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Report No. 9931 (July 1999)

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Abstract

In labor market models as well as in exchange economies with indivisible goods *gross substitutability* is used as a property to guarantee the existence of competitive equilibria. In abstracto this property can be considered as a property of TU-games. This paper develops an easy way to check gross substitutability for general TU-games. Concavity is one of the conditions that has to be satisfied. Only one other type of relation must be checked to guarantee gross substitutability.

1 Introduction

In *Kelsò* and *Crawford* (1982) a model of a labor market is described. There is a set of *firms* and a set of *workers* W. A firm can hire a team of workers $S \subseteq W$ and the profitability (utility) of such an action is given by:

$$u(S,p) := v(S) - \sum_{i \in S} p_i.$$

The number p_i denotes the salary demand of worker i and $v(S)$ is the profit the firm gets from the job done (here, the characteristic function v depends on the firm). Given a fixed salary (price) vector p, a firm maximizes its utility by hiring a team of employees S for which $v(S) - \sum_{i \in S} p_i$ is maximal.
We denote the collection of teams $S \subseteq W$ that maximizes the utility by $\mathcal{B}(p)$ and the union of these teams by $c(p)$.
The paper introduces the following condition on the functions v:
The pair (W, v) satisfies *gross substitutability* (the *GS-property*) if for every salary level $p \in \mathbb{R}^W$, for every worker $i \in c(p)$ and for every alternative salary level $q \geq p$ with $q_i = p_i$, the player is still 'in demand' i.e. $i \in c(q)$.
In words: if hiring a team S with $i \in S$ is an optimal action for some firm at salary level p, and the salary levels of some workers increase, but the one of worker i remains the same, then hiring a (perhaps different) team T containing i is still an optimal action for this firm.
It can be proved that if all characteristic functions satisfy gross substitutability, a salary level can be found that "clears the labor market".

In an other paper (*Bevià*, *Quinzii* and *Silva* (1997)) an exchange economy with indivisible goods and money is considered. Let Q be the set of indivisible goods (it will play the role of W in the previous example). The preference of an agent in the exchange economy is of the form:

$u(B, m) := v(B) + m,$

in which $B \subseteq Q$ is any consumption bundle of indivisible goods and m is an amount of money. The function v depends on the agent. As the paper assumes that the money supply is abundant, the agent is able to purchase every bundle of goods $B \subseteq Q$ and he is willing to do so at price level $p \in \mathbb{R}^Q$ if B maximizes $v(B) - \sum_{i \in B} p_i$. In this model the GS-property reads like:

If a good i is in demand by some agent at price level p, i.e. there is a bundle B containing i that maximizes the agents utility at price level p, then the good is still demanded when the price level rises but the price of good i remains the same.

In both examples gross substitutability can be understood as a property of a TU-game. At the end of the manuscript *Bevià* et al. raised the following problem:

"It would be interesting to characterize all reservation functions which lead to demands satisfying Gross Substitutability, in order to find which interpretable restrictions on the preferences of the agents are compatible with GS."

This is exactly what we do in this paper.

The following example shows a typical situation in which the GS-property is *violated*. We use the terminology of *Kelsò* and *Crawford*.

Suppose that, at some salary level p, it is only optimal for a firm to hire the team $S \cup (ij)$ or the team $S \cup (k)$. For instance when the skills of i, j and k are more or less the same but workers i and j want a part-time job and worker k wants a full-time job. When worker j raises his salary demand (e.g. because he got an offer from an outsider), the firm is no longer indifferent between hiring $S \cup (ij)$ and $S \cup (k)$ and will hire $S \cup (k)$. Gross substitutability is violated, as worker i is no longer in demand, while his salary claim is the same as before.

The example shows that, if the GS-property holds, there cannot be a salary level p such that $\mathcal{B}(p) = \{S \cup (ij), S \cup (k)\}$ for three different players i, j, k not in S. By an analogous example one can show that also $\mathcal{B}(p) = \{S \cup (ij), S\}$ is not possible under the GS-property. This paper proves the more striking converse of these statements:

If a TU-game does not satisfy the GS-property, then there is a price level p such that $\mathcal{B}(p) = \{S \cup (ij), S\}$ or $\mathcal{B}(p) = \{S \cup (ij), S \cup (k)\}$.

