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Abstract

A balanced set of weights infers an inequality that games with a nonempty core obey. This paper gives a generalization of the notion 'balanced set of weights'. Herewith it provides necessary and sufficient conditions to determine whether a TU-game has a population monotonic allocation scheme or not.
Furthermore it shows that every 4-person integer valued game with a population monotonic allocation scheme has an integer valued population monotonic allocation scheme and it gives an example of a 7-person integer valued game that has only non-integer valued population monotonic allocation schemes.

1 Introduction

In *Sprumont* (1990) the concept of a *population monotonic allocation scheme* (*pmas* for short) has been defined as a kind of extension of a core allocation (cf. *Moulin* (1989)). A pmas gives a core allocation for every subgame of a TU-game such that every player gets a weakly higher payoff in larger coalitions.
Games with a pmas have obviously a nonempty core. *Bondareva* (1963) and *Shapley* (1967) independently proved that a game (N, v) has a nonempty core if and only if it is *balanced*, that is, for each balanced set of weights $\{\lambda_S\}_{S\subseteq N}$, the game obeys the corresponding inequality: $$\lambda_N v(N) \geq \sum_{S\subsetneq N} \lambda_S v(S).$$
Here, a balanced set of weights consists of nonnegative numbers with the property that $\lambda_N e_N = \sum_{S\subsetneq N} \lambda_S e_S$, in which e_S denotes the indicator vector of S. The following interpretation can be given to this inequality: if every member of S works λ_S hours in coalition S, which generates a profit of $v(S)$ dollars per hour, and if every player in N works the same number of hours (λ_N) in total then it is more profitable for the whole society to work together all of the time in the grand coalition. The class of balanced games is a finitely generated cone in the space of TU-games. The class of games with a pmas is a subcone of it, also finitely generated. Hence, there exists a collection of inequalities that describes this subcone. This collection is larger than the

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collection of inequalities describing the cone of balanced games and will be described in this paper by introducing the notion 'generalized balanced set of weights'. This description enables us to answer an open question postulated in *Reijnierse* (1995): "Do there exist integer valued games with only non-integer valued pmasses?". In Section 6 we will show that the answer is negative for 4-person games. Section 7 however, gives an example of such a game with 7 players.

2 Preliminaries

Let N be a finite set and let $\mathcal{G}^N$ be the space of TU-games with player set N. Let $M = \{S \subseteq N \mid S \neq \phi\}$.

Definition 1: *A population monotonic allocation scheme or pmas of the game* (N, v) *is a table* $\boldsymbol{x} = \{x_{S,i}\}_{S \in M, i \in S}$ *with the properties:*

(i) $\displaystyle\sum_{i \in S} x_{S,i} = v(S)$ *for all* $S \in M$,

(ii) $x_{S,i} \leq x_{T,i}$ *for all* $S, T \in M,\ i \in S \subset T$.

The class of games with player set N that have a pmas is called PM^N, or PM if no confusion can occur. *Sprumont* (1990) was the first who proved the following result:

Theorem 2: *The class* PM^N *is a cone and it is generated by the collection of all simple monotonic* i*-veto games in* $\mathcal{G}^N$*, united with the games* $-u_i$ $(i \in N)$.

This collection will be called $g(PM)$. A game is called *simple* if all its coalitional values are either 0 or 1, it is called *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T$ and it is called i*-veto* if $v(S) \neq 0$ implies $i \in S$. The simple game u_i is defined by $u_i(S) = 1$ if and only if $i \in S$.
Reijnierse (1995) submits a complete section to pmasses. Other results concerning pmasses can be found in *Derks* (1991).

3 Generalized balanced sets of weights

As stated in the introduction, we would like to find inequalities that games with a pmas obey, by generalizing the notion of a balanced set of weights.

