Fisher information and topological pressure

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Abstract

The relationship between the Fisher information and the topological pressure is explored. The new concept of topological Fisher information is introduced and the relation between the topological pressure and the topological Fisher information is derived. In the special case of expanding maps, a relation between the topological and the dimensional Fisher information is obtained. Relations between the Legendre transforms of the topological and the dimensional Fisher information are also formulated.

I. INTRODUCTION

Fisher information [1] has an important role in physics [2]. Among others, it has been applied in thermodynamics. E. g. a Legendre transform structure [2, 3], Fisher temperature and Fisher thermodynamics [2, 4, 5] were worked out. It can be also mentioned that a thermodynamic formalism utilizing Fisher information [6] has recently been introduced in quantum mechanics based on a thermodynamical transcription [7–10].

Recently a thermodynamics based on Fisher information has been presented [11]. Then the present authors introduced Fisher information into the chaos theory and established a relationship between the Fisher information and the Rényi dimensions [12]. In this paper a link between the Fisher information and the topological pressure is analyzed.

The thermodynamic formalism [13–15] has become very valuable and productive in the chaos theory. The thermodynamic concept takes advantage of the analogy between the statistical mechanics and the chaos theory. This analogy is now extended to the Fisher information by introducing the new concept of topological Fisher information. The relationship between the topological pressure and the topological Fisher information is established. In the special case of expanding maps, a relationship between the topological and the dimensional Fisher information is introduced.

The layout of this paper is as follows: the Fisher thermodynamics [11] is summarized in the following section. The theory of the topological pressure is reviewed in Sections III. The new concept of topological Fisher information is introduced in Section IV. Section V presents the theory for the special case of expanding maps. Section VI provides an illustrative example and discussion. The theory of dimensional Fisher information is summarized in the Appendix.

II. CANONICAL THERMODYNAMICS VIA FISHER INFORMATION

First, the theory of Porporato [11] is summarized. In the macrostate of a system characterized by the volume V, the particle number N and the temperature T, the canonical distribution of a random variable associated with the energy $\hat{U}$, takes the form

$$p(\hat{U};\beta) = \frac{g(\hat{U})}{Z(\beta)} e^{-\beta\hat{U}} = g(\hat{U}) e^{-\Phi(\beta)-\beta\hat{U}}. \tag{1}$$

Applying the unit, where the Boltzmann constant is equal to 1, $\beta = 1/T$ is the inverse temperature and $g(\hat{U})$ is the density of states. The partition function $Z(\beta)$ is

$$Z(\beta) = \int g(\hat{U}) e^{-\beta \hat{U}} d\hat{U} \tag{2}$$

for continuous and

$$Z(\beta) = \sum_r g_r e^{-\beta \hat{U}_r} \tag{3}$$

for discrete probability distributions.

The Helmholtz free energy

$$F(\beta) = -T \ln\left(Z(\beta)\right) = -T\Phi(\beta), \tag{4}$$

where the Massieu function $\Phi(\beta)$

$$\Phi(\beta) = \ln\left(Z(\beta)\right) \tag{5}$$

has an important role in the theory. Namely, the first derivative of the Massieu function $\Phi(\beta)$ with respect to the inverse temperature β gives the average of the energy

$$-\frac{\partial \Phi(\beta)}{\partial \beta} = \langle \hat{U} \rangle = U, \tag{6}$$

while the second derivative provides the variance of the energy $\hat{U}$:

$$\mathcal{F}_U = \frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = (U - \langle \hat{U} \rangle)^2. \tag{7}$$

The latter is identical with the Fisher information:

$$\mathcal{F}_U = \frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = I. \tag{8}$$

The usual form of the Fisher information is

$$I(\beta) = \int \frac{1}{p(x|\beta)} \left[\frac{\partial p(x|\beta)}{\partial \beta}\right]^2 dx \tag{9}$$

and

$$I(\beta) = \sum_i \frac{1}{p_i(\beta)} \left[\frac{\partial p_i(\beta)}{\partial \beta}\right]^2 \tag{10}$$

for continuous and discrete probability distributions, respectively. The heat capacity C can also be related to the Fisher information

$$\mathcal{F}_U = \frac{C(\beta)}{\beta^2} = C(T)T^2. \tag{11}$$

A Legendre structure can be built with the Fisher information. It is well-known that the Legendre transform of entropy $S(U, V, N)$ is the Massieu function

$$\Phi = S - \beta U = -\frac{F}{T} \tag{12}$$

in entropy representation. Note that the entropy $S(U, V, N)$ is a fundamental function of the variables U, V and N, whereas $\Phi(T, V, N)$ is a fundamental function of the variables T, V and N. The following fundamental relations are fulfilled:

$$\frac{\partial S}{\partial U} = \beta, \tag{13}$$

and

$$\frac{\partial \Phi}{\partial \beta} = -U. \tag{14}$$

(For convenience, the dependence on the fixed quantities V and N is not denoted.)

