Analytical solutions for the radial Scarf II potential

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Abstract. The real Scarf II potential is discussed as a radial problem. This potential has been studied extensively as a one-dimensional problem, and now these results are used to construct its bound and resonance solutions for $l = 0$ by setting the origin at some arbitrary value of the coordinate. The solutions with appropriate boundary conditions are composed as the linear combination of the two independent solutions of the Schrödinger equation. The asymptotic expression of these solutions is used to construct the $S_0(k)$ s-wave S-matrix, the poles of which supply the k values corresponding to the bound, resonance and anti-bound solutions. The location of the discrete energy eigenvalues is analyzed, and the relation of the solutions of the radial and one-dimensional Scarf II potentials is discussed. Analogies with the generalized Woods–Saxon and the Rosen–Morse II potential are pointed out.

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1. Introduction

Exactly solvable quantum mechanical potentials proved to be invaluable tools in the understanding of many fundamental quantum mechanical concepts. In particular, they give insight into complex phenomena, like the symmetries of quantum mechanical systems, and they allow the investigation of transitions through critical parameter domains. Besides this, analytical solutions serve as a firm basis for the development of numerical techniques.

The one-dimensional Schrödinger equation

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{1}$$

occurs in many applications. Here the potential function and the energy eigenvalue are defined such that they contain reduced mass m and $\hbar$ as $V(x) = 2mv(x)/\hbar^2$ and $E = 2m\epsilon/\hbar^2$, so their physical dimension is distance^{-2}. In the simplest case (1) is defined on the full x axis, i.e. $x \in (-\infty, \infty)$, while for spherical potentials defined in higher, typically three dimension, Eq. (1) can be obtained after the separation of the angular variables, if only the s-wave ($l = 0$) solutions are considered. In this case the problem is defined on the positive half axis, $r \in [0, \infty)$, and the x variable is denoted by r. Besides these options, (1) can also be defined on finite sections of the real x axis, or even on more complicated trajectories of the complex x plane, but we shall not consider these in the present work.

Being a second-order ordinary differential equation, (1) has two independent solutions, and the physical solutions can be obtained as linear combination of these, satisfying the appropriate boundary conditions. Due to normalizability, bound states have to vanish at the boundaries (i.e. $x = \pm\infty$ in one dimension, and $r = 0$ and $r = \infty$ in the radial case). Unbound solutions, e.g. scattering and resonance solutions also have to satisfy asymptotic boundary conditions, depending on the nature of the potential. If $V(x)$ vanishes exactly or exponentially for $x \to \pm\infty$, then these solutions of the one-dimensional problem have exponential asymptotic components $\exp(\pm \mathrm{i}kx)$, where $E = k^2$. In the radial case the same asymptotics are valid for $r \to \infty$, while for $r = 0$ these solutions have to vanish.

There are some potentials that are defined both as one-dimensional and as radial problems, e.g. the harmonic oscillator. The bound-state solutions of these two problems are related to each other in a special way: the odd wave functions of the one-dimensional potential, which vanish at $x = 0$, are identical for $x \geq 0$ to the s-wave ($l = 0$) radial wave functions, and the energy eigenvalues are also identical.

A rather effective way for the unified discussion of bound, scattering and resonance solutions is the application of the transmission coefficient $T(k)$ (in one dimension) and the s-wave S matrix $S_0(k)$ (in the radial case). These quantities can be constructed from the asymptotic solutions, and their poles correspond to the bound, anti-bound and resonance states. From the exact solutions of these problems $T(k)$ and $S_0(k)$ can also be expressed in closed analytic form.

Here we dicuss the Scarf II potential as a radial problem. This potential has two independent terms and belongs to the shape-invariant [1] subclass of the Natanzon potential class [2], which contains problems with bound-state solutions written in terms of a single hypergeometric function. The first reference to potential (2) in the English literature occurred in 1983 in Ref. [1], so it is sometimes referred to as the Gendenshtein potential. However, it was already mentioned a year before in a Russian monograph [3]. Its detailed description was presented later, e.g. the normalization coefficients of its bound-state solutions have been calculated only recently [4]. The transmission and reflection coefficients have been given in Ref. [5], with corrections added in Ref. [6]. It has been a favourite toy model in $\mathcal{PT}$-symmetric quantum mechanics, where it was used to demonstrate the breakdown of $\mathcal{PT}$ symmetry [7, 8]. Further studies concerned its algebraic [9, 6] and scattering aspects [6], the combined effects of SUSYQM and $\mathcal{PT}$ symmetry [10], the pseudo-norm of its bound states [4], the handedness (chirality) effects in scattering [11], spectral singularities [12], unidirectional invisibility [13] and the accidental crossing of its energy levels [14].