We show that the existence of a price level p with $\mathcal{B}(p) = \{S \cup (ij), S\}$ corresponds to the non-concavity condition:

$$v(S \cup (ij)) - v(S \cup (i)) - v(S \cup (j)) + v(S) > 0$$

and that $\mathcal{B}(p) = \{S \cup (ij), S \cup (k)\}$ corresponds to the situation that:

$$v(S \cup (ij)) + v(S \cup (k)) > \max\{v(S \cup (ik)) + v(S \cup (j)), v(S \cup (jk)) + v(S \cup (i))\}.$$

The paper is organized as follows. The next section gives the necessary preliminaries. Section 3 shows that concavity of a TU-game is implied by the GS-property. Section 4 deals with the converse statement mentioned above. This enables us to give a method to check the GS-property in section 5. It has a complexity of $\mathcal{O}(K \log^3(K))$, in which K is the size of the input, i.e. the number of coalitions of the game (N, v). We conclude with two features of the collection of games with the GS-property (section 6).

2 Preliminaries

A *Transferable Utility game* or *TU-game* is a pair (N, v) in which N is a finite set (of *players*) and $v: 2^N \to \mathbb{R}$ is a map, called the *characteristic function* of the TU-game. It is assumed that $v(\phi) = 0$. In the papers mentioned in the introduction, (W, v) and (Q, v) are examples of TU-games.
For a *price vector* $p \in \mathbb{R}^N$ and a *coalition* $S \subseteq N$ we denote $\sum_{i \in S} p_i$ by $p(S)$.
The *excess* of a coalition S with respect to a price vector p equals $v(S) - p(S)$. For a price vector p we denote the collection of coalitions with *highest excess* by $\mathcal{B}(p)$.
For any collection of coalitions $\mathcal{B}$, not necessarily corresponding to a price vector, the coalition $c(\mathcal{B})$ contains the players that are a member of at least one element of $\mathcal{B}$. Similarly, $d(\mathcal{B})$ contains the players that are in all coalitions in $\mathcal{B}$. The notations $c(p)$ and $d(p)$ are abbreviations of $c(\mathcal{B}(p))$ and $d(\mathcal{B}(p))$ respectively.
The following concept is the key concept of this paper.

Definition 1: *A TU-game (N, v) satisfies the gross substitutability property (GS-property) if for every price vector $p \in \mathbb{R}^N$ and every player $i \in c(p)$, also $i \in c(q)$ holds, if $q \geq p$ with $q_i = p_i$.*

In the characterization of TU-games with the GS-property we need a property for collections of coalitions. For such a collection $\mathcal{B}$ and a player $i \in N$ we define:

$\mathcal{B}_{-i} = \{S \in \mathcal{B} : i \notin S\}$ and $\mathcal{B}_i = \{T \in \mathcal{B} : i \in T\}$.

Definition 2: *A collection of coalitions $\mathcal{B}$ demonstrates violation of the GS-property (has property (DV)) if there is a player $i \in c(\mathcal{B})$ and a coalition $S \in \mathcal{B}_{-i}$ such that, for all $T \in \mathcal{B}_i$, we have $T \not\subseteq S \cup (i)$.*

Note that (DV) is a technical property on collections of coalitions (in N). It does not depend on the characteristic function v. Proposition 3 proves that "the existence of a price vector p such that $\mathcal{B}(p)$ has property (DV)" is truly a signal that the GS-property is violated. Typical examples of collections $\mathcal{B}$ (of size two) satisfying (DV) are:

$\mathcal{B} = \{S \cup (ij), S\}$ and $\mathcal{B} = \{S \cup (ij), S \cup (k)\}$.

Here S is a coalition in N and i, j (and k) are *different* players outside S. More general, for collections $\mathcal{B} = \{S, T\}$ the property (DV) holds if and only if:

$$\max\{|S \backslash T|, |T \backslash S|\} \geq 2. \tag{1}$$

The following proposition characterizes games with the GS-property by the collections $\mathcal{B}(p)$ that occur.

