ON THE DIOPHANTINE EQUATION $1 + x^a + z^b = y^n$

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ABSTRACT. Several classical problems are related to mixed polynomial-exponential equations. Such equations have been also considered recently by many authors. In the present paper, extending a theorem of the second and last authors, we completely solve the title equation in positive integers a, b, y, n with $n \geq 4$ for all values of x, z with $1 \leq x, z \leq 50$ and $x \not\equiv z \pmod 2$. It is interesting to note that apparently deep effective tools (e.g. Baker's method) alone are not sufficient to handle the problem completely. In our arguments we combine local arguments and Baker's method to prove our results.

1. INTRODUCTION

Diophantine equations of mixed polynomial-exponential type are of classical and recent interest. Here we only give a brief introduction on them; for a more detailed description of history and results see e.g. [7] and the references there.

One of the classical examples of such equations is the Ramanujan-Nagell equation (see Ramanujan [11] and Nagell [10]). A problem of recent interest, leading to such type of equations, is to describe perfect powers having only few digits written in some bases. See for example the papers of Szalay [13], Bennett, Bugeaud and Mignotte [2, 3], Corvaja and Zannier [5] and Bennett and Bugeaud [1] and the references there. Among other things, in these papers diophantine equations of type

$$1 + x_1^a + x_2^b = y^n \tag{1}$$

are investigated. However, in these results it is always necessary to assume that $\gcd(x_1, x_2) > 1$. The case $x_1 = x_2$ is of particular interest.

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At this point we mention that in some papers equations of type (1), but with one more term x_3^c appearing on the left hand side are considered. Here we do not give details, only refer to [3] and the references there.

Apparently, the case where $\gcd(x_1, x_2) = 1$ is more difficult. At least, even the special case $x_1 = 2$, $x_2 = 3$ and $n = 2$ could not be handled by deep methods of Corvaja and Zannier [5]. This particular case has been settled by Leitner [9].

In this paper we completely solve the equation

$$1 + x^a + z^b = y^n \tag{2}$$

in positive integers a, b, y, n with $n \geq 4$, for all values of x and z with $1 \leq x, z \leq 50$ and $x \not\equiv z \pmod 2$. In this way we also avoid the condition of non-coprimality of x_1 and x_2 in (1). Our results yield a considerable extension of recent results of the second and last authors [7], where the special case $x = 2$ was considered.

To prove our results, we also need to considerably extend the methods used in [7], and we also have to use serious computational facilities. It seems that classical effective tools, e.g. Baker's method, are not sufficient alone to handle (2). To solve (2) we combine Baker's method by local methods in the following way. First by a local argument we show that in every solution, one of a and n is at most five in (2). If $n \leq 5$ then by involved local arguments using congruences we find all solutions. Note that here it is not automatic to find appropriate moduli; this is done by careful considerations: the prime divisors of our moduli must have rather special properties. If $a \leq 5$ then we have one exponential variables less on the left hand side, and we can apply Baker's method to bound n; in fact, we get that $n < 15000$ in this case. Then for the 'small' values of n the remaining equations are solved through Thue equations of small degrees. In this part the corresponding procedures of the program package Magma [4] are used, too. Finally, for the larger values of n (up to the bound obtained) we apply the above described local method again.

2. The main result

Our main purpose is to solve equation (2) in positive integers a, b, y, n with $n \geq 4$, where x, z are integers with $1 \leq x, z \leq 50$ and $x \not\equiv z \pmod 2$. Clearly, without loss of generality we may assume that x is even, z is odd and $n = 4, 6, 9$ or n is a prime with $n \geq 5$.

The main result of the paper is the following.

