static condensation*. The obtained sparse structure is such that solving systems with $E_{11}^{(k+1)}$ requires: first, local elimination steps along lines of dominated anisotropy; and at the end, solving at most a band system the order and structure of which are similar to that of $A^{(0)}$. This means that the computational cost to solve a system with the current matrix $B_{11}^{(k+1)}$ is proportional to the size of this matrix. The connectivity pattern of the $E_{11}^{(k+1)}$ block related to a given triangle $T \in \mathcal{T}_0$ is illustrated in Figure (4.b). The only difference between the decoupled structure of the additive and the multiplicative preconditioners is in the boundary layer which is parallel to the dominating anisotropy direction of the current coarsest grid triangle $T \in \mathcal{T}_0$.

A similar construction was first introduced and studied in [16] for the particular case of triangulation $\mathcal{T}_0$ consisting of right triangles with legs parallel to the coordinate axes.

As in the previous section, a local spectral analysis will be applied to estimate the relative condition number of the preconditioner under consideration.
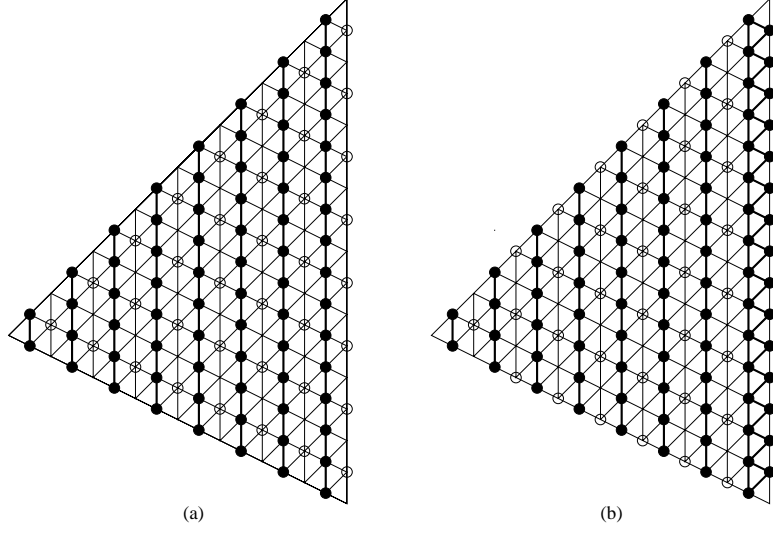
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(a)   (b)

Figure 4: Connectivity pattern of $T \in \mathcal{T}_0$: (a) additive preconditioning of $A_{11}^{(4)}$; (b) multiplicative preconditioning of $A_{11}^{(4)}$.

Lemma 4 *Consider the generalized eigenvalue problem*

$$S_{11:Q}^{(k+1)} v_Q = \lambda_Q E_{11:Q}^{(k+1)} v_Q. \tag{22}$$

Then the minimal eigenvalue λ_Q^{min} is uniformly bounded by

$$\lambda_Q^{min} > \frac{8}{15} \tag{23}$$

and all remaining eigenvalues are equal to 1.

*Proof.*The required superelement matrices read as follows:

$$B_{11:Q}^{(k+1)} = \frac{2\ r}{c} \begin{bmatrix} 2\delta & -\alpha & -\beta & -\alpha & -\beta \\ -\alpha & \delta & -1 & 0 & 0 \\ -\beta & -1 & \delta & 0 & 0 \\ -\alpha & 0 & 0 & \delta & -1 \\ -\beta & 0 & 0 & -1 & \delta \end{bmatrix}, \qquad E_{11:Q}^{(k+1)} = \frac{2\ r}{c} \begin{bmatrix} \delta & -1 & 0 & 0 \\ -1 & \delta & 0 & 0 \\ 0 & 0 & \delta & -1 \\ 0 & 0 & -1 & \delta \end{bmatrix},$$