Proposition 3: *For every TU-game (N, v) exactly one of the following alternatives holds:*

(i) *(N, v) has the GS-property,*

(ii) *there is a price vector $p \in \mathbb{R}^N$ such that $\mathcal{B}(p)$ has property (DV).*

Proof: Suppose that $\mathcal{B}(p)$ has property (DV). Let $i \in c(p)$ and $S \in \mathcal{B}(p)_{-i}$ have the required property. If we define $q_k := p_k + 1$ for $k \notin S \cup (i)$ and $q_\ell = p_\ell$ for $\ell \in S \cup (i)$, the coalition S keeps the highest excess but all coalitions with at least one player outside $S \cup (i)$ are not elements of $\mathcal{B}(q)$. In particular, all coalitions $T \in \mathcal{B}(p)_i$ are no longer in $\mathcal{B}(q)$ because of (DV). Then $i \notin c(q)$ and (N, v) does not satisfy the GS-property.
Conversely, suppose that no collection $\mathcal{B}(p)$ has property (DV). Let p be any price vector, let i be a player in $c(p)$ and let $q \geq p$ with $q_i = p_i$. Because locally the collection of coalitions with highest excess can only decrease, there is an open neighborhood V of p such that if $q \in V$, then $\mathcal{B}(q) \subseteq \mathcal{B}(p)$. For the moment we assume that $q \in V$. Let $S \in \mathcal{B}(q)$. If $i \in S$ we have $i \in c(q)$. If $i \notin S$, let $T \in \mathcal{B}(p)_i$ be a coalition with $T \subseteq S \cup (i)$. Such a coalition exists because $\mathcal{B}(p)$ does not have property (DV). Then also $T \in \mathcal{B}(q)$ as the price of T increases less than the price of S. Also in this case we find $i \in c(q)$. So, for every price vector $p \in \mathbb{R}^N$ there is a neighborhood V such that for alternative price vectors $q \in V$ with $q \geq p$ and $q_i = p_i$ the condition of the GS-property is satisfied. Take now $q \geq p$ with $q_i = p_i$ arbitrarily.
The set of numbers $t \in [0, 1]$ such that $i \in c(q_t)$ for $q_t := p + t(q - p)$ is open (by the previous argument) and closed. It contains 0 and, therefore, 1 too. $\triangleleft$

Corollary 4: *If (N, v) has the GS-property, then no collection $\mathcal{B}(p)$ is a collection of one of the following types:*

type 1 : $\{S \cup (ij), S\}$
type 2 : $\{S \cup (ij), S \cup (k)\}$.

Section 4 proves the converse of this corollary:
If (N, v) does not have the GS-property, then there is a price vector p such that $\mathcal{B}(p)$ is a collection of type 1 *or type* 2.

3 Gross substitutability implies concavity

A TU-game (N, v) is called *concave* if $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$ whenever $S, T \subseteq N$. An equivalent condition is: $v(S \cup (i)) + v(S \cup (j)) \geq v(S \cup (ij)) + v(S)$ whenever $S \subseteq N \backslash (ij)$ and $i \neq j$.
In order to prove the statement in the title of this section we show that the existence of a price level p with $\mathcal{B}(p) = \{S \cup (ij), S\}$ corresponds to the non-concavity condition:

$$v(S \cup (ij)) - v(S \cup (i)) - v(S \cup (j)) + v(S) > 0.$$

Proposition 5: *Let (N, v) be a TU-game, let i and j be different players in N and let $S \subseteq N \backslash (ij)$. The following statements are equivalent:*

(i) *there is a price vector p with $\mathcal{B}(p) = \{S \cup (ij), S\}$,*
(ii) $v(S \cup (ij)) + v(S) > v(S \cup (i)) + v(S \cup (j))$.

Proof: If condition (i) holds, take such a vector p and the addition of:

$$v(S \cup (ij)) - p(S \cup (ij)) > v(S \cup (i)) - p(S \cup (i)) \qquad \text{and}$$
$$v(S) - p(S) > v(S \cup (j)) - p(S \cup (j))$$

generates the non-concavity condition (ii).

If condition (ii) holds, let $p_i := v(S \cup (i)) - v(S) + \delta$ and $p_j := v(S \cup (j)) - v(S) + \delta$ such that $p_i + p_j = v(S \cup (ij)) - v(S)$. Then δ is strictly positive by condition (ii). The prices in S are taken quite low and the prices outside $S \cup (ij)$ are taken quite high, i.e. take a large number K and define $p_k := -K$ for $k \in S$ and $p_\ell := K$ for $\ell \notin S \cup (ij)$. Then coalitions with maximal excess contain S and are contained in $S \cup (ij)$. It is easy to verify that the excesses of S and $S \cup (ij)$ are both equal to $v(S) + K|S|$ and the excesses of $S \cup (i)$ and $S \cup (j)$ are both equal to $v(S) + K|S| - \delta$. Hence, $\mathcal{B}(p) = \{S \cup (ij), S\}$. $\triangleleft$