Definition 3: *A generalized balanced set of weights, or gbw for short, is a tuple* $\langle \{\delta_S\}_{S \in \Delta}, \{\lambda_T\}_{T \in \Lambda} \rangle$ *with the following properties:*

(i) Δ *and* Λ *are disjoint subsets of* M,

(ii) $\delta_S > 0$ *and* $\lambda_T > 0$ *for all* $S \in \Delta$, $T \in \Lambda$,

(iii) *it is possible to assign a nonnegative number* $\mu^i_{S,T}$ *to each triple* $(i, S, T) \in N \times \Delta \times \Lambda$ *with* $i \in T \subset S$*, in such a way that:*

$$\sum_{T \in \Lambda : i \in T \subset S} \mu^i_{S,T} = \delta_S \text{ for each } S \in \Delta \text{ and } i \in S \qquad \text{and}$$

$$\sum_{S \in \Delta : i \in T \subset S} \mu^i_{S,T} = \lambda_T \text{ for each } T \in \Lambda \text{ and } i \in T.$$

It is easy to infer that a balanced set of weights is a *gbw*. Namely, if $\{\lambda_S\}_{S\in M}$ is a balanced set of weights, take $\Delta = \{N\}$, $\Lambda = \{S \subsetneq N \mid \lambda_S > 0\}$ and $\delta_N = \lambda_N$. Then the tuple $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ satisfies the properties (i) and (ii) of the previous definition.

Define for each $T \in \Lambda$ and every $i \in T$: $\mu^i_{N,T} = \lambda_T$. Then, for all $i \in N$:

$$\sum_{T\in\Lambda:T\ni i} \mu^i_{N,T} = \sum_{T\subsetneq N:T\ni i} \lambda_T = \lambda_N = \delta_N.$$

Moreover, for each $T \in \Lambda$ and every $i \in T$:

$$\sum_{S\in\Delta:T\subset S} \mu^i_{S,T} = \mu^i_{N,T} = \lambda_T.$$

Hence, the third property of Definition 3 has been satisfied as well.

Example 4: Let $\Delta = \{(123), (234)\}$ and $\Lambda = \{(12), (23), (34)\}$. Let $\delta_S = \lambda_T = 1$ for all $S \in \Delta$, $T \in \Lambda$. Is $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ a *gbw*? Yes, take:

$$\mu^1_{(123),(12)} = \mu^2_{(123),(12)} = \mu^2_{(234),(23)} = \mu^3_{(123),(23)} = \mu^3_{(234),(34)} = \mu^4_{(234),(34)} = 1$$

and

$$\mu^2_{(123),(23)} = \mu^3_{(234),(23)} = 0.$$

The *gbw* corresponds to the inequality:

$$v(123) + v(234) \geq v(12) + v(23) + v(34).$$

If a game (N, v) has a *pmas* $\boldsymbol{x}$, then it obeys this inequality, since

$$\begin{gathered} v(123) + v(234) = \\ x_{123,1} + x_{123,2} + x_{123,3} + x_{234,2} + x_{234,3} + x_{234,4} \geq \\ x_{12,1} + x_{12,2} + x_{23,3} + x_{23,2} + x_{34,3} + x_{34,4} = \\ v(12) + v(23) + v(34). \end{gathered}$$

Each relation corresponding to a *gbw* is a necessary condition for having a pmas:

Theorem 5: *Let the game (N, v) have a population monotonic allocation scheme, say $\boldsymbol{x}$, and let $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ be a gbw. Then v obeys the inequality:*

$$\sum_{S\in\Delta} \delta_S v(S) \geq \sum_{T\in\Lambda} \lambda_T v(T).$$

Proof: Let for $i \in N$, $S \in \Delta$ and $T \in \Lambda$ with $i \in T \subset S$ the numbers $\mu^i_{S,T}$ be as in Definition 3. We have:

$$\begin{aligned} \sum_{S\in\Delta} \delta_S v(S) &= \sum_{S\in\Delta} \delta_S \sum_{i\in S} x_{S,i} = \sum_{S\in\Delta}\sum_{i\in S} \delta_S x_{S,i} = \\ \sum_{S\in\Delta}\sum_{i\in S}\sum_{T\in\Lambda:i\in T\subset S} \mu^i_{S,T} x_{S,i} &= \sum_{(i,S,T):i\in T\subset S} \mu^i_{S,T} x_{S,i} \geq \sum_{(i,S,T):i\in T\subset S} \mu^i_{S,T} x_{T,i} = \\ \sum_{T\in\Lambda}\sum_{i\in T}\sum_{S\in\Delta:i\in T\subset S} \mu^i_{S,T} x_{T,i} &= \sum_{T\in\Lambda}\sum_{i\in T} \lambda_T x_{T,i} = \sum_{T\in\Lambda} \lambda_T v(T). \end{aligned}$$