Despite its prominent status as a one-dimensional quantum system, the Scarf II potential has not been considered yet as a radial problem. Here we fill this gap by introducing a lower cut at a certain $x = r_0$ value and prescribing the appropriate boundary conditions. We construct the S-matrix for the s-wave solutions, $S_0(k)$, and determine its poles on the complex k plane to identify its bound, anti-bound and resonance solutions. This will be done in Sec. 3, following the discussion of the one-dimensional problem for reference in Sec. 2 In Sec. 4 the analogy with the case of the generalized Woods–Saxon and the Rosen–Morse II potentials will be outlined, finally the results are summarized in Sec. 5.

2. The Scarf II potential in one dimension

A possible parametrization of this potential is [10]

$$V(x) = -\frac{V_1}{\cosh^2(cx)} + \frac{V_2 \sinh(cx)}{\cosh^2(cx)} , \tag{2}$$

where

$$V_1 = c^2 \left(\frac{\alpha^2 + \beta^2}{2} - \frac{1}{4} \right) \qquad\qquad V_2 = \mathrm{i}c^2 \frac{\beta^2 - \alpha^2}{2} . \tag{3}$$

This potential is real if $\alpha^* = \beta$ holds, while it is $\mathcal{PT}$-symmetric if α and β are real or imaginary. In what follows we consider the real version only. Potential (2) is depicted in Fig. 1 for some values of the parameters. It has a minimum x_- and a maximum x_+ at

$$x_{\pm} = c^{-1} \sinh^{-1} \left[\frac{V_1}{V_2} \pm \left[\left(\frac{V_1}{V_2} \right)^2 + 1 \right]^{1/2} \right] . \tag{4}$$

The potential reflected by $x = 0$ can be constructed easily by considering $V_2 \to -V_2$, i.e. $\alpha \leftrightarrow \beta$.

The bound-state wave functions are

$$\psi_n(x) = C_n(1 - \mathrm{i}\sinh(cx))^{\frac{\alpha}{2}+\frac{1}{4}}(1 + \mathrm{i}\sinh(cx))^{\frac{\beta}{2}+\frac{1}{4}} P_n^{(\alpha,\beta)}(\mathrm{i}\sinh(cx)) \ , \quad (5)$$

while the corresponding energy eigenvalues are written as

$$E_n = -c^2\left(n + \frac{\alpha+\beta+1}{2}\right)^2 \ . \quad (6)$$

Normalizability of (5) requires

$$n < -\frac{1}{2}[\mathrm{Re}(\alpha+\beta)+1] \ . \quad (7)$$

C_n in (5) was calculated for the real and the $\mathcal{PT}$-symmetric version of the Scarf II potential in Ref. [4]. In the former case it can be written as

$$C_n = 2^{-\frac{\alpha+\beta}{2}-1}\left[c\frac{\Gamma(-\alpha-n)\Gamma(-\beta-n)(-\alpha-\beta-2n-1)n!}{\Gamma(-\alpha-\beta-n)\pi}\right]^{1/2} \ . \quad (8)$$

Note that although α and β are complex, C_n is real due to $\alpha = \beta^*$ and Eq. (7). It can also be proven that the bound-state wave functions (5) are real for even n, and imaginary for odd n: this can be demonstrated by expressing the complex conjugate of (5), which turns out to be $[\psi_n(x)]^* = (-1)^n\psi_n(x)$, due to Eq. 22.4.1 of Ref. [15].