Theorem 2.1. *Consider the equation*

$$1 + x^a + z^b = y^n \tag{3}$$

in positive integers a, b, y, n with $n = 4, 6, 9$ or $n \geq 5$ prime, where x is even, z is odd and $1 \leq x, z \leq 50$. All solutions to (3) *are given by* $(x, a, z, b, y, n) \in \{(14, 1, 1, b_0, 2, 4), (30, 1, 1, b_0, 2, 5), (2, 3, 7, 1, 2, 4), (2, 2, 11, 1, 2, 4), (2, 1, 13, 1, 2, 4), (4, 1, 11, 1, 2, 4), (6, 1, 3, 2, 2, 4), (6, 1, 9, 1, 2, 4), (6, 3, 39, 1, 4, 4), (8, 1, 7, 1, 2, 4), (10, 1, 5, 1, 2, 4), (12, 1, 3, 1, 2, 4), (12, 1, 3, 5, 4, 4), (30, 1, 15, 2, 4, 4), (2, 2, 3, 3, 2, 5), (2, 4, 15, 1, 2, 5), (2, 3, 23, 1, 2, 5), (2, 2, 27, 1, 2, 5), (2, 1, 29, 1, 2, 5), (4, 1, 3, 3, 2, 5), (4, 2, 15, 1, 2, 5), (4, 1, 27, 1, 2, 5), (6, 1, 5, 2, 2, 5), (6, 1, 25, 1, 2, 5), (8, 1, 23, 1, 2, 5), (10, 1, 21, 1, 2, 5), (10, 3, 23, 1, 4, 5), (12, 1, 19, 1, 2, 5), (14, 1, 17, 1, 2, 5), (16, 1, 15, 1, 2, 5), (18, 1, 13, 1, 2, 5), (20, 1, 11, 1, 2, 5), (22, 1, 3, 2, 2, 5), (22, 1, 9, 1, 2, 5), (24, 1, 7, 1, 2, 5), (26, 1, 5, 1, 2, 5), (28, 1, 3, 1, 2, 5), (2, 5, 31, 1, 2, 6), (2, 4, 47, 1, 2, 6), (4, 2, 47, 1, 2, 6), (6, 2, 3, 3, 2, 6), (6, 2, 27, 1, 2, 6), (14, 1, 7, 2, 2, 6), (14, 1, 49, 1, 2, 6), (16, 1, 47, 1, 2, 6), (18, 1, 45, 1, 2, 6), (20, 1, 43, 1, 2, 6), (22, 1, 41, 1, 2, 6), (24, 1, 39, 1, 2, 6), (26, 1, 37, 1, 2, 6), (28, 1, 35, 1, 2, 6), (30, 1, 33, 1, 2, 6), (32, 1, 31, 1, 2, 6), (34, 1, 29, 1, 2, 6), (36, 1, 3, 3, 2, 6), (36, 1, 27, 1, 2, 6), (38, 1, 5, 2, 2, 6), (38, 1, 25, 1, 2, 6), (40, 1, 23, 1, 2, 6), (42, 1, 21, 1, 2, 6), (44, 1, 19, 1, 2, 6), (46, 1, 17, 1, 2, 6), (48, 1, 15, 1, 2, 6), (50, 1, 13, 1, 2, 6), (2, 1, 5, 3, 2, 7), (6, 1, 11, 2, 2, 7), (10, 2, 3, 3, 2, 7), (10, 2, 27, 1, 2, 7), (46, 1, 3, 4, 2, 7), (46, 1, 9, 2, 2, 7), (22, 2, 3, 3, 2, 9), (22, 2, 27, 1, 2, 9), (22, 1, 45, 2, 2, 11)\}$, *where b_0 is any positive integer.*

3. Some lemmas and the proof of Theorem 2.1

To prove Theorem 2.1 we need some notation and several lemmas. The proof will be obtained as a simple combination of these lemmas.

For a non-zero integer k and a prime p we denote by $\mathrm{ord}_p(k)$ the p-adic order of k, that is, $\mathrm{ord}_p(k)$ is the largest non-negative integer such that $p^{\mathrm{ord}_p(k)}$ divides k.

Lemma 3.1. *Let x, z be integers with $1 \leq x, z \leq 50$, x even, z odd. Then all solutions to equation* (3) *in integers a, b, y, n with $n = 4, 6, 9$ or $n \geq 5$ prime satisfy* $\min(a, n) \leq 5$.