◁

Corollary 6: *Let (N, v) have a population monotonic allocation scheme $\boldsymbol{x}$ and let $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ be a gbw with associated numbers $\{\mu^i_{S,T} \mid i \in T \in \Lambda, T \subset S \in \Delta\}$. Suppose that $\sum_{S\in\Delta} \delta_S v(S) = \sum_{T\in\Lambda} \lambda_T v(T)$. Then $x_{S,i} = x_{T,i}$ for every triple (i, S, T) with $i \in T \subset S$ and $\mu^i_{S,T} > 0$.*

The following interpretation can be given to an inequality corresponding to a *gbw*: if the society N works according to schedule $\{\lambda_T\}_{T\in\Lambda}$ and if a rescheduling $\{\delta_S\}_{S\in\Delta}$ is possible such that all members in the society work the same hours as before but in larger coalitions (not necessarily the grand coalition) then such a rescheduling is profitable for the whole society.

4 Verifying whether a tuple is a *gbw*

Let $\langle\{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ be a tuple with properties (i) and (ii) of Definition 3. How can we find numbers $\mu^i_{S,T}$ such that property (iii) is satisfied or show that such numbers do not exist?
Let $i \in N$. Define $\Delta^i = \{S \in \Delta \mid i \in S\}$ and $\Lambda^i = \{T \in \Lambda \mid i \in T\}$. Because

$$\sum_{S\in\Delta^i} \delta_S = \sum_{S\in\Delta^i} \left(\sum_{T\in\Lambda^i:T\subset S} \mu^i_{S,T} \right) = \sum_{T\in\Lambda^i} \left(\sum_{S\in\Delta^i:S\supset T} \mu^i_{S,T} \right) = \sum_{T\in\Lambda^i} \lambda_T,$$

the first test the tuple has to pass to be a *gbw*, is that $\sum_{S\in\Delta^i} \delta_S = \sum_{T\in\Lambda^i} \lambda_T$. If so, a flow network $\Gamma_i = \langle V, E\rangle$ is constructed as follows. The node set V consists of a *source*, a *sink* and a node for each coalition T in $\Delta^i \cup \Lambda^i$. The nodes will be called So, Si and $node(T)$ ($T \in \Delta^i \cup \Lambda^i$). The arc set E consists of directed arcs. For all $S \in \Delta^i$ there is an arc from the source to $node(S)$, called $arc(S)$. The capacity of this arc is δ_S. For all $T \in \Lambda^i$ there is an arc called $arc(T)$ from $node(T)$ to the sink with capacity λ_T. If $S \in \Delta^i$, $T \in \Lambda^i$ and $S \supset T$, there is an arc called $arc(S,T)$ from $node(S)$ to $node(T)$ with a large capacity, i.e. strictly larger than $\sum_{T\in\Lambda} \lambda_T$.
Find a maximal source to sink flow with the maximal flow algorithm of *Ford* and *Fulkerson* (1956). If its value f equals $\sum_{T\in\Lambda^i} \lambda_T$, take $\mu^i_{S,T}$ equal to the flow in $arc(S,T)$.
On the other hand, if there exist appropriate numbers $\mu^i_{S,T}$ (*for this particular player* i), f will equal $\sum_{T\in\Lambda^i} \lambda_T$. Namely, take the flow which uses the arcs from the source and the arcs to the sink with full capacity and which uses the other arcs $arc(S,T)$ with capacity $\mu^i_{S,T}$.
These observations lead to the following Proposition:

Proposition 7: *Let $\langle\{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ be a tuple with properties (i) and (ii) of Definition 3. Then $\langle\{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ is a gbw if and only if for every player $i \in N$:*

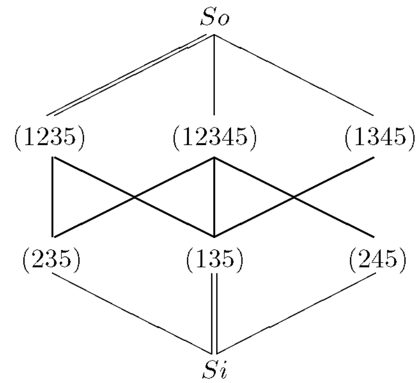
$$\sum_{S\in\Delta^i} \delta_S = \sum_{T\in\Lambda^i} \lambda_T, \text{ and the network } \Gamma_i \text{ has value } \sum_{S\in\Delta^i} \delta_S.$$

Let us give an example of such a network.