$$S_{11:Q}^{(k+1)} = \frac{2\ r}{c} \begin{bmatrix} \delta - \alpha^2\omega & -1 - \alpha\beta\omega & -\alpha^2\omega & -\alpha\beta\omega \\ -1 - \alpha\beta\omega & \delta - \beta^2\omega & -\alpha\beta\omega & -\beta^2\omega \\ -\alpha^2\omega & -\alpha\beta\omega & \delta - \alpha^2\omega & -1 - \alpha\beta\omega \\ -\alpha\beta\omega & -\beta^2\omega & -1 - \alpha\beta\omega & \delta - \beta^2\omega \end{bmatrix},$$

where $\delta = \alpha + \beta + 1$ and $\omega = \frac{1}{2\delta}$. Then, for the solution of the generalized eigenvalue problem (22) we obtain

$$\lambda_Q^{(1)} = 2\frac{\alpha\beta + \alpha + \beta + 1}{(\alpha + \beta + 1)(\alpha + \beta + 2)}$$

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and $\lambda_Q^{(2)} = \lambda_Q^{(3)} = \lambda_Q^{(3)} = 1$. It is easy to see that $\lambda_Q^{(1)}$ is really the minimal eigenvalue because the inequality $\lambda_Q^{(1)} < 1$ is equivalent to the obviously satisfied inequality $\alpha^2 + \beta^2 + \alpha + \beta > 0$. Finally we have to show that $\lambda_Q^{(1)} > \frac{8}{15}$ which follows immediately from (12).

The major result of this subsection is given in the theorem.

Theorem 4 *The multiplicative preconditioner of $A_{11}^{(k+1)}$ has an optimal order computational complexity with a relative condition number uniformly bounded by*

$$\kappa \left(B_{11}^{(k+1)^{-1}} A_{11}^{(k+1)} \right) < \frac{15}{8} = 1.875 \tag{24}$$

This is proved in the same way as used in Theorem 3 applying the estimate from Lemma 4.

Based on the above estimates and estimates in Section 2 we conclude that the condition numbers of the preconditioners $D^{(k_0,0)}$ and $M^{(k_0)}$ have optimal orders, uniformly in size and shape of the elements and in the coefficients of the differential operator.

4.3 An example

Finally we consider briefly the particular case when the coefficient matrix of the differential problem is diagonal, i.e., $[a_{ij}(x)] = diag\,[a_{11}(x), a_{22}(x)]$, and the initial triangulation $\mathcal{T}_0$ consists of right triangles with legs parallel to the coordinate axes. The goal of this consideration is better to illustrate the behaviour of the related condition numbers. This model problem was studied during the years by various authors, applying different preconditioning techniques (see, e.g., in [1, 16, 17]), and the results we present here will allow better to recognize the advantages of the here reported results. Here $\cot \theta_T^{(1)} = 0$ and consequently $a_T = \alpha_T = 0$ for the problem under consideration. The parameter $\beta_T \in (0, 1]$ is referred as a *ratio of anisotropy*. Then the estimates (19) and (24) of the additive and multiplicative preconditioners take the following explicit forms.

$$\kappa^{(add)} \left(B_{11}^{(k+1)^{-1}} A_{11}^{(k+1)} \right) \leq \max_{T \in \mathcal{T}_0} \left\{ 1 + \beta_T + \sqrt{\beta_T (\beta_T + 2)} \right\} < 2 + \sqrt{3} \approx 3.73 \tag{25}$$

$$\kappa^{(mult)} \left(B_{11}^{(k+1)^{-1}} A_{11}^{(k+1)} \right) < \max_{T \in \mathcal{T}_0} \left\{ 1 + \frac{\beta_T}{2} \right\} < \frac{3}{2} \tag{26}$$

It is important to stress here that the model problem considered in this subsection includes the interesting case when the direction of dominating anisotropy varies in different $T \in \mathcal{T}_0$.

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