POWER VALUES OF SUMS OF PRODUCTS OF CONSECUTIVE INTEGERS

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Abstract. We investigate power values of sums of products of consecutive integers. We give general finiteness results, and also give all solutions when the number of terms in the sum considered is at most ten.

1. Introduction

For $k = 0, 1, 2, \dots$ put

$$f_k(x) = \sum_{i=0}^{k} \prod_{j=0}^{i} (x + j).$$

For the first few values of k we have

$$f_0(x) = x, \quad f_1(x) = x + x(x + 1) = x(x + 2),$$

$$f_2(x) = x + x(x + 1) + x(x + 1)(x + 2) = x(x + 2)^2.$$

In general, $f_k(x)$ is a monic polynomial of degree $k+1$. Further, the coefficients of the $f_k(x)$ are positive integers, which could easily be expressed as sums of consecutive Stirling numbers of the first kind.

In this paper we are interested in the equation

$$f_k(x) = y^n \tag{1.1}$$

in integers x, y, k, n with $k \geq 0$ and $n \geq 2$. Without loss of generality, throughout the paper we shall assume that n is a prime.

Equation (1.1) is closely related to several classical problems and results. Here we only briefly mention some of them.

When we take only one block (i.e. consider the equation $f_{k+1}(x) - f_k(x) = y^n$), then we get a classical problem of Erdős and Selfridge [14]. For related results one can see e.g. [17, 30], and the references there. An important generalization of this problem is when instead of products of consecutive

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integers one takes products of consecutive terms of an arithmetic progression. For this case, see e.g. the papers [5, 19, 20, 22, 31, 33, 38] and the references there.

If instead of sums, we take products of blocks of consecutive integers, we get classical questions of Erdős and Graham [12, 13]. For results into this direction, see e.g. [3, 10, 37, 39] and the references there.

Finally, if in (1.1) the products of blocks of consecutive integers are replaced by binomial coefficients, then we arrive at classical problems again. In case of one summand see the papers Erdős [11] and Győry [18]. In case of more summands, we mention a classical problem of Mordell [26] p. 259, solved by Ljunggren [25] (see Pintér [27] for a related general finiteness theorem).

In this paper we obtain a general finiteness result concerning (1.1). Further, we provide all solutions to this equation for $k \leq 10$. These results are given in the next section. Our first theorem is proved in Section 3. To prove our result describing all solutions for $k \leq 10$, we need more preparation. We introduce the tools needed in Section 4. Then we give the proof of our second theorem in Section 5 for the case $n \geq 3$, and in Section 6 for the case $n = 2$. Altogether, in our proofs we need to combine several tools and techniques, including Baker's method, local arguments, Runge's method, and a method of Gebel, Pethő, Zimmer [15] and Stroeker, Tzanakis [34] to find integer points on elliptic curves.

2. New results

Our first theorem gives a general effective finiteness result for equation (1.1).

Theorem 2.1. *For the solutions of equation* (1.1) *we have the following:*

i) *if* $k \geq 1$ *and* $y \neq 0, -1$ *then* $n < c_1(k)$*,*
ii) *if* $k \geq 1$ *and* $n \geq 3$ *then* $\max(n, |x|, |y|) < c_2(k)$*,*
iii) *if* $k \geq 1$*,* $k \neq 2$*, and* $n = 2$ *then* $\max(|x|, |y|) < c_3(k)$*.*

Here $c_1(k), c_2(k), c_3(k)$ *are effectively computable constants depending only on* k*.*

The following theorem describes all solutions of equation (1.1) for $k \leq 10$.

Theorem 2.2. *Let* $1 \leq k \leq 10$ *such that* $k \neq 2$ *if* $n = 2$*. Then equation* (1.1) *has the only solutions* $(x, y) = (-2, 0), (0, 0)$*,* k, n *arbitrary;* $(x, y) = (-1, -1)$*,* k, n *arbitrary with* $n \geq 3$*;* $(x, y, k, n) = (-4, 2, 1, 3), (2, 2, 1, 3), (2, 2, 2, 5)$*.*

Remark. Note that the assumptions in Theorems 2.1 and 2.2 are necessary: equation (1.1) has infinitely many solutions (x, y, k, n) with $k = 0$, with $y = 0$ or -1, and with $k = 2$, $n = 2$. These solutions can be described easily.

3. Proof of Theorem 2.1

To prove Theorem 2.1 we need three lemmas. To formulate them, we have to introduce some notation. Let $g(x)$ be a non-zero polynomial with integer coefficients, of degree d and height H. Consider the diophantine equation

$$g(x) = y^n \tag{3.1}$$

in integers x, y, n with n being a prime.

The next lemma is a special case of a result of Tijdeman [38]. For a more general version, see [32].

Lemma 3.1. *If $g(x)$ has at least two distinct roots and $|y| > 1$, then in equation* (3.1) *we have $n < c_4(d, H)$, where $c_4(d, H)$ is an effectively computable constant depending only on d, H.*

The next lemma is a special case of a theorem of Brindza [8]. For predecessors of this result see [1, 2], and for an earlier ineffective version [24].

Lemma 3.2. *Suppose that one of the following conditions holds:*

i) *$n \geq 3$ and $g(x)$ has at least two roots with multiplicities coprime to n,*

ii) *$n = 2$ and $g(x)$ has at least three roots with odd multiplicities.*

Then in equation (3.1) *we have $\max(|x|, |y|) < c_5(d, H)$, where $c_5(d, H)$ is an effectively computable constant depending only on d, H.*

The last assertion needed to prove Theorem 2.1 describes the root structure of the polynomial family $f_k(x)$.