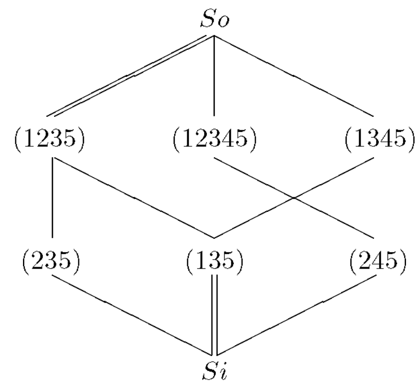
Example 8: Consider the tuple that corresponds to the inequality:

$$2v(1235) + v(12345) + v(1345) \geq v(12) + v(235) + 2v(135) + v(245) + v(134).$$

Let $i = 5$. The following figure illustrates the corresponding flow network (a node is represented by So, Si or its corresponding coalition):



Thick arcs have large capacity, single tiny arcs have capacity 1, double tiny arcs have capacity 2. The network has a unique maximal flow, depicted in the following figure:



Hence, the value f of the maximal flow equals 4, which is necessary for the tuple to be a *gbw*. The flow shows how the numbers $\mu^5_{S,T}$ can be chosen:

$\mu^5_{(1235),(235)} = \mu^5_{(1235),(135)} = \mu^5_{(12345),(245)} = \mu^5_{(1345),(135)} = 1$ and
$\mu^5_{(12345),(235)} = \mu^5_{(12345),(135)} = 0$.

For each player in N we can perform this test. If all tests have a positive answer, the tuple is a *gbw*.

5 The converse statement

The converse of Theorem 5 is also true:

Theorem 9: *Let the game (N, v) obey all inequalities that arise from gbw's. Then v has a population monotonic allocation scheme.*

Proof: The dual of the cone PM, called PM^*, is defined by: $\{w \in \mathcal{G}^N \mid \langle w, v\rangle \geq 0$ for all $v \in PM\}$. Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathcal{G}^N$, i.e. $\langle w, v\rangle = \sum_{S \in M} w(S)v(S)$.
Because the cone PM is generated by the finite collection $g(PM)$ (Theorem 2), we have $PM = PM^{**} = \{v \in \mathcal{G}^N \mid \langle w, v\rangle \geq 0$ for all $w \in PM^*\}$.

Hence, to prove the Theorem, it is sufficient to prove that $v \in PM^{**}$, i.e. that $\langle w, v \rangle \geq 0$ for every $w \in PM^*$. In order to show this last statement, it suffices to prove that every $w \in PM^*$ induces an inequality that corresponds to some *gbw*. Therefore, let $w \in PM^*$ and define $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda} \rangle$ as follows:

$\Delta = \{S \in M \mid w(S) > 0\}$, $\Lambda = \{T \in M \mid w(T) < 0\}$,

$\delta_S = w(S)$ for every $S \in \Delta$ and $\lambda_T = -w(T)$ for every $T \in \Lambda$.

For each $i \in N$ we must show that the tuple has the properties described in Proposition 7. Let $i \in N$.

We have $\langle w, u \rangle \geq 0$ for all $u \in g(PM)$. Since both u_i and $-u_i$ are elements of $g(PM)$, we have $\langle w, u_i \rangle = 0$. Hence, $\sum_{S\in\Delta^i} w(S) = \sum_{T\in\Lambda^i} -w(T) =: d$, in which Δ^i and Λ^i have been defined as in Section 4. We have to prove that the value f of the network $\Gamma_i = \langle V, E \rangle$ equals d.

A *cut* in the network is a 2-partition $(P, V\backslash P)$ of the node set V, such that $So \in P$ and $Si \in V\backslash P$. The capacity of $(P, V\backslash P)$, denoted by $c(P, V\backslash P)$, is the sum of the capacities of all arcs with begin-point in P and end-point in $V\backslash P$.

