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# Denesting certain nested radicals of depth two

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# Denesting certain nested radicals of depth two

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**Abstract**

Using elementary methods we derive a simple criterion to decide if a nested radical of the form $\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$ can be denested. We present two methods to compute such a denesting.

## 1 Introduction

As for many subjects in number theory the history of denesting nested radicals leads us to Srinivasa Ramanujan (1887-1920). One of the questions he sent to the *Journal of the Indian Mathematical Society* was *Question 525* [7]:

> *Shew how to find the square roots of surds of the form* $\sqrt[3]{A}+\sqrt[3]{B}$ *and hence prove that*
>
> (i) $\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} = \frac{1}{3}\left(\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}\right)$,
>
> (ii) $\sqrt{\sqrt[3]{28}-\sqrt[3]{27}} = \frac{1}{3}\left(\sqrt[3]{98}-\sqrt[3]{28}-1\right)$.

In one of his notebooks we can find the theorem this question must have been based upon.

**Theorem.** *If $m, n$ are arbitrary, then*

$$\sqrt{m\sqrt[3]{4m-8n}+n\sqrt[3]{4m+n}}$$
$$= \pm\frac{1}{3}\left(\sqrt[3]{(4m+n)^2}+\sqrt[3]{4(m-2n)(4m+n)}-\sqrt[3]{2(m-2n)^2}\right).$$

It is easy to verify that this equation is valid by squaring both sides, but it is harder to understand why the denesting is of this form. And it is not clear at all, when a nested radical of the above form can be denested, if the denesting will have to be of this form

In this paper we will describe how to denest nested radicals of the form

$$\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}},$$

for rational numbers $\alpha$ and $\beta$. We will compare this method to the theorem above and prove the following theorem that shows the generality of Ramanujan's formula.

**Theorem.** *Let $\alpha, \beta \in \mathbb{Q}^*$, such that $\beta/\alpha$ is not a cube in $\mathbb{Q}$. The nested radical*

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$$

*can be denested if and only if there exist integers $m, n$ such that*

$$\frac{\beta}{\alpha} = \frac{(4m+n)n^3}{4(m-2n)m^3}.$$


## Acknowledgments

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# 2 Definitions

In this section we will give some definitions concerning the denesting of nested radicals. As will be clear from the examples in the introduction, nested radicals are expressions involving root signs. Denesting nested radicals means rewriting these expressions in such a way that less root signs appear inside each other.

We will follow the definitions of [2], [3], [5], [6] and [10].

**Definition 1.** A *nested radical* over a field $K$ is an expression inductively defined by

- For every $a \in K$, '$a$' is a nested radical over $K$ (where we mean by '$a$', the expression representing $a$).
- Given nested radicals $A, B$ over $K$ also the expressions $A + B$, $A - B$, $A \times B$ and $A/B$ are nested radicals over $K$.
- Given a nested radical $A$ over $K$ and a natural number $n \geq 2$, the expression $\sqrt[n]{A}$ is a nested radical over $K$ too.

Nested radicals are multi valued; one nested radical can represent different values, like $\sqrt{4}$ can have the values 2 and $-2$. For the denesting of nested radicals these values are important as we will see in example 4. In this paper we will fix the values for the expressions according to the following two rules. Given a real number $\alpha$, the nested radical $\sqrt[3]{\alpha}$ will represent the unique *real* root of $\alpha$. When $\alpha$ is a positive real number then $\sqrt{\alpha}$ represents the *positive* square root of $\alpha$.

**Definition 2.** Given a nested radical over $K$ we define its *nesting depth* over $K$, $\text{depth}_K$, inductively as

- if $a \in K$, then $\text{depth}_K(\text{'}a\text{'}) = 0$.
- $\text{depth}_K(A + B) = \text{depth}_K(A - B) = \text{depth}_K(A \times B) = \text{depth}_K(A/B) = \max(\text{depth}_K(A), \text{depth}_K(B))$.

- $\operatorname{depth}_K\left(\sqrt[n]{A}\right) = 1 + \operatorname{depth}_K(A)$.

Note that by $\operatorname{depth}_K(A \times B)$ we mean the depth of the expression $A \times B$ itself, not the depth of some evaluated expression. For example, when we take $A = B = \sqrt{2}$, then the expression $A \times B$ is $\sqrt{2}\sqrt{2}$ and not 2, therefor its nesting depth is 1.

This allows us to define the denesting of a nested radical.

**Definition 3.** Let $A$ be a nested radical over $K$ representing the value $\alpha$. We call $B$ a *denesting* of $A$ over $K$ for the value $\alpha$, if $B$ also is a nested radical over $K$ representing $\alpha$ and $\operatorname{depth}_K(B) < \operatorname{depth}_K(A)$.