With a reparametrization, the notation of Ref. [6] can be obtained, in which the scattering aspects of the one-dimensional Scarf II potential have been discussed. Taking

$$\alpha = -s - \frac{1}{2} - \mathrm{i}\lambda \ , \qquad \beta = -s - \frac{1}{2} + \mathrm{i}\lambda \quad (9)$$

one obtains

$$V_1 = c^2[s(s+1) - \lambda^2] \qquad V_2 = c^2(2s+1)\lambda \quad (10)$$

in Eq. (2). According to (9), the Scarf II potential will be real for real values of s and λ. Note that this potential remains invariant if the signs of $s + 1/2$ and λ are reversed simutaneously. This means that without the loss of generality one can require $s > -1/2$. Condition (7) is now $n < \mathrm{Re}(s) = s$, so in order to obtain normalizable states one needs $s > 0$. It is notable that E_n in (6) depends only on $s = -(\alpha+\beta+1)/2$, and is independent of $\lambda = \mathrm{i}(\alpha-\beta)/2$.

In Ref. [6] the general solutions of the Schrödinger equation with the potential (2) and (10) are expressed in terms of hypergeometric functions as

$$\begin{aligned} F_1(x) \quad &= (1 - \mathrm{i}\sinh(cx))^{-\frac{s+\mathrm{i}\lambda}{2}}(1 + \mathrm{i}\sinh(cx))^{-\frac{s-\mathrm{i}\lambda}{2}} \\ &\times {}_2F_1(-s - \mathrm{i}k, -s + \mathrm{i}k; \mathrm{i}\lambda - s + 1/2; (1 + \mathrm{i}\sinh(cx))/2) \end{aligned} \quad (11)$$

and

$$\begin{aligned} F_2(x) \quad &= A(1 - \mathrm{i}\sinh(cx))^{-\frac{s+\mathrm{i}\lambda}{2}}(1 + \mathrm{i}\sinh(cx))^{\frac{s+1-\mathrm{i}\lambda}{2}} \\ &\times {}_2F_1(1/2 - \mathrm{i}\lambda - \mathrm{i}k, 1/2 - \mathrm{i}\lambda + \mathrm{i}k; s + 3/2 - \mathrm{i}\lambda; (1 + \mathrm{i}\sinh(cx))/2) \ . \end{aligned} \quad (12)$$

Note that (12) is obtained from (11) by Eq. 15.5.4 of Ref. [15], where $A = 2^{\mathrm{i}\lambda-s-1/2}$.

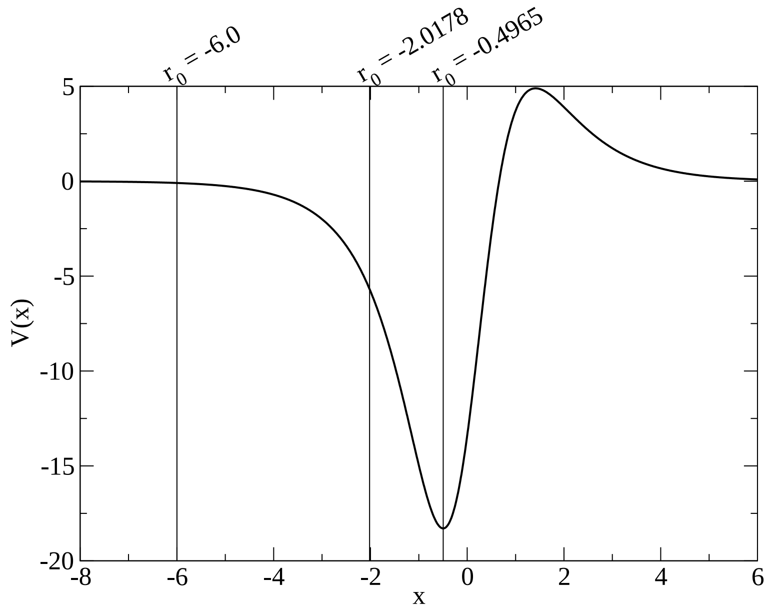

Figure 1. The structure of the one-dimensional Scarf II potential for $V_1 = 13.4$ and $V_2 = 18.92$ corresponding to $c = 1$, $\alpha = -4.3 - 2.2\mathrm{i} = \beta^*$, $s = 3.8$ and $\lambda = 2.2$. This potential has four bound states at $E_0 = -14.44$, $E_1 = -7.84$, $E_2 = -3.24$ and $E_3 = -0.64$. The vertical lines at $x = r_0 = -0.4965$, -2.017825 (abbreviated in the plot) and -6.0 define three radial potentials (see Sec. 3) with different location of the origin. In the first case $r_0 = x_-$, while the second r_0 corresponds to the first node of $\psi_2(x)$ in (5).