Taking the Legendre transformation of U

$$\mathcal{F}_S = U + \beta \mathcal{F}_U \tag{15}$$

we can notice the resemblance between the functions $S(U)$ and $\mathcal{F}_S(\mathcal{F}_U)$. The fundamental relation

$$\frac{\partial \mathcal{F}_S}{\partial \mathcal{F}_U} = \beta \tag{16}$$

corresponds to Eq. (13). The well-known Legendre transformation of U in energy representation: $F = U - TS$ leads to the free energy F. A comparable relation is

$$\mathcal{F}_F = \mathcal{F}_U - T\mathcal{F}_S \tag{17}$$

with the fundamental relation

$$\frac{\partial \mathcal{F}_F}{\partial T} = -\mathcal{F}_S \tag{18}$$

resembling to the expression $\partial F/\partial T = -S$.

In [12] we introduced the 'Fisher' heat capacity as the derivative of the Fisher information $\mathcal{F}_U$ with respect to T

$$\frac{\partial \mathcal{F}_U}{\partial T} = \mathcal{C}_F. \tag{19}$$

in accordance with the heat capacity of the traditional thermodynamics.

III. THE TOPOLOGICAL PRESSURE

In this paper the geometrical interpretation of the topological pressure is utilised [13–15]. Assume that there is a generating partition of the phase space containing a finite number of cells of variable size. Each initial value x_0 generates a symbol sequence $i_0, i_1, i_2....$ Consider a finite symbol sequence $i_0, ..., i_{N-1}$. Let $J[i_0, ..., i_{N-1}]$ denote the set of all initial values x_0 that generate this sequence. This set $J[i_0, ..., i_{N-1}]$ is called N-cylinder and denoted by $J_j^{(N)}$. Of course, for $N = 1$ we obtain the cells of the generating partition. E. g. in one-dimensional case the N-cylinders are intervals having length $l_j^{(N)}$. As we have a generating partition, the N-cylinder become infinitesimally small for large N. The length of the cells can be given as

$$l_j^{(N)} \sim e^{-NE_N(x_0^{(j)})}, \tag{20}$$

where E_N is the local expansion rate that depends on the initial value $x_0^{(j)}$ in the cylinder $J_j^{(N+1)}$. Raising these lengths to the power β, in the limit $N \to \infty$, the sum

$$Z_N^{top}(\beta) = \sum_j \left(l_j^{(N)} \right)^\beta \tag{21}$$

gives a partition function. The topological pressure is defined as

$$\mathcal{P}(\beta) = \lim_{N \to \infty} \frac{1}{N} \ln \left(Z_N^{top}(\beta) \right). \tag{22}$$

As the partition function includes the sum of powers β of length scales of the cylinders, the topological pressure incorporates geometrical information on the cylinders in thermodynamic limit. We mention in passing that the name "topological pressure" is used in the literature, though the quantity $\mathcal{P}(\beta)$ is really free energy as the definition (22) shows. A detailed explanation of the name can be found in [13].

The sum in (21) can be replaced by an integral

$$Z_N^{top}(\beta) \sim \int e^{Ns(E) - NE\beta} dE \tag{23}$$

with the local expansion rate $E = E_N(x_0^{(j)})$. We have an asymptotic scaling behaviour $e^{Ns(E)}$, that is, the number of cylinders having a given E is $e^{Ns(E)}$. As for $N \to \infty$, the maximum value of the integrand dominates. Therefore, the Massieu function takes the form

$$\Phi^{top}(\beta) = -\frac{F^{top}}{T} = \ln\left(Z_N^{top}(\beta)\right) \sim N(s(E) - E\beta), \tag{24}$$

where F^{top} is the topological free energy analogous with (4).

In the thermodynamic limit we obtain

$$\lim_{N\to\infty} \frac{S}{N} = s(E), \tag{25}$$

$$\lim_{N\to\infty} \frac{U}{N} = E \tag{26}$$

and

$$\lim_{N\to\infty} \frac{\Phi^{top}}{N} = -\tau^{top} = \mathcal{P}(\beta), \tag{27}$$

that is,

$$\tau^{top} = \beta E - s = -\mathcal{P} \tag{28}$$

is the Legendre transform of s with the fundamental relations

$$\frac{\partial \tau^{top}}{\partial \beta} = E = -\frac{\partial \mathcal{P}}{\partial \beta} \tag{29}$$

and

$$\frac{\partial s}{\partial E} = \beta. \tag{30}$$

Note that the function $s(E)$ is analogous to the well-known spectrum $f(\alpha)$ of the local dimensions. Consulting the summary in the Appendix one can find the correspondence between the quantities s and f (Eqs. (25) and (87)), E and α (Eqs. (26) and (88)), τ^{top} and τ (Eqs. (27) and (88) and the relations (29) and (90) and (30) and (86)).