Lemma 3.3. *We have*

$$f_0(x) = x, \quad f_1(x) = x(x+2), \quad f_2(x) = x(x+2)^2.$$

Beside this, for $k \geq 3$ all the roots of the polynomial $f_k(x)$ are simple. In particular, 0 is a root of $f_k(x)$ for all $k \geq 0$, and -2 is a root of $f_k(x)$ for all $k \geq 1$.

Proof. For $k = 0, 1, 2$ the statement is obvious. In the rest of the proof we assume that $k \geq 3$.

It follows from the definition that x is a factor of $f_k(x)$ (or, 0 is a root of $f_k(x)$) for all $k \geq 0$. Further, since

$$x + x(x+1) = x(x+2),$$

the definition clearly implies that $x+2$ is a factor (or, -2 is a root) of $f_k(x)$ for $k \geq 1$. So it remains to prove that all the roots of $f_k(x)$ ($k \geq 3$) are simple.

For this observe that by the definition we have

$$f_k(1) > 0, \quad f_k(-1) = -1 < 0, \quad f_k(-1.5) > 0.$$

The last inequality follows from the fact that writing

$$P_i(x) = x(x+1)\dots(x+i)$$

for $i = 0, 1, 2, \dots,$ we have that $P_i(-1.5) > 0$ for $i \geq 1$. Hence $f_k(-1.5) \geq -1.5 + 0.75 + 0.375 + 0.5625 > 0$ for $k \geq 3$. Further, as one can easily check, for $i = -3, \dots, -k-1$ we have

$$(-1)^i f_k(i) > 0.$$

These assertions (by continuity) imply that $f_k(x)$ has roots in the intervals

$$(-1,1),\ (-1.5,-1),\ (-3,-1.5),\ (-4,-3),\ (-5,-4),\ \dots,\ (-k-1,-k).$$

(Note that in the first and third intervals the roots are 0 and -2, respectively.) Hence $f_k(x)$ has $\deg(f_k(x)) = k+1$ distinct real roots, and the lemma follows.

□

Now we are ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. i) Let $k \geq 1$. By Lemma 3.3 we have that $f_k(x)$ is divisible by $x(x+2)$ in $\mathbb{Z}[x]$. In particular, the polynomial $f_k(x)$ has two distinct roots, namely 0 and -2. Further, observe that $f_k(x)$ does not take the value 1 for $x \in \mathbb{Z}$. Indeed, since $x(x+2)$ divides $f_k(x)$, it would be possible only for $x = -1$. However, for that choice by definition we clearly have $f_k(-1) = -1$ for any $k \geq 0$. Hence equation (1.1) has no solution with $y = 1$, and our claim follows by Lemma 3.1.

ii) Let $k \geq 1$ and $n \geq 3$. Recall that n is assumed to be a prime. By the explicit form of $f_1(x)$ and $f_2(x)$ we see that 0 and -2 are roots of these polynomials of degrees coprime to n. Hence the statement follows from part i) of Lemma 3.2 in these cases. Let $k \geq 3$. Then by Lemma 3.3, all the roots of $f_k(x)$ are simple. Since now the degree $k+1$ of $f_k(x)$ is greater than two, our claim follows from part i) of Lemma 3.2.

iii) Let $k \geq 1$, $k \neq 2$ and $n = 2$. In case of $k = 1$, equation (1.1) now reads as

$$x(x+2) = y^2.$$

Since $x(x+2) = (x+1)^2 - 1$, our claim obviously follows in this case. Let now $k \geq 3$. Then by Lemma 3.3, all the roots of $f_k(x)$ are simple. As now the degree $k+1$ of $f_k(x)$ is greater than two, by part ii) of Lemma 3.2 the assertion follows also in this case. $\square$

4. Linear forms in logarithms

In this section, we use linear forms in logarithms to give a bound for n for the solution (u, v, n) of equations of the form

$$au^n - bv^n = c$$

under certain conditions. These bounds will be used in the proof of Theorem 2.2 for $n \geq 3$. Such equations have been studied by many authors. Note that bounds for such equations were obtained in [4, 21]. We refer to [4] for earlier results. However, in these papers the restrictions put on the coefficients a, b, c are not valid in the cases we need later on.

We begin with some preliminaries for linear forms in logarithms. For an algebraic number α of degree d over $\mathbb{Q}$, the *absolute logarithmic height* $h(\alpha)$ of α is given by

$$h(\alpha) = \frac{1}{d}\left(\log|a| + \sum_{i=1}^{d} \log\max(1, |\alpha^{(i)}|)\right)$$

where a is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$ and the $\alpha^{(i)}$'s are the conjugates of α. When $\alpha = \frac{p}{q} \in \mathbb{Q}$ with $(p, q) = 1$, we have $h(\alpha) = \max(\log|p|, \log|q|)$.

The following result is due to Laurent [23, Theorem 2].