The asymptotic behavior of (11) and (12) can be obtained by applying 15.3.4, 15.3.5 and 15.3.6 of Ref. [15], and the results are

$$\lim_{x\to\infty} F_1(x) = a_{1+}\exp(\mathrm{i}kx) + b_{1+}\exp(-\mathrm{i}kx) \tag{13}$$
$$\lim_{x\to-\infty} F_1(x) = a_{1-}\exp(\mathrm{i}kx) + b_{1-}\exp(-\mathrm{i}kx) \tag{14}$$
$$\lim_{x\to\infty} F_2(x) = a_{2+}\exp(\mathrm{i}kx) + b_{2+}\exp(-\mathrm{i}kx) \tag{15}$$
$$\lim_{x\to-\infty} F_2(x) = a_{2-}\exp(\mathrm{i}kx) + b_{2-}\exp(-\mathrm{i}kx)\ , \tag{16}$$

where

$$a_{1+} = D_1 2^{-s-2\mathrm{i}k/c}\mathrm{e}^{\pi(k/c-\lambda-\mathrm{i}s)/2} \qquad b_{1+} = C_1 2^{-s+2\mathrm{i}k/c}\mathrm{e}^{\pi(-k/c-\lambda-\mathrm{i}s)/2} \tag{17}$$
$$a_{1-} = b_{1+}\mathrm{e}^{\pi(k/c+\lambda+\mathrm{i}s)} \qquad b_{1-} = a_{1+}\mathrm{e}^{\pi(-k/c+\lambda+\mathrm{i}s)} \tag{18}$$
$$a_{2+} = D_2 2^{-s-2\mathrm{i}k/c}\mathrm{e}^{\pi(k/c+\lambda+\mathrm{i}(s+1))/2} \qquad b_{2+} = C_2 2^{-s+2\mathrm{i}k/c}\mathrm{e}^{\pi(-k/c+\lambda+\mathrm{i}(s+1))/2} \tag{19}$$
$$a_{2-} = b_{2+}\mathrm{e}^{\pi(k/c-\lambda-\mathrm{i}(s+1))} \qquad b_{2-} = a_{2+}\mathrm{e}^{\pi(-k/c-\lambda-\mathrm{i}(s+1))}\ , \tag{20}$$

and

$$C_1 = \frac{\Gamma(\mathrm{i}\lambda + 1/2 - s)\Gamma(-2\mathrm{i}k/c)}{\Gamma(\mathrm{i}\lambda + 1/2 - \mathrm{i}k/c)\Gamma(-s - \mathrm{i}k/c)} \qquad D_1 = \frac{\Gamma(\mathrm{i}\lambda + 1/2 - s)\Gamma(2\mathrm{i}k/c)}{\Gamma(\mathrm{i}\lambda + 1/2 + \mathrm{i}k/c)\Gamma(-s + \mathrm{i}k/c)}$$

$$(21)$$

$$C_2 = \frac{\Gamma(-\mathrm{i}\lambda + 3/2 + s)\Gamma(-2\mathrm{i}k/c)}{\Gamma(s+1-\mathrm{i}k/c)\Gamma(-\mathrm{i}\lambda + 1/2 - \mathrm{i}k/c)} \quad D_2 = \frac{\Gamma(-\mathrm{i}\lambda + 3/2 + s)\Gamma(2\mathrm{i}k/c)}{\Gamma(s+1+\mathrm{i}k/c)\Gamma(-\mathrm{i}\lambda + 1/2 + \mathrm{i}k/c)} \tag{22}$$