Proposition 5 makes the following theorem easy to prove:

Theorem 6: *TU-games satisfying gross substitutability are concave.*

Proof: If (N, v) satisfies the GS-condition, there is no price p such that $\mathcal{B}(p)$ has type 1. So, by proposition 5 no non-concavity condition of the form $v(S \cup (ij)) + v(S) > v(S \cup (i)) + v(S \cup (j))$ is valid and the game is concave. $\triangleleft$

Proposition 7: *If (N, v) satisfies concavity, and S and T are elements of $\mathcal{B}(p)$ for some vector $p \in \mathbb{R}^N$, then $U \in \mathcal{B}(p)$ for all U with $S \subset U \subset T$.*
Proof: By concavity we have: $v(U) + v((T \backslash U) \cup S) \geq v(S) + v(T)$. Hence, if S and T have maximal excess, U and $(T \backslash U) \cup S$ have maximal excess as well. $\triangleleft$

4 Characterization of games satisfying the GS-property

This section proves the key theorem of the paper:

Theorem 8: *A TU-game (N, v) has the GS-property if and only if none of the collections $\mathcal{B}(p)$ is a collection of type 1 or type 2.*

Proof: Corollary 4 proves the 'only if'-part of the theorem already. If the 'if'-part were false, by proposition 5 and theorem 6 there must exist *concave* TU-games violating the GS-property, such that $\mathcal{B}(p)$ is never a collection of type 2. Let us assume that (N, v) is such a game with a *minimal* number of players n, where $n := |N|$.
The proof consists of three steps. The first two steps prove that there is a price vector p with $\mathcal{B}(p) = \{S, N \backslash S\}$.

Step 1. *For every $p \in \mathbb{R}^N$ such that $\mathcal{B}(p)$ has property* (DV), *$c(p) = N$ and $d(p) = \phi$.*

Let $p \in \mathbb{R}^N$. Suppose, on the contrary, that $\mathcal{B}(p)$ has property (DV), but $c(p) \neq N$ or $d(p) \neq \phi$. Define the concave TU-game $(\bar{N}, \bar{v})$ by:

$$\bar{N} := c(p) \backslash d(p) \qquad \text{and} \qquad \bar{v}(T) := v(T \cup d(p)) - v(d(p)) \text{ for } T \subseteq \bar{N}.$$

Clearly $|\bar{N}| < |N|$. Let $\bar{p} := p_{\bar{N}}$. Then for all $S, T \subseteq \bar{N}$:

$$\bar{v}(S) - p(S) > \bar{v}(T) - p(T) \iff v(S \cup d(p)) - p(S \cup d(p)) > v(T \cup d(p)) - p(T \cup d(p)).$$

Since all coalitions in $\mathcal{B}(p)$ contain $d(p)$, we have: $\quad S \in \bar{\mathcal{B}}(\bar{p}) \iff S \cup d(p) \in \mathcal{B}(p)$.
Hence, $\bar{\mathcal{B}}(\bar{p}) = \{S \backslash d(p) : S \in \mathcal{B}(p)\}$.
Furthermore, $\bar{\mathcal{B}}(\bar{p})$ has property (DV) because $\mathcal{B}(p)$ has property (DV). If, namely, $i \in c(p)$ and $S \in \mathcal{B}(p)_{-i}$ show (DV) in $\mathcal{B}(p)$, then $i \in \bar{c}(\bar{p})$ and $S \backslash d(p) \in \bar{\mathcal{B}}(\bar{p})_{-i}$ show (DV) in $\bar{\mathcal{B}}(\bar{p})$. Hence, $(\bar{N}, \bar{v})$ is a concave game without the GS-property.

Because n has been chosen minimal, there is a price vector $\bar{q} \in \mathbb{R}^{\bar{N}}$ such that $\bar{\mathcal{B}}(\bar{q})$ is a collection of type 2. Extend $\bar{q}$ to q by $q_k := -K$ if $k \in d(p)$ and $q_\ell := K$ if $\ell \notin c(p)$. If K has been chosen sufficiently large, all coalitions in $\mathcal{B}(q)$ contain $d(p)$ and are contained in $c(p)$. In fact, $\mathcal{B}(q)$ equals $\{T \cup d(p) : T \in \bar{\mathcal{B}}(\bar{q})\}$. Then $\mathcal{B}(q)$ is also of type 2. This was supposed to be false: $c(p) = N$ and $d(p) = \phi$.