IV. TOPOLOGICAL FISHER INFORMATION

Utilizing the resemblance of Eqs. (3), (24) and (28), we are led to

$$\tau^{top} \sim -\Phi \tag{31}$$

and

$$E \sim U \tag{32}$$

owing to Eqs. (14) and (29). Now, the topological Fisher information is defined as

$$\mathcal{F}_E^{top} = -\frac{d^2\tau^{top}(\beta)}{d\beta^2}. \tag{33}$$

It can also be written as

$$\mathcal{F}_E^{top} = \frac{d^2\mathcal{P}(\beta)}{d\beta^2} = \frac{\partial E}{\partial \beta} \tag{34}$$

considering Eqs. (27) - (29). The Legendre transformation resulting $\mathcal{F}_{\mathcal{S}}$ in Eq. (15), in this case takes the form

$$\mathcal{F}_s^{top} = E - \beta\frac{dE}{d\beta} = E + \beta\mathcal{F}_E^{top} \tag{35}$$

and satisfies the fundamental relation

$$\frac{\partial \mathcal{F}_s^{top}}{\partial \mathcal{F}_E^{top}} = \beta. \tag{36}$$

Another Legendre transformation leads to the quantity analogous to the free energy of the usual thermodynamic formalism:

$$\mathcal{F}_F^{top} = \mathcal{F}_E^{top} - T\mathcal{F}_s^{top}. \tag{37}$$

According to Eq. (7) the Fisher information is the variance of the energy U, correspondingly the topological Fisher information gives the variance of E:

$$\mathcal{F}_E^{top} = (E - \langle E \rangle)^2. \tag{38}$$

The heat capacity analogous to Eq. (11) has the form

$$\mathcal{F}_E^{top} = \frac{C^{top}(\beta)}{\beta^2} = C^{top}(T)T^2. \tag{39}$$

Finally, according to Eq. (19) topological the Fisher heat capacity is defined as the derivative of the topological Fisher information $\mathcal{F}_E^{top}$ with respect to T:

$$\frac{\partial \mathcal{F}_E^{top}}{\partial T} = \mathcal{C}_F^{top}. \tag{40}$$

V. A SPECIAL CASE: THERMODYNAMICS OF EXPANDING MAPS

Consider now special classes of chaotic maps: expanding or hyperbolic maps. In this case [13] the probability of a cylinder is proportional to the power q of the length scale:

$$p_j^{(N)} \sim (l_j^{(N)})^q \tag{41}$$

First the relation between the Rényi dimension and the topological pressure is reviewed, then the link between the dimensional and the topological Fisher information is explored. Consider a multifractal and cover it by disjoint cells of variable size. Denote p_i the probability attributed to cell σ_i. The cells are completely covered by spherical balls. The smallest possible radius of the spherical ball covering σ_i is denoted by l_i, $l_i < l$. Define a generating function Γ [18] as

$$\Gamma(\beta, \tau) = \inf_{\{\sigma\}} \sum_{i=1}^{r} \frac{p_i^\beta}{l_i^\tau}, \qquad \text{if } \beta \leq 1,\ \tau \leq 0 \tag{42}$$

and

$$\Gamma(\beta, \tau) = \sup_{\{\sigma\}} \sum_{i=1}^{r} \frac{p_i^\beta}{l_i^\tau}, \qquad \text{if } \beta > 1,\ \tau > 0. \tag{43}$$

The infimum or supremum is obtained over all possible coverings. Select that value of τ for which $\Gamma(\beta, \tau)$ neither diverges nor goes to zero in the limit $l \to 0$. Then the Rényi dimension $D(\beta)$ is defined by

$$\tau(\beta) = (\beta - 1) D(\beta), \tag{44}$$

Now we select p_j the Gibbs measure

$$p_j = \frac{[l_j^{(N)}]^q}{\sum_{j'} [l_{j'}^{(N)}]^q} \tag{45}$$

and take into account that the infimum or supremum has been reached owing to the generating partition. Using the expression of the topological pressure (Eq. (22)) we obtain

$$p_j \sim [l_j^{(N)}]^q e^{-N\mathcal{P}(q)} \tag{46}$$

for large N. Consequently, Eqs. (42)- (43)) lead to

$$e^{-N\beta\mathcal{P}(q)} \sum_j [l_j^{(N)}]^{q\beta - \tau} = O(1) \tag{47}$$

if $N \to \infty$, that is,

$$\beta \mathcal{P}(q) - \mathcal{P}(q\beta - \tau) = 0. \tag{48}$$

The case $q = 1$ gives the relation

$$\beta \mathcal{P}(1) - \mathcal{P}(\beta - \tau) = 0. \tag{49}$$

Bearing in mind that $\mathcal{P}(1)$ gives the escape rate κ

$$\kappa = -\mathcal{P}(1), \tag{50}$$

from the definition (44) follows a relation between the escape rate [13], the topological pressure and the Rényi dimension:

$$\beta\kappa + \mathcal{P}[\beta - (\beta - 1)D(\beta)] = 0. \tag{51}$$

At the value $\beta = 0$, we obtain the Bowen-Ruelle expression

$$\mathcal{P}[D(0)] = 0, \tag{52}$$

that is, at the fractal dimension the topological pressure disappears. Differentiating Eq. (51) with respect to β we obtain

$$\mathcal{P}'[\beta - \tau]\left(1 - \frac{d\tau}{d\beta}\right) + \kappa = 0. \tag{53}$$

Selecting the value $\beta = 1$, we arrive at the Kantz-Grassberger relation

$$\lambda[1 - D(1)] = \kappa, \tag{54}$$

where

$$\lambda = -\mathcal{P}'(1) \tag{55}$$

is the Lyapunov exponent.

The starting point to establish a relation between the dimensional and topological Fisher information is equation (49). The second derivative with respect to β leads to

$$\mathcal{P}"[\beta - \tau]\left(1 - \frac{d\tau}{d\beta}\right)^2 - \mathcal{P}'[\beta - \tau]\frac{d^2\tau}{d\beta^2} = 0. \tag{56}$$

The dimensional Fisher information can then be immediatelly obtained

$$\mathcal{F}_{\alpha}^{dim} = -\frac{d^2\tau}{d\beta^2} = -\frac{\mathcal{P}"[\beta-\tau]\left(1-\frac{d\tau}{d\beta}\right)^2}{\mathcal{P}'[\beta-\tau]}. \tag{57}$$

Taking into account that the second derivative of the topological pressure is the topological Fisher information (Eq. (34)), the dimensional Fisher information is

$$\mathcal{F}_{\alpha}^{dim} = \frac{\mathcal{F}_{E}^{top}[\beta-\tau](1-\alpha)^2}{E[\beta-\tau]}, \tag{58}$$

where Eqs. (29) and (90) were utilised. Applying Eq. (53), Eq. (58) can also be written as

$$\mathcal{F}_{\alpha}^{dim} = \frac{\mathcal{F}_{E}^{top}[\beta-\tau]\kappa^2}{(E[\beta-\tau])^3}. \tag{59}$$

Expression (58) states that dimensional Fisher information is 'proportional' to the topological Fisher information taken at a shifted argument $\beta - \tau$ and the factor contains the first derivative of the topological pressure $(E[\beta-\tau]))$ taken at the same shifted argument and the escape rate κ. Keeping in mind that the factor incorporates the escape rate κ and that the first derivative of the topological pressure at the argument 1 gives the Lyapunov exponent (Eq. (55)), we might say that the factor includes certain knowledge concerning the expanding property of the map and sensitive dependence on initial condition.

The Legendre transforms of $\mathcal{F}_{\alpha}^{dim}$and $\mathcal{F}_{E}^{top}$

$$\mathcal{F}_{f}^{dim} = \alpha + \beta\mathcal{F}_{\alpha}^{dim}. \tag{60}$$

$$\mathcal{F}_{s}^{top} = E + \beta\mathcal{F}_{E}^{top} \tag{61}$$

can be related as

$$\mathcal{F}_{f}^{dim} = \alpha + \frac{\mathcal{F}_{s}^{top} - E}{(E[\beta-\tau])^3}\kappa^2. \tag{62}$$

We can also find the link between the Legendre transforms

$$\mathcal{F}_{F}^{dim} = -\frac{\alpha}{\beta} \tag{63}$$

and

$$\mathcal{F}_{F}^{top} = -\frac{E}{\beta}, \tag{64}$$

as

$$\frac{1}{\alpha}\mathcal{F}_{F}^{dim} = \frac{1}{E}\mathcal{F}_{F}^{top}. \tag{65}$$