Theorem 4.1. *Let a_1, a_2, h, ϱ and μ be real numbers with $\varrho > 1$ and $1/3 \leq \mu \leq 1$. Set*

$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda = \sigma \log \varrho, \quad H = \frac{h}{\lambda} + \frac{1}{\sigma},$$
$$\omega = 2\left(1 + \sqrt{1 + \frac{1}{4H^2}}\right), \quad \theta = \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H}.$$

Let α_1, α_2 be non-zero algebraic numbers and let $\log \alpha_1$ *and* $\log \alpha_2$ *be any determinations of their logarithms. Without loss of generality we may assume that $|\alpha_1| \geq 1, |\alpha_2| \geq 1$. Let*

$$\Lambda = |b_2 \log \alpha_1 - b_2 \log \alpha_2| \quad b_1, b_2 \in \mathbb{Z}, b_1 > 0, b_2 > 0,$$

where b_1, b_2 are positive integers. Suppose that α_1 and α_2 are multiplicatively independent. Put $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}]$ and assume that

$$\begin{aligned} &h \geq \max\left\{D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.75\right) + 0.06, \lambda, \frac{D\log 2}{2}\right\}, \\ &a_i \geq \max\left\{1, \varrho\log|\alpha_i| - \log|\alpha_i| + 2Dh(\alpha_i)\right\}, \ (i = 1, 2), \\ &a_1a_2 \geq \lambda^2. \end{aligned} \tag{4.1}$$

Then

$$\log\Lambda \geq -C\left(h + \frac{\lambda}{\sigma}\right)^2 a_1a_2 - \sqrt{\omega\theta}\left(h + \frac{\lambda}{\sigma}\right) - \log\left(C'\left(h + \frac{\lambda}{\sigma}\right)^2 a_1a_2\right)$$

with

$$\begin{aligned} C &= \frac{\mu}{\lambda^3\sigma}\left(\frac{\omega}{6} + \frac{1}{2}\sqrt{\frac{\omega^2}{9} + \frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1a_2H^{1/2}}} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\lambda\omega}{H}}\right)^2, \\ C' &= \sqrt{\frac{C\sigma\omega\theta}{\lambda^3\mu}}. \end{aligned}$$

We use Theorem 4.1 to give a bound for n for the equation $au^n - bv^n = c$. For this, we need the following lemma.

Lemma 4.2. *Let a, b, c be positive integers with $b > a > 0$ and $abc \leq 4 \cdot 2018957 \cdot 99 \cdot 467$. Then the equation $au^n - bu^n = \pm c$ with $u > v > 1$ implies*

$$\frac{u}{v} \leq \begin{cases} 1.00462 & \text{if } b \leq 100 \text{ and } n \geq 1000 \\ 1.00462 & \text{if } b \leq 10000 \text{ and } n \geq 2000 \\ 1.00267 & \text{if } n \geq 10000 \end{cases} \tag{4.2}$$

and

$$u > v \geq \begin{cases} 217 & \text{if } b \leq 100 \text{ and } n \geq 1000 \\ 217 & \text{if } b \leq 10000 \text{ and } n \geq 2000 \\ 375 & \text{if } n \geq 10000. \end{cases} \tag{4.3}$$

Proof. From $au^n - bv^n = \pm c$, we get $(\frac{u}{v})^n = \frac{b}{a} \pm \frac{c}{av^n} \leq b + 1/4$ since $n \geq 1000$ and $c \leq 2^{100}a$. Therefore

$$\frac{u}{v} \leq \begin{cases} \sqrt[1000]{100 + 1/4} & \text{if } b \leq 100 \text{ and } n \geq 1000 \\ \sqrt[2000]{10000 + 1/4} & \text{if } b \leq 10000 \text{ and } n \geq 2000 \\ \sqrt[10000]{4 \cdot 2018957 \cdot 99 \cdot 467 + 1/4} & \text{if } n \geq 10000 \end{cases}$$

implying (4.2). The assertion (4.3) follows easily from (4.2) by observing that $1 \leq u - v \leq 0.00462v, 0.00267v$ according as $b \leq 100, n \geq 1000$ or $b \leq 10000, n \geq 2000$, or $n \geq 10000$, respectively. □

Proposition 4.3. *Let a, b, c be positive integers with $c \leq 2ab$. Then the equation*

$$au^n - bv^n = \pm c \tag{4.4}$$

in integer variables $u > v > 1, n > 3$ implies

$$n \leq \begin{cases} \max\{1000, 824.338 \log b + 0.258\} & \text{if } b \leq 100 \\ \max\{2000, 769.218 \log b + 0.258\} & \text{if } 100 < b \leq 10000 \\ \max\{10000, 740.683 \log b + 0.234\} & \text{if } b > 10000. \end{cases} \tag{4.5}$$

In particular, $n \leq 3796, 7084, 19736$ when $b \leq 100, 10000, 4 \cdot 9 \cdot 11 \cdot 467 \cdot 2018957$, respectively.

Remark. We note here that when $c \leq 3$, we can get a much better bound, see [6]. However, we will follow a more general approach.

Proof. We can rewrite (4.4) as

$$\left| \frac{b}{a} \left(\frac{u}{v} \right)^n - 1 \right| = \frac{c}{au^n}.$$

Let

$$\Lambda = \left| n \log \frac{u}{v} - \log \frac{b}{a} \right|.$$

Then $\Lambda \leq \frac{2c}{au^n}$ implying

$$\log \Lambda \leq -n \log u + \log \left(\frac{2c}{a} \right) \leq -n \log u + \log(4b) \tag{4.6}$$

since $c \leq 2ab$. We now apply Theorem 4.1 to get a lower bound for Λ. We follow the proof of [23, Corollary 1, 2]. Let

$$\alpha_1 = \frac{u}{v}, \ \alpha_2 = \frac{b}{a}, \ b_1 = n, \ b_2 = 1$$

so that $h(\alpha_1) = \log u, h(\alpha_2) = \log b$ and $D = 1$. Let $m = 8$ and we choose $\varrho, \mu, q_0, u_0, b_0$ as follows:

b	ϱ	μ	q_0	u_0	b_0
$b \leq 100$	5.7	0.54	$\log 1.00462$	218	$\log 4$
$b \leq 10000$	5.6	0.57	$\log 1.00462$	218	$\log 5$
$b > 10000$	5.6	0.59	$\log 1.00267$	$\log 376$	$\log 10000$