For a wave traveling to the right the transmission and reflection coefficients are expressed as [5, 6]

$$\begin{aligned} T(k) &= \frac{a_{1+}b_{2+} - b_{1+}a_{2+}}{a_{1-}b_{2+} - b_{1+}a_{2-}} \\ &= \frac{\Gamma(-s-\mathrm{i}k/c)\Gamma(s+1-\mathrm{i}k/c)\Gamma(\mathrm{i}\lambda+1/2-\mathrm{i}k/c)\Gamma(-\mathrm{i}\lambda+1/2-\mathrm{i}k/c)}{\Gamma(-\mathrm{i}k/c)\Gamma(1-\mathrm{i}k/c)\Gamma^2(1/2-\mathrm{i}k/c)} \end{aligned} \tag{23}$$

$$\begin{aligned} R(k) &= \frac{b_{1-}b_{2+} - b_{1+}b_{2-}}{a_{1-}b_{2+} - b_{1+}a_{2-}} \\ &= T(k)\left(\frac{\cos(\pi s)\sinh(\pi\lambda)}{\cosh(\pi k/c)} + \mathrm{i}\frac{\sin(\pi s)\cosh(\pi\lambda)}{\sinh(\pi k/c)}\right) . \end{aligned} \tag{24}$$

The poles of $T(k)$ are located at $-n = -s - \mathrm{i}k/c$, $-n = s + 1 - \mathrm{i}k/c$, $-n = -\mathrm{i}\lambda + 1/2 - \mathrm{i}k/c$ and $-n = \mathrm{i}\lambda + 1/2 - \mathrm{i}k/c$. The first choice corresponds to the energy eigenvalues

$$E_n = -c^2(s-n)^2 \tag{25}$$

in accordance with (7), and converts $F_1(x)$ in (11) into (5) (up to the constant factor $(-1)^n n!\Gamma(\beta+1)[\Gamma(\beta+n+1)C_n]^{-1}$) after applying Eqs. 15.3.6 and 22.5.42 of Ref. [15]. The second one stands for anti-bound or virtual states with k located on the negative imaginary axis, while the last two poles correspond to non-normalizable complex-energy states, i.e. resonances with $E_n = k^2 = -c^2(n - 1/2 \pm \mathrm{i}\lambda)^2$.

3. The Scarf II potential as a radial problem

In this case the general wave function is constructed from the linear combination of the two independent solutions (11) and (12) with boundary condition that it should vanish at the origin. The position of the origin need not be chosen at $x = 0$, rather one can cut the one-dimensional potential (2) at an arbitrary finite value. Let us thus define $x = r + r_0$, where $r \in [0, \infty)$, i.e. $x \in [r_0, \infty)$. Figure 1 displays three possible radial Scarf II potential with origin corresponding to various values of $x = r_0$.

The general solution

$$\psi(r) = F_1(x) + CF_2(x) \tag{26}$$

should vanish at $x = r_0$, which defines the constant C as

$$C = -F_1(r_0)/F_2(r_0) . \tag{27}$$

The asymptotic behavior of the solution has to be inspected only for $r \to \infty$, and the S-matrix for $l = 0$ can be obtained from

$$\lim_{r\to\infty} \psi(r) = \exp(-\mathrm{i}kr) - S_0(k)\exp(\mathrm{i}kr) \ . \tag{28}$$

Making use of Eqs. (13) and (15) of the one-dimensional problem the S-matrix can be constructed as

$$S_0(k) = -\frac{a_{1+} + Ca_{2+}}{b_{1+} + Cb_{2+}} \tag{29}$$

After some straightforward algebra one obtains

$$\begin{aligned} S_0(k) = \quad & -2^{-4\mathrm{i}k/c}\exp(\pi k/c)\frac{\Gamma(2\mathrm{i}k/c)}{\Gamma(-2\mathrm{i}k/c)} \\ & \times \left[\frac{\Gamma(\mathrm{i}\lambda+1/2-s)}{\Gamma(\mathrm{i}\lambda+1/2+\mathrm{i}k/c)\Gamma(-s+\mathrm{i}k/c)} + \mathrm{i}C\frac{\Gamma(-\mathrm{i}\lambda+3/2+s)\exp(\pi(\lambda+\mathrm{i}s))}{\Gamma(s+1+\mathrm{i}k/c)\Gamma(-\mathrm{i}\lambda+1/2+\mathrm{i}k/c)}\right] \\ & \times \left[\frac{\Gamma(\mathrm{i}\lambda+1/2-s)}{\Gamma(\mathrm{i}\lambda+1/2-\mathrm{i}k/c)\Gamma(-s-\mathrm{i}k/c)} + \mathrm{i}C\frac{\Gamma(-\mathrm{i}\lambda+3/2+s)\exp(\pi(\lambda+\mathrm{i}s))}{\Gamma(s+1-\mathrm{i}k/c)\Gamma(-\mathrm{i}\lambda+1/2-\mathrm{i}k/c)}\right]^{-1} \end{aligned} \tag{30}$$