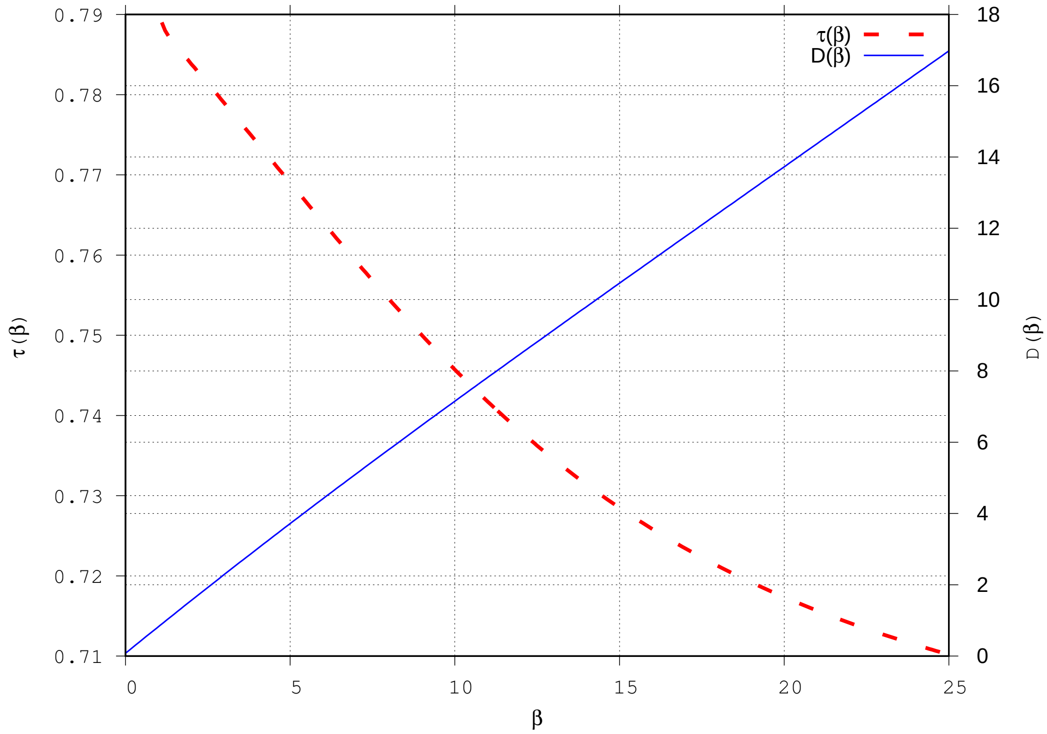

FIG. 1: (Color online) $\tau(\beta)$ (red dotted) and the Rényi dimension $D(\beta)$ (blue solid) as a function of β for the map (66) for the parameter values $p_0 = 0.25$ and $p_1 = 0.6$.

VI. ILLUSTRATIVE EXAMPLE AND DISCUSSION

We select the following map [19] as an illustration.

$$x_{n+1} = \begin{cases} \frac{x_n}{p_0}, & \text{for } 0 \le x_n, \le \frac{1}{2} \\ \frac{x_n - 1}{p_1} + 1, & \text{for } \frac{1}{2} \le x_n \le 1. \end{cases} \tag{66}$$

with $p_0, p_1 < \frac{1}{2}$ and $p_0 + p_1 < 1$. The points between p_0 and $1 - p_1$ leave the unit interval at each iteration. The set of points that never leave the unit interval is a Cantor set. The total length of the intervals containing the points that remain in the unit interval after n iterations:

$$l_n = (p_0 + p_1)^n = e^{n \ln(p_0 + p_1)} = e^{-n\kappa}, \tag{67}$$

where

$$\kappa = \ln\left(\frac{1}{p_0 + p_1}\right) \tag{68}$$

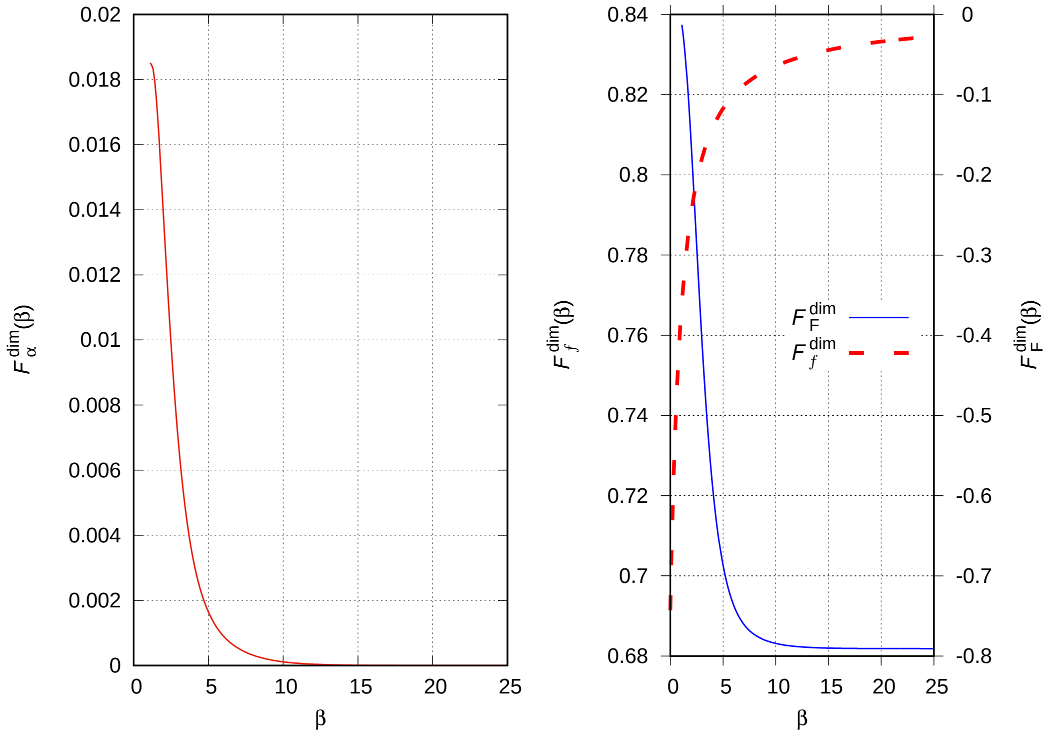

FIG. 2: (Color online) The dimensional Fisher information(left, red) $\mathcal{F}_{\alpha}^{dim}$ and the Legendre transforms $\mathcal{F}_{f}^{dim}$ (right,dotted red) and $\mathcal{F}_{F}^{dim}$ (right,blue solid) of the Fisher information as a function of β for the map (66) for the parameter values $p_0 = 0.25$ and $p_1 = 0.6$.

is the escape rate. The topological partition function and the topological pressure have the form

$$Z_n^{top}(\beta) = \sum_{j=1}^{2^n} l_j^{\beta} \tag{69}$$

and

$$\mathcal{P}(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln\left(Z_n^{top}(\beta)\right). \tag{70}$$

The summation results [19]

$$Z_n^{top}(\beta) = (p_0^{\beta} + p_1^{\beta})^n \tag{71}$$

and

$$\mathcal{P}(\beta) = \ln\left(p_0^{\beta} + p_1^{\beta}\right). \tag{72}$$

From Eqs. (49) and (50) we are led to

$$\ln\left(p_0^{\beta-\tau} + p_1^{\beta-\tau}\right) + \beta\kappa = 0 \tag{73}$$

or

$$p_0^{\beta-\tau} + p_1^{\beta-\tau} = e^{-\beta\kappa} \tag{74}$$

and $\kappa = -\mathcal{P}(1) = -\ln(p_0 + p_1)$. Eq. (74) can be solved numerically to obtain the function $\tau(\beta)$. Then the Rényi dimension $D(\beta) = \tau(\beta)/(\beta - 1)$ can be immediatelly computed. Fig. 1 presents $\tau(\beta)$ and $D(\beta)$ for $p_0 = 0.25$ and $p_1 = 0.6$. Differentiating Eq. (74) with respect to β we arrive at

$$\frac{d\tau}{d\beta} = 1 + \kappa \frac{p_0^{\beta-\tau} + p_1^{\beta-\tau}}{p_0^{\beta-\tau}\ln p_0 + p_1^{\beta-\tau}\ln p_1} = \alpha. \tag{75}$$

Another derivation with respect to β leads to the dimensional Fisher information:

$$\mathcal{F}_\alpha^{dim} = -\frac{d^2\tau(\beta)}{d\beta^2} = \frac{\kappa^2 e^{-\beta\kappa} - (1-\alpha)^2[p_0^{\beta-\tau}(\ln p_0)^2 + p_1^{\beta-\tau}(\ln p_1)^2}{p_0^{\beta-\tau}\ln p_0 + p_1^{\beta-\tau}\ln p_1}. \tag{76}$$

Fig. 2 presents the dimensional Fisher information $\mathcal{F}_\alpha^{dim}$ (left), the Legendre transforms (right) $\mathcal{F}_f^{dim} = \alpha + \beta\mathcal{F}_\alpha^{dim}$ and $\mathcal{F}_F^{dim} = \mathcal{F}_\alpha^{dim} - T\mathcal{F}_f^{dim}$.

Applying Eq. (29) we have

$$\frac{\partial\tau^{top}}{\partial\beta} = E = -\frac{\partial\mathcal{P}}{\partial\beta} = -\frac{p_0^\beta\ln p_0 + p_1^\beta\ln p_1}{p_0^\beta + p_1^\beta}. \tag{77}$$

The topological Fisher information (33) takes the form

$$\mathcal{F}_E^{top} = -\frac{d^2\tau^{top}(\beta)}{d\beta^2} = \frac{p_0^\beta(\ln p_0)^2 + p_1^\beta(\ln p_1)^2}{p_0^\beta + p_1^\beta} - \left(\frac{p_0^\beta\ln p_0 + p_1^\beta\ln p_1}{p_0^\beta + p_1^\beta}\right)^2. \tag{78}$$