By the famous Theorem of *Ford* and *Fulkerson* (1956), the value of a maximal source to sink flow equals the minimum of the capacities of all cuts. The capacities of the cuts $(\{So\}, V\backslash\{So\})$ and $(V\backslash\{Si\}, \{Si\})$ equal d. Hence, it is sufficient to show that $c(P, V\backslash P) \geq d$ for every cut $(P, V\backslash P)$.

Take a cut $(P, V\backslash P)$. Let $\mathcal{S} \subset \Delta^i$ be the collection of coalitions in Δ^i of which the nodes are elements of $V\backslash P$. Let $\mathcal{T} \subset \Lambda^i$ be the collection of coalitions in Λ^i of which the nodes are elements of $V\backslash P$. Let $u_{\mathcal{T}}$ be the smallest simple monotonic i-veto game such that all elements of $\mathcal{T}$ are winning. Then the winning coalitions of $u_{\mathcal{T}}$ are the ones that have at least one element of $\mathcal{T}$ as a subset.

If there exists a coalition $S \in \Delta^i$ such that $node(S) \in P$ and $u_{\mathcal{T}}(S) = 1$, then there exists a $T \in \mathcal{T}$ such that the capacity of $arc(S,T)$ contributes to the capacity of $(P, V\backslash P)$. The capacity of this arc alone exceeds d already.

Hence, we can assume that such a coalition does not exist, i.e. all elements of $\Delta^i\backslash\mathcal{S}$ have coalitional value 0 with respect to $u_{\mathcal{T}}$ and the capacity of each arc with begin-point in Δ^i and end-point in Λ^i does not contribute to $c(P, V\backslash P)$. Therefore:

$$c(P, V\backslash P) = \sum_{S\in\mathcal{S}} \delta_S + \sum_{T\in\Lambda^i\backslash\mathcal{T}} \lambda_T = \sum_{S\in\mathcal{S}} w(S) - \sum_{T\in\Lambda^i\backslash\mathcal{T}} w(T) =$$

$$d + \sum_{S\in\mathcal{S}} w(S) + \sum_{T\in\mathcal{T}} w(T) \geq d + \langle w, u_{\mathcal{T}} \rangle \geq d. \qquad \triangleleft$$

6 Four person games

Consider for a (characteristic function of a) 4-person game v the following inequalities, which correspond to gbw's:

(A)	$v(ij) \geq v(i) + v(j)$	(6 inequalities)
(B)	$v(ijk) \geq v(i) + v(jk)$	(12 inequalities)
(C)	$v(1234) \geq v(i) + v(jkl)$	(4 inequalities)
(D)	$v(ijk) + v(jkl) \geq v(ij) + v(jk) + v(kl)$	(12 inequalities)
(E)	$2v(ijk) \geq v(ij) + v(ik) + v(jk)$	(4 inequalities)

(F)	$v(ijk) + v(1234) \geq v(ij) + v(jk) + v(ikl)$	(12 inequalities)
(G)	$v(1234) \geq v(ij) + v(kl)$	(3 inequalities)
(H)	$2v(1234) \geq v(ij) + v(jkl) + v(ikl)$	(6 inequalities)
(I)	$3v(1234) \geq v(123) + v(124) + v(134) + v(234)$	(1 inequality)

Different characters are used to denote different players. If a 4-person game has a pmas then this game satisfies all conditions (A)-(I). In this section we prove that these conditions are sufficient conditions in order to guarantee that a game has a pmas. As a byproduct we get that an integer-valued 4-person game with a pmas has an integer-valued pmas. Note that the conditions (A), (B), (C) and (G) imply superadditivity, the conditions (C), (G), (H) and (I) imply balancedness and the conditions (A), (B), (C), (E), (G), (H) and (I) imply totally balancedness. Note moreover that for every condition in (A)-(I) the following statement is true: if v is monotonic and $v(S) = 0$ for some coalition occuring in the right-hand side of this condition then v satisfies this condition.