**Example 4.** Let

$$A = \sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}.$$

Fixing the values as agreed upon above, the denesting of $A$ is 1. When we allow all the possible values, assuming that $\sqrt{5}$ in both occurrences represents the same value, then the minimal polynomials belonging to the values of $A$ are,

$$x-1, \quad x^2+x+1, \quad x^2+x+4, \quad \text{and} \quad x^4-x^3-3x^2-4x+16.$$

All values in one minimal polynomial have the same denesting, we find the denestings

$$1, \quad -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}, \quad -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-15} \quad \text{and} \quad \tfrac{1}{4}\left(3+\sqrt{-3}\right)\left(\sqrt{5}-1\right)+1.$$

## 3 Field of denesting

In this section we will determine a field in which a nested radical

$$\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}},$$

for rationals $\alpha$ and $\beta$, can be denested if a denesting exists at all. To be able to apply results from Galois theory we will need some roots of unity, but as we will see later, these won't appear in the denesting.

Let $Q_m$ denote the $m$-th cyclotomic field, so $Q_m = \mathbb{Q}(\zeta_m)$ for a primitive $m$-th root of unity $\zeta_m$. By $Q_\infty$ we denote the field extension of $\mathbb{Q}$ containing all roots of unity: $Q_\infty = \mathbb{Q}\left(\bigcup\{\zeta_m\}\right)$.

**Lemma 5.** *Let $K$ be a field containing the field $Q_\infty$. Let $a, b_1, b_2, \ldots, b_t \in K$ and $n, m_1, m_2, \ldots, m_t \in \mathbb{N}$. Suppose that*

$$\sqrt[n]{a} \in L = K\left(\sqrt[m_1]{b_1}, \sqrt[m_2]{b_2}, \ldots, \sqrt[m_t]{b_t}\right),$$

*then there exists an element $\gamma \in L$ with*

$$\gamma = \prod_{i=1}^{t} \sqrt[m_i]{b_i}^{\,l_i},$$

*for integers $0 \le l_i < m_i$, such that $\sqrt[n]{a} \in \gamma \cdot K$.*

*Proof.* A basis of the field $L$ is given by the different elements $c_j$ of the form

$$\prod_{i=1}^{t} \sqrt[m_i]{b_i}^{\,l_i},$$

for integers $0 \leq l_i < m_i$ for all $i \in \{1, \dots, t\}$. As $\sqrt[n]{a} \in L$, there are elements $a_j \in K$ such that

$$\sqrt[n]{a} = \sum_j a_j c_j.$$

Because $K$ contains all roots of unity the extension $L$ over $K$ is a Galois extension, let its Galois group be $G$. For $\sigma \in G$ we then have

$$\sigma\left(\sqrt[n]{a}\right) = \sum_j a_j \sigma(c_j).$$

That is

$$\zeta_\sigma \sqrt[n]{a} = \sum_j a_j \zeta_{\sigma,j} c_j,$$

for an $n$-th root of unity $\zeta_\sigma$ and roots of unity $\zeta_{\sigma,j}$ for all $j$.

The elements $c_j$ are lineairly independent over $K$ and because $a_j\zeta_{\sigma,j} \in K$ for all $j$ it follows from the equations 3 and 3 that $\zeta_\sigma = \zeta_{\sigma,j}$ for all $j$. This holds for every $\sigma \in G$, hence there can be at most one index $j$ for which $a_j \neq 0$. Thus $\sqrt[n]{a} \in c_j \cdot K$ for some $j$. ☐

**Proposition 6.** *Let $\alpha, \beta \in \mathbb{Q}^*$. If the nested radical*

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$$

*can be denested over $\mathbb{Q}$ then there exist $x, y, z \in Q_\infty$, $m \in \mathbb{N}$ and $b \in \mathbb{Q}$ such that*

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}} = \frac{\sqrt{\alpha}}{\sqrt[3]{\alpha}} \sqrt[m]{b} \left( x + y\sqrt[3]{\beta/\alpha} + z\sqrt[3]{(\beta/\alpha)^2} \right).$$