where

$$C = -\frac{(1+\mathrm{i}\sinh(cr_0))^{-s+\mathrm{i}\lambda-1/2}\ {}_2F_1(-s-\mathrm{i}k/c, -s+\mathrm{i}k; \mathrm{i}\lambda-s+1/2; (1+\mathrm{i}\sinh(cr_0))/2)}{A_2F_1(1/2-\mathrm{i}\lambda-\mathrm{i}k/c, 1/2-\mathrm{i}\lambda+\mathrm{i}k/c; s+3/2-\mathrm{i}\lambda; (1+\mathrm{i}\sinh(cr_0))/2)} \tag{31}$$

The S-matrix of Ref. [16] is recovered in the special case of $c = 1$, $\lambda = 0$ and $r_0 = 0$. In that case the radial wave functions are obtained from the odd-n solutions of the one-dimensional problem that vanish at the origin.

The poles of the S-matrix displayed for the various r_0 used in Fig. 1 are shown in Fig. 2.

The solutions of the one-dimensional and the radial problems can be related to each other by various ways. First, if r_0 is defined to be at a node of a particular wave function $\psi_n(x)$ of the one-dimensional problem, then Eq. (26) implies that $\psi_n(r_0) = 0$ can occur only for $C = 0$, i.e. the solution of the radial problem will be the corresponding solution of the one-dimensional problem, defined for $x \geq r_0$. Furthermore, the energy eigenvalues (and k) will also be the same. This scenario is illustrated by the E_3 excited state of the one-dimensional problem in Table 1: $r_0 = -2.017825$ coincides with the first of the three nodes of $\psi_3(x)$ in (5), so this function will also act as the second excited state ($n = 2$) of the radial problem with the same energy eigenvalue ($E_2 = -0.64$), since it has two more nodes. The unnormalized bound-state wave functions of this potential are displayed in Fig. 3. Obviously, the remaining solutions of the radial problem cannot be calculated in the same way. The situation is analogous to the case of the harmonic oscillator, where some solutions of the radial problem can be generated from those of the one-dimensional problem.

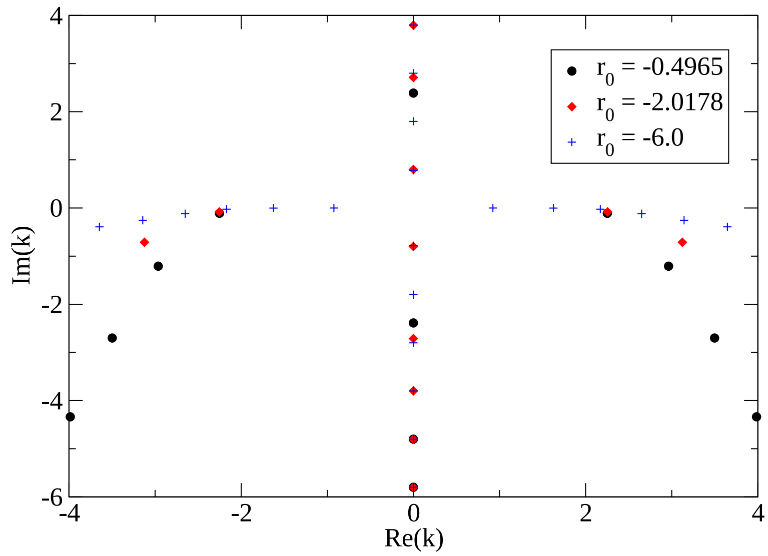

Figure 2. The poles of $S_0(k)$ for the parameters used in Fig. 1. Poles located on the positive and negative imaginary axis correspond to bound and anti-bound states, quadrant