By Lemma 4.2, we have $u \geq u_0, \log u/v \leq q_0$ and $b \geq b_0$. We take

$$a_1 = (\varrho - 1)q_0 + 2 \log u, \ a_2 = (\varrho + 1) \log b,$$

and

$$h = \max \left\{ m, \log \left(\frac{n}{a_2} + \frac{1}{a_1} \right) + 1.81 + \log \lambda \right\}.$$

Then (4.1) is satisfied. In fact, we have

$$h \geq m,\ a_1 \geq (\varrho - 1)q_0 + 2\log u_0,\ a_2 \geq (\varrho + 1)\log b_0.$$

As in the proof of [23, Corollary 1, 2], we get

$$\log \Lambda \geq -C''_m(\varrho + 1)(\log b)((\varrho - 1)q_0 + 2\log u), h^2$$

where C''_m is the constant C'' obtained in [23, Section 4, (28)] by putting $h = m, a_1 = (\varrho-1)q_0+2\log u_0$ and $a_2 \geq (\varrho+1)\log b_0$. Putting $C_m = C''_m(\varrho+1)$, we get

$$\log \Lambda \geq -C_m(\log b)((\varrho - 1)q_0 + 2\log u)(\max(m, h_n))^2,$$

where

$$h_n = \log\left(\frac{n}{(\varrho+1)\log b} + \frac{1}{2\log u + (\varrho - 1)q_0}\right) + \varepsilon_m,$$

and

$$(C_m, \varepsilon_m) = \begin{cases} (5.8821, 2.2524) & \text{if } b \leq 100, \\ (5.4890, 2.2570) & \text{if } b \leq 10000, \\ (5.3315, 2.2662) & \text{if } b > 10000. \end{cases}$$

Comparing this lower bound of $\log \Lambda$ with the upper bound (4.6), we obtain

$$\begin{aligned} n &\leq C_m(\max(m, h_n))^2(\log b)\left(2 + \frac{(\varrho - 1)q_0}{\log u}\right) + \frac{\log 4b}{\log u} \\ &\leq C_m(\max(m, h_n))^2(\log b)\left(2 + \frac{(\varrho - 1)q_0}{\log u_0} + \frac{1}{\log u_0}\right) + \frac{\log 4}{\log u_0} \end{aligned} \tag{4.7}$$

since $u \geq u_0$. Recall that $m = 8$. We now consider two cases.
Assume $h_n \geq 8$. Then

$$n \geq n_0 := \left\{\exp(m - \varepsilon_m) - \frac{1}{2\log u + (\varrho - 1)q_0}\right\}(\varrho + 1)\log b$$

and $h_{n_0} = 8$. Since the last expression of (4.7) is a decreasing function of n, we have for $n \geq n_0$ that

$$\begin{aligned}
0 \leq & \frac{C_m h_n^2 (\log b) \left(2 + \frac{(\varrho-1)q_0}{\log u_0} + \frac{1}{\log u_0}\right) + \frac{\log 4}{\log u_0} - n}{\log b} \\
\leq & \frac{C_m h_{n_0}^2 (\log b) \left(2 + \frac{(\varrho-1)q_0}{\log u_0} + \frac{1}{\log u_0}\right) + \frac{\log 4}{\log u_0} - n_0}{\log b} \\
\leq & C_m m^2 \left(2 + \frac{(\varrho-1)q_0}{\log u_0} + \frac{1}{\log u_0}\right) + \frac{\log 4}{(\log u_0)(\log b)} \\
& - (\varrho+1)\exp(m - \varepsilon_m) + \frac{\varrho+1}{2\log u + (\varrho-1)q_0} \\
\leq & C_m m^2 \left(2 + \frac{(\varrho-1)q_0}{\log x_0} + \frac{1}{\log u_0}\right) + \frac{\log 4}{(\log u_0)(\log b_0)} \\
& - (\varrho+1)\exp(m - \varepsilon_m) + \frac{\varrho+1}{2\log u_0 + (\varrho-1)q_0} < 0
\end{aligned}$$

since $u \geq u_0$ and $b \geq b_0$. This is a contradiction.

Therefore $h_n < 8$. Then from (4.7), we get

$$n \leq C_m m^2 (\log b) \left(2 + \frac{(\varrho-1)q_0}{\log u_0} + \frac{1}{\log u_0}\right) + \frac{\log 4}{\log u_0},$$

where $m = 8$. Hence we get the assertion (4.5) by putting explicit values of $m = 8, C_m, \varrho, \mu, q_0, u_0, b_0$ in the above inequality. The statement following (4.5) is clear. □

5. Proof of Theorem 2.2 for $n \geq 3$

Throughout this section we assume that $n \geq 3$ is a prime.

Suppose first that $k = 1$ or 2. Then equation (1.1) can be rewritten as

$$x(x+2)^k = y^n.$$

We see that for every n odd, $(x, n) = (-1, n)$ is a solution. Hence we may suppose that $x \notin \{-2, -1, 0\}$. Hence $\gcd(x, x+2) \leq 2$ gives

$$x = 2^\alpha u^n, \quad x + 2 = 2^\beta v^n$$

with non-negative integers α, β and coprime integers u, v. This implies

$$2^\beta v^n - 2^\alpha u^n = 2(1)^n.$$

Using now results of Darmon and Merel [9] and Ribet [28], our statement easily follows in this case.