Step 2. *There is a price vector p such that $\mathcal{B}(p)$ has property* (DV) *and has size* two.

By proposition 3 there are price vectors q such that $\mathcal{B}(q)$ has property (DV). Let p be one of them such that the size of $\mathcal{B}(p)$ is minimal among the collections $\mathcal{B}(q)$ with property (DV).
Take a player $i \in c(p) = N$ and a coalition $S \in \mathcal{B}(p)_{-i}$ such that, for all $T \in \mathcal{B}(p)_i$, we have $T \not\subseteq S \cup (i)$. If we choose $T \in \mathcal{B}(p)_i$ arbitrarily, the collection $\{S, T\}$ also has property (DV). If we prove that $\mathcal{B}(p) = \{S, T\}$, we are done. Note that $T \neq (i)$.

First we prove that $\{S, T\}$ is a partition of N. Suppose this is not the case. Let q be the price vector obtained from p by *increasing* the prices outside $S \cup T$ and *decreasing* the prices inside $S \cap T$ with the amount $\varepsilon > 0$. Take ε so small that $\mathcal{B}(q) \subseteq \mathcal{B}(p)$. Coalitions $U \in \mathcal{B}(p)$ with $S \cap T \not\subseteq U$ or $U \not\subseteq S \cup T$ drop out. The coalitions S and T, however, survive. Then $|\mathcal{B}(q)| < |\mathcal{B}(p)|$ and we are done if also $\mathcal{B}(q)$ has property (DV). The collection $\mathcal{B}(q)$ has the property (DV) because $S \in \mathcal{B}(q)_{-i}$ and no $V \in \mathcal{B}(p)_i$ is a part of $S \cup (i)$. This is certainly true for $V \in \mathcal{B}(q)_i$, as it is a subset of $\mathcal{B}(p)_i$. So $T = N \backslash S$. Because $(i) \subsetneq T$, we have $|T| \geq 2$.
As $T \in \mathcal{B}(p)_i$ was chosen *arbitrarily*, an other choice T^* must also be the complement of S, i.e. there is no other choice and $\mathcal{B}(p)_i = \{T\}$.
If $S^* \in \mathcal{B}(p)_{-i}$ and $S^* \neq S$, we must have $T \subseteq S^* \cup (i)$, otherwise $\{S^*, T\}$ is also a partition and $S^* = S$. If we take a new price vector q obtained from p by *decreasing* the price p_i with ε and *increasing* the prices in $T \backslash (i)$ with $|T \backslash (i)|^{-1}\varepsilon$ (this is possible because $T \neq (i)$!), the excesses of S and T remain the same and the excess of S^* decreases. If ε has been chosen sufficiently small, the collection $\mathcal{B}(q)$ is of smaller size than $\mathcal{B}(p)$, and has property (DV). This is not possible because the size of $\mathcal{B}(p)$ has been chosen minimal, which proves step 2.

Combining steps 1 and 2, we find a price vector p with $\mathcal{B}(p) = \{S, N \backslash S\}$ for some coalition $S \subseteq N$. Moreover, $\mathcal{B}(p)$ has property (DV). Observe that $\mathcal{B}(p)$ cannot be $\{\phi, N\}$ because of proposition 7.

Step 3. *There is a price vector r such that:*
$\mathcal{B}(r)$ has property (DV) *and*
$c(r) \neq N$ or $\mathcal{B}(r)$ is a collection of type 2.

First we handle low values of n. If $n = 2$, $\mathcal{B}(p)$ must be $\{(i), (j)\}$, but this partition does not have property (DV). Therefore, n is at least equal to 3.
For $n = 3$, $\mathcal{B}(p)$ must be of the form $\{(ij), (k)\}$, i.e. $\mathcal{B}(p)$ is of type 2. Hence, we can take r equal to p.
For $n = 4$, the partition $\{(ij), (k\ell)\}$ will require a special treatment.
So, for the time being we will assume that $n \geq 4$, that $1 \leq |N \backslash S| \leq |S|$ and $|S| \geq 3$.
Take $j \in T := N \backslash S$, $k \in S$ and increase p_j and p_k with the same amount till the collection of coalitions with maximal excess changes. We get a price vector q with $\mathcal{B}(q) = \{S, T, W_1, \ldots, W_s\}$ with $W_m \subseteq N \backslash (jk)$ for $m = 1, 2, \ldots, s$.