If a 4-person game satisfies the conditions (A)-(I) then the corresponding 0-normalized game also satisfies these conditions. This statement is an immediate consequence of the fact that linear games satisfy all conditions (A)-(I) with equality. Moreover, due to conditions (A)-(C), this 0-normalized game is monotonic. Let v be a 0-normalized monotonic game. A 0-normalized monotonic simple veto game is *subtractable* from v if $v - \varepsilon u$ is monotonic for some $\varepsilon > 0$. Note that $v(N) > v(N \backslash i)$ is a necessary and sufficient condition for the existence of a monotonic simple i-veto game which is subtractable from v. Moreover, if u_1 and u_2 are both monotonic simple i-veto games which are subtractable from v then also $u := \max\{u_1, u_2\}$ is subtractable from v. This enables us (in case $v(N) > v(N \backslash i)$) to define u_i^v as the maximal monotonic simple i-veto game which is subtractable from v. Moreover, the positive number $a_i^v := \min\{v(S) - v(T) : S \supset T, u_i^v(S) = 1, u_i^v(T) = 0\}$ indicates how many times u_i^v can be subtracted from v at most such that the remainder is still monotonic. If $v(N) = v(N \backslash i)$ then $u_i^v := 0$.

Lemma 10: *If a 0-normalized 4-person game v satisfies conditions (A)-(I) and $v(S) > 0$ for some $S \subseteq N$ then there is an $i \in S$ such that $u_i^v(S) = 1$.*

Proof: Without loss of generality we may assume that $v(S) > v(S \backslash j)$ for every $j \in S$ (if there is a $j \in S$ with $v(S) = v(S \backslash j) > 0$ it is sufficient to prove the statement for $S \backslash j$). We distinguish between three cases: i) $|S| = 4$; ii) $|S| = 3$; iii) $|S| = 2$.

Case i): $|S| = 4$. Then $S = (1234)$. Since $v(S) > v(S \backslash j)$ for every $j \in S$ the game u_{1234} (which is the monotonic simple veto game with (1234) as unique (minimal) winning coalition) is subtractable from v. Hence $u_i^v(S) = 1$ for every $i \in S$.

Case ii): $|S| = 3$. Without loss of generality assume that $S = (123)$. Since v satisfies condition (I) there is at least one $j \in S$ with $v(N) > v(N \backslash j)$, say $j = 1$. Then at least one of the games u_{123}, $u_{123,124}$, $u_{123,134}$, $u_{123,124,134}$ or $u_{123,14}$ is subtractable from v (the subscripts refer to the minimal winning coalitions in the corresponding monotonic simple veto games). Hence $u_1^v(S) = 1$.

Case iii): $|S| = 2$. Without loss of generality assume that $S = (12)$. Either $v(T) > v(T \backslash 1)$ for every $T \supset S$ or $v(T) > v(T \backslash 2)$ for every $T \supset S$ (otherwise there is a

$T_1 \supset S$ with $v(T_1) = v(T_1\backslash 1)$ and a $T_2 \supset S$ with $v(T_2) = v(T_2\backslash 2)$ which contradicts condition $v(T_1) + v(T_2) \geq v(12) + v(T_1\backslash 1) + v(T_2\backslash 2)$, which is one of the conditions in (D)-(F) and (H)). Assume $v(T) > v(T\backslash 1)$ for every $T \supset S$. If $v(134) = v(34)$ then $v(123) > v(13)$ (because $v(123)+v(134) \geq v(12)+v(13)+v(34)$) and $v(124) > v(14)$ (because $v(124) + v(134) \geq v(12) + v(14) + v(34)$) and hence u_{12} is subtractable. If $v(134) > v(34)$ then the monotonic simple 1-veto game u defined by $u(S) := 1$ iff $1 \in S$ and $v(S) > 0$ is subtractable from v. Anyhow, $u_1^v(S) = 1$. ◁

Lemma 11: *Let v be a 0-normalized 4-person game that satisfies conditions (A)-(I) and let $N^v \subseteq N$ be the set of players i with $u_i^v \neq 0$. Let $i^* \in N^v$ be such that $u_{i^*}^v$ has a minimal number of veto players. Then $v - a_{i^*}^v u_{i^*}^v$ also satisfies conditions (A)-(I).*