Let $k \geq 3$. Then equation (1.1) can be rewritten as

$$y^n = f_k(x) = x(x+2)g_k(x)$$

where $g_k(x)$ is a polynomial of degree $k-1$. We see that for every k, $(x,n) = (-1,n)$ is a solution. Hence we may suppose that $x \notin \{-2,-1,0\}$. Then we have either $x > 0$ or $x < x+2 < 0$.

We see that $(x, x+2) = 1, 2$ with 2 only if x is even, $(x, g_k(x))|g_k(0)$ and $(x+2, g_k(x))|g_k(-2)$. Also $g_k(x)$ is odd for every x. The values of $g_k(0)$ and $-g_k(-2)$ are given in Table 1.

k	3	4	5	6	7	8	9	10
$g_k(0)$	5	17	$7 \cdot 11$	$19 \cdot 23$	2957	23117	204557	2018957
$-g_k(-2)$	1	3	3^2	$3 \cdot 11$	$3^2 \cdot 17$	$3^2 \cdot 97$	$3^4 \cdot 73$	$3^2 \cdot 11 \cdot 467$

TABLE 1. Values of $g(0)$ and $-g(-2)$ for $3 \le k \le 10$

If $x = v^n$, $x+2 = u^n$ are both n-th powers, then we have $u^n - v^n = 2$ giving the trivial solution $x+2 = 1$, $x = -1$ which is already excluded. Hence we can suppose that either x or $x+2$ is not n-th power. Thus we can write

$$x = 2^{\delta_1} s_1 t_1^{n-1} u_1^n,\ x+2 = 2^{\delta_2} 3^{\nu_2} s_2 t_2^{n-1} u_2^n,\ g_k(x) = 3^{\nu_3}(s_1 s_2)^{n-1} t_1 t_2 u_3^n,$$

where

$$s_1 t_1 | g_k(0),\ s_2 t_2 | g_k(-2) \text{ with } (s_1, t_1) = (s_2, t_2) = 1, 3 \nmid s_1 s_2 t_1 t_2,$$

and

$$\delta_1, \delta_2 \in \{(0,0), (1, n-1), (n-1, 1)\},$$

and $(\nu_2, \nu_3) = (0,0)$ or

$$\nu_2 \in \{1, \cdots, \mathrm{ord}_3(g_k(-2))\},\ \nu_3 = n - \nu_2 \text{ or vice versa.}$$

Further, each of s_i, t_i is positive and u_1, u_2 are of the same sign. From $x + 2 - x = 2$, we get

$$\begin{aligned}
3^{\nu_2} s_2 t_1 (t_2 u_2)^n - s_1 t_2 (t_1 u_1)^n &= 2 t_1 t_2 \text{ if } \delta_1 = \delta_2 = 0, \nu_2 \le \mathrm{ord}_3(g_k(-2));\\
s_2 t_1 (3 t_2 u_2)^n - 3^{\nu_3} s_1 t_2 (t_1 u_1)^n &= 2 \cdot 3^{\nu_3} t_1 t_2 \text{ if } \delta_1 = \delta_2 = 0, \nu_2 > \mathrm{ord}_3(g_k(-2));\\
3^{\nu_2} s_2 t_1 (2 t_2 u_2)^n - 4 s_1 t_2 (t_1 u_1)^n &= 4 t_1 t_2 \text{ if } \delta_1 = 1, \nu_2 \le \mathrm{ord}_3(g_k(-2));\\
4 \cdot 3^{\nu_2} s_2 t_1 (t_2 u_2)^n - s_1 t_2 (2 t_1 u_1)^n &= 4 t_1 t_2 \text{ if } \delta_2 = 1, \nu_2 \le \mathrm{ord}_3(g_k(-2));\\
s_2 t_1 (6 t_2 u_2)^n - 4 \cdot 3^{\nu_3} s_1 t_2 (t_1 u_1)^n &= 4 \cdot 3^{\nu_3} t_1 t_2 \text{ if } \delta_1 = 1, \nu_2 > \mathrm{ord}_3(g_k(-2));\\
4 s_2 t_1 (3 t_2 u_2)^n - 3^{\nu_3} s_1 t_2 (2 t_1 u_1)^n &= 4 \cdot 3^{\nu_2} t_1 t_2 \text{ if } \delta_2 = 1, \nu_2 > \mathrm{ord}_3(g_k(-2)).
\end{aligned}$$

These equations are of the form $au^n - bv^n = c$ with u, v of the same sign. Note that from the equation $au^n - bv^n = c$, we can get back $x, x+2$ by

$$x = \frac{2bv^n}{c},\quad x + 2 = \frac{2au^n}{c}.$$

We see from Table 1 that the largest value of $\max(a,b)$ is given by $k=10$ and equation

$$(6\cdot 11\cdot 467u_2)^n - 4\cdot 3^2\cdot 11\cdot 467\cdot 2018957u_1^n = 4\cdot 3^2\cdot 11\cdot 467.$$

We observe that $|c|\le \frac{2ab}{s_1s_2}\le 2ab$. Further, from $(g_k(0),g_k(-2))=1$, we get $(s_2t_1,s_1t_2)=1$ giving $(a,b)=1$. We first exclude the trivial cases.

1. Let $a=b$. Then $a=b=1$ since $\gcd(a,b)=1$. Further $s_1t_2=s_2t_1=1$ and $3^{\nu_2}=1$ or $3^{\nu_3}=1$ implying $c=2$ and we have $u^n-v^n=2$ for which we have the trivial solution $u=1,v=-1$. Then $x=-1,x+2=1$ which gives $f_k(x)=(-1)^n$ for all odd n which is a trivial solution. Thus we now assume $a\ne b$ and further $x\ne -1$.