The left curve in Fig. 3 presents the topological Fisher information $\mathcal{F}_E^{top}$ for the same p_0 and p_1. Making use of the first (Eq. (77)) and second (Eq. (78)) derivatives, the Legendre transforms of the topological Fisher information: $\mathcal{F}_s^{top} = E^{top} + \beta\mathcal{F}_E^{top}$ and $\mathcal{F}_F^{top} = \mathcal{F}_E^{top} - T\mathcal{F}_s^{top}$ were calculated (from Eqs. (35) and (37)) and are plotted also on Fig. 3. The relation between the dimensional and topological Fisher information is given by Eqs. (58) or (59). Figs. 2 and 3 (left curves) clearly show the similar behaviour of $\mathcal{F}_\alpha^{dim}$ and $\mathcal{F}_E^{top}$ as suggested by expressions (58) or (59). The link between the Legendre transforms $\mathcal{F}_f^{dim}$ and $\mathcal{F}_s^{top}$ is given by Eq.(62). As we can observe (right dotted curves in Figs. 2 and 3) $\mathcal{F}_f^{dim}$ is an increasing function of β, while $\mathcal{F}_s^{top}$ is decreasing because of other terms appearing in Eq. (62). On the other hand, the other Legendre transforms $\mathcal{F}_F^{dim}$ and $\mathcal{F}_F^{top}$ are simply proportional (see right blue curves in Figs. 2 and 3) as it can be seen in Eq. (65).

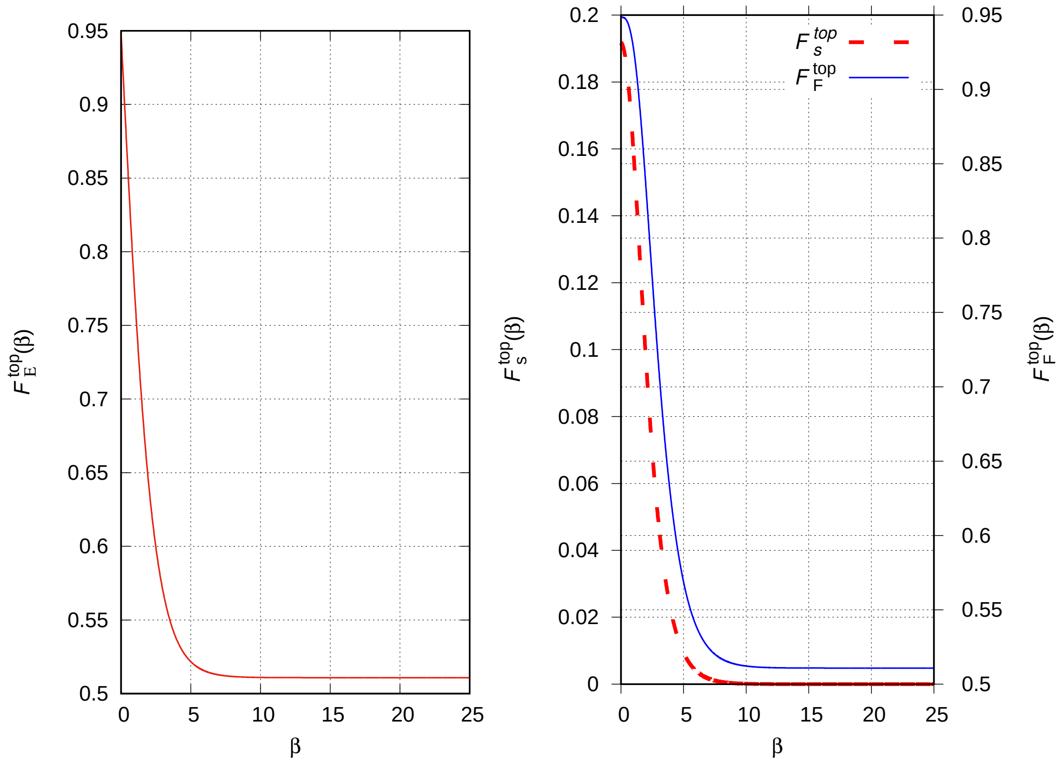

FIG. 3: (Color online) The topological Fisher information (left, red) $\mathcal{F}_E^{top}$ and the Legendre transforms $\mathcal{F}_s^{top}$ (right, red dotted) and $\mathcal{F}_F^{top}$ (right,blue solid) of the Fisher information as a function of β for the map (66) for the parameter values $p_0 = 0.25$ and $p_1 = 0.6$.

In summary, the link between the Fisher information and the topological pressure is explored. In the thermodynamic formalism the topological pressure has an important role. Utilizing the analogy between the statistical mechanics and the chaos theory the Fisher information is involved in the description of dynamical systems. The new concept of topological Fisher information is introduced. The relation between the topological pressure and the topological Fisher information is derived. In the special case of expanding maps, a relation between the topological and the dimensional Fisher information is formulated. Relations between the Legendre transforms of the topological and the dimensional Fisher information are also derived.