Proof. Assume that $a > 5$. A short computation shows that

$$1 \leq \mathrm{ord}_2(1 + z^b) \leq 5$$

for every odd z with $1 \leq z < 50$. Indeed, if b is even then $z^b \equiv 1 \pmod 4$, so $\mathrm{ord}_2(1 + z^b) = 1$. Further, we have $z^{16} \equiv 1 \pmod{64}$ for any odd integer z. Thus for z odd we have $z^k \equiv z^{k+16} \pmod{64}$ for every $k \in \mathbb{Z}$. So if b is odd, we only have to check that $1 + z^b \not\equiv 0 \pmod{64}$ for z odd with $1 \leq z < 50$, and $0 \leq b \leq 15$. This can be done in Magma [4] in a few seconds.

Now, by $\mathrm{ord}_2(1+z^b) \le 5$, x even and $a > 5$ we get

$$1 \le \mathrm{ord}_2(1+x^a+z^b) \le 5,$$

which together with equation (3) shows that y is even and $n \le 5$. This completes the proof of Lemma 3.1. □

3.1. **The case** $a \le 5$**.** First we give sharp upper bounds for n in equation (3) in this case. For this, we need to introduce some notation. Put $t = 1 + x^a$. Since $a \le 5$ we find that

$$t = 1 + x^a \le 1 + 50^5 = 312,500,001.$$

Then, our original equation (3) takes the form

$$z^b + t = y^n, \tag{4}$$

where b, y are positive integers with $y \ge 2, n \ge 4$ and $t \le 1 + 50^5$. In Lemma 3.3 below we give upper bounds for n in equation (4). For this purpose, we use Baker's method of linear forms in logarithms of two algebraic numbers. For an algebraic number α of degree d over $\mathbb{Q}$, we define the *absolute logarithmic height* of α by the following formula:

$$\mathrm{h}(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\max\{1, |\alpha^{(i)}|\}\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and $\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(d)}$ are the conjugates of α in the field of complex numbers.

Let α_1 and α_2 be multiplicatively independent algebraic numbers with $|\alpha_1| \ge 1$ and $|\alpha_2| \ge 1$. Consider the linear form in two logarithms

$$\Lambda = b_2 \log\alpha_2 - b_1 \log\alpha_1,$$

where $\log\alpha_1, \log\alpha_2$ are any determinations of the logarithms of α_1, α_2 respectively, and b_1, b_2 are positive integers. We rely on the following result due to Laurent [8].

Lemma 3.2 ([8], Corollary 2)**.** *Suppose that α_1 and α_2 are multiplicatively independent positive rational numbers greater than one. Then we have*

$$\log|\Lambda| \ge -25.2H(\alpha_1)H(\alpha_2)\max\{\log h' + 0.38, 10\}^2,$$

where

$$H(\alpha_i) = \max\{\mathrm{h}(\alpha_\mathrm{i}), \log\alpha_\mathrm{i}, 1\} \quad (\mathrm{i} = 1, 2),$$

and

$$h' = \frac{b_1}{H(\alpha_2)} + \frac{b_2}{H(\alpha_1)}.$$

Using Lemma 3.2 we show the following.

Lemma 3.3. *Let z, t be odd positive integers with $3 \le z \le 49$ and $3 \le t \le 1 + 50^5$. Let (b, y, n) be a solution of equation (4) with $n \ge 5$ an odd prime. Then we have*

$$n < \begin{cases} 15,000 & \text{if } y = 2, \\ 10,000 & \text{if } y > 2. \end{cases}$$

Proof. In the proof we will closely follow the method used in the first part of Lemma 3.3 of [7]. Note, that for a non-zero rational number $\alpha = \frac{u}{v}$ we always have $H(\alpha) \le \max\{\log|u|, \log|v|, 1\}$.