*Proof.* It is clear that $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$ can be denested over $\mathbb{Q}$ if and only if $\sqrt{1 + \sqrt[3]{\beta/\alpha}}$ can be denested over $\mathbb{Q}$. That means if and only if there exists an element $B$ in the field $\mathbb{Q}^{(1)}$ generated by the set

$$\{a \in \mathbb{C} \mid a^n \in \mathbb{Q} \text{ for some } n \in \mathbb{N}^*\},$$

such that $B^2 = 1 + \sqrt[3]{\beta/\alpha}$. It is easy to verify that $\mathbb{Q}^{(1)}$ over $Q_\infty$ is a Galois extension with a basis of real elements of the form $\sqrt[m_i]{b_i}$, where $m_i$ is an integer, $m_i \geq 2$, and $b_i \in \mathbb{Q}$. There is a finite subset of these basis elements such that

$$B \in Q_\infty\left(\sqrt[3]{\beta/\alpha}\right)\left(\sqrt[m_1]{b_1}, \sqrt[m_2]{b_2}, \dots, \sqrt[m_t]{b_t}\right).$$

Now apply lemma 5 with $K = Q_\infty\left(\sqrt[3]{\beta/\alpha}\right)$, $n = 2$ and $a = 1 + \sqrt[3]{\beta/\alpha}$. We conclude that $\sqrt{1 + \sqrt[3]{\beta/\alpha}} = B \in \sqrt[m]{b} \cdot Q_\infty\left(\sqrt[3]{\beta/\alpha}\right)$ for some $m \in \mathbb{N}$ and some $b \in \mathbb{Q}$. ☐

Let us have a closer look at the factor $\sqrt[m]{b}$ in the previous proposition. As we know that $B^2 \in Q_\infty\left(\sqrt[3]{\beta/\alpha}\right)$ we see that $\sqrt[m]{b^2} \in Q_\infty\left(\sqrt[3]{\beta/\alpha}\right)$. Once again applying lemma 5, we see that $\sqrt[m]{b^2} \in \sqrt[3]{\beta/\alpha}^k \cdot Q_\infty$ for some $k \in \{0,1,2\}$, and thus $\sqrt[m]{b} = \sqrt{f}\,\sqrt[3]{\beta/\alpha}^l$ for some $f \in Q_\infty$ and some $l \in \{0,1,2\}$.

After renaming $x, y$ and $z$ we find the following result.

**Corollary 7.** *Let $\alpha, \beta \in \mathbb{Q}^*$. The nested radical*

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$$

*can be denested over $\mathbb{Q}$ if and only if there exist $k \in \mathbb{N}$, $f \in Q_\infty$ and $x, y$ and $z \in \sqrt{f} \cdot Q_\infty$ such that*

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}} = \frac{\sqrt{\alpha}}{\sqrt[3]{\alpha}}\left(x + y\sqrt[3]{\beta/\alpha} + z\sqrt[3]{(\beta/\alpha)^2}\right).$$

**Remark:** in the corollary we only state that $f \in Q_\infty$, in section 5 we will see that in fact $f \in \mathbb{Q}$.

# 4 Existence

In the previous section we proved that denesting $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$, for rationals $\alpha$ and $\beta$, is equivalent to finding special elements $x$, $y$ and $z$ in $\sqrt{f} \cdot Q_\infty$. In this section we will determine conditions under which such elements exist. We will use that the quotients $x/z$ and $y/z$ are elements of $Q_\infty$.

**Definition 8.** Let $\alpha, \beta \in \mathbb{Q}^*$, such that $\beta/\alpha$ is not a cube in $\mathbb{Q}$. The polynomial $F_{\beta/\alpha} \in \mathbb{Q}[t]$ is defined by

$$F_{\beta/\alpha} = t^4 + 4t^3 + 8\frac{\beta}{\alpha}t - 4\frac{\beta}{\alpha} = 0.$$

**Proposition 9.** *Let $\alpha, \beta \in \mathbb{Q}^*$, such that $\beta/\alpha$ is not a cube in $\mathbb{Q}$. The nested radical*

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$$

*can be denested over $\mathbb{Q}$ if and only if the polynomial $F_{\beta/\alpha}$ has a root in $Q_\infty$.*

*Proof.* As we have seen above, for $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$ to be denested over $\mathbb{Q}$, there must be $x, y, z \in \sqrt{f} \cdot Q_\infty$, with $f \in Q_\infty$ such that

$$1 + \sqrt[3]{\beta/\alpha} = \left(x + y\sqrt[3]{\beta/\alpha} + z\sqrt[3]{(\beta/\alpha)^2}\right)^2.$$