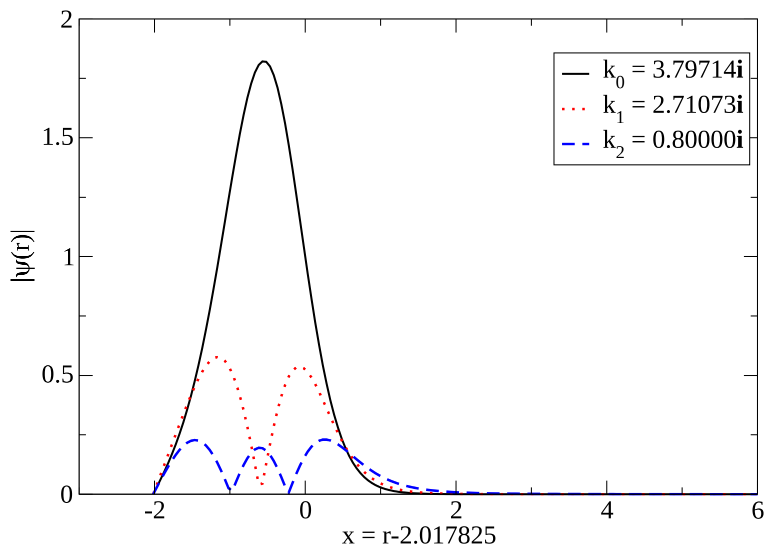

Figure 3. Modulus of the unnormalized $l = 0$ bound-state wave functions of the radial Scarf II potential for $r_0 = -2.017825$. The remaining parameters are the same as those used in Fig. 1. The second ($n = 2$) excited wave function with two nodes coincides with the wave function of the one-dimensional problem belonging to the same energy eigenvalue, -0.64 (see Table 1). For the normalization of the wave functions, integrals containing the product of hypergeometric functions defined on $x \in [r_0, \infty)$ would have to be calculated using numerical methods.

Another relation follows in situations when r_0 is defined at a large enough negative value, where the bound-state wave functions of the one-dimensional problem are close to zero. In this case the boundary condition at r_0 implies that the second term of (26) should also be small in magnitude (C will be small), so the solution will be dominated by $F_1(x)$, i.e. the bound-state solution of the one-dimensional problem. This also means that the energy eigenvalues of the radial problem will also be close to those of the one-dimensional problem. A simple test for some parameters is displayed in Table 1. It is seen that the energy eigenvalues match reasonably well, and the agreement gets better with $r_0 \to -\infty$ and for lower values of the n quantum number, i.e. in situations when the magnitude of $\psi_n(r_0)$ is smaller.

Table 1. Bound-state energies of the one-dimensional Scarf II potential and those of the radial one defined with various r_0 in Fig. 1.

	1D case	$r_0 = -0.4965$	$r_0 = -2.017825$	$r_0 = -6.0$
E_0	-14.44	-5.69802	-14.41825	-14.44000
E_1	-7.84	–	-7.34803	-7.84000
E_2	-3.24	–	-0.64000	-3.23998
E_3	-0.64	–	–	-0.61366

Considering that the resonance solutions do not vanish asymptotically, the same argumentation cannot be applied to them. Consequently, the resonance energies of the one-dimensional and the radial Scarf II potential differ from each other significantly.

Possible applications of the radial Scarf II potential can be envisaged in nuclear physics, for example. The barrier in Fig. 1 can simulate the effects of the Coulomb barrier that occurs when charged particles (e.g. protons or α-particles) interact with a nucleus. The situation is qualitatively similar to the case of the generalized Woods–Saxon potential, which has a similar barrier, and the difference occurs within the nucleus, where the latter potential is constant, while the radial Scarf II potential has a clear minimum, and depending on r_0 it can increase close to the origin.

It is also possible to shift the barrier *inside* the nucleus by formally reflecting the potential curve in Fig. 1 about $x = 0$ and defining the origin near the potential maximum. Staying with the original formalism, this corresponds to considering $V_2 < 0$, i.e. taking $\alpha \leftrightarrow \beta$ in (3) or $\lambda \to -\lambda$ in (10). Potentials with such shape occur in hypernuclei, where the interaction of Λ particles with α particles or nucleons requires the presence of a soft repulsive core with variable heigth [17].

Finally, the formalism can be extended to the case of optical potentials with complex values of V_1 and V_2: the calculation of the wave functions, S-matrix, energy eigenvalues, including that of the resonances will be the same.