If in equation (4) we have $n \mid b$ then by writing $b = nb_1$ ($b_1 \ge 1$), we get from (4) that

$$t = y^n - (z^{b_1})^n = (y - z^{b_1})(y^{n-1} + y^{n-2}z^{b_1} + \cdots + (z^{b_1})^{n-1}). \tag{5}$$

Since $t \ge 3 > 0$, equation (5) implies that $y - z^{b_1} \ge 1$, which together with $y \ge 2$ and (5) gives

$$t > 2^{n-1}.$$

Therefore by $t \le 1 + 50^5$ we obtain $n \le 29$, which is much better bound for n than stated. So, in what follows, we may write b occurring in (4) in the form $b = nB + r$, where B and r are integers for which $B \ge 0$ and $0 < r \le n - 1$. Thus, equation (4) yields

$$\left| z^r \left(\frac{z^B}{y} \right)^n - 1 \right| = \frac{t}{y^n}. \tag{6}$$

Set

$$\Lambda := r \log z - n \log \left(\frac{y}{z^B} \right). \tag{7}$$

By (6) one can easily see that $\Lambda \ne 0$. Now, if $\left| z^r \left(\frac{z^B}{y} \right)^n - 1 \right| > \frac{1}{3}$ we get by $y \ge 2$ and (6) that

$$n < \frac{\log 3t}{\log 2},$$

which by $t \le 1 + 50^5$ yields $n \le 29$. So, in what follows we may assume that

$$\left| z^r \left(\frac{z^B}{y} \right)^n - 1 \right| \le \frac{1}{3}.$$

It is well-known that for every $w \in \mathbb{C}$ for which $|w - 1| \le 1/3$ we have $|\log w| < 2|w - 1|$. Hence by (6) and (7) we get that

$$|\Lambda| < \frac{2t}{y^n}. \tag{8}$$

Now, we are going to derive a lower bound for $|\Lambda|$ occurring in (7). Since z and t are odd, it follows that in equation (4) we have $z \not\equiv y$

(mod 2). Thus the rational numbers $\frac{y}{z^B}$ and z are multiplicatively independent. Hence we may apply Lemma 3.2 on taking

$$\alpha_1 = \frac{y}{z^B}, \alpha_2 = z, b_1 = n, b_2 = r.$$

By the inequality $\left| z^r \left(\frac{z^B}{y}\right)^n - 1 \right| \leq \frac{1}{3}$ we obviously have that $y > z^B$, thus we may choose

$$H(\alpha_1) = H\left(\frac{y}{z^B}\right) \leq \begin{cases} 1, & \text{if} \quad y = 2, \\ \log y, & \text{if} \quad y > 2 \end{cases} \quad \text{and} \quad H(\alpha_2) = H(z) = \log z.$$

Since $r < n$ and $H(\alpha_1) \geq 1$ we obtain

$$h' < \frac{n}{\log z} + n,$$

whence we get the lower bound

$$|\Lambda| > -25.2 \log z H(\alpha_1) \max\left\{ \log\left(\frac{n}{\log z} + n\right) + 0.38, 10 \right\}^2. \tag{9}$$

On comparing (8) and (9) we infer that

(10)

$$n < 25.2 \max\left\{ \log\left(\frac{n}{\log z} + n\right) + 0.38, 10 \right\}^2 \log z \frac{H(\alpha_1)}{\log y} + \frac{\log 2t}{\log y},$$

which, by $y \geq 2$, implies that either

$$n < 2520 \log z \frac{H(\alpha_1)}{\log y} + \frac{\log 2t}{\log 2}, \tag{11}$$

or

$$n < 25.2 \left(\log\left(\frac{n}{\log z} + n\right) + 0.38 \right)^2 \log z \frac{H(\alpha_1)}{\log y} + \frac{\log 2t}{\log 2}. \tag{12}$$

Since

$$\frac{H(\alpha_1)}{\log y} = \begin{cases} \frac{1}{\log 2}, & \text{if} \quad y = 2, \\ 1, & \text{if} \quad y > 2 \end{cases}$$

then by (11) and (12) a simple calculation gives an upper bound for n valid for every $t \leq 1 + 50^5$ and $y \geq 2$. Namely, we may write $n < 15,000$ if $y = 2$, and $n < 10,000$ if $y > 2$. $\square$

In the following we shall need Carmichael's lambda function $\lambda(m)$, which (for $m \geq 2$) is the smallest positive integer such that $k^{\lambda(m)} \equiv 1 \pmod{m}$ for all integers k coprime to m. For some properties of $\lambda(m)$, see e.g. [6] and the references given there.