2. Suppose $uv=1$. Then $c|2a$ and $c|2b$ giving $c=2$ since $(a,b)=1$ and hence we have $a-b=\pm 2$. This implies $3^{\nu_2}s_2(\pm1)-s_1(\pm1)=2$ as in other cases, $c>2$. We find that the only such possibilities are $3(1)-1(1)=2, 9(-1)-11(-1)=2, 9(1)-7(1)=2$. Hence $x\in\{1,-11,7\}$. This with $x=2^{\delta_1}s_1t_1^{n-1}u_1^n=s_1(\pm1)$ gives $x=1,k\le 10$ or $(x,k)\in\{(-11,5),(7,5)\}$ and we check that $x=1,k=2$ is the only solution. Thus we now suppose that $uv>1$.

3. Suppose $u=v$. Then $(a-b)v^n=c$ implying $\frac{c}{a-b}\in\mathbb{Z}$. Further $\frac{c}{a-b}=v^n$ is an n-th power. We can easily find such triples (a,b,c) and exponents n. For such triples, we have $x=\frac{bc}{a-b}$ and we check for $f_k(x)$ being an n-th power. There are no solutions. Thus we can now suppose $u\ne v$.

4. Suppose $u=\pm1$. Then $c|2a$, $v\ne\pm1$ and $v^n=\frac{\pm a-c}{b}\in\mathbb{Z}$. We find all such triplets (a,b,c) and the exponents n. Then $x+2=\pm\frac{2a}{c}$ or $x=\pm\frac{2a}{c}-2$. We check for $f_k(x)$ being an n-th power. We find that there are no solutions. Hence we now assume $u\ne\pm1$.

5. Suppose $v=\pm1$. Then $c|2b$ and $u^n=\frac{c-\pm b}{a}\in\mathbb{Z}$ is a power. We find such triples (a,b,c) and the exponent n. Then $x=\pm\frac{2b}{c}$ and we check for $f_k(x)$ being an n-th power. There are no solutions.

Hence from now on, we consider the equation $au^n-bv^n=c$ with

$$a\ge 1, b\ge 1, c>1, |u|>1, |v|>1 \text{ and } a\ne b, u\ne v.$$

If u,v is a solution of $au^n-bv^n=c$ with u,v negative, then we have $a(-u)^n-b(-v)^n=-c$ with $-u,-v$ positive. Therefore it is sufficient to consider the equation $au^n-bv^n=\pm c$ with $u>1$, $v>1$. Recall that $abc\le 4\cdot 9\cdot 11\cdot 467\cdot 2018957$. Hence we have for $n\ge 40$ that

$$\left(\frac{u}{v}\right)^n=\frac{b}{a}\pm\frac{c}{v^n}\ge\frac{b}{a}-\frac{c}{2^n}\ge 1+\frac{1}{a}-\frac{c}{2^{40}}>1 \text{ if } a<b;$$
$$\left(\frac{v}{u}\right)^n=\frac{a}{b}\pm\frac{c}{u^n}\ge\frac{a}{b}-\frac{c}{2^n}\ge 1+\frac{1}{b}-\frac{c}{2^{40}}>1 \text{ if } a>b.$$

Thus for $n > 37$, we have $u > v$ if $a < b$ and $v > u$ if $a > b$. By Proposition 4.3, we get

$$n \leq \begin{cases} \max\{1000, 824.338 \log b + 0.258\} & \text{if } b \leq 100 \\ \max\{2000, 769.218 \log b + 0.258\} & \text{if } 100 < b \leq 10000 \\ \max\{10000, 740.683 \log b + 0.234\} & \text{if } b > 10000. \end{cases} \tag{5.1}$$

when $a < b$. We now exclude these values of n.

For every prime n, let r be the least positive integer such that $nr+1 = p$ is a prime. Then both u^n and v^n are r-th roots of unity modulo p. Since $f_k(x) = y^n$, $f_k(x)$ is also an r-th root of unity modulo p. Let $U(p, r)$ be the set of r-th roots of unity modulo p. Recall that $x = \frac{2bv^n}{c}$.

For every $3 \leq k \leq 10$, we first list all possible triples (a, b, c). Given a triple (a, b, c), we have a bound $n \leq n_0 := n_0(a, b, c)$ given by (5.1). For every prime $n \leq n_0$, we check for solutions $a\alpha - b\beta \equiv \pm c$ modulo p for $\alpha, \beta \in U(p, r)$. We now restrict to such pairs (α, β). For any such pair (α, β), we check if $f_k(\frac{2\beta}{c})$ modulo p is in $U(p, r)$. We find that there are no such pairs (α, β). The case $a > b$ can be handled similarly, and now new solutions arise.

Therefore, we have no further solutions (k, x, y) of the equation $f_k(x, y)$. Hence the proof of Theorem 2.2 is complete for $n \geq 3$. $\square$

6. Proof of Theorem 2.2 for $n = 2$

For $k = 1$ equation (1.1) reads as

$$f_1(x) = (x+1)^2 - 1 = y^2.$$

Hence the statement trivially follows in this case.

Let $k = 3$. Equation (1.1) has the form

$$x^4 + 7x^3 + 15x^2 + 10x = x(x+2)(x^2+5x+5) = y^2.$$

Here we use the MAGMA [7] procedure

```
IntegralQuarticPoints([1,7,15,10,0])
```

to determine all integral points. We only obtain the solutions with $x = 0, -2$ and $y = 0$.