Note that 1, $\sqrt[3]{\beta/\alpha}$ and $\sqrt[3]{(\beta/\alpha)^2}$ are linearly independent over $\sqrt{f} \cdot Q_\infty$, as they are linearly independent over $Q_\infty$ and $\sqrt{f}$ is of degree at most two over $Q_\infty$. Therefore coefficients of like powers on both sides must be equal. This leads to the following equations:

$$1 = x^2 + 2yz\frac{\beta}{\alpha} \tag{1}$$

$$0 = y^2 + 2xz \tag{2}$$

$$1 = \frac{\beta}{\alpha}z^2 + 2xy \tag{3}$$

We may assume that $z \neq 0$ (since $z = 0$ implies $y = 0$ and this gives a contradiction in (3)). Substitution of

$$\frac{x}{z} = -\frac{1}{2}\left(\frac{y}{z}\right)^2,$$

in (1) and (3) yields

$$\left(\frac{y}{z}\right)^4 + 8\frac{\beta}{\alpha}\frac{y}{z} = \frac{4}{z^2},$$

$$\left(\frac{y}{z}\right)^3 = \frac{\beta}{\alpha} - \frac{1}{z^2}.$$

The combination of which leads to the quartic equation

$$\left(\frac{y}{z}\right)^4 + 4\left(\frac{y}{z}\right)^3 + 8\frac{\beta}{\alpha}\left(\frac{y}{z}\right) - 4\frac{\beta}{\alpha} = 0$$

in $y/z$. □

For $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$ to be denestable now is equivalent to the existence of a root of $F_{\beta/\alpha}$ in $Q_\infty$.

**Proposition 10.** *If $F_{\beta/\alpha}(t) \in \mathbb{Q}[t]$ is irreducible, then it has no root in $Q_\infty$.*

*Proof.* Consider the number field $\mathbb{Q}(v)$, where $v \in Q_\infty$ satisfies $F_{\beta/\alpha}(v) = 0$. By assumption this is a quartic field that is contained in $Q_\infty$. Hence it is a Galois extension over $\mathbb{Q}$ with an Abelian Galois group, for which the only possibilities are $V_4$ and $C_4$.

The discriminant of $F_{\beta/\alpha}$ is

$$-2^{12}3^3\left(\frac{\beta}{\alpha}\right)^2\left(\frac{\beta}{\alpha}+1\right)^2,$$

which is not a square in $\mathbb{Q}$ and hence the Galois group is not a subgroup of $A_4$, so that rules out $V_4$. To prove that the Galois group cannot be $C_4$ either, we apply the following criterion from [8] (see also [4] and [9]):

**Theorem 11.** *Let $K$ be a number field. The polynomial*

$$t^4 - 4(At^2 + 2Bt + C) \in K[t]$$

*has Galois group $C_4$ if and only if the roots $r_1, r_2, r_3$ of the associated resolvent polynomial $R = t^3 - 2At^2 + (A^2 + C)t - B^2$ satisfy the following properties:*

*(i) exactly one of the roots is an element of $K$, say $r_1 \in K$*

*(ii) the root in $K$ is not a square in $K$; $r_1 \notin K^2$*

*(iii) for the other two roots, $r_2$ and $r_3$, it holds that $\frac{(r_2-r_3)^2}{4r_2r_3} \in K^2$.*

In our situation the polynomial

$$\begin{aligned} F_{\beta/\alpha}(t-1) &= t^4 - 6t^2 + 8\left(\frac{\beta}{\alpha}+1\right)t - 12\frac{\beta}{\alpha} - 3 \\ &= t^4 - 4\left(\tfrac{3}{2}t^2 - 2\left(\frac{\beta}{\alpha}+1\right)t + 3\frac{\beta}{\alpha} + \tfrac{3}{4}\right) \end{aligned}$$

generates the same field as $F_{\beta/\alpha}$. Its resolvent polynomial is

$$R = t^3 - 3t^2 + 3\left(\frac{\beta}{\alpha}+1\right)t - \left(\frac{\beta}{\alpha}+1\right)^2.$$

Let $r_1, r_2$ and $r_3$ be the roots of $R$, with $r_1 \in \mathbb{Q}$. Then $r_1r_2r_3 = (\beta/\alpha+1)^2$, and so $(r_2-r_3)^2/(4r_2r_3)$ is a square in $\mathbb{Q}$ if and only if $r_1$ is a square in $\mathbb{Q}$. Thus by Theorem 11 the Galois group cannot be $C_4$ either, which proves the proposition. □

**Proposition 12.** *Suppose that $F_{\beta/\alpha}(t) = G_1G_2 \in \mathbb{Q}[t]$, with $G_1$ and $G_2$ quadratic polynomials. Then $\beta/\alpha = \delta^3$ for some $\delta \in \mathbb{Q}$.*

*Proof.* As we have seen above

$$F_{\beta/\alpha}(t-1) = t^4 - 6t^2 + 8\left(\frac{\beta}{\alpha}+1\right)t - 12\frac{\beta}{\alpha} - 3.$$