The next Lemma shows that for $a \leq 5$ and $10 < n < n_0$ with the bound n_0 coming from Lemma 3.3 equation (3) possesses only one solution. Namely, we have the following.

Lemma 3.4. *Suppose that* $1 \leq a \leq 5$ *and* $n > 10$. *Then equation* (3)
(i) for $y \neq 2$ *and* $n < 10000$ *prime has no solution,*
(ii) for $y = 2$ *and* $n < 15000$ *prime has the only solution*

$$(x, a, z, b, y, n) = (22, 1, 45, 2, 2, 11).$$

Proof. (i) For fixed values of x, a, z, n we proceed as follows. For the least prime m_0 of the form $m_0 = 2in + 1$ with $i \in \mathbb{N}$ we put $o := \lambda(m_0)$ and find all exponents $b \pmod{o}$ for which

$$1 + x^a + z^b = y^n \pmod{m_0}$$

is solvable. If the list is empty, then there is no solution for the actual choice of x, a, z, n. Otherwise, we put $M := o$ and we take the next prime of the form $m_1 = 2ni + 1$ with $i \in \mathbb{N}$, put $o_1 := \lambda(m_1)$ and $M_1 := \mathrm{lcm}(M, o_1)$. We list all possible exponents $b \pmod{M_1}$ for which equation (3) is solvable modulo $m_0 \cdot m_1$ and we exclude all those values of $b \pmod{M_1}$ for which (3) is unsolvable modulo M_1.

So we may assume that we have a list of possible exponents $b \pmod{M_1}$ for which the equation (3) is solvable modulo a suitable modulus μ which is the product of some primes.

We continue the above process, taking new and new moduli of the form $m_j = 2in + 1$ with $i \in \mathbb{N}$ until we end up with an empty list of possible values of exponents $b \pmod{M_j}$, where $M_j := \mathrm{lcm}(M, \lambda(m_j))$. Then we have proved that the original equation is unsolvable for the given values of x, a, z, n. This method worked successfully for all possible tuples x, a, z, n with $1 \leq a \leq 5$, $1 \leq x, z \leq 50$, x even, z odd, and $10 < n \leq 10000$, except for $(x, a, z, n) = (22, 1, 45, 11)$. Our algorithm has been implemented in Magma [4]. It has worked efficiently enough, the total running time was about 250 hours on an Intel Xeon X5680 (Westmere EP) processor. We mention that clearly, for fixed x and z instead of Carmichael's λ function one could work with the orders of these numbers modulo the appropriate moduli. However, the use of λ was more convenient, and it is still efficient enough.

It is clear because of the identity $1 + 22^1 + 45^2 = 2^{11}$, for $(x, a, z, n) = (22, 1, 45, 11)$ the above method does not work. In this case we reduce (3) to a Thue equation of the form

$$y^{11} - 45^j \cdot w^{11} = 23$$

where $w = 45^t$ and $b = 11t + j$, with j running through the values $0, 1, 2, \ldots, 10$. Then we solve all such equations by Magma [4]. For each fixed j the solution of the corresponding Thue equation took less than 4 hours on an Intel Xeon X5680 (Westmere EP) processor. Since these Thue equations have no solution with w being a power of 45 and $y \neq 2$, the proof of statement (i) of the lemma is complete.

(ii) In the case when $y = 2$, for all possible fixed x, a, z, n we check whether $2^n - x^a - 1$ is a perfect power of z or not. The only case when we get a positive answer is when $(x, a, z, y, n) = (22, 1, 45, 2, 11)$. This gives the solution $(x, a, z, b, y, n) = (22, 1, 45, 2, 2, 11)$.

These computations were done in Magma [4] and they took around 4 minutes of CPU time on an Intel Xeon X5680 (Westmere EP) processor. $\square$

The next lemma is concerned with equation (3) for some 'small' values of n in the case $a \leq 5$.