4. Relation to the generalized Woods–Saxon potential

It can be noted that the radial Scarf II potential can be brought to a form that is close to the notation of the generalized Woods–Saxon potential. Applying the variable transformation

$$x = \frac{r-R}{2a} \tag{32}$$

in (2) with $c = (2a)^{-1}$, the equivalent form

$$\begin{aligned} V(x) = & -4V_1 \frac{\exp((r-R)/a)}{[1+\exp((r-R)/a)]^2} \\ & + 2V_2 \left[\frac{\exp((r-R)/(2a))}{1+\exp((r-R)/a)} - 2\frac{\exp((r-R)/(2a))}{[1+\exp((r-R)/a)]^2} \right] \end{aligned} \tag{33}$$

is obtained. r_0 corresponds to $-R$, so the radial version of the potential can be obtained by defining the origin at the negative value of $r_0 = -R$.

In fact, the same variable transformation relates the generalized Woods–Saxon potential with the Rosen–Morse II potential [18]

$$V(x) = -\frac{U_1}{\cosh^2(cx)} + U_2 \tanh(cx) \ . \tag{34}$$

After applying the (32) transformation one obtains

$$V(r) = -4U_1 \frac{\exp((r-R)/(a))}{[1+\exp((r-R)/a)]^2} - U_2 \frac{2}{1+\exp((r-R)/a)} + U_2 \ , \tag{35}$$

which is the Woods–Saxon potential with a shifted energy scale. Note that the first terms of (33) and (35) are the same.

It should be noted that the Rosen–Morse II potential is defined on the full real x axis, so in order to obtain the Woods–Saxon potential from it, one should consider $r \in [0, \infty)$ in (32). This means that the two solutions have to be matched at $r = 0$ in a way similar to that considered previously for the Scarf II potential. In fact, the procedure applied there is the same as that described in the notable work of Bencze [19], where the S matrix of the Woods–Saxon potential was constructed in an analytical form. See also Ref. [20] as a more accessible source of the formulas.

Note that in the one-dimensional Rosen–Morse II potential the normalizable states are expressed in terms of only one of the two independent solutions (similarly to the case of the one-dimensional Scarf II potential), and they exist only for $U_1 > 0$ [18]. In contrast with this, in the radial problem, this "surface" term usually plays the role of a barrier, i.e. $U_1 < 0$, and the attractive component of the potential is represented by the "volume" term with $U_2 > 0$. The relation between the Rosen–Morse II and the generalized Woods–Saxon potential has been pointed out in Ref. [21], where analytical expressions were given for the $l = 0$ bound-state wave functions and the corresponding energy eigenvalues. However, those wave functions do not vanish at the origin (their structure is similar to the wave functions of the one-dimensional Rosen–Morse II potential), so that approach can be considered as an approximation only.

5. Summary

Based on the results of the one-dimensional Scarf II potential, the radial version of this potential was studied. For this, the origin was defined at an arbitrary value on the real

x axis, and the s-wave solutions were constructed from the two independent solutions of the one-dimensional Schrödinger equation, after prescribing the appropriate boundary conditions. The asymptotic form of these solutions was used to construct the $S_0(k)$ S-matrix. The poles of $S_0(k)$ were located, and were identified with the bound, anti-bound and resonance solutions.

It was shown that by selecting the origin far enough from the potential minimum, the bound-state energy eigenvalues and wave functions of the radial potential tended to those of the one-dimensional potential. Furthermore, selecting the origin at the node of some bound-state wave function of the one-dimensional potential, this wave function appeared as a bound-state wave function of the radial potential with the same energy eigenvalue.

With a slightly modified parametrization, the radial Scarf II potential could be compared with the generalized Woods–Saxon potential, and it was shown that they share a term (the "surface" term of the latter potential). In fact, it was demonstrated that the radial Scarf II potential can be generated from the one-dimensional Scarf II potential in the same way as the generalized Woods–Saxon potential is generated from the one-dimensional Rosen–Morse II potential. The connection between the latter two potentials has been known before [21], however, the bound-state wave functions generated from this connection did not satisfy the appropriate boundary conditions.

Based on its similarity with the generalized Woods–Saxon potential, the radial Scarf II potential could be applied in nuclear physics, for example. One possibility is considering problems, which are characterized by a barrier at the surface of the nucleus, but in which the flat potential inside the nucleus is replaced with a potential well with a clear minimum. Another option is placing the barrier inside the nucleus near the origin, simulating a repulsive interaction there. The formalism can be extended to the case of complex values of V_1 and V_2, i.e. to optical potentials.

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