Suppose there exist $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ such that $F_{\beta/\alpha}(t-1) = G_1G_2$, where $G_i = t^2 + a_it + b_i$, for $i = 1, 2$. By equating coefficients this is equivalent to the existence of $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ such that

$$a_1 + a_2 = 0,$$

$$b_1 + b_2 + a_1a_2 = -6,$$

$$a_2b_1 + a_1b_2 = 8\left(\frac{\beta}{\alpha}+1\right),$$

and

$$b_1b_2 = -12\frac{\beta}{\alpha} - 3.$$

Combining the first two of these equations we find that

$$b_1 + b_2 = a_1^2 - 6.$$

With the first and the third equation this leads to

$$\begin{aligned} b_1 &= \tfrac{1}{2}a_1^2 - 3 - \frac{4}{a_1}\left(\frac{\beta}{\alpha}+1\right), \\ b_2 &= \tfrac{1}{2}a_1^2 - 3 + \frac{4}{a_1}\left(\frac{\beta}{\alpha}+1\right). \end{aligned}$$

And with

$$a_1^4 = \left(2b_1 + 6 + \frac{8}{a_1}\left(\frac{\beta}{\alpha}+1\right)\right)\left(2b_2 + 6 - \frac{8}{a_1}\left(\frac{\beta}{\alpha}+1\right)\right),$$

we find the equation

$$t^3 - 12t^2 + 48\left(\frac{\beta}{\alpha}+1\right) - 64\left(\frac{\beta}{\alpha}+1\right)^2$$

which $a_1^2$ should satisfy. When we define $a = a_1^2$ and solve the equation for $\beta/\alpha + 1$, which is rational, we get

$$\frac{\beta}{\alpha} + 1 = \frac{48a \pm \sqrt{256a^2(a-3)}}{128} = \frac{3a \pm a\sqrt{a-3}}{8}.$$

So $a - 3$ must be a square in $\mathbb{Q}$, but then

$$\frac{\beta}{\alpha} = \frac{3a - 8 \pm a\sqrt{a-3}}{8}.$$

When $a = u^2/v^2$ and $a - 3 = w^2/v^2$ with $u, v, w \in \mathbb{Q}$ this gives

$$\frac{\beta}{\alpha} = \frac{3u^2v - 8v^3 \pm u^2 w}{8v^3} = \left(\frac{v \pm w}{2v}\right)^3.$$

Hence if $F_{\beta/\alpha}$ is the product of two polynomials of degree two, then $\beta/\alpha$ is the third power of $\delta = (v+w)/2v \in \mathbb{Q}$. □

**Corollary 13.** *Let $\alpha, \beta \in \mathbb{Q}^*$ such that $\beta/\alpha$ is not a cube in $\mathbb{Q}$. Then the nested radical*

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$$

*can be denested over $\mathbb{Q}$ if and only if $F_{\beta/\alpha}(t) \in \mathbb{Q}[t]$ has a rational root.*

*If $\alpha$ or $\beta$ is equal to $0$ or $\beta/\alpha$ is a cube in $\mathbb{Q}$ then this nested radical can always be denested. If $\beta/\alpha = c^3$ for some $c \in \mathbb{Q}$ then*

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}} = \sqrt{\sqrt[3]{\alpha} + c\sqrt[3]{\alpha}} = \sqrt[6]{\alpha}\sqrt{(c+1)}.$$

## 5 Computations

Now we can compute the denesting of $\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$ if it exists. By corollary 7 there are $x$, $y$ and $z$ such that

$$\begin{aligned}\sqrt{1+\sqrt[3]{\beta/\alpha}} &= x+y\sqrt[3]{\beta/\alpha}+z\sqrt[3]{(\beta/\alpha)^2}\\ &= z\left(\tfrac{x}{z}+\tfrac{y}{z}\sqrt[3]{\beta/\alpha}+\sqrt[3]{(\beta/\alpha)^2}\right)\\ &= z\left(-\tfrac{1}{2}\left(\tfrac{y}{z}\right)^2+\tfrac{y}{z}\sqrt[3]{\beta/\alpha}+\sqrt[3]{(\beta/\alpha)^2}\right).\end{aligned}$$