Lemma 3.5. *Suppose that* $1 \leq a \leq 5$ *and* $n \in \{4, 5, 6, 7, 9\}$. *Then all solutions* (x, a, z, b, y, n) *of equation* (3) *are those listed below*
$(x, a, z, b, y, n) \in \{(14, 1, 1, b_0, 2, 4), (30, 1, 1, b_0, 2, 5), (2, 3, 7, 1, 2, 4),$
$(2, 2, 11, 1, 2, 4), (2, 1, 13, 1, 2, 4), (4, 1, 11, 1, 2, 4), (6, 1, 3, 2, 2, 4),$
$(6, 1, 9, 1, 2, 4), (6, 3, 39, 1, 4, 4), (8, 1, 7, 1, 2, 4), (10, 1, 5, 1, 2, 4),$
$(12, 1, 3, 1, 2, 4), (12, 1, 3, 5, 4, 4), (30, 1, 15, 2, 4, 4), (2, 2, 3, 3, 2, 5),$
$(2, 4, 15, 1, 2, 5), (2, 3, 23, 1, 2, 5), (2, 2, 27, 1, 2, 5), (2, 1, 29, 1, 2, 5),$
$(4, 1, 3, 3, 2, 5), (4, 2, 15, 1, 2, 5), (4, 1, 27, 1, 2, 5), (6, 1, 5, 2, 2, 5),$
$(6, 1, 25, 1, 2, 5), (8, 1, 23, 1, 2, 5), (10, 1, 21, 1, 2, 5), (10, 3, 23, 1, 4, 5),$
$(12, 1, 19, 1, 2, 5), (14, 1, 17, 1, 2, 5), (16, 1, 15, 1, 2, 5), (18, 1, 13, 1, 2, 5),$
$(20, 1, 11, 1, 2, 5), (22, 1, 3, 2, 2, 5), (22, 1, 9, 1, 2, 5), (24, 1, 7, 1, 2, 5),$
$(26, 1, 5, 1, 2, 5), (28, 1, 3, 1, 2, 5), (2, 5, 31, 1, 2, 6), (2, 4, 47, 1, 2, 6),$
$(4, 2, 47, 1, 2, 6), (6, 2, 3, 3, 2, 6), (6, 2, 27, 1, 2, 6), (14, 1, 7, 2, 2, 6),$
$(14, 1, 49, 1, 2, 6), (16, 1, 47, 1, 2, 6), (18, 1, 45, 1, 2, 6), (20, 1, 43, 1, 2, 6),$
$(22, 1, 41, 1, 2, 6), (24, 1, 39, 1, 2, 6), (26, 1, 37, 1, 2, 6), (28, 1, 35, 1, 2, 6),$
$(30, 1, 33, 1, 2, 6), (32, 1, 31, 1, 2, 6), (34, 1, 29, 1, 2, 6), (36, 1, 3, 3, 2, 6),$
$(36, 1, 27, 1, 2, 6), (38, 1, 5, 2, 2, 6), (38, 1, 25, 1, 2, 6), (40, 1, 23, 1, 2, 6),$
$(42, 1, 21, 1, 2, 6), (44, 1, 19, 1, 2, 6), (46, 1, 17, 1, 2, 6), (48, 1, 15, 1, 2, 6),$
$(50, 1, 13, 1, 2, 6), (2, 1, 5, 3, 2, 7), (6, 1, 11, 2, 2, 7), (10, 2, 3, 3, 2, 7),$
$(10, 2, 27, 1, 2, 7), (46, 1, 3, 4, 2, 7), (46, 1, 9, 2, 2, 7), (22, 2, 3, 3, 2, 9),$
$(22, 2, 27, 1, 2, 9)\}$, *where* b_0 *is any positive integer.*

Proof. To prove this lemma for all fixed tuples (x, a, z, n) which fulfill the assumptions of our lemma, for all values $j = 0, 1, \ldots, n-1$ we solve the Thue equations

$$y^n - z^j \cdot w^n = 1 + x^a \qquad \text{in } y, w \in \mathbb{Z}.$$

We check whether among the solutions we find some pairs (y, w) with w being a perfect power of z. In this way we list all solutions of (3) which fulfill the conditions of our lemma. The computations were done in Magma [4] and took around 340 hours CPU time on an Intel Xeon X5680 (Westmere EP) processor. □

3.2. **The case $a > 5$.** In this case Lemma 3.1 yields that in equation (3) we must have $n \leq 5$. However, by our assumprions on n this means that $n = 4, 5$.