Consider the case $k = 4$. The hyperelliptic curve is as follows

$$x(x+2)(x^3 + 9x^2 + 24x + 17) = y^2.$$

k	d	$P_1(x), P_2(x)$	I_k
5	1	$P_1(x) = x^3 + 8x^2 + 16x + 5$ $P_2(x) = x^3 + 8x^2 + 16x + 6$	$[-10, 3]$
7	16	$P_1(x) = 16x^4 + 232x^3 + 1070x^2 + 1693x + 473$ $P_2(x) = 16x^4 + 232x^3 + 1070x^2 + 1693x + 474$	$[-282, 148]$
9	2	$P_1(x) = 2x^5 + 46x^4 + 378x^3 + 1331x^2 + 1819x + 528$ $P_2(x) = 2x^5 + 46x^4 + 378x^3 + 1331x^2 + 1819x + 530$	$[-291, 278]$

TABLE 2. Data corresponding to the values $k = 5, 7, 9$

We obtain that

$$\begin{aligned} x &= d_1u_1^2, \\ x+2 &= d_2u_2^2, \\ x^3+9x^2+24x+17 &= d_3u_3^2, \end{aligned}$$

where $d_3 \in \{\pm1, \pm3, \pm17, \pm3\cdot17\}$. It remains to determine all integral points on certain elliptic curves defined by the third equation, that is we use the MAGMA procedure

$$\texttt{IntegralPoints(EllipticCurve(}[0, 9d_3, 0, 24d_3^2, 17d_3^3]\texttt{))}.$$

We note that these procedures are based on methods developed by Gebel, Pethő and Zimmer [15] and independently by Stroeker and Tzanakis [34]. Once again, we obtain the solutions with $x = 0, -2$ and $y = 0$.

We apply Runge's method [16, 29, 40] in the cases $k = 5, 7, 9$. We follow the algorithm described in [35]. First we determine the polynomial part of the Puiseux expansions of $\sqrt{f_k(x)}$. These expansions yield polynomials $P_1(x), P_2(x)$ such that either

$$\begin{aligned} d^2f_k(x) - P_1(x)^2 &> 0, \\ d^2f_k(x) - P_2(x)^2 &< 0 \end{aligned}$$

or

$$\begin{aligned} d^2f_k(x) - P_1(x)^2 &< 0, \\ d^2f_k(x) - P_2(x)^2 &> 0 \end{aligned}$$

for some $d \in \mathbb{Z}$ and $x \notin I_k$, where I_k is a finite interval. We summarize some data in Table 2.

We only provide details of the method in case of $k = 9$, the other two cases can be solved in a similar way. We obtain that

$$4f_9(x) - P_1(x)^2 = 4x^5 - 1045x^4 - 17958x^3 - 108973x^2 - 284408x - 278784,$$
$$4f_9(x) - P_2(x)^2 = -4x^5 - 1229x^4 - 19470x^3 - 114297x^2 - 291684x - 280900.$$

If $x > 278$, then

$$(P_1(x) - 2y)(P_1(x) + 2y) < 0 < (P_2(x) - 2y)(P_2(x) + 2y).$$

If $P_2(x)-2y < 0$ and $P_2(x)+2y < 0$, then $P_1(x)-2y < -2$ and $P_1(x)+2y < -2$, which implies that $(P_1(x) - 2y)(P_1(x) + 2y) > 0$, a contradiction. If $P_2(x) - 2y > 0$ and $P_2(x) + 2y > 0$, then $P_1(x) - 2y > -2$ and $P_1(x) + 2y > -2$. It follows that

$$P_1(x) - 2y = -1 \text{ or } P_1(x) + 2y = -1.$$

Consider the case $x < -291$. Here we get that

$$(P_2(x) - 2y)(P_2(x) + 2y) < 0 < (P_1(x) - 2y)(P_1(x) + 2y).$$

If $P_1(x) - 2y > 0$ and $P_1(x) + 2y > 0$, then we have a contradiction. If $P_1(x) - 2y < 0$ and $P_1(x) + 2y < 0$, then $P_2(x) - 2y < 2$ and $P_2(x) + 2y < 2$, therefore

$$P_2(x) - 2y = 1 \text{ or } P_2(x) + 2y = 1.$$

Thus if we have a solution $(x, y) \in \mathbb{Z}^2$, then either $x \in I_9$ (provided in Table 2.) or $y = \pm(x^5 + 23x^4 + 189x^3 + 1331/2x^2 + 1819/2x + 529/2)$. We obtain only the trivial integral solutions $(x, y) = (-2, 0), (0, 0)$.

It remains to handle the cases $k = 6, 8, 10$. Observe that since in this case the degree of $f_k(x)$ is odd, the solutions to (1.1) with $x \le 0$ can be easily found. In fact, we get that all such solutions have $x = 0, -2$. So in what follows, without loss of generality we may assume that $x > 0$.

Consider the equation related to $k = 6$. We have

$$\begin{aligned} x &= d_1u_1^2, \\ x + 2 &= d_2u_2^2, \\ x^5 + 20x^4 + 151x^3 + 529x^2 + 833x + 437 &= d_3u_3^2, \end{aligned}$$

with some positive integers d_1, d_2, d_3. Checking the possible values of d_1, d_2, d_3, we get that

$$\begin{aligned} x &= 2^{\alpha_1}19^{\alpha_4}23^{\alpha_5}u_1^2, \\ x + 2 &= 2^{\alpha_1}3^{\alpha_2}11^{\alpha_3}u_2^2, \\ x^5 + 20x^4 + 151x^3 + 529x^2 + 833x + 437 &= 3^{\alpha_2}11^{\alpha_3}19^{\alpha_4}23^{\alpha_5}u_3^2, \end{aligned}$$

where $\alpha_i \in \{0, 1\}$ and $u_i \in \mathbb{Z}$. Working modulo 720 it follows that the above system of equations has solutions only if $(\alpha_2, \alpha_3, \alpha_4, \alpha_5) \in$

$$\begin{aligned} &\{(0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 0), \\ &(0, 1, 0, 1), (0, 1, 1, 1), (1, 0, 0, 0), (1, 0, 0, 1), \\ &(1, 0, 1, 1), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1)\}. \end{aligned}$$