Defining $s=y/z$, the rational root of $F_{\beta/\alpha}$, and using the relations between $x$, $y$ and $z$, we find

$$\begin{aligned}\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}} &= \pm z\sqrt{\sqrt[3]{\alpha}}\left(-\tfrac{1}{2}s^2+s\sqrt[3]{\beta/\alpha}+\sqrt[3]{(\beta/\alpha)^2}\right)\\ &= \pm z\frac{\sqrt[3]{\alpha^2}}{\sqrt{\alpha}}\left(-\tfrac{1}{2}s^2+s\sqrt[3]{\beta/\alpha}+\sqrt[3]{(\beta/\alpha)^2}\right)\\ &= \pm\frac{1}{\sqrt{\beta-s^3\alpha}}\left(-\tfrac{1}{2}s^2\sqrt[3]{\alpha^2}+s\sqrt[3]{\alpha\beta}+\sqrt[3]{\beta^2}\right). \qquad (4)\end{aligned}$$

**Theorem 14.** *Let $\alpha$, $\beta\in\mathbb{Q}^*$ such that $\beta$ is not a cube in $\mathbb{Q}$. Then the nested radical*

$$\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$$

*can be denested over $\mathbb{Q}$ if and only if there exist elements $f,s\in\mathbb{Q}$ such that*

$$\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}=\pm\frac{1}{\sqrt{f}}\left(-\tfrac{1}{2}s^2\sqrt[3]{\alpha^2}+s\sqrt[3]{\alpha\beta}+\sqrt[3]{\beta^2}\right),$$

*where $s$ will be the rational root of $F_{\beta/\alpha}$ and $f=\beta-s^3\alpha$.*

**Example 15.** Computing the rational roots of $F_{-4/5}$ and $F_{-27/28}$ we now are able to compute the denestings from the introduction. As

$$F_{-4/5}(-2)=F_{-27/28}(-3)=0$$

we find that

$$\begin{aligned}\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} &= \frac{1}{\sqrt{36}}\left(-2\sqrt[3]{25}+2\sqrt[3]{20}+2\sqrt[3]{2}\right)\\ &= \tfrac{1}{3}\left(-\sqrt[3]{25}+\sqrt[3]{20}+\sqrt[3]{2}\right)\end{aligned}$$

and

$$\begin{aligned}\sqrt{\sqrt[3]{28}-\sqrt[3]{27}} &= -\frac{1}{\sqrt{27^2}}\left(-\tfrac{1}{2}9\sqrt[3]{28^2}-3\sqrt[3]{-27\cdot 28}+\sqrt[3]{27^2}\right)\\ &= -\tfrac{1}{3}\left(-\sqrt[3]{98}+\sqrt[3]{28}+1\right)\end{aligned}$$

**Example 16.** The nested radicals we denest do not have to be real. For example, when we take $\alpha = -2^7$ and $\beta = 7$ we get

$$\sqrt{-4\sqrt[3]{2} + \sqrt[3]{7}}.$$

For this nested radical, the polynomial

$$F_{-7/2^7} = t^4 + 4t^3 - \frac{56}{2^7}t + \frac{28}{2^7},$$

has rational root $-1/2$. Substituting this in the denesting (4) gives

$$\begin{aligned}\sqrt{-4\sqrt[3]{2} + \sqrt[3]{7}} &= \pm\frac{1}{\sqrt{7-16}}\left(-\tfrac{1}{8}\sqrt[3]{2^{14}} - \tfrac{1}{2}\sqrt[3]{-7\cdot 2^7} + \sqrt[3]{49}\right) \\ &= \pm i \cdot \tfrac{1}{3}\left(2\sqrt[3]{4} + 2\sqrt[3]{14} + \sqrt{49}\right)\end{aligned}$$

**Example 17.** Until now, we only denested nested radicals for which both $\alpha$ and $\beta$ were integers. Now let $\alpha = 2$ and $\beta = -1/2$, then $\beta/\alpha = -1/4$. The rational root of $F_{-1/4}$ is $-1$. This leads to the denesting

$$\sqrt{\sqrt[3]{2} - \sqrt[3]{1/2}} = \frac{1}{\sqrt{3/2}}\left(-\tfrac{1}{2}\sqrt[3]{4} - \sqrt[3]{-1} + \sqrt[3]{1/4}\right),$$

which we simplify to

$$\sqrt{\sqrt[3]{2} - \sqrt[3]{1/2}} = \frac{\sqrt{2}}{\sqrt{3}}\left(1 + \tfrac{1}{2}\sqrt[3]{2} - \tfrac{1}{2}\sqrt[3]{4}\right).$$

## 6 Comparison to Ramanujan's denesting

The denesting given by equation (4) seems to have no similarity with Ramanujan's denesting, but we will show that they give a denesting for exactly the same class of nested radicals of the form

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}},$$

where $\alpha, \beta \in \mathbb{Q}^*$ and $\beta/\alpha$ is not a cube in $\mathbb{Q}$.