Proof: Let $v' := v - a_{i^*}^v u_{i^*}^v$. We will distinguish between four cases.
Case i): $u_{i^*}^v$ has only one veto player, say $i^* = 1$. Then $u_1^v(123) = u_1^v(124) = u_1^v(134) = 1$. In order to show that v' satisfies all conditions (A)-(I), consider an arbitrary condition in (A)-(I), to be referred to as condition $(*)$. If u_1^v satisfies condition $(*)$ with equality then clearly v' satisfies condition $(*)$. If u_1^v satisfies condition $(*)$ with strict inequality then in the right-hand side of this inequality occurs some coalition S with $1 \in S$ and $u_1^v(S) = 0$. So $|S| \leq 2$. If $|S| = 1$ then clearly $v(S) = 0$ and hence $v'(S) = 0$. If $|S| = 2$ then $u_1^v(S) = 0$ implies $v(S) = 0$ and we also get $v'(S) = 0$. Now v' satisfies condition $(*)$ because of monotonicity of v'.
Case ii): $u_{i^*}^v$ has two veto players, say 1 and 2. So, $u_{i^*}^v = u_{12}$ or $u_{i^*}^v = u_{123,124}$. If $u_1^v(13) = 1$ then u_1^v has (13) and (124) as winning coalitions and therefore only one veto player, giving a contradiction. Hence, $u_1^v(13) = 0$. If $u_3^v(13) = 1$ then $u_3^v = u_{13}$ is subtractable and hence $u_1^v(13) = 1$ leading, as before, to a contradiction. So, $u_1^v(13) = u_3^v(13) = 0$ and hence, according to Lemma 10, $v(13) = 0$. Analogously we get $v(14) = v(23) = v(24) = 0$. Hence $v'(13) = v'(14) = v'(23) = v'(24) = 0$ and v' satisfies all conditions (A)-(F) by monotonicity. Condition (G) with $(ij) = (13)$ or (14) is satisfied by v' due to monotonicity. Condition (G) with $(ij) = (12)$ is satisfied by v' due to monotonicity in case $v(12) = v'(12) = 0$ and due to the fact that $u_{i^*}^v = u_{12}$ satisfies this condition with equality in case $v(12) > 0$. Condition (H) with $(ij) \in \{(13), (14), (23), (24)\}$ is satisfied by monotonicity of v', condition (H) with $(ij) = (34)$ is satisfied because $u_{i^*}^v$ satisfies this condition with equality and condition (H) with $(ij) = (12)$ is satisfied because $v(134) = v(34)$ (and hence $v'(134) = v'(34)$) and $v'(1234) \geq v'(12) + v'(34)$. Condition (I) is satisfied by v' because $v'(134) = v'(34)$, v' satisfies condition (H) with $(ij) = (34)$ and monotonicity of v'.
Case iii): $u_{i^*}^v$ has three veto players, say 1, 2 and 3. Then $v(S) = 0$ if $S \neq (123)$ and $S \neq (1234)$ and the statement is trivial.
Case iv): $u_{i^*}^v$ has four veto players. Then $v(S) = 0$ if $S \neq (1234)$ and the statement is trivial. ◁

Lemmas 10 and 11 provide the basis for an algorithm in order to determine whether a 0-normalized 4-person game v has a pmas or not: compute in each step the games u_i^v and subtract that game u_i^v which has a minimal number of veto players (a_i^v times). If the game v satisfies conditions (A)-(I) then Lemma 11 guarantees that after such a step we are left with a game $v' := v - a_i^v u_i^v$ which also satisfies conditions (A)-(I). Moreover, if $v' \neq 0$, Lemma 10 guarantees that there is at least one player i with $u_i^{v'} \neq 0$ and hence the algorithm does not stop. Note also that if v is integer valued, all a_i^v's are integer. We have proved the following theorem.