Appendix: Dimensional Fisher information

Recently, dimensional Fisher information has been introduced into the thermodynamics of chaotic systems [12]. A summary is now presented in order to reveal the link with the topological Fisher information. Suppose we have a multifractal and the phase spase is divided into boxes of side length ϵ. The partition function is defined as

$$Z^{dim}(\beta) = \sum_j e^{-\beta b_j} = \sum_j (p_j)^\beta, \tag{79}$$

where

$$b_i = -\ln p_i = -\alpha_i \ln \epsilon \tag{80}$$

and $\alpha_i(\epsilon)$ is the crowding index. The Massieu function takes the form

$$\Phi(\beta) = \ln (Z(\beta)) = \ln (\sum_i e^{-\beta b_i}). \tag{81}$$

The limit $\epsilon \to 0$ can be considered as a thermodynamic limit with defining the 'volume' [13]

$$V = -\ln \epsilon \tag{82}$$

and V going infinity. We can replace the sum in Eq. (81) by an integral,

$$\Phi(\beta) = \ln \int_{\alpha_{min}}^{\alpha_{max}} d\alpha \gamma(\alpha) e^{-\beta\alpha V}, \tag{83}$$

where $\gamma(\alpha)$ is the number of boxes with the crowing index α in the range between α and $\alpha + d\alpha$. Asymptotically, $\gamma(\alpha)$ behaves as

$$\gamma(\alpha) \sim \epsilon^{-f(\alpha)} \tag{84}$$

with $f(\alpha)$ being the so-called spectrum of singularities. According to the saddle point method the maximum value dominates the integrand

$$\Phi \sim [f(\alpha) - \beta\alpha]V, \tag{85}$$

therefore

$$\frac{\partial f(\alpha)}{\partial \alpha} = \beta. \tag{86}$$

Comparing Eqs. (85) and (12) one can observe that $f(\alpha)$ corresponds to the entropy density S/V

$$\lim_{V\to\infty} \frac{S}{V} = f(\alpha) \tag{87}$$

in the thermodynamic limit. On the other hand, for the Legendre transform we obtain

$$\lim_{V\to\infty}\frac{\Phi}{V} = -\tau(\beta) \tag{88}$$

with

$$\tau(\beta) = \beta\alpha - f(\alpha) \tag{89}$$

and

$$\frac{\partial\tau}{\partial\beta} = \alpha. \tag{90}$$

Instead of the function $\tau(\beta)$ the Rényi dimension $D(\beta)$ is often used:

$$\tau(\beta) = (\beta-1)D(\beta), \tag{91}$$

with

$$D(\beta) = \lim_{V\to\infty}\frac{R^{(\beta)}}{V} = \lim_{V\to\infty}\frac{1}{V}\frac{1}{1-\beta}\ln\left(Z(\beta)\right). \tag{92}$$

To obtain the dimensional Fisher information first take into account the resemblance (Eqs. (6), (12), (89)- (90)):

$$\tau \sim -\Phi \tag{93}$$

and

$$\alpha \sim U. \tag{94}$$

The dimensional Fisher information is defined as

$$\mathcal{F}_{\alpha}^{dim} = -\frac{d^2\tau(\beta)}{d\beta^2} = -\frac{\partial\alpha}{\partial\beta}. \tag{95}$$

The Legendre transformation akin to $\mathcal{F}_{\mathcal{S}}$ in Eq. (15) can be written

$$\mathcal{F}_{f}^{dim} = \alpha - \beta\frac{d\alpha}{d\beta} = \alpha + \beta\mathcal{F}_{\alpha}^{dim} \tag{96}$$

with the fundamental relation

$$\frac{\partial\mathcal{F}_{f}^{dim}}{\partial\mathcal{F}_{\alpha}^{dim}} = \beta. \tag{97}$$

On the other hand, the free energy of the usual thermodynamic harmonizes with the Legendre transform

$$\mathcal{F}_F^{dim} = \mathcal{F}_\alpha^{dim} - T\mathcal{F}_f^{dim}. \tag{98}$$

The dimensional Fisher information can be interpreted as the variance of α (see Eq. (7))

$$\mathcal{F}_\alpha^{dim} = (\alpha - \langle\alpha\rangle)^2. \tag{99}$$

The dimensional heat capacity can be introduced as the analogue of the heat capacity (Eq. (11))

$$\mathcal{F}_\alpha^{dim} = \frac{C^{dim}(\beta)}{\beta^2} = C^{dim}(T)T^2. \tag{100}$$

Finally, the dimensional Fisher heat capacity can also be defined as the derivative of the dimensional Fisher information $\mathcal{F}_\alpha^{dim}$ with respect to T:

$$\frac{\partial\mathcal{F}_\alpha^{dim}}{\partial T} = \mathcal{C}_F^{dim}. \tag{101}$$

(compare with Eq.(19)).

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