**Theorem 18.** *$F_{\beta/\alpha}(t) \in \mathbb{Q}[t]$ has a root in $\mathbb{Q}$ if and only if there exist integers $m, n$ such that*

$$\frac{\beta}{\alpha} = \frac{(4m+n)n^3}{4(m-2n)m^3}.$$

*Proof.* Let $s$ be a rational root of $F_{\beta/\alpha}$. Then

$$s^4 + 4s^3 + 8\frac{\beta}{\alpha}s - 4\frac{\beta}{\alpha} = 0,$$

so $s^3(s+4) = 4\beta/\alpha(-2s+1)$. That is:

$$\frac{\beta}{\alpha} = \frac{s^3(s+4)}{4(1-2s)}.$$

Writing $s = n/m$ for integers $n, m$ gives

$$\frac{\beta}{\alpha} = \frac{(4m+n)n^3}{4(m-2n)m^3}.$$

On the other hand, if $\beta/\alpha$ can be written in the above form then it is easy to verify that $F_{\beta/\alpha}(n/m) = 0$. $\square$

**Corollary 19.** *Let $\alpha, \beta \in \mathbb{Q}^*$, such that $\beta/\alpha$ is not a cube in $\mathbb{Q}$. The nested radical*

$$\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$$

*can be denested over $\mathbb{Q}$ if and only if there exist integers $m, n$ such that*

$$\frac{\beta}{\alpha} = \frac{(4m+n)n^3}{4(m-2n)m^3}.$$

Now we can use Ramanujan's denesting

$$\begin{aligned} &\sqrt{m\sqrt[3]{4(m-2n)} + n\sqrt[3]{4m+n}} \\ &\qquad = \pm\tfrac{1}{3}\left(\sqrt[3]{(4m+n)^2} + \sqrt[3]{4(m-2n)(4m+n)} - \sqrt[3]{2(m-2n)^2}\right) \qquad (5) \end{aligned}$$

to find a denesting for $\sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}}$, as

$$\begin{aligned} \sqrt{\sqrt[3]{\alpha} + \sqrt[3]{\beta}} &= \sqrt{\sqrt[3]{\alpha}}\sqrt{1 + \sqrt[3]{\beta/\alpha}} \\ &= \sqrt[6]{\alpha}\sqrt{1 + \frac{n}{m}\sqrt[3]{\frac{4m+n}{4(m-2n)}}} \\ &= \frac{\sqrt[6]{\alpha}}{\sqrt{m\sqrt[3]{4(m-2n)}}}\sqrt{m\sqrt[3]{4(m-2n)} + n\sqrt[3]{4m+n}} \\ &= \frac{1}{\sqrt{m}}\sqrt[6]{\frac{\alpha}{4(m-2n)}}\sqrt{m\sqrt[3]{4(m-2n)} + n\sqrt[3]{4m+n}}. \end{aligned}$$

This formula looks pretty awkward, and in the case that both $\alpha$ and $\beta$ are integers the question arises if we are not able to find integers $m$, $n$ such that

$$\beta = (4m+n)n^3 \quad \text{and} \quad \alpha = 4(m-2n)m^3.$$

If this were possible, we could immediately substitute this pair $m$, $n$ in Ramanujan's denesting formula. Unfortunately, this will not always work as the following examples show.

**Example 20.** Let us try to denest the expression

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}.$$

When we take $\alpha=-4$ and $\beta=5$, we see that for $m=n=1$ we have $\alpha=4(m-2n)m^3$ and $\beta=(4m+n)n^3$. So

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}=-\tfrac{1}{3}\left(\sqrt[3]{25}-\sqrt[3]{20}-\sqrt[3]{2}\right).$$

However, when we take $\alpha=5$ and $\beta=-4$, which is a more obvious thing to do, we will not be able to find integers $m,n$ as described above, although

$$\frac{\beta}{\alpha}=\frac{(-4+2)2^3}{4(-1-4)(-1)^3}.$$

This is the case because numerator and denominator at the right hand side are not relatively prime.

**Example 21.** Now look at

$$\sqrt{8-5\sqrt[3]{7}}.$$

For this nested radical, for each of the choices $\alpha=\pm 7^3 5$ and $\alpha=\pm 8^3$, we are not able to find integers $m,n$ such that $\alpha=(4m+n)n^3$ and $\beta=4(m-2n)m^3$. For in both

$$\frac{\beta}{\alpha}=\frac{(16+5)5^3}{4(4-10)4^3} \qquad \text{and} \qquad \frac{\beta}{\alpha}=\frac{(-20+8)8^3}{4(-5-16)(-5)^3}$$

the numerator and the denominator are not coprime integers.