Our last lemma shows that equation (3) has no solutions with $n \in \{4, 5\}$ and $a > 5$.

Lemma 3.6. *Suppose that $n \in \{4, 5\}$. Then equation* (3) *has no solution with $a > 5$.*

Proof. First we recall that for any integer q the sequence $1, q, q^2, q^3, \dots$ is ultimately periodic modulo m for any positive integer modulus m. More precisely we have

$$q^k \equiv q^{k+\lambda(m)} \pmod{m},$$

where k is the tail length (i.e. the length of the part of the sequence before the repeating part). The tail length is strictly smaller than the maximal exponent in the prime factorization of the modulus. For more detailed description of related properties we refer to [12].

For all possible values of x, z, n we reduce our equation modulo several moduli. In fact our method is similar to that used in the proof of Lemma 3.4. First take a modulus m (we shall list the used moduli explicitly later on). We take $o = \lambda(m)$ and list all possible pairs (a, b) modulo o for which equation (3) is solvable modulo m, i.e. for which $1 + x^a + z^b$ is a perfect n-th power modulo m. If this list is empty, then we are done for the given x, z, n. Otherwise, we choose another modulus, say m', and we write m_1 for the least common multiple of m, m' and M for the least common multiple of the values $o, \lambda(m')$. We extend the list of possible pairs (a, b) modulo o to a list of possible pairs (a, b) modulo M, and exclude from this list all those pairs for which $1 + x^a + z^b$ is not a perfect n-th power modulo m'. If the list becomes empty, then we are done, otherwise we take the next modulus, and repeat the above procedure. Here we mention that at the extension step of the list of possible pairs (a, b), whenever in a pair we have $a \leq 5$ we replace it by $a + M$. The reason is that we have the assumption $a > 5$, and since we are also using composite moduli it may happen that for $a \leq 5$ we have $x^a \not\equiv x^{a+M} \pmod{m'}$. Thus a solution of the equation $1 + x^a + z^b = y^n$ with $a \leq 5$ would make impossible to exclude the corresponding pair (a, b), although for $a > 5$, as assumed in this lemma there is no solution, and this can be checked locally.

Taking the moduli $11, 8, 3, 5, 32, 9, 13, 32, 31, 41, 25, 61, 64, 181, 241, 49, 281, 631, 2521, 3361, 131, 2081, 2341, 8191$ this procedure proves for all possible values of x, z, n fulfilling the conditions of our lemma that equation (3) has no solutions for $a > 5$.

We can see that in the list the moduli are not pairwise co-prime, which from the theoretical point of view is a redundance. However, using the moduli in this order makes the computation feasible, meanwhile if we would use 64 on the second place instead of 8 it would make the computation extremely lengthy. Further, we note that (similarly to the proof of Lemma 3.5) the use of λ is because of convenience, and it could be avoided.

These computations were done in Magma [4] and they required about 40 seconds of CPU time on an Intel Xeon X5680 (Westmere EP) processor. $\square$

Now we are ready to prove our main result.

Proof of Theorem 2.1. Assume first that $a \leq 5$ holds. In view of that $1 + x^a \leq 1 + 50^5$ in this case, the solutions with $n \geq 10$ are given by the combination of Lemmas 3.3 and 3.4. Further, the solutions to (3) with $n < 10$ are obtained by Lemma 3.5.

Suppose next that $a > 5$. Then by Lemma 3.1 we know that $n \leq 5$ must be valid. Hence $n = 4, 5$, and the solutions to (3) are given by Lemma 3.6. $\square$

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