Case 1. *There is a coalition $W = W_m \neq N\backslash(jk)$.*
Then there is a player $\ell \notin W_m \cup (jk)$. Suppose that $\ell \in T$ (the other choice leads to the same argument). If we increase p_k slightly, we get a new price vector r with $\mathcal{B}(r) = \mathcal{B}(q)\backslash\{S\}$. Then $\mathcal{B}(r)$ has property (DV) by $W \in \mathcal{B}(r)_{-j}$, $\mathcal{B}(r)_j = \{T\}$ and $T \not\subseteq W \cup (j)$ (since $\ell \in T\backslash W$). Then $k \notin c(r)$, so $c(r) \neq N$.

Case 2. $\mathcal{B}(q) = \{S, T, N\backslash(jk)\}$.
Increase the price of k slightly and get a new price vector r with $\mathcal{B}(r) = \{T, N\backslash(jk)\}$. Then $c(r) \neq N$ since $k \notin T$ and $\mathcal{B}(r)$ has property (DV) because $N\backslash(jk)$ contains at least two players not in T (by $(N\backslash(jk))\backslash T = (N\backslash(jk)) \cap S = S\backslash(k)$ and $|S| \geq 3$), hence formula (1) in the introduction can be applied.

So, we are left with the case $n = 4$ and $\mathcal{B}(p) = \{(ij), (k\ell)\}$. Increase all prices with the same amount till the collection of coalitions with maximal excess changes. We get $\mathcal{B}(q) = \mathcal{B}(p) \cup \{W_1, \ldots, W_s\}$. Then $|W_m| \leq 1$ for all $m \leq s$ and, by proposition 7, one of the new coalitions is a singleton. Suppose $W_1 = (i)$. We will change q into r in such a way that $\mathcal{B}(r) = \{(i), (k\ell)\}$. This happens if $r_i = q_i - 2\varepsilon$, $r_j = q_j + \varepsilon$, $r_k = q_k - \varepsilon$ and $r_\ell = q_\ell - \varepsilon$. The prices of (i) and $(k\ell)$ decrease with 2ε and the prices of other coalitions in $\mathcal{B}(q)$ decrease less. Take, once again, $\varepsilon > 0$ sufficiently small to ensure that $\mathcal{B}(r) \subseteq \mathcal{B}(q)$. Then $\mathcal{B}(r)$ is of type 2. This proves the statement of step 3.

By assumption, there is no price vector r with $\mathcal{B}(r)$ of type 2. Hence, the statements of steps 1 and 3 are contradicting. This finishes the proof of theorem 8. $\lhd$

5 Checking gross substitutability

The following theorem shows that checking whether a TU-game has the GS-property concerns two types of inequalities. In the formulation of the theorem, i, j, k are different players, not in S.

Theorem 9: *Let (N, v) be a TU-game. The following statements are equivalent:*
- (i)-a *there is a price vector p with $\mathcal{B}(p) = \{S \cup (ij), S\}$ for some S, i, j,*
- (ii)-a $v(S \cup (ij)) + v(S) > v(S \cup (i)) + v(S \cup (j))$ *for some S, i, j,*
- (iii)-a *the game (N, v) is not concave.*

If the game (N, v) is concave, the following statements are equivalent:
- (i)-b *there is a price vector p with $\mathcal{B}(p) = \{S \cup (ij), S \cup (k)\}$ for some S, i, j, k,*
- (ii)-b $v(S\cup(ij))+v(S\cup(k)) > \max\{v(S\cup(ik))+v(S\cup(j)), v(S\cup(jk))+v(S\cup(i))\}$ *for some S, i, j, k,*
- (iii)-b *the game (N, v) does not have the GS-property*

Proof: The a-part of the theorem holds by proposition 5 and the definition of concavity. So, assume that (N, v) is a concave game. The equivalence of (i)-b and (iii)-b follows by proposition 5 and theorem 8.