We describe an argument which works for all cases except the one with $(\alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 0, 1)$. Combining the first two equations yields

$$(x+1)^2 - 3^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4} 23^{\alpha_5} (2^{\alpha_1} u_1 u_2)^2 = 1,$$

a Pell equation. Computing the fundamental solution of the Pell equation provides a formula for x. Substituting it into the equation

$$x^5 + 20x^4 + 151x^3 + 529x^2 + 833x + 437 = 3^{\alpha_2} 11^{\alpha_3} 19^{\alpha_4} 23^{\alpha_5} u_3^2$$

we get a contradiction modulo some positive integer m. The following table contains the possible tuples and the corresponding integer m.

$(\alpha_2, \alpha_3, \alpha_4, \alpha_5)$	m	$(\alpha_2, \alpha_3, \alpha_4, \alpha_5)$	m
$(0,0,1,0)$	11	$(0,0,1,1)$	13
$(0,1,0,0)$	13	$(0,1,0,1)$	29
$(0,1,1,1)$	37	$(1,0,0,0)$	5
$(1,0,0,1)$	11	$(1,0,1,1)$	29
$(1,1,0,1)$	13	$(1,1,1,0)$	29
$(1,1,1,1)$	43		

As an example we deal with $(\alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 1, 1, 1)$. The fundamental solution of the Pell equation is

$$208 - 3\sqrt{11 \cdot 19 \cdot 23}.$$

If there exists a solution, then

$$x = \frac{(208 - 3\sqrt{11 \cdot 19 \cdot 23})^k + (208 + 3\sqrt{11 \cdot 19 \cdot 23})^k}{2} - 1$$

for some $k \in \mathbb{N}$. If x satisfies the above equation, then

$$x^5 + 20x^4 + 151x^3 + 529x^2 + 833x + 437 \pmod{37} \in \{17, 20, 22, 29\}$$

and $11 \cdot 19 \cdot 23 u_3^2 \pmod{37} \in$

$$\{0, 1, 3, 4, 7, 9, 10, 11, 12, 16, 21, 25, 26, 27, 28, 30, 33, 34, 36\},$$

a contradiction. It remains to resolve the equation corresponding to the tuple $(\alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 0, 1)$. Here we have that

$$F(x) = x(x^5 + 20x^4 + 151x^3 + 529x^2 + 833x + 437) = (23u_1u_3)^2$$

a Diophantine equation satisfying Runge's condition. Define

$$\begin{aligned} P_1(x) &= 2x^3 + 20x^2 + 51x + 18, \\ P_2(x) &= 2x^3 + 20x^2 + 51x + 20. \end{aligned}$$

The two cubic polynomials

$$4F(x) - P_1(x)^2 = 4x^3 + 11x^2 - 88x - 324$$

and

$$4F(x) - P_2(x)^2 = -4x^3 - 69x^2 - 292x - 400$$

have opposite signs if $x \notin [-12, 5]$. The inequalities

$$\begin{aligned} P_1(x)^2 - 4y^2 < & \quad 0 \quad < P_2(x)^2 - 4y^2, \\ P_2(x)^2 - 4y^2 < & \quad 0 \quad < P_1(x)^2 - 4y^2 \end{aligned}$$

imply that if there exists a solution, then $y = x^3 + 10x^2 + \frac{51}{2}x + \frac{19}{2}$. The polynomial

$$(x+2)F(x) - \left(x^3 + 10x^2 + \frac{51}{2}x + \frac{19}{2}\right)^2$$

has no integral root. Thus it remains to check the cases $x \in [-12, 5]$. We obtain only the trivial solutions.

The above procedure also works in the cases $k = 8$ and 10. For $k = 8$ we get that

$$\begin{aligned} x &= 2^{\alpha_1} 23117^{\alpha_4} u_1^2, \\ x + 2 &= 2^{\alpha_1} 3^{\alpha_2} 97^{\alpha_3} u_2^2, \\ \frac{f_8(x)}{x(x+2)} &= 3^{\alpha_2} 97^{\alpha_3} 23117^{\alpha_4} u_3^2 \end{aligned}$$

for some $\alpha_i \in \{0, 1\}$ and $u_i \in \mathbb{Z}$, and in case of $k = 10$ we can write

$$\begin{aligned} x &= 2^{\alpha_1} 2018957^{\alpha_5} u_1^2, \\ x + 2 &= 2^{\alpha_1} 3^{\alpha_2} 11^{\alpha_3} 467^{\alpha_4} u_2^2, \\ \frac{f_{10}(x)}{x(x+2)} &= 3^{\alpha_2} 11^{\alpha_3} 467^{\alpha_4} 2018957^{\alpha_5} u_3^2 \end{aligned}$$

for some $\alpha_i \in \{0, 1\}$ and $u_i \in \mathbb{Z}$. After that, we exclude as many putative exponent tuples working modulo 720 as we can. The remaining exponent tuples are treated via Pell equations and congruence arguments. Everything worked in a similar way as previously. The largest modulus used to eliminate tuples is 37. $\square$

Remark. We note that the total running time of our calculations was only half an our on an Intel Core i5 2.6GHz PC. The most time consuming part was the computation of fundamental solutions of Pell equations and appropriate moduli to eliminate tuples. It took approximately twenty minutes.

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