Theorem 12: *If v is a 4-person game satisfying conditions (A)-(I) then v has a pmas. If, moreover, v is integer valued then v has an integer valued pmas.*

Example 13: Let v be the 0-normalized game, given by $v(1234) = 7$, $v(123) = v(124) = v(134) = 4$, $v(234) = 6$, $v(12) = v(13) = v(14) = 2$, $v(23) = v(24) = 3$, and $v(34) = 4$. Computing the u_i^v's we get, e.g., $u_2^v = u_{21,23,24}$ (one veto player) with $a_2^v = 2$. Determination of $v' = v - a_i^v u_i^v$ yields $v'(1234) = 5$, $v'(123) = v'(124) = 2$, $v'(134) = v'(234) = 4$, $v'(12) = 0$, $v'(13) = v'(14) = 2$, $v(23) = v(24) = 1$, and $v'(34) = 4$. Proceeding in the same way we subtract $u_{31,32,34}$ in the second step, $u_{31,34}$ in the third step, $u_{41,42,43}$ in the fourth step, $u_{41,43}$ in the fifth step, and u_{1234} in the sixth step, after which the algorithm ends. So, $v = 2u_{21,23,24} + u_{31,32,34} + u_{31,34} + u_{41,42,43} + u_{41,43} + u_{1234}$.

7 An integer game with only non-integer pmasses

The previous results enable us to find an integer game that has only non-integer pmasses. Let $N = (1234567)$ and consider the inequality:

$$2v(12345) + 2v(12346) + 2v(12347) \geq$$
$$3v(1234) + v(125) + v(136) + v(147) + v(237) + v(246) + v(345).$$

It is easy to verify that it arises from a *gbw*, namely take:

$$\begin{aligned}
\Delta &= \{(12345), (12346), (12347)\},\\
\Lambda &= \{(1234), (125), (136), (147), (237), (246), (345)\},\\
\delta_S &= 2 \text{ for all } S \in \Delta,\ \lambda_{(1234)} = 3,\\
\lambda_T &= 1 \text{ for all } T \in \Lambda\backslash\{(1234)\},\\
\mu^i_{S,T} &= 1 \text{ for all } (i, S, T) \in N \times \Delta \times \Lambda \text{ with } i \in T \subset S.
\end{aligned}$$

Suppose that we have a game v with a pmas $\boldsymbol{x}$ such that $v(S) = 2$ for $S \in \Delta \cup \{(1234)\}$ and $v(T) = 1$ for $T \in \Lambda\backslash\{(1234)\}$. Then the inequality is tight. By Corollary 6 we can infer that there exist numbers $\alpha_1, \ldots, \alpha_7$, such that for all $S \in \Delta \cup \Lambda$: $x_{S,i} = \alpha_i$ for all $i \in S$.

We have: $\alpha_5 = \sum_{i=1}^{5} \alpha_i - \sum_{i=1}^{4} \alpha_i = \sum_{i=1}^{5} x_{(12345),i} - \sum_{i=1}^{4} x_{(1234),i} = v(12345) - v(1234) = 0.$

Because of the symmetric roles of the players 5, 6 and 7, $\alpha_6 = \alpha_7 = 0$ as well.

Let i and j be players in (1234). Then there is a 3-person coalition $T \in \Lambda$ which contains (ij) and one player of the coalition (567). Therefore $\alpha_i + \alpha_j = v(T) = 1$.

This makes $\alpha_i = \frac{1}{2}$ for every $i \in (1234)$.

Hence, in order to find an example we have to find a game v with the previous properties. This can be done by defining:

$v(S) = 0$ if there is no $T \in \Delta \cup \Lambda$ with $T \subseteq S$,
$v(S) = 1$ if there are $T \in \Lambda\backslash\{(1234)\}$, $U \in \Delta$ such that $T \subseteq S \subsetneq U$,
$v(S) = 2$ if $S \in \Delta \cup \{(1234)\}$ and

$v(S) = |S \cap (1234)| \quad$ else.

Let $\boldsymbol{x}$ be defined as follows:

$x_{S,i} = 0 \qquad$ if $v(S) = 0$ or $i \in (567)$,
$x_{S,i} = 0 \qquad$ if $v(S) = 1$, $i \in S \cap (1234)$ and $S \backslash i \in \Lambda$,
$x_{S,i} = \frac{1}{2} \qquad$ if $v(S) = 1$, $i \in S \cap (1234)$ and $S \backslash i \notin \Lambda$,
$x_{S,i} = \frac{1}{2} \qquad$ if $S \in \Delta \cup \{(1234)\}$ and $i \in (1234)$ and
$x_{S,i} = 1 \qquad$ else.

Then $\boldsymbol{x}$ is a pmas of v.

We have not been able to find examples with less than 7 players.

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