To determine a denesting, we take $\alpha=8^3$ and $\beta=-7\cdot 5^3$, as we have seen

$$\frac{\beta}{\alpha}=\frac{-7\cdot 5^3}{8^3}=\frac{(16+5)5^3}{4(4-10)4^3}.$$

Using that the greatest common divisor of the numerator and the denominator is 3, we see that

$$\sqrt{-\sqrt[3]{3}}\,\sqrt{8-5\sqrt[3]{7}}=\sqrt{4\sqrt[3]{4(4-10)}+5\sqrt[3]{16+5}}.$$

And this last expression we can denest with Ramanujan's formula for $m=4$ and $n=5$. So the denesting becomes

$$\sqrt{8-5\sqrt[3]{7}}=\frac{1}{3\sqrt[6]{-3}}\left(-\sqrt[3]{21^2}+2\sqrt[3]{63}+2\sqrt[3]{9}\right).$$

The most difficult steps in the two denesting methods we have seen are computing the rational root of the polynomial $F_{\beta/\alpha}$ and determining integers $m$, $n$ such that

$$\frac{\beta}{\alpha}=\frac{(4m+n)n^3}{4(m-2n)m^3}.$$

For computing the integers $m,n$ as meant above we have the following theorem.

**Theorem 22.** *Given $\alpha, \beta \in \mathbb{Q}^*$ and coprime integers $m, n$ such that*

$$\frac{\beta}{\alpha} = \frac{(4m+n)n^3}{4(m-2n)m^3},$$

*then $n^3$ is a divisor of the the numerator of $4\beta/\alpha$ and $m^3$ is a divisor of the denominator of $\beta/\alpha$.*

*Proof.* Define $d = \gcd(\text{numerator}(\beta/\alpha), \text{denominator}(\beta/\alpha))$. Let $p$ be a prime such that $p \mid d$. If $p = 2$ then $n$ must be even and then $4 \mid d$ and $8 \nmid d$. If $p$ is odd, then $p \mid 4m + n$ and $p \mid m - 2n$, that is $p \mid 9$. We conclude that $d$ is a divisor of 36.

Now we can derive that $n^3$ is a divisor of the numerator of $4\beta/\alpha$ because if $3 \mid n$ then $3 \nmid d$. Similarly $m^3$ is a divisor of the denominator of $\beta/\alpha$ because if $2 \mid m$ or $3 \mid m$ then these primes will not divide $d$. □

Computing the rational root of $F_{\beta/\alpha}$ is equally difficult, as we can see when we look at prime divisors of the numerator and the denominator of $s$ in the integer equation that we can derive from $F_{\beta/\alpha}(s) = 0$:

$$s^4 + 4s^3 = 4\frac{\beta}{\alpha} - 8\frac{\beta}{\alpha}s.$$

Finally, one could wonder, as Ramanujan's denesting formula lacks obvious symmetry, what happens if the roles of $\alpha$ and $\beta$ interchange. The answer is that we find the same simplification, putting $p = -n/\sqrt{2}$ and $q = m\sqrt{2}$. (This explains why in example 20 we were not able to find integers $m$ and $n$ such that $\alpha = 4(m-2n)m^3$ and $\beta = (4m+n)n^3$ for the choice $\alpha = 5$ and $\beta = -4$.) If we define $p$ and $q$ in this way, it holds that

$$\begin{aligned} \alpha &= 4(m-2n)m^3 = (4p+q)q^3 \\ \beta &= (4m+n)n^3 = 4(p-2q)p^3. \end{aligned}$$

Since

$$\begin{aligned} (4m+n)^2 &= \left(2\sqrt{2}q - \sqrt{2}p\right)^2 = 2(p-2q)^2 \\ 2(m-2n)^2 &= 2\left(\tfrac{1}{2}q^2 + 4pq + 8p^2\right) = (4p+q)^2 \\ 4(m-2n)(4m+n) &= 4\left(2q^2 + 7pq - 4p^2\right) = -4(p-2q)(4p+q), \end{aligned}$$

we may replace $m$ and $n$ simply by $p$ and $q$ in the right hand side of Ramanujan's formula:

$$\sqrt{\sqrt[3]{\beta} + \sqrt[3]{\alpha}} = \pm\tfrac{1}{3}\left(-\sqrt[3]{(4p+q)^2} - \sqrt[3]{4(p-2q)(4p+q)} + \sqrt[3]{2(p-2q)^2}\right).$$

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