If (i)-b holds, we have the inequalities:

$$v(S \cup (ij)) - p(S \cup (ij)) > v(S \cup (ik)) - p(S \cup (ik)) \text{ and}$$
$$v(S \cup (k)) - p(S \cup (k)) > v(S \cup (j)) - p(S \cup (j)),$$

which sum up to:

$$v(S \cup (ij)) + v(S \cup (k)) \ > v(S \cup (ik)) + v(S \cup (j)).$$

The other inequality follows by interchanging the roles of i and j.

If (ii)-b is given, we have to find a price vector p such that $\mathcal{B}(p) = \{S \cup (ij), S \cup (k)\}$. As in the proof of proposition 5, take the prices in S quite low: $p_k := -K$ for $k \in S$, and outside $S \cup (ijk)$ quite high: $p_\ell := K$ if $\ell \notin S \cup (ijk)$. Take K so large that only the coalitions T with $S \subseteq T \subseteq S \cup (ijk)$ can have the highest excess. Only the prices for i, j and k still have to be defined.

Let $((ijk), w)$ be the 3-person game defined by: $w(T) := v(S \cup T) - v(S)$ for $T \subseteq (ijk)$. We are left with the problem to find a vector (p_i, p_j, p_k) such that $w(ij) - p(ij) = w(k) - p(k) > w(T) - p(T)$ for all $T \notin \{(ij), (k)\}$. By concavity of $((ijk), w)$, we have:

$$w(ij) = w(i) + w(j) - a \text{ with } a \geq 0.$$

Statement (ii)-b gives:

$$w(ik) - w(i) - w(k) = w(ij) - w(i) - w(j) - b \text{ with } b > 0 \text{ and}$$
$$w(jk) - w(j) - w(k) = w(ij) - w(i) - w(j) - c \text{ with } c > 0.$$

Without loss of generality we assume that $c \geq b$. Define $p_i := w(i) - a - 0.5b$, $p_j := w(j) - a - 0.5b$ and $p_k := w(k) - a - b$. Then the coalitions (ij) and (k) have excess $a+b$ (with respect to w) and the coalitions (i), (j) and (ik) have excess $a+0.5b$. The excess of (jk) is $a + 1.5b - c \leq a + 0.5b$. Finally, to compute the excess of (ijk), take the concavity condition:

$$w(ijk) + w(k) \leq w(ik) + w(jk),$$

subtract $p_i + p_j + 2p_k$:

$$w(ijk) - p(ijk) + (a + b) \leq (a + 0.5b) + (a + 1.5b - c).$$

Therefore, $w(ijk) - p(ijk) \leq a + b - c < a + b$. ◁

Corollary 10: *Gross substitutability can be tested by inspecting* $\binom{n}{2}2^{n-2} + 3\binom{n}{3}2^{n-3}$ *inequalities.*

6 Final remarks

Definition 11: *The dual game of a TU-game v is defined by $v^*(S) := v(N) - v(N \backslash S)$ for all $S \subseteq N$.*

We conclude with two results, derived from theorem 9:

Proposition 12: *If (N, v) has the GS-property, then:*

(i) *the game $(N, -v^*)$ has the GS-property too,*

(ii) *the game $(N, v + w)$ has the GS-property if (N, w) is a* symmetric *concave game.*

Proof: (i) Let i, j, k be different players outside S and define $\tilde{S} := N \backslash (S \cup (ijk))$. By using the definition of $\tilde{S}$ and theorem 9b we get

$$\begin{aligned}
-v^*(S \cup (ij)) - v^*(S \cup (k)) &= v(N \backslash (S \cup (ij))) + v(N \backslash (S \cup (k))) - 2v(N) \\
&= v(\tilde{S} \cup (k)) + v(\tilde{S} \cup (ij)) - 2v(N) \\
&\leq v(\tilde{S} \cup (ik)) + v(\tilde{S} \cup (j)) - 2v(N) \\
&= v(N \backslash (S \cup (j))) + v(N \backslash (S \cup (ik))) - 2v(N) \\
&= -v^*(S \cup (j)) - v^*(S \cup (ik))
\end{aligned}$$

By interchanging the roles of i and j and using theorem 9b again we find that $(N, -v^*)$ has the GS-property.

(ii) The addition of a concave (symmetric) game (N, w) does not change the concavity. Also the other conditions are not violated, because at both sides of the inequality the same number $w(s+2) + w(s+1)$ is added. Here, $w(t)$ is the coalition value of each coalition of size t. $\triangleleft$

7 References

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